Bendikov, A. and Saloff-Coste, L.
Osaka J. Math.
42 (2005), 677–722

ON THE REGULARITY OF SAMPLE PATHS OF SUB-ELLIPTIC DIFFUSIONS ON MANIFOLDS

ALEXANDER BENDIKOV and LAURENT SALOFF-COSTE[1]

(Received March 4, 2004)

Abstract
Using heat kernel Gaussian estimates and related properties, we study the intrinsic regularity of the sample paths of the Hunt process associated to a strictly local regular Dirichlet form. We describe how the results specialize to Riemannian Brownian motion and to sub-elliptic symmetric diffusions.

1. Introduction

The present work is concerned with regularity properties of the sample paths of symmetric diffusion processes. We will work in the context of regular strictly local Dirichlet forms and their associated Hunt processes under some additional assumptions. Without such assumptions, one cannot hope to obtain the results we will describe, see [5]. Our goal is to cover such cases as Brownian motions on Riemannian manifolds and left-invariant symmetric sub-elliptic diffusions on Lie groups.

On $\mathbb{R}^n$, any translation invariant, symmetric, non-degenerate diffusion process X is, up to a change of coordinates, the classical Brownian motion whose distribution at time $t > 0$ has density

$$x \mapsto \left(\frac{1}{2\pi t}\right)^{n/2} \exp\left(-\frac{\|x\|^2}{2t}\right)$$

with respect to Lebesgue measure, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Here we have been following the classical notation according to which Brownian motion is driven by $(1/2)\Delta$ where $\Delta = \sum_1^n \partial_i^2$ is the Laplacian of the given Euclidean structure. The reason for the prevalence of this choice is that the covariance matrix of X_1 equals the identity matrix. In the more general context of interest to us, this choice is not very natural and we will instead consider that the canonical Brownian motion associated to a given Euclidean structure is driven by Δ itself. In this normalization, the distribution at time $t > 0$ has density

$$x \mapsto \left(\frac{1}{4\pi t}\right)^{n/2} \exp\left(-\frac{\|x\|^2}{4t}\right).$$

[1]Research partially supported by NSF grant DMS 0102126.

Our aim is to discuss generalizations of the following celebrated properties of the Brownian sample paths:
(i) The Lévy-Khinchine law of the iterated logarithm asserts that, almost surely,

$$\limsup_{t\to 0} \frac{\|X_t\|}{\sqrt{4t\log\log(1/t)}} = 1.$$

See, e.g., [30, 33, 40].
(ii) Lévy's result on the modulus of continuity of Brownian paths asserts that, almost surely,

$$\lim_{\epsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\epsilon}} \frac{\|X_t - X_s\|}{\sqrt{4(t-s)\log(1/(t-s))}} = 1.$$

See, e.g., [30, 33, 40].
(iii) If $n \geq 3$, the theorem of Dvoretski and Erdös [15] concerning the "rate of escape" of Brownian motion asserts that, for any continuous increasing positive function ψ, one has

$$\liminf_{t\to 0} \frac{\|X_t\|}{\psi(t)\sqrt{t}} = \begin{cases} +\infty \\ 0 \end{cases} \quad \text{almost surely iff} \quad \sum_k [\psi(2^{-k})]^{n-2} \begin{cases} \text{converges} \\ \text{diverges.} \end{cases}$$

The two dimensional version of this result was obtained by Spitzer [44] and reads

$$\liminf_{t\to 0} \frac{\|X_t\|}{\phi(t)} = \begin{cases} +\infty \\ 0 \end{cases} \quad \text{almost surely iff} \quad \sum_k \left(\log \frac{1}{\phi(2^{-k})}\right)^{-1} \begin{cases} \text{converges} \\ \text{diverges.} \end{cases}$$

Here, we have followed our convention that Brownian motion is driven by Δ. If instead we consider that Brownian motion is driven by $(1/2)\Delta$, the factor 4 in Lévy-Khinchine's law of the iterated logarithm and in Lévy's modulus of continuity should be changed to a factor 2.

The techniques used in this paper are robust and apply without essential changes to some other settings. The papers [21, 26, 52] contain some long time results that are closely related in spirit to the short time results described below. Similar techniques have been used by several authors to prove analogs of the law of iterated logarithm and related results in various settings including fractals. See, e.g., [1, 2, 3]. Still, it is important to realize that such results are not entirely universal (compare with Takeda's inequality stated in 3.8 below) and that hypotheses of some sort are needed for a Hunt process associated with a strictly local regular Dirichlet form to satisfy, say, the law of iterated logarithm (see, e.g., [5]).

We close this introduction with a short description of the content of the paper.

Section 2 contains background information concerning Dirichlet spaces.

Section 3 describes the relations between several properties that play a crucial role in this paper. For instance, the doubling property (3.3) of the volume function (and some variants of it) play an important role throughout the paper.

Section 4 contains upper bounds related to the law of iterated logarithm and the modulus of continuity and a lower bound on the rate of escape. All these are obtained by assuming some type of upper bound on the heat kernel (or related mean value properties).

Section 5 contains lower bounds for the law of iterated logarithm and the modulus of continuity and an upper bound for the rate of escape. The lower bounds for the law of the iterated logarithm and the modulus of continuity are obtained by assuming lower bound on the heat kernel (no heat kernel upper bounds are needed). The upper bound on the rate of escape is obtained under a two sided heat kernel bound (equivalently, the parabolic Harnack inequality (3.19)).

Section 6 describes explicitly how these results apply to a number of basic examples including Brownian motion on Riemannian manifolds and symmetric sub-elliptic diffusions.

2. Background and notation

2.1. Dirichlet spaces. One of the natural settings for the results of this paper is that of regular, strictly local Dirichlet spaces. Thus, let M be a connected locally compact separable metric space and let μ be a positive Radon measure on M with full support. For any open set $\Omega \subset M$, let $\mathcal{C}_0(\Omega)$ be the set of all continuous functions with compact support in Ω. Consider a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(M, d\mu)$ and core $\mathcal{C} \subset \mathcal{D}$: a core is a subset of $\mathcal{D} \cap \mathcal{C}_0(M)$ which is dense in $\mathcal{D}$ for the norm $(\|f\|_2^2 + \mathcal{E}(f, f))^{1/2}$ and dense in $\mathcal{C}_0(M)$ for the uniform norm. A Dirichlet form is regular if it admits a core. See [17]. We also assume that $\mathcal{E}$ is strictly local: for any $u, v \in \mathcal{D}$ such that the supports of u and v are compact and v is constant in a neighborhood of the support of u, we have $\mathcal{E}(u, v) = 0$. See [17, p.6] where such Dirichlet forms are called "strong local." Any such Dirichlet form $\mathcal{E}$ can be written in terms of an "energy measure" Γ so that $\mathcal{E}(u, v) = \int_M d\Gamma(u, v)$ where $d\Gamma(u, v)$ is a signed Radon measure for $u, v \in \mathcal{D}$. Moreover, Γ satisfies the Leibniz rule and the chain rule. See [17, pp.115–116].

