A FUNDAMENTAL DOMAIN FOR SOME QUASI-FUCHSIAN GROUPS

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1. Introduction

Let $A$ and $B$ be two loxodromic elements of $\text{SL}(2, \mathbb{C})$ having no common fixed point and subjected to the condition that $ABA^{-1}B^{-1}$ is parabolic. Writing $tr \ A = x$, $tr \ B = y$ and $tr \ AB = z$, one sees that the condition on the commutator of $A$ and $B$ reduces to $x^2 + y^2 + z^2 = xyz$. In [1] Keen observed the case $z > 2$, which is a special case considered there, and obtained a sufficient condition on traces $x$, $y$, $z$ for which the group $\langle A, B \rangle$ generated by $A$ and $B$ is Kleinian. Her idea is first to take a symmetric normalization of the fixed points of $A$ and $B$ and then to construct a symmetric fundamental polygon for $\langle A, B \rangle$. In this article we shall take another normalization and construct a fundamental domain to show the following.

Theorem. Let $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$, be loxodromic elements of $\text{SL}(2, \mathbb{C})$ such that $ABA^{-1}B^{-1}$ is parabolic and let $G = \langle A, B \rangle$. If, for each integer $n$, the inequality

\begin{equation}
\frac{|\alpha^n a| + |\beta^n d|}{|\alpha^n a + \beta^n d|} < \frac{|\alpha| + |\beta|}{|\alpha - \beta|}
\end{equation}

holds, then $G$ is quasi-Fuchsian and represents a pair of once punctured tori.

Restricting to the case $z > 2$, one might wish to compare (1) with the Keen condition (***) of Theorem 5.1 in [1]. This will be discussed elsewhere.

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Remark 1. Since $A$ is an element of $\text{SL}(2, \mathbb{C})$, $\beta = 1/\alpha$. Also since $A$ is loxodromic, $|\alpha| \neq 1$ and we may assume $|\alpha| > 1 > |\beta|$. Hence the left hand side of (1) tends to 1 as the absolute value of $n$ tends to $\infty$. If one examines (1) for $n = 0$ and it holds, then the right hand side of (1) must be greater than 1. Therefore, for given $A$ and $B$, we may examine (1) only for a finite number of integers $n$. 
**Remark 2.** Geometric meaning of (1) appears in the course of the proof of Theorem. It is given stepwise in Propositions 2 and 4 and in Section 3. In Propositions 2 and 4 we examine some restricted cases and use some simple conditions instead of (1).

2. Preliminary and special cases

Let \( A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \) and \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( bc \neq 0 \), be loxodromic elements of \( SL(2, \mathbb{C}) \) such that \( ABA^{-1}B^{-1} \) is parabolic. We first note that the parabolicity of \( ABA^{-1}B^{-1} \) and the inequalities (1) are invariant under the conjugations by \( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) and \( \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \), where \( i = \sqrt{-1} \) and \( k \) is a non-zero complex number. They are also invariant under the change of generators \( A \) and \( B \) to \( A \) and \( BA^n \). Now, conjugating \( A \) and \( B \) by \( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \), if necessary, and by \( \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \) with a suitable \( k \), we may assume and do hereafter that

\[
|\alpha| > 1 \quad \text{and} \quad c = \frac{\alpha + \beta}{\alpha - \beta}.
\]

The explicit form of the matrix \( ABA^{-1}B^{-1} \) is

\[
\begin{pmatrix}
(ad - \alpha^2 bc) & (\alpha^2 - 1) ab \\
(\beta^2 - 1) cc' & ad - \beta^2 bc
\end{pmatrix},
\]

so from the condition that (3) is parabolic (or from the equality \( x^2 + y^2 + z^2 = xyz \) with \( x = \alpha + \beta, y = a + d, z = \alpha a + \beta d \)) we obtain easily

\[
ad = \left(\frac{\alpha + \beta}{\alpha - \beta}\right)^2.
\]

Using (2) and (4) we have the following equalities:

\[
\begin{align*}
ad &= c^2, \\
bc &= c^2 - 1, \\
c/d &= a/c, \\
\alpha^2 &= (c+1)(c-1), \\
\alpha^2 - \beta^2 &= 4c/(c^2 - 1) \quad \text{and} \\
\beta^2 - 1 &= -2/(c+1).
\end{align*}
\]

Using these equalities, we see from (3) that the fixed point of \( ABA^{-1}B^{-1} \) is

\[-(\alpha^2 - \beta^2) bc/2(\beta^2 - 1) cd = (c+1)/d = a/c + 1/d.\]

We denote it by \( p_1 \). We put \( p_2 = A^{-1}(p_1), p_3 = B^{-1}A^{-1}(p_1) \) and \( p_4 = AB^{-1}A^{-1}(p_1) \). By the aid of (5) we calculate them and obtain

\[
\begin{align*}
p_1 &= a/c + 1/d, \\
p_2 &= a/c - 1/d, \\
p_3 &= -d/c + 1/a, \\
p_4 &= -d/c - 1/a.
\end{align*}
\]
Let \( C_1 = \{ z \mid \frac{z-a}{c} = 1/|d| \} \) and \( C_2 = \{ z \mid \frac{z+d}{c} = 1/|a| \} \). Note that the center of \( C_1 \) (resp. \( C_2 \)) coincides with that of the isometric circle of \( B^{-1} \) (resp. \( B \)) and that (6) implies that the segment connecting \( p_1 \) with \( p_2 \) (resp. \( p_3 \) with \( p_4 \)) is a diameter of \( C_1 \) (resp. \( C_2 \)). Denoting by \( D_i \) the inside of \( C_i \) \( (i = 1, 2) \), we have the following.

**Proposition 1.** \( B \) maps \( C_2 \) to \( C_1 \) such that \( B \) maps the exterior of \( D_2 \) to \( D_1 \).

Proof. Let \( l \) be the line passing through \( p_3 \) and \( p_4 \). Since \( B \) maps \( p_3, p_4 \) and \( \infty \) to \( p_2, p_1 \) and \( a/c \), respectively, \( B(l) \) is a line passing through \( p_2, p_1 \) and the center of \( C_1 \). Since \( C_2 \) and \( C_1 \) are orthogonal to \( l \) and \( B(l) \), respectively, \( B \) maps \( C_2 \) to \( C_1 \). Since the center of \( C_1 \) is \( B(\infty) \), \( B \) maps the exterior of \( D_2 \) onto \( D_1 \).

