



Title	On a local property of absolute neighbourhood retracts
Author(s)	Yajima, Takeshi
Citation	Osaka Mathematical Journal. 1950, 2(1), p. 59-62
Version Type	VoR
URL	<a href="https://doi.org/10.18910/11557">https://doi.org/10.18910/11557</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## ***On a Local Property of Absolute Neighbourhood Retracts***

By Takeshi YAJIMA

1. In this note we shall prove the following theorem and derive from it several corollaries.

**THEOREM.** *In order that a separable metric space  $Y$  is an absolute neighbourhood retract<sup>1)</sup> it is necessary and sufficient that  $Y$  is compact and has the following property (L):*

(L)  $\left\{ \begin{array}{l} \text{For each point } p \in Y \text{ and its neighbourhood } U \text{ there exists a} \\ \text{neighbourhood } V \subset U \text{ of } p \text{ such that every continuous mapping } f \text{ of} \\ \text{a closed subset } A \text{ of a given metric space } X \text{ into } V \text{ can be extended} \\ \text{over } X \text{ with respect to } U. \end{array} \right.$

This theorem is an extension of the C. Kuratowski's characterization<sup>2)</sup> of absolute neighbourhood  $n$ -retracts and gives a local property of absolute neighbourhood retracts.

2. First of all let us prove<sup>3)</sup> the necessity of the condition. Suppose  $Y$  is an absolute neighbourhood retract.  $Y$  may be considered<sup>4)</sup> as a neighbourhood retract of Hilbert parallelotope  $Q$ . Let  $r$  be the retraction. Then  $r^{-1}(U)$  is an open set of  $Q$  containing  $p$ . Let  $\varepsilon$  be a positive number and

$$K_\varepsilon = \bigcup_{x \in Q} \{x \mid \rho(x, p) < \varepsilon\}.$$

Take  $\varepsilon > 0$  sufficiently small such that  $\bar{K}_\varepsilon \subset r^{-1}(U)$  and put  $V = K_\varepsilon Y$ . Each mapping  $f \in V^A$  has an extension  $f_1 \in Q^{X^5)}$ . Let  $\pi$  be the projection of  $Q$  onto  $\bar{K}_\varepsilon$  such that

<sup>1)</sup> In the sense of K. Borsuk. See, K. Borsuk: Über eine Klasse von lokal zusammenhängenden Räumen. Fund. Math. 19 (1932), pp. 220-242.

<sup>2)</sup> C. Kuratowski: Sur les espaces localement connexes et péaniens de dimension  $n$ . Fund. Math. 24 (1935), p. 273, Théorème 1.

<sup>3)</sup> The proof of C. Kuratowski also holds in the case  $n = \infty$  without the assumption of compactness of  $Y$ . But if  $Y$  is compact we can prove it more simply as in the text.

<sup>4)</sup> K. Borsuk, loc. cit. p. 223, Section 3.

<sup>5)</sup> W. Hurewicz and H. Wallman: Dimension Theory, p. 82, Cor. 1.

$$\begin{aligned}\pi(x) &= \text{the intersection of the segment } \overline{px} \text{ with the boundary of } K_\varepsilon \\ &\quad \text{for } x \notin \overline{K}_\varepsilon, \\ &= x \quad \text{for } x \in \overline{K}_\varepsilon.\end{aligned}$$

The continuous mapping

$$f^*(x) = r \pi f_1(x)$$

maps  $X$  into  $U$  and we have  $f^*(x) = f(x)$  for  $x \in A$ . Hence  $f^* \in U^X$  is the required extension of  $f \in V^A$ .

3. To prove the sufficiency of the condition we need the following Definition and Lemma.

DEFINITION. Let  $U, V$  be two open sets of  $Y$  such that  $V \subset U$ . If for every continuous mapping  $f \in V^A$ , where  $A$  is an arbitrary closed subset of  $X$ , there exists an open set  $E$  containing  $A$  such that  $f$  has an extension  $f^* \in U^E$ , then we say that  $V$  is an *associated neighbourhood* of  $U$ .

LEMMA. Let  $V_1$  and  $V_2$  be associated neighbourhoods of  $U_1, U_2$  respectively and let  $W_2$  be an open set such that  $\overline{W}_2 \subset V_2$ . Then  $V_1 + W_2$  is an associated neighbourhood of  $U_1 + U_2$ .