It is a simple but remarkable fact that the data above suffices to introduce a pseudo-distance d on M often called the *intrinsic distance* and defined as follows. Let $\mathcal{L}_1$ be the set of all functions f in the core $\mathcal{C}$ such that $d\Gamma(f, f) \le d\mu$, i.e., $\Gamma(f, f)$ is absolutely continuous with respect to μ with Radon-Nikodym derivative bounded by 1. Thus $\mathcal{L}_1$ is, in some sense, a set of compactly supported Lipschitz functions with Lipschitz constant 1. For each $x, y \in M$, define $d(x, y)$ by

$$d(x, y) = \sup\{f(x) - f(y) \colon f \in \mathcal{L}_1\}. \tag{2.1}$$

Note that d is always a lower semi-continuous function and satisfies the triangle inequality. For $x \neq y$, it might happen that $d(x, y) = 0$ (e.g., on fractals) or $+\infty$ (this actually happens in some interesting cases (see e.g., [4]) but we will not be concerned with such cases in this paper. The idea of the intrinsic distance (or at least its usefulness) seems to have emerged in the eighties in connection with E.B. Davies' work on Gaussian heat kernel bounds, see, e.g., [12, Theorem 3.2.7]. Details concerning the intrinsic distance in the case of general regular strictly local Dirichlet spaces are found in [6, 7] and [46, 47, 51].

We now make a couple of crucial hypotheses about the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$, in terms of the intrinsic distance d. Throughout the paper we assume that the following properties are satisfied.

- The pseudo-distance d is finite everywhere and the topology induced by d is equivalent to the initial topology of M. In particular, $(x, y) \mapsto d(x, y)$ is a continuous function.
- (M, d) is a complete metric space.

These hypotheses imply that (M, d) is a path metric space (i.e., d can be defined in terms of "shortest paths"). See e.g., [10, 24] and [46]. Path metric spaces are also called length spaces or inner metric spaces. It also implies that the cut-off functions

$$u\colon y \mapsto \sup\{d(x, y) - r, 0\} = (d(x, y) - r)_+$$

are in $\mathcal{D} \cap \mathcal{C}_0(M)$ and satisfy $d\Gamma(u, u) \leq d\mu$. This is a crucial fact, see [7, 47]. It allows us to extend classical arguments from the Riemannian setting to the present more general framework.

We will denote by $B(x, r) = \{y \in M\colon d(x, y) < r\}$ the ball of radius r around x. Given a ball $B = B(x, r)$ we let $r(B) = r$ be its radius and $\mu(B)$ be its volume relative to the measure μ. Our basic assumptions on (M, d) implies that the closure of the open ball $B(x, r)$ is the closed ball $\{y\colon d(x, y) \leq r\}$ and that any closed ball is compact. See [10, 46, 47, 51].

2.2. The heat semigroup. Fix a Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ as above. As is well-known, there is a self-adjoint semigroup of contractions of $L^2(M, d\mu)$, call it $(H_t)_{t>0}$, uniquely associated with this Dirichlet space. Moreover, $(H_t)_{t>0}$ is (sub-)Markovian. Let $-L$ be the infinitesimal generator of $(H_t)_{t>0}$ so that $H_t = e^{-tL}$ and $\mathcal{E}(f, g) = \langle f, Lg\rangle,\ f, g \in \mathcal{D}$.

We assume throughout the paper that the transition function of the semigroup $(H_t)_{t>0}$ is absolutely continuous with respect to μ, that is, there exists a non-negative measurable function $(t, x, y) \mapsto h(t, x, y)$, the heat diffusion kernel, such that

$$\forall x \in M,\ \ t > 0, \qquad H_t f(x) = \int_M h(t, x, y) f(y)\, d\mu(y).$$

In the present context it is useful to be a little more precise since the above formula does not uniquely define $h(t, x, y)$. In what follows, we assume that $(t, y) \mapsto h(t, x, y)$

is *the unique excessive density* of $(H_t)_{t>0}$. See [8, Chapter 6]. (The reader unfamiliar with this notion can make the a priori restrictive assumption that $h(t, x, y)$ is continuous.)

By [17, Lemma 4.2.4], the assumption that $h(t, x, y)$ exists implies that

$$G_1(x, y) = \int_0^\infty e^{-t} h(t, x, y)\, dt \tag{2.2}$$

is a 1-excessive function (i.e., excessive with respect to the semigroup $(e^{-t} H_t)_{t>0}$).

2.3. The Hunt process. By the general theory presented in [17], there exists a Hunt process $X = ((X_t)_{t>0}, \mathbb{P}_x)$ with continuous paths $t \to X_t(\omega) \in M$ associated to our fixed strictly local regular Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$. In particular, this process X is such that

$$\forall f \in \mathcal{C}, \qquad \mathbb{E}^x(f(X_t)) = H_t f(x).$$

Since we assume in this paper that the transition function of the semigroup $(H_t)_{t>0}$ is absolutely continuous with respect to μ (i.e., the existence of the heat kernel), the Hunt process $((X_t)_{t>0}, \mathbb{P}_x)$ is well defined for any starting point $x \in M$.

As the basic goal of this paper is to study properties of the sample paths of X, it would of course be very natural to start from the Hunt process X having continuous paths and associate to it the corresponding local Dirichlet space as in [17, Chapter 4].

3. Local properties

This section introduces a number of well-known properties such as the doubling property for volume growth and Poincaré, Sobolev and Harnack inequalities. These properties may or may not hold on a given Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$. They will play an important role in the sequel. When they hold, these properties yield some control of the local geometry and analysis on M. We will consider two local versions of these various properties. In the first version, we ask that the given property holds uniformly for all balls contained in a fixed ball $2B_0$ (if B is a ball of radius r and $\lambda > 0$, λB is the concentric ball of radius λr). In this case we say the property holds around B_0. In the second version, we ask that the property holds uniformly for all balls of radius less than a fixed r_0. In this case we say that the property holds *up to scale* r_0.

3.1. The doubling property. We say that the doubling property holds around a fixed ball B_0 if there exists a constant D_0 such that, for all balls $B \subset 2B_0$,

$$\mu(2B) \le D_0\, \mu(B) \tag{3.3}$$

For later references, we note a few consequences of this property.

- If (3.3) holds then, for all $x, y \in M$ and $0 < s < r$ such that $B(x,r), B(y,s) \subset B_0$, we have

$$\mu(B(x,r)) \over \mu(B(y,s)) \leq D_1 \left(\frac{d(x,y)+r}{s}\right)^{\nu} \tag{3.4}$$

for any $\nu \geq \log_2 D_0$. Actually, one can take $D_1 = D_0^2$.
- If (3.3) holds and $2B_0 \neq M$ then there exist $\beta, \gamma > 0$ such that, for all balls $B \subset B_0$,

$$\forall t \in (0,1), \qquad \frac{\mu(B)}{\mu(tB)} \geq \beta \left(\frac{1}{t}\right)^{\gamma}. \tag{3.5}$$

See, e.g., [43, Lemma 5.2.8] or [20, Lemma 7.16].
We say that the doubling property holds uniformly up to scale r_0 if there exists a constant D_0 such that (3.3) holds for all balls B of radius at most r_0. In that case, (3.4) holds uniformly over all x, y, $0 < s \leq r$ with $d(x,y) + r \leq r_0$, and (3.5) holds uniformly for all balls B of radius at most r_0 such that $2B \neq M$.