In case \( A \) is hyperbolic, a suitable position of \( C_1 \) and \( C_2 \) gives us some fundamental domain for \( G \), the picture of which is easy to draw. As a fundamental idea of this article we show the following.

**Proposition 2.** Let \( A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \) and \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( bc \neq 0 \), be hyperbolic and loxodromic elements of \( SL(2, \mathbb{C}) \), respectively, such that \( ABA^{-1}B^{-1} \) is parabolic and let \( G = \langle A, B \rangle \). If \( C_1 \) and \( C_2 \) are separated by the imaginary axis, then \( G \) is quasi-Fuchsian and represents a pair of once punctured tori.

Proof. By Proposition 1 we see that the assumption on \( C_1 \) and \( C_2 \) implies that the exterior of \( D_1 \cup D_2 \) is a fundamental domain for \( \langle A \rangle \). We shall construct a fundamental domain \( R \) for \( \langle A \rangle \) such that \( R \supset D_1 \cup D_2 \). Since \( A \) is hyperbolic, the action of \( A \) is a stretching from the origin so that \( p_1 \) and \( p_2 \) (resp. \( p_3 \) and \( p_4 \)) lie on a ray emanating from the origin. Let \( H_i \) be one of the half planes \( \{ z \mid \text{Re} \ z > 0 \} \) and \( \{ z \mid \text{Re} \ z < 0 \} \) containing \( D_i \) \( (i = 1, 2) \). We put

\[
R_1 = \{ z \mid |p_2| < |z| < |p_1| \} \cap H_1 \quad \text{and} \quad R_2 = \{ z \mid |p_3| < |z| < |p_4| \} \cap H_2 .
\]

Then \( R_i \supset D_i \ (i = 1, 2) \). If the set \( \overline{R}_1 \cap \overline{R}_2 \) consists of two closed intervals on the imaginary axis, then we put

\[
R = R_1 \cup R_2 \cup \text{Int}(\overline{R}_1 \cap \overline{R}_2) ,
\]

where \( \overline{R}_i \) denotes the closure of \( R_i \) and \( \text{Int}(\overline{R}_1 \cap \overline{R}_2) \) denotes two open intervals lying on the imaginary axis. If \( \overline{R}_1 \cap \overline{R}_2 \) consists of at most two points, then we change the generators for \( G \) so that we are in the same situation just above. This is done simply by changing \( B \) to \( BA^n \) with a suitable integer \( n \). Since the ring \( \{ z \mid |p_2| \leq |z| < |p_1| \} \) is a fundamental set for \( \langle A \rangle \), there is an integer \( n \) such that \( |p_2| \leq |A^{-n}(p_3)| < |p_1| \). For this \( n \) we see that \( \overline{R}_1 \cap A^{-n}(\overline{R}_2) \) contains more than two points. Changing \( B \) by \( BA^n \) and conjugating it with a
suitable matrix of the form \( \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \), we are in the same situation cited above. In any case we have a fundamental domain \( R \) for \( \langle A \rangle \) such that \( R \supseteq D_1 \cup D_2 \). Let \( J_i \) be the closed boundary curve of \( R \) passing through \( p_i \) \((i = 3, 4)\). Then putting \( J_i = C_i \) \((i = 1, 2)\), the proof of the proposition is completed by the following.

**Proposition 3.** Let \( J_1 \) and \( J_2 \) be disjoint circles passing through \( p_1, p_2 \) and \( p_3, p_4 \), respectively, such that \( B(J_2) = J_1 \) and that \( J_1 \) and \( J_2 \) bound a fundamental domain for \( \langle B \rangle \), where \( p_i \) are the points in (6). Assume that there are disjoint Jordan curves \( J_3 \) and \( J_4 \) passing through \( p_2 \) and \( p_3 \) and \( p_4 \), respectively, such that \( A(J_3) = J_4 \), \( J_1 \cap J_3 = \{ p_3 \} \), \( J_1 \cap J_4 = \{ p_4 \} \), \( J_2 \cap J_3 = \{ p_2 \} \), \( J_2 \cap J_4 = \{ p_4 \} \) and that \( J_3 \) and \( J_4 \) bound a fundamental domain for \( \langle A \rangle \). Then \( G \) is quasi-Fuchsian and represents a pair of once punctured tori.

Our proof of Proposition 3 is somewhat long, so it is given in Section 4.

Next, assuming \( A \) is hyperbolic, we treat the case where \( C_1 \) and \( C_2 \) are not necessarily separated by the imaginary axis. As is shown in the proof of Proposition 2, we may assume and do hereafter that the center of the isometric circle of \( B \) lies in a ring \( \{ z \mid |z_2| \lesssim |z| < |A(p_2)| \} \).

**Proposition 4.** Let \( A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \) and \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), be hyperbolic and loxodromic elements of \( \text{SL}(2, \mathbb{C}) \), respectively, such that \( ABA^{-1}B^{-1} \) is parabolic and let \( G = \langle A, B \rangle \). Assume that \( |p_2| \leq \frac{-d}{c} < |p_1| \), where \( p_1 \) and \( p_2 \) are in (6). If the inequality

\[
\frac{|a| + |d|}{|a+d|} < \frac{\alpha + \beta}{\alpha - \beta}
\]

holds, then \( G \) is quasi-Fuchsian and represents a pair of once punctured tori.