Proof. Put  $A_1 = f^{-1}(V_1)$ ,  $A_2 = f^{-1}(W_2)$  and  $F_1 = A - A_2$ ,  $F_2 = A - A_1$ . Then  $F_1$  and  $F_2$  are mutually disjoint closed subsets of  $X$ . Therefore there exist a closed set  $F$  and two open sets  $G_1, G_2$  such that

$$X - F = G_1 + G_2, \quad G_1 G_2 = 0, \quad F_1 \subset G_1, \quad F_2 \subset G_2.$$

Let  $f_1$  be the partial mapping  $f|_{(A-G_2)}$ . Since  $f_1$  maps the closed subset  $(A-G_2)$  of  $X$  into  $V_1$ , there exists an open set  $E_1 \supset A-G_2$  such that  $f_1$  can be extended to  $f_1^* \in U_1^{E_1}$ . Take an open set  $E_0$  containing  $FA$  sufficiently small such that  $\overline{E}_0 \subset E_1$  and  $f_1^*(\overline{E}_0) \subset V_2$ . The existence of such an open set  $E_0$  is guaranteed since  $\overline{W}_2 \subset V_2$  and  $f_1^*(FA) = f(FA) \subset V_1 W_2$ . Consider the continuous mapping  $f_2$  defined as follows:

$$\begin{aligned}f_2(x) &= f_1^*(x) & \text{for } x \in \overline{E}_0 F, \\ &= f(x) & \text{for } x \in A - G_1 - F.\end{aligned}$$

Then  $f_2$  maps the closed subset  $(A-G_1) + \overline{E}_0 F$  into  $V_2$ . Therefore there exists an open set  $E_2 \subset \overline{E}_0 F + (A-G_1)$  such that  $f_2$  can be extended to  $f_2^* \in U_2^{E_2}$ .

Define the mapping  $f^*$  as follows:

$$\begin{aligned}f^*(x) &= f_1^*(x) & \text{for } x \in G_1 E_1 + F E_0, \\ &= f_2^*(x) & \text{for } x \in G_2 E_2.\end{aligned}$$

It is obvious that  $f^*$  is continuous over  $G_1 E_1 + F E_0 + G_2 E_2$ , and  $G_1 E_1 + F E_0 + G_2 E_2$  is an open subset of  $X$  containing  $A$ . Moreover if  $x \in A$  then  $f^*(x) = f(x)$ . Hence  $V_1 + W_2$  is an associated neighbourhood of  $U_1 + U_2$ .

4. Now we prove the sufficiency of the condition. Let  $p$  be an arbitrary point of  $Y$ . There exists by supposition an associated neighbourhood  $V_p$  of the open set  $Y$ . Let  $W_p$  be an open set such that  $\overline{W_p} \subset V_p$ . Since  $Y$  is compact, there exist a finite number of points  $p_1, p_2, \dots, p_k$  such that

$$Y \subset \sum_{i=1}^k W_{p_i}.$$

By virtue of Lemma  $V_{p_1} + W_{p_2}$  is an associated neighbourhood of  $Y + Y = Y$ ; in general  $V_{p_1} + W_{p_2} + \dots + W_{p_j}$  is an associated neighbourhood of  $Y$ . Consequently  $V_{p_1} + W_{p_2} + \dots + W_{p_k} = Y$  is an associated neighbourhood of  $Y$ .

5. From our Theorem follows directly the following

**COROLLARY.** *If every point of a compact metric space  $Y$  has a neighbourhood homeomorphic with an open set of an absolute neighbourhood retract, then  $Y$  is also an absolute neighbourhood retract.*

6. If  $X$  is a compact metric space, then its closed subset  $A$  is also compact and so is  $f(A)$ . Therefore  $f(A)$  may be covered with a finite number of  $W$ 's of section 4. Hence we have the following theorem by virtue of Lemma.

**THEOREM.** *Let  $X$  be a compact metric space and let  $Y$  be a separable metric space of the property (L). Then every continuous mapping  $f$  of a closed subset  $A$  of  $X$  into  $Y$  can be extended over some neighbourhood of  $A$  with respect to  $Y$ .*

This theorem and the remark of <sup>3)</sup> shows that under the restriction of the space  $X$  in the class of compact metric spaces, absolute neighbourhood retracts in the sense of C. Kuratowski<sup>6)</sup> and separable metric space of the property (L) are equivalent.

7. The local contractibility is characterized as follows.

*In order that a separable metric space  $Y$  is locally contractible it is necessary and sufficient that for every point  $p \in Y$  and for every neighbourhood  $U_p \ni p$  there exists an associated neighbourhood  $V_p \ni p$  of  $U_p$  of the following property:*

*Let  $X$  be an arbitrary metric space and let  $f_0, f_1$  be two continuous*

<sup>6)</sup> C. Kuratowski, loc. cit. p. 276, Remarques,

mappings which map  $X$  into  $V_p$ . Then the mapping  $\varphi(x, t)$  defined on the closed subset  $X \times (0) + X \times (1)$  of  $X \times \langle 0, 1 \rangle$  such that

$$\varphi(x, 0) = f_0(x), \quad \varphi(x, 1) = f_1(x)$$

can be extended to  $\varphi * \in U_p^{X \times \langle 0, 1 \rangle}$ .

By virtue of this characterization and Theorem of Section 1, the following well known theorem is an immediate consequence.

THEOREM. *If  $Y$  is an absolute neighbourhood retract then it is locally contractible (Borsuk)<sup>7)</sup>.*

(Received December 23 (1949))

---

<sup>7)</sup> K. Borsuk, loc. cit. p. 237, Section 27.