3.2. Poincaré inequality. We say that a (scale-invariant) Poincaré inequality holds around B_0 if there exists a constant P_0 such that, for any ball $B \subset 2B_0$,

$$\forall f \in \mathcal{D}, \qquad \int_B |f - f_B|^2 \, d\mu \leq P_0 r^2 \int_{2B} d\Gamma(f,f), \tag{3.6}$$

where $f_B = \mu(B)^{-1} \int_B f d\mu$ and $r = r(B)$.

It is known (see, e.g., [31] or [43, Corollary 5.3.5]) that (3.3) and (3.6) together imply the stronger inequality

$$\forall B \subset 2B_0, \ \forall f \in \mathcal{D}, \qquad \int_B |f - f_B|^2 \, d\mu \leq P_1 r^2 \int_B d\Gamma(f,f). \tag{3.7}$$

This inequality is equivalent to say that the lowest non-zero Neumann eigenvalue $\lambda^N(B)$ in the ball B is bounded below by $\lambda^N(B) \geq (P_1 r^2)^{-1}$.

We say that the Poincaré inequality holds uniformly up to scale r_0 if there exists a constant P_0 such that (3.6) holds for all balls B of radius at most r_0. If the doubling property and the Poincaré inequality hold uniformly up to scale r_0, then (3.7) holds for all balls B of radius at most r_0.

3.3. Sobolev type inequalities. We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a (scale-invariant) Sobolev inequality around the ball B_0 if there exists a constant S_0 and a real $\nu > 2$ such that, for any ball $B \subset 2B_0$,

$$\forall f \in \mathcal{D} \cap \mathcal{C}_0(B), \quad \left(\int_B |f|^q d\mu\right)^{2/q} \leq \frac{S_0\, r(B)^2}{\mu(B)^{2/\nu}} \left(\int_B d\Gamma(f,f) + r(B)^{-2} \int_B |f|^2 d\mu\right) \tag{3.8}$$

where $q = 2\nu/(\nu - 2)$. The exact values of q and ν will play no role in what follows.

We say that a local Sobolev inequality holds up to scale r_0 if (3.8) holds true for all balls B of radius at most r_0.

A crucial observation that first appeared in [41] is that the doubling property (3.3) and the Poincaré inequality (3.6) together imply the Sobolev inequality (3.8).

Theorem 3.1. *Fix a ball $B_0 \subset M$. Assume that* (3.3) *and* (3.6) *holds around B_0. Then the Sobolev inequality* (3.8) *holds around B_0.*

See [36, 43] for proofs that can be adapted to the present setting.

For completeness, we recall that (3.8) can be characterized in terms of what is called a Faber-Krahn inequality, i.e., an inequality relating the lowest Dirichlet eigenvalue on an open set to the volume of that open set. More precisely, let $\lambda(U)$ denotes the lowest Dirichlet eigenvalue in the open set U. A (scale-invariant) Faber-Krahn inequality holds around B_0 if there are positive constants k_0 and ν such that for any ball $B \subset 2B_0$ and any open set $U \subset B$,

$$\lambda(U) \geq \frac{k_o}{r^2}\left(\frac{\mu(B)}{\mu(U)}\right)^{\nu}. \tag{3.9}$$

Theorem 3.2. *Given a ball B_0, the following two properties are equivalent.*
1. *The lowest Dirichlet eigenvalue $\lambda(2B_0)$ is positive and the scale-invariant local Sobolev inequality* (3.8) *holds around B_0.*
2. *The scale-invariant Faber-Krahn inequality* (3.9) *holds around B_0.*

This equivalence holds with the same ν for both inequalities if $\nu > 2$.

Next we recall the characterization of (3.8) in terms of heat kernel upper bounds. Proofs that can be adapted to the present setting can be found in [43]. See also [20].

Theorem 3.3. *Fix a ball $B_0 \subset M$.*
1. *Assume that the scale-invariant local Sobolev inequality* (3.8) *holds around B_0. Then the doubling property* (3.3) *holds around B_0 and there exists a constant C such that for all $x \in M$, $t > 0$ with $B(x, \sqrt{t}) \subset 2B_0$, we have*

$$h(t, x, x) \leq \frac{C}{\mu(B(x, \sqrt{t}))}. \tag{3.10}$$

Moreover, for any $\delta > 0$ there exists a constant C_δ such that, for all $x, y \in M$, $t > 0$ with $B(x, \sqrt{t}) \subset 2B_0$ and $B(y, \sqrt{t}) \subset 2B_0$, we have

$$h(t, x, y) \leq \frac{C_\delta}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{4(1+\delta)t}\right). \tag{3.11}$$

2. *Assume that the doubling property* (3.3) *holds around B_0. Assume also that there*

exists a constant C such that (3.10) *holds for all $x \in M$, $t > 0$ satisfying $B(x, \sqrt{t}) \subset 2B_0$. Then the scale-invariant local Sobolev inequality* (3.8) *holds around B_0.*

These three theorems admit "up-to-scale-r_0" versions. See [20], [41, 43] and [49, 50].

3.4. Harnack and mean value inequalities. Fix an open set Ω. We say that a function u belongs to $\mathcal{D}_{\Omega,\mathrm{loc}}$ if for any relatively compact open set A with $\overline{A} \subset \Omega$, there exists a function $f \in \mathcal{D}$ such that $u = f$ almost everywhere in A.

A *solution* u of the equation $(\partial_t + L)u = 0$ on $I \times \Omega$ (where $I \subset \mathbb{R}$ is an open interval and $\Omega \subset M$ is an open subset of M) is a measurable function $u \colon I \times \Omega \to \mathbb{R}$ such that $(t, x) \mapsto \partial_t u(t, x) \in L^{\infty,2}_{\mathrm{loc}}(I \times \Omega, dt \otimes d\mu)$, $x \mapsto u(t, x) \in \mathcal{D}_{\Omega,\mathrm{loc}}$ and

$$\int_M \partial_t u(t, \cdot)\phi \, d\mu + \int_M d\Gamma(u(t, \cdot), \phi) = 0 \tag{3.12}$$

for all $\phi \in \mathcal{C} \cap \mathcal{C}_0(\Omega)$. It is possible to deal with solutions in a weaker sense but we will not pursue this here. For instance, for any $k = 0, 1, 2, \dots,$ the functions $(t, x) \mapsto \partial_t^k h(t, x, y)$ and $(t, y) \mapsto \partial_t^k h(t, x, y)$ are solutions of $(\partial_t + L)u = 0$ in $(0, +\infty) \times M$.

A *subsolution* is a measurable function $u \colon I \times \Omega \to \mathbb{R}$ such that $(t, x) \mapsto \partial_t u(t, x) \in L^{\infty,2}_{\mathrm{loc}}(I \times \Omega, dt \otimes d\mu)$, $x \mapsto u(t, x) \in \mathcal{D}_{\Omega,\mathrm{loc}}$ and

$$\int_M \partial_t u(t, \cdot)\phi \, d\mu + \int_M d\Gamma(u(t, \cdot), \phi) \le 0 \tag{3.13}$$

for all non-negative $\phi \in \mathcal{C} \cap \mathcal{C}_0(\Omega)$. For instance, for any $k = 0, 1, 2, \dots,$ the functions $(t, x) \mapsto |\partial_t^k h(t, x, y)|$ are subsolutions of the equation $(\partial_t + L)u = 0$ in $(0, +\infty) \times M$.