Proof. First we show that (7) means that two discs \( D_1 \) and \( D_2 \) lie in a positive distance. The centers and the radii of \( D_1 \) and \( D_2 \) are \( a/c, -d/c \) and \( 1/|d|, 1/|a| \), respectively. Hence the condition that \( D_1 \) and \( D_2 \) lie in a positive distance is equivalent to

\[
1/|a| + 1/|d| < |(a+d)/c| .
\]

By (2) and (4) we see easily that this inequality is equivalent to (7). If \( |p_3| = |p_2| \), then by (5) and (6) we have \( |a| = |d| = c \). Hence we see that the circles \( C_1 \) and \( C_2 \) are symmetric with respect to the imaginary axis so that we are in a situation of Proposition 2. So assume \( |p_3| < |p_2| \). Draw a circle \( C = \{ z \mid |z| = |p_2| \} \). Then \( C \) intersects \( C_2 \) at two points. Denoting by \( D \) the inside of \( C \), we put

\[
J = (C \cap D_2) \cup (C_2 \cap D) .
\]
Then the ring domain bounded by \( f \) and \( A(f) \) is a fundamental domain for \( \langle A \rangle \) and contains \( D_1 \) and \( D_2 \) in it. Since \( f \cap C_2 = \{ p_3 \} \), we need a slight deformation of a part of \( f \) lying on \( C_2 \) to apply Proposition 3. We put \( C_f^\varepsilon = \{ z \mid |z+(1+\varepsilon)/|a|\} \) with \( a \) so small positive number \( \varepsilon \) that \( C_f^\varepsilon \cap C_1 = \emptyset \). This is possible, for we have shown that the condition (7) is equivalent to \( C_1 \cap C_2 = \emptyset \). We put

\[
J_3 = (C \cap (D_f^\varepsilon)) \cup (C_f \cap D),
\]
where \( D_f^\varepsilon \) denotes the inside of \( C_f^\varepsilon \). Now it is easy to see that the curves \( J_1 = C_1, J_2 = C_2, J_3 \) and \( J_4 = A(J_3) \) satisfy the conditions of Proposition 3. Thus we have Proposition 4.

**Remark 3.** Let \( A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \) and \( B = \begin{pmatrix} (5/6) e^{i\theta} & 16/15 \\ 5/3 & (10/3) e^{-i\theta} \end{pmatrix} \), \( 0 < |\theta| < \pi/2 \).

Since the square of the absolute value of the trace of \( B \) is \( 225/36 + (100/9) \cos^2 \theta \), \( B \) is loxodromic. We see by (2) and (4) that \( ABA^{-1}B^{-1} \) is parabolic. Since \( \cos \theta \neq 0 \), the square of the left hand side of (7) is smaller than 25/9. The right hand side of (7) is 5/3 so that (7) holds. Thus we have seen that \( A \) and \( B \) satisfy the assumptions of Proposition 4 except for \( |p_4| \leq |d/c| < |p_1| \). Since \( AB = \begin{pmatrix} (5/3) e^{i\theta} & 32/15 \\ 5/6 & (5/3) e^{-i\theta} \end{pmatrix} \), the trace of \( AB \) is \( (10/3) \cos \theta \). Hence we can find a \( \theta \) such that \( AB \) is elliptic of the infinite order. This implies that \( \langle A, B \rangle \) is not necessarily Kleinian. Hence the assumption on the absolute value of \( -d/c \) can not be dropped.

### 3. Proof of Theorem

Finally we shall go into the case where \( A \) and \( B \) are not necessarily hyperbolic. Recall that when \( A \) is hyperbolic we are in a good situation in which the line tangent to \( C_1 \) at \( p_2 \) (resp. \( C_2 \) at \( p_3 \)) is mapped by \( A \) to the line tangent to \( C_1 \) at \( p_4 \) (resp. \( C_2 \) at \( p_4 \)), which enables us easily to draw fundamental domains for \( \langle A \rangle \) and \( \langle B \rangle \) by which the condition of the Klein combination theorem is satisfied. We are now not in the same situation. That is, for a non-hyperbolic \( A \), the line tangent to \( C_1 \) at \( p_2 \) is mapped by \( A \) to a line not tangent to \( C_1 \) at \( p_1 \). So we shall introduce new circles \( C'_1 \) and \( C'_2 \) such that the exterior of both \( D'_1 \) and \( D'_2 \) is a fundamental domain for \( \langle B \rangle \) and that the line tangent to \( C'_1 \) at \( p_4 \) (resp. \( C'_2 \) at \( p_4 \)) is mapped by \( A \) to the line tangent to \( C'_1 \) at \( p_4 \) (resp. \( C'_2 \) at \( p_4 \)), where \( D'_i \) denotes the disc bounded by \( C'_i \) \((i=1,2)\). Then we will be in a similar situation as before.

Let \( \theta \) be the argument of \( \alpha \) lying in the interval \((-\pi/2, \pi/2)\). The left hand side of (1) is not smaller than 1, so the right hand side of (1) is greater than 1. This implies that \( \theta \in (-\pi/2, \pi/2) \). The action of \( A \) is a stretching from the
origin of magnitude \(|\alpha|^2\) followed by a rotation about the origin of the angle \(2\theta\). We define the new circles as follows:

\[
\begin{align*}
C_1' &= \{z \mid |z-a|/|d| = 1/|a| \cos \theta\} \quad \text{and} \\
C_2' &= \{z \mid |z+d|/|a| = 1/|a| \cos \theta\},
\end{align*}
\]

where \(i = \sqrt{-1}\). Note that by (2) and (4) we see that the centers of \(C_1'\) and \(C_2'\) are respectively

\[
\begin{align*}
\text{and}
\end{align*}
\]

Since the vector from \(p_1\) to the center of \(C_1'\) is \(-d(a-\beta) \cos \theta / \beta\), the line \(l_1\) tangent to \(C_1'\) at \(p_1\) has the form with a real parameter \(s\)

\[
l_1: p_1 + is/\beta|\cos \theta.
\]

Likewise we see that the line \(l_2\) tangent to \(C_1'\) at \(p_3\) has the form with a real parameter \(t\)

\[
l_2: p_2 + it/\beta|\cos \theta.
\]