Fix a ball $B_0 \subset M$. We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a *scale-invariant mean value inequality around* B_0 if there exists a constant C such that for any reals s, r with $r > 0$, for any $x \in M$ such that $B = B(x, r) \subset 2B_0$, and for any non-negative subsolution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$, we have

$$u(s, x) \le \frac{C}{r^2 \mu(B(x, r))} \int_{s-r^2}^{s} \int_{B(x,r)} u(t, y) \, d\mu(y) \, dt. \tag{3.14}$$

We say that $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a *scale-invariant mean value inequality up to scale* r_0 if (3.14) holds for all balls B of radius at most r_0.

The following known result relates (3.14) to heat kernel upper bounds and to the local Sobolev inequality (3.8).

Theorem 3.4. *Fix a ball $B_0 \subset M$.*

1. *Assume that the mean value inequality* (3.14) *holds around B_0. Then for all $x \in$*

M and $t > 0$ such that $B(x, \sqrt{t}) \subset 2B_0$, we have

$$h(t, x, x) \le \frac{C}{\mu(B(x, \sqrt{t}))}.$$

Moreover, for all $x, y \in M$ and $t > 0$ such that $B(x, \sqrt{t}), B(y, \sqrt{t}) \subset 2B_0$, we have

$$h(t, x, y) \le \frac{C}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \exp\left(-\frac{d(x, y)^2}{4t} + \frac{d(x, y)}{\sqrt{t}}\right). \tag{3.15}$$

2. *Assume that the Sobolev inequality* (3.8) *holds around B_0. Then the mean value inequality* (3.14) *holds around B_0.*

Proof. The first part of the first statement follows from applying (3.14) to the function $u(t, y) = h(t, x, y)$ which is a solution in $(0, \infty) \times M$. The proof of (3.15) follows from the proof of [43, Theorem 5.2.10]. For the proof of the second statement based on the classical Moser iteration argument [37], see, e.g., [43, Theorem 5.2.9].

One can easily state a version of Theorem 3.4 for the case where the various properties are considered "up to scale r_0." See [41, 42, 43]. □

On occasion, we will also consider a weaker type of mean value inequality. We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a *C-mean value inequality around B_0* (resp. up to scale r_0) if there exists a *function* $B \mapsto C(B)$ defined on the set of all metric balls such that for any reals s, r with $r > 0$, for any $x \in M$ such that $B = B(x, r) \subset 2B_0$ (resp. for all balls B of radius at most r_0), and for any non-negative subsolution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$, we have

$$u(s, x) \le \frac{C(B)}{r^2\mu(B(x, r))} \int_{s-r^2}^{s} \int_{B(x,r)} u(t, y)\, d\mu(y)\, dt. \tag{3.16}$$

To give an example, assume that the Sobolev-type inequality

$$\forall f \in \mathcal{D} \cap \mathcal{C}_0(2B_0), \qquad \left(\int_{2B_0} |f|^q d\mu\right)^{2/q} \le S_0 \left(\int_{2B_0} d\Gamma(f, f) + |f|^2 d\mu\right) \tag{3.17}$$

holds with $q = 2\nu/(\nu - 2)$ for some $\nu > 2$. Then Moser's iteration can be used to prove for any reals s, r with $r > 0$, for any $x \in M$ such that $B = B(x, r) \subset 2B_0$, and for any non-negative subsolution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$, we have

$$u(s, x) \le \frac{C}{r^{2+\nu}} \int_{s-r^2}^{s} \int_{B(x,r)} u(t, y)\, d\mu(y)\, dt. \tag{3.18}$$

That is, (3.17) implies a C-mean value inequality with $B \mapsto C(B) = C\mu(B)r(B)^{-\nu}$.

We say that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies a *scale-invariant parabolic Harnack inequality around* B_0 (resp. *up to scale* r_0) if there exists a constant C such that, for any reals s, r with $r > 0$, for any $x \in M$ such that $B = B(x, r) \subset 2B_0$ (resp. for all balls B of radius r less or equal to r_0), and for any non-negative solution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$, we have

$$\sup_{Q_-}\{u\} \leq C \inf_{Q_+}\{u\} \tag{3.19}$$

where

$$\begin{aligned} Q_+ &= \left(s - \frac{r^2}{4}, s\right) \times B\left(x, \frac{r}{2}\right), \\ Q_- &= \left(s - \frac{3r^2}{4}, s - \frac{r^2}{2}\right) \times B\left(x, \frac{r}{2}\right). \end{aligned}$$

Moser's iteration technique [37] adapted as in [41, 43] and Theorem 3.1 give the following important result (see also [19] for a different proof).

Theorem 3.5. *Fix a ball $B_0 \subset M$ (resp. $r_0 > 0$).*
1. *Assume that* (3.3) *and* (3.6) *hold around B_0 (resp. up to scale r_0). Then the parabolic Harnack inequality* (3.19) *holds around B_0 (resp. up to scale r_0).*
2. *There exists $\delta > 0$ such that if* (3.19) *holds around B_0 (resp. up to scale r_0) then* (3.3) *and* (3.6) *holds around δB_0 (resp. up t scale δr_0). One can take take $\delta = 1/8$.*

The parabolic Harnack inequality (3.19) is a powerful tool. It yields good two-sided heat kernel estimates as stated in the next Theorem (see, e.g., [43, Theorem 5.4.11]).

Theorem 3.6. *Fix a ball $B_0 \subset M$ (resp. $r_0 > 0$). Assume that the parabolic Harnack inequality* (3.19) *holds around B_0 (resp. up to scale r_0). Then for all x, y, t such that $B(x, \sqrt{t}), B(y, \sqrt{t}) \subset B_0$ (resp. $x, y \in M$, $t \in (0, r_0^2)$) we have*

$$\frac{c_1}{\mu(B(x, \sqrt{t}))} \exp\left(-C_1 \frac{d(x, y)^2}{t}\right) \leq h(t, x, y) \leq \frac{C_2}{\mu(B(x, \sqrt{t}))} \exp\left(-c_2 \frac{d(x, y)^2}{t}\right).$$

This theorem admits a converse (see, e.g., [27, Theorem 5.3]). Another applications of (3.19) is that it yields a certain regularity of the solutions of $(\partial_t + L)u = 0$. This is especially noteworthy in the present framework since these solutions are not even continuous, a priori. The following are well known results. For divergence form operators in $\mathbb{R}^n$, they are due to J. Moser [37] and the proofs go over to the present setting without change.

Theorem 3.7. *Fix a ball $B_0 \subset M$. Assume that the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$ satisfies the scale-invariant parabolic Harnack inequality* (3.19) *around B_0. Then there exist two positive reals A and α such that for any $s \in \mathbb{R}$, any $r > 0$, any ball $B \subset 2B_0$ of radius r, and any solution u of the equation $(\partial_t + L)u = 0$ in $Q = (s - r^2, s) \times B$,*

$$|u(t, x) - u(\tau, y)| \leq A \left[\frac{d(x, y) + \sqrt{|t - \tau|}}{r} \right]^\alpha \sup_Q\{|u|\} \tag{3.20}$$

for all $(t, x), (\tau, y) \in (s - 3r^2/4, s - r^2/4) \times (1/2)B$.

3.5. Takeda's inequality. We will make use of the following inequality due to Takeda [52]. The precise form of this inequality stated below is taken from [35]. For any set $A \subset M$ and $r \geq 0$, set

$$d(x, A) = \inf\{d(x, y) \colon y \in A\}, \qquad A_r = \{y \in M \colon d(y, A) < r\}.$$

Theorem 3.8 ([35, 52]). *Let K be a compact set. Then*

$$\int_K \mathbb{P}_x \left(\sup_{s \in (0,t)} d(x, X_s) > r \right) d\mu(x) \leq \frac{16\mu(K_r)}{\sqrt{\pi}} \left(\frac{\sqrt{t}}{r} \right) \exp\left(-\frac{r^2}{4t} \right).$$

The remarkable feature of this inequality is that it holds without any further assumption on $(\mathcal{E}, \mathcal{D}, L^2(M, d\mu))$. In [52], Takeda gives some applications to the long time behavior of the sample paths of the associated diffusion. See also [21]. However, the averaging over K makes this inequality inappropriate for studying the short time behavior of sample paths. To become efficient in this context, Takeda's inequality must be complemented with some local mean value inequality requiring further local assumptions.

Theorem 3.9. *Fix a ball $B_0 \subset M$ and assume that the C-mean value inequality* (3.16) *holds around B_0. Then there exists a constant A such that, for any $x \in M$, $\epsilon, r > 0$ with $B = B(x, \epsilon) \subset B_0$, we have*

$$\mathbb{P}_x \left(\sup_{s \in (0,t)} d(x, X_s) > r \right) \leq AC(B) \frac{\mu(B(x, \epsilon + r))}{\mu(B(x, \epsilon))} \left(\frac{\sqrt{t}}{r} \right) \exp\left(-\frac{r^2}{4t} \right). \tag{3.21}$$

In particular, if the Sobolev inequality (3.8) *holds around B_0 then, for any $\delta > 0$ there exists a constant C_δ such that for any $x \in M$ and $t, r > 0$ such that $0 < t < r^2$ and $B = B(x, r) \subset B_0$, we have*

$$\mathbb{P}_x \left(\sup_{s \in (0,t)} d(x, X_s) > r \right) \leq C_\delta \exp\left(-\frac{r^2}{4(1 + \delta)t} \right). \tag{3.22}$$

Proof. The first inequality follows immediately from Theorem 3.8 and (3.16) because

$$u(t,x) = \mathbb{P}_x\left(\sup_{s\in(0,t)} d(x,X_s) > r\right)$$

is a non-negative subsolution of $(\partial_t + L)u = 0$.

Setting $\epsilon = \sqrt{t}$, the second inequality follows from the first by Theorems 3.3 and 3.4. Indeed, note that the doubling property (3.3) implies that there exist D_1 and $\nu > 0$ such that

$$\frac{\mu(B(x,\sqrt{t}+r))}{\mu(B(x,\sqrt{t}))} \le D_1\left(1+\frac{r}{\sqrt{t}}\right)^{\nu}$$

since $B(x,\sqrt{t}+r) \subset 2B_0$.

We leave to the reader the easy task to state the "up to scale r_0" version of Theorem 3.9. The "up to scale r_0" version will be used in Section 4.2. $\square$

3.6. Visiting probabilities. Consider the process $X^1 = (X^1_t)$ associated with the semigroup $(e^{-t}H_t)_{t>0}$. This process takes values in $M \cup \{\infty\}$ where ∞ is an isolated point added to M. We set $\rho(x,\infty) = +\infty$ for any $x \in M$. The process $X^1 = (X^1_t)$ can be obtained from X in the following way. Let ξ be a real random variable, independent of the process X and with $\mathbb{P}(\xi > t) = e^{-t}$. Then

$$X^1_t = \begin{cases} X_t & \text{if } t < \xi \\ \infty & \text{if } t \ge \xi. \end{cases}$$

Thus X^1_t is X_t killed at the exponential time ξ. In what follows, we will need good estimates on the probability $\psi^1_K(t,x)$ that, starting at x, X^1_t visits the compact set $K \subset M$ after time t. We have

$$\begin{aligned}\psi^1_K(t,x) &= \mathbb{P}_x(\text{there exists } s > t \text{ such that } X^1_s \in K)\\ &= \mathbb{P}_x(\text{there exists } s > t \text{ such that } X_s \in K\,;\ s < \xi).\end{aligned}$$

Of course, one can also consider $\psi_K(t,x) = \mathbb{P}_x$ (there exists $s > t$ such that $X_s \in K$). However, $\psi_K \equiv 1$ if the process X is recurrent and K has non-empty interior, e.g., for Brownian motion on compact Riemannian manifolds with $K = B(x,r)$. In such cases, ψ_K contains no information whereas ψ^1_K does.

Let us also set

$$\psi^1_K(x) = \mathbb{P}_x(\text{there exists } s > 0 \text{ such that } X^1_s \in K).$$

It is well known that ψ^1_K is a 1-excessive function (i.e., excessive relative to X^1) which is 1-harmonic outside K. Hence, there exists a positive Radon measure ν_K

(the equilibrium measure) supported by K such that

$$\psi_K^1(x) = \int_M G^1(x, y)\, d\nu_K(y) \tag{3.23}$$

where $G^1(x, y)$ is the Green function defined at (2.2). The 1-capacity of a given compact set K is defined by setting

$$\mathrm{Cap}^1(K) = \inf\left\{\int_M d\Gamma(f, f) + |f|^2\, d\mu : \phi \in \mathcal{D} \cap \mathcal{C},\ \ \phi|_K \geq 1\right\}.$$

The 1-capacity of K is related to the equilibrium measure ν_K by

$$\mathrm{Cap}^1(K) = \nu_K(K). \tag{3.24}$$

For all of this, see [8].

We will need the following estimate which is in the spirit of [21, 23] and involves the notion introduced above.

Theorem 3.10. *For any compact K, and any $x \in M$, $t > 0$, we have*

$$\psi_K^1(t, x) \leq \mathrm{Cap}^1(K) \int_t^\infty \sup_{y \in K} h(s, x, y) e^{-s}\, ds.$$

Proof. By the Markov property,

$$\psi_K^1(t, x) = e^{-t} H_t \psi_K^1(x) = \int_M e^{-t} h(t, x, y) \psi_K^1(y)\, d\mu(y).$$

Using (3.23), (2.2) and the semigroup property, we obtain

$$\begin{aligned}
\psi_K^1(t, x) &= \int_M \int_M e^{-t} h(t, x, y) G^1(y, z)\, d\nu_K(z)\, d\mu(y) \\
&= \int_M \int_M \int_0^\infty e^{-t} h(t, x, y) e^{-s} h(s, y, z)\, ds\, d\nu_K(z)\, d\mu(y) \\
&= \int_M \int_0^\infty e^{-t+s} h(t + s, x, z)\, ds\, d\nu_K(z) \\
&= \int_M \int_t^\infty e^{-s} h(s, x, z)\, ds\, d\nu_K(z).
\end{aligned}$$

Together with (3.24) and the fact that ν_K is supported in K, this gives the desired result. □

4. Using the mean value inequality

This section explores what can be said about sample paths starting in a ball B_0 around which a certain C-mean value inequality (3.14) holds.