Since \(A(z) = \alpha^2 z\) and \(A(p_2) = p_1\), we have \(A(l_2) = l_1\). We can also see that the line tangent to \(C_1'\) at \(p_3\) is mapped by \(A\) to the line tangent to \(C_1'\) at \(p_1\). Hence we have our first requirement. Another requirement is that the exterior of both \(D_1'\) and \(D_2'\) is a fundamental domain for \(\langle B \rangle\). To see this we may only to show that \(B(C_2') = C_1'\), because the center of the isometric circle of \(B\) is the mid point of \(p_3\) and \(p_4\) so that it lies in \(D_2'\) and is mapped by \(B\) to \(\infty\). Since \(B(p_3) = p_2\) and \(B(p_4) = p_1\), it suffices to show that a point on \(C_2'\) \(\{p_3, p_4\}\) is mapped by \(B\) to a point on \(C_1'\). From (8) we see that the points \(q = d(c + i(\tan \theta + 1/\cos \theta))/\beta\) and \(q = a(c - i(\tan \theta - 1/\cos \theta))/\beta\) lie on \(C_1'\) and on \(C_2'\), respectively. A direct calculation shows \(B(q) = q'\) and we have our another requirement.

Now we have two chains of circles \(\{A^n(C_1')\}\) and \(\{A^n(C_2')\}\) such that, for each integer \(n\), \(A^n(C_1')\) and \(A^{n+1}(C_1')\) are tangent at \(A^n(p_1)\) if \(i = 1\) and at \(A^n(p_4)\) if \(i = 2\) and that they tend to 0 and \(\infty\) as \(n\) tends to \(-\infty\) and \(\infty\), respectively. We show that the inequalities (1) imply that two chains of circles \(\{A^n(C_1')\}\) are disjoint. Since \(\{A^n(C_1')\}\) are invariant under \(A\), it suffices to show that \(C_1'\) and \(\{A^n(C_2')\}\) are disjoint. From (8) and (9) we see that the center and the radius of \(A^n(C_2')\) are respectively

\[
\begin{align*}
-\alpha^2(|\alpha| + |\beta|)/\beta|\cos \theta \quad \text{and} \\
|\alpha|^{2n}/|a| \cos \theta.
\end{align*}
\]

By (9) and (10), the distance between the centers of \(C_1'\) and \(A^n(C_2')\) is

\[
(((|\alpha| + |\beta|)/|\alpha - \beta| \cos \theta)\{|d + \alpha^2|/a|}
\]

and by (8) and (10) the sum of the radii of \(C_1'\) and \(A^n(C_2')\) is
Hence the condition that $C'$ and $A^n(C')$ are disjoint becomes

$$\frac{1}{|d|} + |\alpha^{2n}/a| < \left(\frac{|\alpha| + |\beta|}{|\alpha - \beta|}\right)\frac{1}{|d| + \alpha^{2n}/a},$$

which is clearly equivalent to (1).

Taking $f_i = C'_i$ ($i = 1, 2$), we shall draw a simple closed curve $f_3$ satisfying the condition of Proposition 3. As we noted after Proposition 3, we may assume that $|p_2| \leq |d/c| < |p_1|$. Let $c = \{z | |z| = |p_2|\}$ and denote by $D$ the disc enclosed by $C$. We put

$$J = (C \cap (\cup \cup A^n(D_i \cup D_i')) \cup \cup (*) \cup \cup (**)),$$

(11)

$$(*) = \cup \cup A^n(C'_1 \cup C'_2) \cap D$$

and

$$(**) = \cup \cup A^n(C'_1 \cup C'_2) \cap D'.$$

That is, $J$ is a Jordan curve separating $\{A^n(D_i)\}_{n = 0}^{n =\infty}$ from $\{A^n(D'_i)\}_{n = \infty - 1}^{\infty - 1}$. A slight deformation of parts of $J$ which lie on $\{A^n(C'_i)\}$ will give us $f_3$. We need a finite number of replacements as follows:

Let $\varepsilon$ be a positive number. Let $m$ be the maximum number of $n$ such that $C \cap A^n(C'_1 \cup C'_2)$ is not null. If $m > 0$, then we put

$$C'_{i,m,m} = \{z | |z - q_i| = r_i + \varepsilon n/m\}$$

and replace

$$A^n(C'_i)$$

by $A^n(C'_{i,m,m})$ ($n = 1, \ldots, m$),

where $q_i$ and $r_i$ denote the center and the radius of $C'_i$, respectively. Replace

$$C'_i$$

by $C'_{i}^{*} = \{z | |z - (q_i + \delta(q_i - p_{i+1}))| = (1 + \delta) r_i\}$,

where $\delta = \varepsilon/2mr_i$. Let $m_i$ be the minimum number of $n$ such that $C \cap A^n(C'_i)$ is not null. We put

$$C'_{i,m_i,m_i} = \{z | |z - q_i| = r_i + \varepsilon n/m_i\}.$$
and replace
\begin{equation}
A^{-1}(C_i') \text{ by } A^{-1}(C_i^{**}),
\end{equation}
where \( \delta_i = \varepsilon/2|m_i| r_i \). In the case where \(|p_3| = |p_2|\), we need a modification. If so, then we put
\begin{equation}
C_i^{**} = \{x \mid x-(q_3+\delta_4(q_3-p_4)) = (1+\delta_2) r_2\}
\end{equation}
and replace
\begin{equation}
A^{-1}(C_i') \text{ by } A^{-1}(C_i^{**}),
\end{equation}
where \( \delta_2 = \varepsilon/2|m_2| r_2 \). The result of these replacements of \( A^n(C_i') \) and the corresponding \( A^n(D_i') \) in (11) denote we by \( J_3 \).

Now, we show that, for a small \( \varepsilon \), \( J_1, J_2, J_3 \) and \( J_4 = A(J_3) \) satisfy the conditions of Proposition 3. It is already shown that \( J_1 \) and \( J_2 \) are disjoint circles passing through \( p_1, p_2 \) and \( p_3, p_4 \), respectively, such that \( B(J_3) = J_1 \) and that \( J_1 \) and \( J_2 \) bound a fundamental domain for \( \langle B \rangle \). Since the replacements above are the deformations of a finite number of circular arcs on \( J \) in small neighborhoods of them and do not move \( p_2 \) and \( p_3 \), we see that, for a small \( \varepsilon’, J_3 \) is a Jordan curve passing through \( p_2 \) and \( p_3 \). By (11), (13), (16) and (17) we see that, except for one point \( p_{i+1} \), \( C_i’ \) and \( A^{-i}(C_i’) \) lie outside and inside of \( J_3 \), respectively, \( (i=1, 2) \) so that \( J_1 \cap J_3 = \{p_2\}, J_1 \cap J_4 = \{p_3\}, J_2 \cap J_3 = \{p_3\} \) and \( J_2 \cap J_4 = \{p_3\} \).

To see that \( J_3 \) and \( J_4 \) are disjoint and that they bound a fundamental domain for \( \langle A \rangle \) it suffices to show the former, because this and two facts that \( J_3 \) separates the fixed points of \( A \) and \( J_4 = A(J_3) \) imply the latter. Assume that \( J_3 \) and \( J_4 \) are not disjoint and let \( p \) be a point lying on both \( J_3 \) and \( J_4 \). This implies that both \( p \) and \( A^{-1}(p) \) lie on \( J_3 \).

Assume that \(|p| > |p_2|\). Then by (11), (14), (15), (16) and (17) we see that \( p \) lies on one of the circles \( A^{-1}(C_i^{**}), A^n(C_i’_{m_i,n}) \) and \( A^{-1}(C_i^{**}) \). Assume that \( p \) lies on \( A^{-1}(C_i^{**}) \). There are two cases. The first case is that \( m_i < -1 \). It is easy to see that \( C_i^{**} \) is contained in the inside of \( C_i’_{m_i-2} \) so that \( A^{-2}(C_i^{**}) \) is contained in the inside of \( A^{-2}(C_i’_{m_i-2}) \). Since \( J_3 \) does not pass through the inside of \( A^{-2}(C_i’_{m_i-2}) \), we see that this case does not occur. The second case is that \( m_i = -1 \). Then \( A^{-2}(C_i’) \) is disjoint to \( C \). Hence the distance between \( J_2 \) and \( A^{-2}(C_i’) \) is positive. Since \( A^{-2}(C_i^{**}) \) lies in a small neighborhood of \( A^{-2}(C_i’) \), we see that, for a small \( \varepsilon \), \( A^{-2}(C_i^{**}) \) does not meet \( J_3 \). Hence this case does not occur, too. Assume that \( p \) lies on \( A^n(C_i’_{m_i,n}) \). There are two cases. The first case is that \( m_i < n \). Then \( A^{-1}(p) \) lies on \( A^{-n}(C_i’_{m_i,n}) \). Since \( C_i’_{m_i,n} \) is a circle concentric with \( C_i’_{m_i,n-1} \) and the radius of the former is smaller than that of the latter, we see that \( A^{-1}(C_i’_{m_i,n}) \) is contained in the inside of \( A^{-n}(C_i’_{m_i,n-1}) \). Since \( J_3 \) does not pass through the inside of \( A^{-n}(C_i’_{m_i,n-1}) \), we see that this case does not occur. The second case is that \( n = m_i \). Then \( A^{-n}(C_i’) \) is disjoint to
C and \( A^*(C'_m,n) \) lies near \( A^*(C'_1) \). By the same argument in the case where \( p \) lies on \( A^{-1}(C'_m) \) and \( m_i=-1 \) we see that this case does not occur, too. The rested two cases where \( p \) lies on \( A^*(C'_m,n) \) and on \( A^*(C'_m) \) are treated similarly with a remark on the minor change (17) and we see that those two cases do not occur. Hence we conclude that \( |p| \leq |p_2| \).

Assume that \( |p| \leq |p_2| \). By (11), (12) and (13) we see that \( A^{-1}(p) \) lies on one of the circles \( C^*_1 \) and \( A^*(C'_m,n) \). Assume that \( A^{-1}(p) \) lies on \( C^*_1 \). Then \( p \) lies on \( A(C^*_1) \). There are two cases. The first case is that \( m>1 \). It is easy to see that \( C^*_1 \) is contained in the inside of \( C'_m,1 \) so that \( A(C^*_1) \) is contained in the inside of \( A(C'_m,1) \). Since \( J_3 \) does not pass through the inside of \( A(C'_m,1) \), we see that this case does not occur. The second case is that \( m=1 \). Then \( A(C'_1) \) is disjoint to \( A(C^*_1) \). Hence the distance between \( J_3 \) and \( A(C'_1) \) is positive. Since \( A(C^*_1) \) lies in a small neighborhood of \( A(C'_1) \), we see that, for a small \( \varepsilon \), \( A(C^*_1) \) does not meet \( J_3 \). Hence this case does not occur, too. Assume that \( A^{-1}(p) \) lies on \( A^*(C'_m,n) \). There are two cases. The first case is that \( n<m \). Then \( p \) lies on \( A^{*+1}(C'_m,n) \). Since \( C'_m,n \) is a circle concentric with \( C'_m,n+1 \) and the radius of the former is smaller than that of the latter, we see that \( A^{*+1}(C'_m,n+1) \) does not pass through the inside of \( A^{*+1}(C'_m,n+1) \), we see that this case does not occur. The second case is that \( n=m \). That this case does not occur is shown similarly by the method used above.

Thus we have shown that \( J_3 \) and \( J_4 \) are disjoint. Therefore we have checked all the conditions of Proposition 3. Hence Proposition 3 completes the proof of Theorem.

4. Proof of Proposition 3

Let \( D_1 \) and \( D_2 \) be the fundamental domains for \( \langle A \rangle \) and \( \langle B \rangle \) bounded by \( J_3, J_4 \) and by \( J_1, J_2 \), respectively. Let \( Q_1 \) (esp. \( Q_2 \)) be the component of \( D_1 \cap D_2 \) on whose boundary four points \( p_1, p_2, p_3 \) and \( p_4 \) lie clockwise (resp. counterclockwise). The Klein combination theorem (see, for example, page 139 of [3]) tells us that \( Q_1 \) and \( Q_2 \) are contained in the region of discontinuity \( \Omega(G) \) of \( G \) and no two points of \( Q_1 \cup Q_2 \) are equivalent under \( G \). To see \( G \) is quasi-Fuchsian it suffices to show that the component \( \Omega_i \) of \( G \) which contains \( Q_i \) is an invariant one \((i=1,2)\) and that \( \Omega_1 \) and \( \Omega_2 \) are disjoint (for a proof of this fact see [2]).