4.1. Iterated logarithm upper-bound.

Proposition 4.1. *Fix a ball $B_0 \subset M$. Assume that a C-mean value inequality* (3.16) *holds around B_0 with a function $B \mapsto C(B)$ such that, for any $\eta > 0$, there exists a constant C_η such that*

$$C(B) \le C_\eta \left(1 + \log\left(1 + \frac{1}{r(B)}\right)\right)^\eta. \tag{4.25}$$

Assume further that for any $\eta > 0$ there exist constant D_η, N_η such that, for any ball $B \subset 2B_0$ with radius r, we have

$$\forall \epsilon \in (0,1), \qquad \frac{\mu(B)}{\mu(\epsilon B)} \le D_\eta \epsilon^{-N_\eta} \left(1 + \log\left(1 + \frac{1}{r(B)}\right)\right)^\eta. \tag{4.26}$$

Then, for all $x \in B_0$,

$$\limsup_{t \to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} \le 1, \qquad \mathbb{P}_x\text{-almost surely}.$$

Proof. Let $\eta > 0$ to be fixed later. Let $x \in B_0$ and $0 < t < r^2$ be such that $B(x, 2r) \subset 2B_0$. Use the hypotheses with $B = B(x, \sqrt{t} + r)$, $\epsilon = \sqrt{t}/(\sqrt{t} + r)$, together with (3.21) to obtain

$$\begin{aligned} &\mathbb{P}_x \left(\sup_{s \in (0,t)} d(x, X_s) > r \right) \\ &\le C(\eta) \left(1 + \frac{r}{\sqrt{t}}\right)^{N_\eta} \left(\frac{\sqrt{t}}{r}\right) \left(1 + \log\left(1 + \frac{1}{r}\right)\right)^{2\eta} \exp\left(-\frac{r^2}{4t}\right) \end{aligned}$$

for some constant $C(\eta)$. In particular, for any $\delta > 0$, there exists a constant $C(\eta, \delta)$ such that

$$\mathbb{P}_x \left(\sup_{s \in (0,t)} d(x, X_s) > r \right) \le C(\eta, \delta) \left(1 + \log\left(1 + \frac{1}{r}\right)\right)^{2\eta} \exp\left(-\frac{r^2}{4(1+\delta)t}\right). \tag{4.27}$$

Fix $\sigma \in (0, 1)$ and consider the events

$$A_i = \left\{ \sup_{t \in [0, \sigma^i]} d(x, X_t) > (1 + \delta)\sqrt{4\sigma^i \log\log \sigma^{-i}} \right\}.$$

By (4.27), we have

$$\mathbb{P}_x(A_i) \le C'(\eta, \delta)(1 + i)^{2\eta} \exp(-(1 + \delta) \log\log \sigma^{-i}) \le C''(\eta, \delta)(1 + i)^{2\eta - 1 - \delta}.$$

Given any $\delta > 0$, pick $\eta \in (0, \delta/2)$. Then the series $\sum \mathbb{P}_x(A_i)$ converges and the Borel-Cantelli lemma shows that, almost surely, for all n large enough,

$$\sup_{t\in[0,\sigma^n]} d(x, X_t) \le (1+\delta)\sqrt{4\sigma^n \log\log \sigma^{-n}}.$$

It follows that, $\mathbb{P}_x$ almost surely, for all t small enough,

$$d(x, X_t) \le (1+\delta)\sqrt{4\frac{t}{\sigma}\log\log\frac{\sigma}{t}}.$$

Hence, almost surely,

$$\limsup_{t\to 0} \frac{d(x, X_t)}{\sqrt{4t\log\log(1/t)}} \le (1+\delta)\sigma^{-1/2}.$$

Since this holds for all $\sigma \in (0,1)$ and $\delta > 0$, the conclusion of Proposition 4.1 follows. □

4.2. Lévy's modulus of continuity. There is an obvious difference in nature between the law of iterated logarithm and Lévy's result on the modulus of continuity of Brownian paths. The former is a purely local statement whereas the latter is not. Indeed, in Lévy's modulus of continuity result, one has to let the Brownian path run up to time 1. It should be clear that there is no hope to control the uniform modulus of continuity of sample paths without some uniform local hypothesis on the geometry of our Dirichlet space. Thus, in contrast with what was done in the previous section where we worked under hypotheses localized around a fixed ball B_0, we will work here under uniform hypotheses "up to a fixed scale r_0."

Proposition 4.2. *Fix $r_0 > 0$. Assume that the mean value inequality* (3.16) *holds in M up to scale r_0 with a function $B \mapsto C(B)$ such that, for any $\eta > 0$, there exists a constant C_η for which*

$$C(B) \le C_\eta\left(1+\frac{1}{r(B)}\right)^{\eta}. \tag{4.28}$$

Assume further that for any $\eta > 0$ there exist constant D_η, N_η such that, for any ball $B \subset M$ with radius $r \in (0, r_0)$,

$$\forall \epsilon \in (0,1), \qquad \frac{\mu(B)}{\mu(\epsilon B)} \le D_\eta \epsilon^{-N_\eta}\left(1+\frac{1}{r}\right)^{\eta}.$$

Then, for any $x \in M$, we have

$$\lim_{\epsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{4(t-s)\log(1/(t-s))}} \le 1, \qquad \mathbb{P}_x \textit{ almost surely}.$$

Proof. Let $\eta > 0$ to be fixed later. For any $y \in M$ and $0 < t, r < r_0/2$, use the hypotheses with $B = B(x, \sqrt{t} + r)$, $\epsilon = \sqrt{t}/(\sqrt{t} + r)$, together with (3.21) to obtain

$$\mathbb{P}_y\left(\sup_{s\in(0,t)} d(y, X_s) > r\right) \le C(\eta)\left(1 + \frac{r}{\sqrt{t}}\right)^{N_\eta}\left(\frac{\sqrt{t}}{r}\right)\left(1 + \frac{1}{r}\right)^{2\eta}\exp\left(-\frac{r^2}{4t}\right)$$

for some constant $C(\eta)$. In particular, for any $\delta > 0$, there exists a constant $C(\eta, \delta)$ such that

$$\mathbb{P}_y\left(\sup_{s\in(0,t)} d(y, X_s) > r\right) \le C(\eta, \delta)\left(1 + \frac{1}{r}\right)^{2\eta}\exp\left(-\frac{r^2}{4(1+\delta)t}\right). \tag{4.29}$$

Fix $x \in M$ and set

$$h(s) = 4s\log\frac{1}{s}.$$

Let also $\delta \in (0, 1)$ be fixed and set

$$K_n = \{(i, j) \in \mathbb{N}^2 : 0 \le i < j \le 2^n,\ \ j - i \le 2^{\delta n}\}.$$