First we show that \( \Omega_1 \) is an invariant component of \( G \). Denote by \( a \) the open arc \( J_3 \cap \partial Q_1 \setminus \{p_3, p_4\} \) and let \( a'=A(a) \), where \( \partial Q_1 \) denotes the boundary of \( Q_1 \). Then we see that \( a'=J_1 \cap \partial Q_1 \setminus \{p_1, p_2\} \). Also denote by \( b \) the open arc \( J_2 \cap \partial Q_1 \setminus \{p_3, p_4\} \) and we see that \( b'=J_2 \cap \partial Q_1 \setminus \{p_1, p_2\} \), where \( b'=B(b) \).

**Lemma 1.** Four arcs \( a, a', b \) and \( b' \) on \( \partial Q_1 \) are contained in \( \Omega(G) \).

**Proof.** If there is a limit point \( p \) on one of the four arcs, say \( a \), then there
is a fixed point of a loxodromic element of \( G \) in any neighborhood of \( p \). Since \( p \) is an interior point of \( Q_1 \cup a \cup A^{-1}(Q_1) \), this implies that there is a loxodromic element \( g \) of \( G \) with the attractive fixed point on \( a \). Since \( g(Q_1) \cap Q_1 = \emptyset \), we see that \( g(Q_1) \cap A^{-1}(Q_1) \neq \emptyset \). Hence \( Ag(Q_1) \cap Q_1 \neq \emptyset \). Since \( Ag \) is not the identity, this contradicts the fact that no two points of \( Q_1 \) are equivalent. Thus we have Lemma 1.

By Lemma 1 we see that \( A(Q_1), A^{-1}(Q_1), B(Q_1) \) and \( B^{-1}(Q_1) \) which are adjacent \( Q_1 \) along \( a', a, b' \) and \( b \), respectively, are contained in \( \Omega_1 \). Writing each element of \( G \) by a word in generators \( A \) and \( B \) and appealing to the induction for lengths of words, we see easily that, for each \( g \) of \( G \), \( g(Q_1) \) is contained in \( \Omega_1 \) so that \( \Omega_1 \) is an invariant component of \( G \). Likewise we see \( \Omega_2 \) is so, too.

It remains to show that \( \Omega_1 \) and \( \Omega_2 \) are disjoint. In order to show this we need several lemmas. We shall consider the \( G \)-orbit of the circle \( J_1 \). It is easy to show the following lemma.

**Lemma 2.** The set of circles \( \{A^n(J_1) \mid i=1, 2; n \in \mathbb{Z} \} \) has the following properties:

i) \( A^n(J_1) \) and \( A^{n+1}(J_1) \) are tangent at \( A^n(p_1) \),

ii) \( A^n(J_2) \) and \( A^{n+1}(J_2) \) are tangent at \( A^n(p_1) \),

iii) \( A^n(J_1) \) and \( A^m(J_1) \) are disjoint whenever \( |n-m| > 1 \),

iv) \( \lim_{n \to \infty} A^n(J_1) = \infty \).

v) \( \lim_{n \to -\infty} A^n(J_1) = 0 \) and

vi) for any \( n \) and \( m \), \( A^n(J_1) \) and \( A^m(J_2) \) are disjoint.

We denote by \( K_1 \) the set of circles \( \{A^n(J_1) \mid i=1, 2; n \in \mathbb{Z} \} \).

**Lemma 3.** The set \( K_1 \cup \{0, \infty\} \) separates \( Q_1 \) from \( Q_2 \).

Proof. By Lemma 2 we see that \( \{A^n(J_1)\} \) constitutes a chain of circles which are tangent successively in the sense of i) and iii) of Lemma 2 and that its ends tend toward \( 0 \) and \( \infty \). This is also true for \( \{A^n(J_2)\} \). These and vi) of Lemma 2 imply the lemma.

Now consider the set of circles \( \{B(c) \mid c \in K_1 \setminus \{J_2\}\} \) (resp. \( \{B^{-1}(c) \mid c \in K_1 \setminus \{J_1\}\} \). There are two circles in this set one of which is tangent to \( J_1 \) at \( p_1 \) (resp. \( J_2 \) at \( p_3 \)) and the other at \( p_2 \) (resp. \( p_4 \)). Except for these two, each circle of the set is contained in the inside of \( J_1 \) (resp. \( J_2 \)). Let \( K_2 \) be the set of circles

\[ \{A^nB(c) \mid c \in K_1 \setminus \{J_2\}; n \in \mathbb{Z} \} \cup \{A^nB^{-1}(c) \mid c \in K_1 \setminus \{J_1\}; n \in \mathbb{Z} \} \]

It is easy to see that

\[ K_2 = (\{A^nB(c) \mid c \in K_1; n \in \mathbb{Z} \} \cup \{A^nB^{-1}(c) \mid c \in K_1; n \in \mathbb{Z} \}) \setminus K_1 \]
It is easy to see that Lemma 3 holds when we change $K_1$ by $K_2$. Inductively we put

$$K_i = \left((\ast) \cup (\ast\ast)\right) \setminus \left(K_1 \cup \cdots \cup K_{i-1}\right),$$

$$\ast = \{A^nB(c) \mid c \in K_{i-1}; \ n \in \mathbb{Z}\} \quad \text{and}$$

$$\ast\ast = \{A^nB^{-1}(c) \mid c \in K_{i-1}; \ n \in \mathbb{Z}\}$$

and have the following.

**Lemma 4.** The set $K_1 \cup \{0, \infty\}$ separates $Q_1$ from $Q_2$.

**Lemma 5.** Let $\{k_i\}$ be a sequence of circles such that $k_i \in K_i$ and that, for each $i$, $k_i$ is contained in the closed disc bounded by $k_{i-1}$. Then the sequence of the diameters of $k_i$'s converges to 0.

The proof of Lemma 5 is given later. Now we show that $\Omega_1$ and $\Omega_2$ are disjoint. Assume contrarily that $\Omega_1 = \Omega_2$. Then there is an arc $J$ lying in $\Omega(G)$ and connecting a point of $Q_1$ to a point of $Q_2$. By Lemmas 3 and 4 we see that, for each positive integer $i$, there is a circle $k_i$ of $K_i$ which intersects $J$. Since each circle of $K_i$ is contained in the closed disc bounded by a circle of $K_{i-1}$, we may assume that $\{k_i\}$ satisfies the condition of Lemma 5. On the other hand, there is a positive number $r$ such that each open disc centered at a point of $J$ with radius $r$ is contained in $\Omega(G)$. By Lemma 5 we see that there is a circle $k_i$ which is contained entirely in $\Omega(G)$. But $k_i$ is an element of $G$-orbit of $J_i$ and there is a fixed point $p_1$ of a parabolic element $ABA^{-1}B^{-1}$ on $J_1$, we have a contradiction. Therefore $\Omega_1$ and $\Omega_2$ are distinct.

Before going to prove Lemma 5 we need two more lemmas.

**Lemma 6.** For each positive integer $n$, $(ABA^{-1}B^{-1})^n(J_1)$, $(A^{-1}BAB^{-1})^n(J_1)$, $(A^{-1}B^{-1}AB)^n(J_2)$ and $(AB^{-1}A^{-1}B)^n(J_2)$ are circles of $K_{2n}$ passing through $p_1$, $p_2$, $p_3$ and $p_4$, respectively. Moreover, they are disjoint to both inside of $J_1$ and $J_2$.

Proof. We show the lemma for $ABA^{-1}B^{-1}$. The other cases are shown similarly. Since $ABA^{-1}B^{-1}(J_1) = ABA^{-1}(J_2)$ and $A^{-1}(J_2)$ is an element of $K_1$ different from $J_2$, we see that $ABA^{-1}B^{-1}(J_1)$ is an element of $K_2$ lying in the closed disc bounded by $A(J_1)$. The fixed point of $ABA^{-1}B^{-1}$ is $p_1$, and it is a point on $J_1$, so $ABA^{-1}B^{-1}(J_1)$ passes through $p_1$. This describes completely the action of the parabolic transformation $ABA^{-1}B^{-1}$, so we have Lemma 6.

**Lemma 7.** There exists a positive constant $M$ such that, for any two disjoint circles $k$ and $k'$ of $K \cup K_1$, the module of the ring domain bounded by $k$ and $k'$ is not smaller than $M$.

Proof. We denote by $R(k, k')$ the ring domain bounded by $k$ and $k'$. We recall that the modules are conformally invariant and satisfy the comparison
principle as follows: if \( R(k, k') \) is contained in \( R(k'', k''') \) and if \( k \) separates the complementary components of \( R(k'', k''') \), then the module of \( R(k, k') \) is not greater than that of \( R(k'', k''') \). Since each element of \( K \) is equivalent to \( J_{f_1} \), we may assume that \( k' = J_{f_1} \). There are three cases.

Case 1. \( k \) is an element of \( K \setminus \{ J_{f_1}, A(J_{f_1}), A^{-1}(J_{f_1}) \} \) or lies in the closed disc bounded by it.

Case 2. \( k \) is contained in one of the closed discs bounded by \( A(J_{f_1}) \) and \( A^{-1}(J_{f_1}) \).

Case 3. \( k \) is contained in the inside of \( J_{f_1} \).

In Case 1, by the comparison principle of the modules we may assume that \( k \) is an element of \( K \setminus \{ J_{f_1}, A(J_{f_1}), A^{-1}(J_{f_1}) \} \). By iii)~vi) of Lemma 2, we see that there is a positive number \( d \) such that no element of \( K \setminus \{ J_{f_1}, A(J_{f_1}), A^{-1}(J_{f_1}) \} \) meets the open disc concentric with \( J_{f_1} \) and having the radius \( r + d \), where \( r \) is the radius of \( J_{f_1} \). Also by the comparison principle we see that the module of \( R(k, J_{f_1}) \) is not smaller than \( \log \frac{r + d}{r} \). We denote this number by \( M_1 \).

Case 2. Let \( k \) be contained in the closed disc bounded by \( A(J_{f_1}) \). Another case with respect to \( A^{-1}(J_{f_1}) \) is treated similarly. It is easy to see that \( ABA(J_{f_2}) \) and \( ABA^{-1}(J_{f_2}) \) are the elements of \( K_2 \) which are contained in the closed disc bounded by \( A(J_{f_1}) \) and tangent to \( A(J_{f_1}) \) at \( p_1 \), respectively, and that all other elements of \( K_2 \) lying inside of \( A(J_{f_1}) \) are \( \{ ABA(\xi) \mid \xi \in K_1 \setminus \{ ABA(J_{f_2}), ABA^{-1}(J_{f_2}), A(J_{f_1}) \} \}. \) If \( k \) lies in the closed disc bounded by an element of the last set, then the images of \( k \) and \( J_{f_1} \) under \( B^{-1}A^{-1} \) are contained in the closed discs bounded by an element of \( K \setminus \{ A(J_{f_2}), J_{f_2}, A^{-1}(J_{f_2}) \} \) and by \( J_{f_2} \), respectively. The comparison principle implies that this case is reduced to Case 1 with a change \( J_{f_1} \) by \( J_{f_2} \). We denote by \( M_2 \) the lower bound of modules of the present case. If \( k \) lies in the closed disc bounded by \( ABA(J_{f_2}) \), then the module of \( R(k, J_{f_1}) \) is not smaller than that of \( R(ABA(J_{f_2}), J_{f_1}) \). We denote this module by \( M_2 \). If \( k \) lies in the closed disc bounded by \( ABA^{-1}(J_{f_2}) \), then there is a positive integer \( n \) such that \( k \) lies in the closed domain bounded by \( (ABA^{-1}B^{-1})^n(J_{f_1}) \) and \( (ABA^{-1}B^{-1})^{n+1}(J_{f_1}) \). Operating \( (ABA^{-1}B^{-1})^{-n} \), we see that \( k \) is transformed in the closed disc bounded by an element of either \( K \setminus \{ A(J_{f_1}), J_{f_1} \} \) or \( \{ ABA(\xi) \mid \xi \in K_1 \setminus \{ ABA^{-1}(J_{f_2}), A(J_{f_1}) \} \) and \( J_{f_1} \) is transformed to a circle lying in the closed disc bounded by \( J_{f_1} \) and being tangent to \( J_{f_1} \) at \( p_1 \). If the transformed \( k \) lies in the closed disc bounded by \( A^{-1}(J_{f_1}) \), then operating \( A \) once more, we can reduce this case to the previously considered case with respect to \( R(ABA(J_{f_2}), J_{f_1}) \). If \( k \) is transformed in the closed disc bounded by an element of \( K \setminus \{ A(J_{f_1}), J_{f_1}, A^{-1}(J_{f_1}) \} \), then by the comparison principle we reduce this case to Case 1. If \( k \) is transformed in the closed disc bounded by an element of \( \{ ABA(\xi) \mid \xi \in K_1 \setminus \{ ABA^{-1}(J_{f_2}), A(J_{f_1}) \} \), then replacing \( J_{f_1} \) by the transformed \( J_{f_1} \) and using the comparison principle, we reduce this case to a case already considered in this paragraph.