It is clear that $\#K_n \le 2^{(1+\delta)n}$, and for $(i, j) \in K_n$, we have $(j - i)2^{-n} \le 2^{-n(1-\delta)}$. Consider the events

$$A_n = \left\{\sup_{(i,j)\in K_n} \frac{d(X_{i2^{-n}}, X_{j2^{-n}})}{\sqrt{k(1+\delta)h((j-i)2^{-n})}} \ge 1\right\}$$

where $k > 0$ will be chosen later. For n large enough and all $(i, j) \in K_n$, we have

$$(j - i)2^{-n} \le \frac{r_0}{2}, \qquad k(1+\delta)h((j-i)2^{-n}) \le \frac{r_0}{2},$$

and

$$\begin{aligned}
\mathbb{P}_x(A_n) &\le \sum_{(i,j)\in K_n} \mathbb{P}_x\left(\frac{d(X_{i2^{-n}}, X_{j2^{-n}})}{\sqrt{k(1+\delta)h((j-i)2^{-n})}} \ge 1\right)\\
&\le C(\eta, \delta)\sum_{(i,j)\in K_n}\left(1 + \sqrt{\frac{2^n}{j-i}}\right)^{2\eta}\exp\left(-k\log\frac{2^n}{(j-i)}\right)\\
&\le 2C(\eta, \delta)2^{(1+\delta)n}2^{\eta n}\exp(-k\log 2^{(1-\delta)n})\\
&= 2C(\eta, \delta)2^{-n(k(1-\delta)-1-\delta-\eta)}.
\end{aligned}$$

Note that, in order to obtain the second inequality above, we have used the Markov property and the fact that (4.29) holds uniformly for all $y \in M$. Now, we choose k to be given by

$$k(1-\delta) = 1 + 2\delta + \eta. \tag{4.30}$$

For this choice of k, we get

$$\mathbb{P}_x(A_n) \le 2C(\eta,\delta)2^{-\delta n}.$$

Hence $\sum \mathbb{P}_x(A_n) < \infty$ and the Borel-Camtelli lemma implies that, for $\mathbb{P}_x$ almost surely all $\omega \in \Omega$, there exists an integer $m(\omega)$ such that for all $n \ge m(\omega)$ and for all $(i, j) \in K_n$,

$$d(X_{i2^{-n}}, X_{j2^{-n}}) \le \sqrt{k(1+\delta)h((j-i)2^{-n})}. \tag{4.31}$$

Fix $\omega \in \Omega$ such that (4.31) holds. Fix $0 \le s < t \le 1$ with $0 < t - s < 2^{-(1-\delta)m(\omega)}$ and let $n \ge m(\omega)$ be such that $2^{-(1-\delta)(n+1)} \le t - s < 2^{-(1-\delta)n}$. Let i be the smallest integer such that $s \le i2^{-n}$ and j be the largest integer such that $j2^{-n} \le t$. Then we have $i < j$ provided that $2^{-(1-\delta)(n+1)} \ge 2^{-n+2}$. This is certainly satisfied if n is greater than $3\delta^{-1}$, which we can assume without loss of generality. Under this condition, we have

$$0 < j - i \le 2^n(t-s) \le 2^{\delta n}$$

and thus, $(i, j) \in K_n$. It follows that (4.31) applies. In particular,

$$d(X_{i2^{-n}}, X_{j2^{-n}}) \le \sqrt{k(1+\delta)h(t-s)}. \tag{4.32}$$

Write

$$\begin{aligned} s &= i2^{-n} - 2^{-u_1} - 2^{-u_2} - \cdots, \\ t &= j2^{-n} + 2^{-v_1} + 2^{-v_2} + \cdots \end{aligned}$$

where $(u_i), (v_i)$ are increasing sequences of integers greater than n. Observe that, for each l, the pairs

$$\begin{gathered} (i2^{-n} - 2^{-u_1} - \cdots - 2^{-u_{l+1}}, i2^{-n} - 2^{-u_1} - \cdots - 2^{-u_l}), \\ (j2^{-n} + 2^{-v_1} + \cdots + 2^{-v_l}, i2^{-n} + 2^{-v_1} + \cdots + 2^{-v_{l+1}}) \end{gathered}$$

are in K_n. Using (4.31) and the fact that $t \mapsto X_t(\omega)$ is continuous, we obtain

$$\begin{aligned} d(X_s, X_{i2^{-n}}) &\le \sum_{l>n} \sqrt{k(1+\delta)h(2^{-l})} \le \sqrt{k(1+\delta)h(2^{-n})} \sum_{\alpha \ge 1} 2^{-\alpha/2} \left(1 + \frac{\alpha}{n}\right)^{1/2} \\ &\le A\sqrt{\frac{2^{-\delta n}}{1-\delta}} \sqrt{k(1+\delta)h(2^{-(1-\delta)(n+1)})} \end{aligned}$$

$$\leq A\sqrt{\frac{2^{-\delta n}}{1-\delta}}\sqrt{k(1+\delta)h(t-s)}$$

for all $n \geq \max\{3\delta^{-1}, m(\omega)\}$ and with $A = \sum_{\alpha\geq 0} 2^{-\alpha/2}(2+\alpha)^{1/2}$. The same argument yields the same upper bound for $d(X_{j2^{-n}}, X_t)$. Thus, for $0 < t - s < 2^{-(1-\delta)m(\omega)}$, we have

$$d(X_s, X_t) \leq \left(1 + 2A\sqrt{\frac{2^{-\delta n}}{1-\delta}}\right)\sqrt{k(1+\delta)h(t-s)}.$$

It follows that

$$\lim_{\epsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{h(t-s)}} \leq \sqrt{k(1+\delta)}.$$

Using the definition of k at (4.30) and the fact that $\delta, \eta \in (0, 1)$ are arbitrary, we see that we can now let η, δ tend to 0 to obtain

$$\lim_{\epsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{h(t-s)}} \leq 1$$

as desired. □

4.3. Local rate of escape. The aim of this section is to prove Proposition 4.4 which complements [21, Theorem 5.1]. The proof is adapted from [15, 21, 26]. First we relate the hypothesis of this theorem to the notion of mean value inequality.

Lemma 4.3. *Fix a ball $B_0 = B(x_0, r_0)$. Assume that the doubling property* (3.3) *and the mean value inequality* (3.14) *hold around B_0. Then there exists a constant C_0 such that, for all $x, y \in B_0$ and all $t \in (0, \infty)$, we have*

$$h(t, x, y) \leq \frac{C_0}{\mu\left(B\left(x, \sqrt{t \wedge r_0^2}\right)\right)}.$$

Proof. For $t \in (0, r_0^2)$, see (3.15) and (3.4). For $t \geq r_0^2$, use [13, Lemma 1] and the result for $t < r_0^2$. □

Proposition 4.4. *Fix a $x_0 \in M$ and set $B(r) = B(x_0, r)$, $V(r) = \mu(B(r))$. Assume that there exists $r_0 > 0$ such that*:

1. *The doubling property holds at x_0 up to scale r_0, that is, there exists a constant D_0 such that*

$$\forall 0 < r < r_0, \qquad V(2r) \leq D_0 V(r). \tag{4.33}$$

2. *There exists a constant* C_0 *such that, for all* $y \in B(r_0)$ *and all* $t \in (0, \infty)$, *the heat kernel is bounded by*

$$h(t, x_0, y) \le \frac{C_0}{V\left(\sqrt{t \wedge r_0^2}\right)}. \tag{4.34}$$

For $r \in (0, r_0)$, *set*

$$m(r) = \int_r^{2r_0} \frac{s\,ds}{V(s)}$$

and assume that $m(r)$ *tends to* ∞ *as* r *tends to* 0. *Let* $\phi(t)$ *be an increasing positive function on* $(0, r_0^2)$ *such that*

$$\int_0^{r_0^2} \frac{1}{m(\phi(s))V(\sqrt{s})}\,ds < \infty. \tag{4.35}$$

Then, $\mathbb{P}_{x_0}$*-almost surely*,

$$\liminf_{t \to 0} \frac{d(x_0, X_t)}{\phi(t)} = \infty.$$

REMARK. The function m is decreasing and satisfies $m(2s) \ge \epsilon m(s)$ (see (4.42) below). It follows that changing ϕ to $k\phi$ where k is a positive constant has no effect on the result of the integral test (4.35). Hence, to prove Proposition 4.4, it suffices to show that (4.35) implies

$$\liminf_{t \to 0} \frac{d(x_0, X_t)}{\phi(t)} \ge 1 \qquad \mathbb{P}_{x_0}\text{-almost surely.}$$