Case 3. By the action of \( B \) we see that \( B^{-1}(J_{f_1}) = J_{f_2} \) and \( B^{-1}(k) \) lies in the
closed disc bounded by an element of \( K \setminus \{ J, A(J), A^{-1}(J) \} \). By the same reasoning in Case 1 we see that the module of \( R(B^{-1}(J), B^{-1}(k)) \) is not smaller than a positive constant. We denote it by \( M_4 \).

Now, we conclude that \( M = \min (M_1, M_2, M_3, M_4) \) is the desired constant.

We return to Lemma 5.

Proof of Lemma 5. If there is a subsequence of \( \{ k_i \} \) such that any two of which are disjoint, then Lemma 7 implies that the diameters of elements of the subsequence must converge to 0, so that that of \( \{ k_i \} \) must do, too. Hence we consider the case where there is no subsequence of \( \{ k_i \} \) such that any two of which are disjoint. We assert that there are a positive integer \( i_0 \) and a point \( p \) equivalent to \( \lambda \) such that, for each \( i > i_0 \), \( k_i \) passes through \( p \). Assuming the contrary of the assertion, we shall construct a subsequence having the property that any two of which are disjoint. We first note that only two points \( p_1 \) and \( p_2 \) on \( J \) are the points at which elements of \( K \setminus \{ J \} \) can meet \( J \) and that each \( k_i \) is equivalent to \( J \). Put \( k_1 = k_i \). If \( k_1 \) and \( k_2 \) are disjoint, then put \( k_1 = k_2 \). If \( k_1 \) and \( k_2 \) are not disjoint, then denote by \( p_{1,2} \) the tangent point of them. Since \( p_{1,2} \) is equivalent to \( p_1 \), there is a number \( i_2 > 2 \) such that \( k_{i_2} \) does not pass through \( p_{1,2} \) so that \( k_1 \) and \( k_{i_2} \) are disjoint. If \( k_{i_2} \) and \( k_{i_2+1} \) are disjoint, then put \( k_{i_3} = k_{i_2+1} \). If \( k_{i_2} \) and \( k_{i_2+1} \) are not disjoint, then denote by \( p_{i_2,i_2+1} \) the tangent point of them. Since \( p_{i_2,i_2+1} \) is equivalent to \( p_1 \), there is a number \( i_3 > i_2 + 1 \) such that \( k_{i_2} \) does not pass through \( p_{i_2,i_2+1} \), and so on. Then the subsequence \( \{ k_i \} \) has the property stated above, a contradiction. Thus we have our assertion. Let \( g \) be an element of \( G \) such that \( g(p) = p \). Then \( \{ g(k_i) \} \) is an infinite subset of \( K \) passing through \( p \). Lemma 6 tells us that the diameter of \( g(k_i) \) tends to 0. Hence that of \( k_i \) tends to 0, too.

Now, we have shown that \( \Omega_1 \) and \( \Omega_2 \) are disjoint so that \( G \) is quasi-Fuchsian. To complete the proof of Proposition 3 we must show that \( G \) represents a pair of once punctured tori. Let \( G' = \langle A', B' \rangle \) be a Fuchsian group keeping the real axis invariant and \( f \) be a quasiconformal formal such that \( A = fA'f^{-1} \) and \( B = fB'f^{-1} \). Then we see that the commutator of \( A' \) and \( B' \) is parabolic. Writing \( A' = \left( \begin{array}{cc} \alpha' & 0 \\ 0 & \beta' \end{array} \right) \) and \( B' = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \), we may assume that the equalities (2)~(6) hold for the letters with a prime \( \alpha', \ldots, d' \) and \( p' \). Then we have that

\[
\begin{align*}
p_1' p_2' &= ((c')^2 - 1)((d')^2) > 0, \\
p_3' p_4' &= ((c')^2 - 1)((a')^2) > 0 & \text{and} \\
p_1' p_3' &= -((c')^2 - 1)((c')^2) < 0.
\end{align*}
\]

Hence we see that two circles orthogonal to the real axis and passing through \( p', p_2' \) and \( p_3', p_4' \) are separated by the imaginary axis. Now it is clear that two quadrilateral domains bounded by four circles which are orthogonal to the real
axis and pass through \( p'_4, p'_2; p'_4, p'_1; p'_4, p'_2 \) and \( p'_4, p'_1 \), respectively, form a fundamental domain for \( G' \) so that \( G' \) represents a pair of once punctured tori. As \( G \) is a quasiconformal deformation of \( G' \), \( G \) also represents a pair of once punctured tori. Thus we have completed the proof of Proposition 3.

References


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