EXAMPLE. Referring to the setting of Proposition 4.4, assume that $V(r) \approx r^n$ as r tends to 0, with $n > 2$. Then $m(r) \approx r^{-n+2}$. The integral test (4.35) becomes

$$\int_0^{r_0^2} \frac{\phi(s)^{n-2}}{s^{n/2}}\,ds < \infty.$$

In particular, the function $\phi(t) = \sqrt{t}\,[\log(1/t)]^{-\gamma}$ satisfies (4.35) if and only if $\gamma > 1/(n-2)$.

Proof of Proposition 4.4. Set

$$A_n = \{d(x_0, X_t) < \phi(t_n) \text{ for some } t \in [t_{n+1}, t_n]\}$$

where (t_n) is a decreasing sequence tending to 0 to be chosen later. Our aim is to show that, assuming that ϕ satisfies the integral test (4.35), (t_n) can be chosen so that

$\sum \mathbb{P}_{x_0}(A_n) < \infty$. If this is the case, then the Borel-Cantelli lemma shows that, $\mathbb{P}_{x_0}$-almost surely,

$$\liminf_{t\to 0} \frac{d(x_0, X_t)}{\phi(t)} \geq 1$$

as desired.

Consider the process $X^1 = (X_t^1)$ introduced in Section 3.6. By definition, X_t^1 is X_t killed at an independent exponential time ξ. For any positive $s < s'$ and r, we have

$$\begin{aligned} &\mathbb{P}_{x_0}(d(x_0, X_t) < r \text{ for some } t \in [s, s']) \\ &= \mathbb{P}_{x_0}(d(x_0, X_t) < r \text{ for some } t \in [s, s'];\, t \geq \xi) \\ &\quad + \mathbb{P}_{x_0}(d(x_0, X_t) < r \text{ for some } t \in [s, s'];\, t < \xi) \\ &\leq \mathbb{P}_{x_0}(s' \geq \xi) + \mathbb{P}_{x_0}(d(x_0, X_t^1) < r \text{ for some} t \geq s) \\ &= 1 - e^{-s'} + \mathbb{P}_{x_0}(d(x_0, X_t^1) < r \text{ for some} t \geq s). \end{aligned} \tag{4.36}$$

Since $1 - e^{-s'} \sim s'$ at 0, (4.36) shows that the series $\sum \mathbb{P}_{x_0}(A_n)$ converges if and only if the series $\sum t_n$ and $\sum \mathbb{P}_{x_0}(B_n)$ converge, where

$$B_n = \left\{d(x_0, X_t^1) < \phi(t_n) \text{ for some } t \geq t_{n+1}\right\}.$$

Thus it suffices to show that we can choose (t_n) so that the two series

$$\sum t_n, \qquad \sum \mathbb{P}_{x_0}(B_n)$$

converge.

Next, in the notation of Section 3.6, we have

$$\mathbb{P}_{x_0}(B_n) = \psi^1_{B(\phi(t_n))}(t_{n+1}, x_0).$$

Hence Theorem 3.10 gives

$$\mathbb{P}_{x_0}(B_n) \leq \mathrm{Cap}^1(B(\phi(t_n))) \int_{t_{n+1}}^{\infty} \sup_{y\in B(\phi(t_n))} h(s, x_0, y)e^{-s}\, ds. \tag{4.37}$$

We need to estimate

$$\mathrm{Cap}^1(B(r)) \qquad \text{and} \qquad \int_t^\infty \sup_{y\in B(r)} h(s, x_0, y)e^{-s}\, ds.$$

We start with the second term. For any $t \leq r_0^2$ and $r \leq r_0$, the hypothesis gives

$$\int_t^\infty \sup_{y\in B(r)} h(s, x_0, y)e^{-s}\, ds \leq C_0 \left(\int_t^{r_0^2} \frac{ds}{V(\sqrt{s})} + \frac{r_0^2}{V(r_0)} \right) \leq C_1 m(\sqrt{t}) \tag{4.38}$$

with $C_1 = 2C_0(1 + D_0/3)$.

For $\mathrm{Cap}^1(B(r))$ we use the very general estimate from [48, Theorem 2] (adapted to the case of Cap^1) which gives, for all $r' \in (r, r_0)$,

$$\mathrm{Cap}^1(B(r)) \le 2\left(\int_r^{r'} \frac{(s-r)\,ds}{V(s)-V(r)}\right)^{-1} + V(r'). \tag{4.39}$$

Following [21, p.85], write

$$\begin{aligned} \int_r^{r'} \frac{(s-r)\,ds}{V(s)-V(r)} &\ge \frac{1}{2}\int_{2r}^{r'} \frac{s\,ds}{V(s)} = 2\int_r^{r'/2} \frac{s\,ds}{V(2s)} \\ &\ge \frac{2}{D_0}\int_r^{r'/2} \frac{s\,ds}{V(s)} \\ &= \frac{2}{D_0}\left(m(r) - m\left(\frac{r'}{2}\right)\right). \end{aligned} \tag{4.40}$$

Next, by the doubling property (4.33), for $s \in (0, r_0/4)$, we have

$$\begin{aligned} m(s) - m(2s) &= \int_s^{2s} \frac{\tau\,d\tau}{V(\tau)} = \frac{1}{4}\int_{2s}^{4s} \frac{\tau\,d\tau}{V(\tau/2)} \\ &\le \frac{D_0}{4}\int_{2s}^{4s} \frac{\tau\,d\tau}{V(\tau)} \le \frac{D_0}{4}\int_{2s}^{r_0} \frac{\tau\,d\tau}{V(\tau)} \\ &\le \frac{D_0}{4} m(2s). \end{aligned} \tag{4.41}$$

Hence

$$m(2s) \ge \epsilon m(s) \tag{4.42}$$

where $\epsilon = (1 + D_0/4)^{-1}$. This implies that if we define $\rho = \rho(r)$ by $m(\rho/2) = \epsilon m(r)$, we have

$$\rho \ge 4r.$$

Using $r' = \rho = \rho(r)$ in (4.39) and (4.40), we obtain

$$\mathrm{Cap}^1(B(r)) \le \frac{D_0}{(1-\epsilon)m(r)} + V(\rho).$$

By the definition of m, we have

$$m(s) = \int_s^{2r_0} \frac{\tau\,d\tau}{V(\tau)} \le \frac{2r_0^2}{V(s)}.$$

Thus

$$V(\rho) \le D_0 V\left(\frac{\rho}{2}\right) \le \frac{2r_0^2 D_0}{m(\rho/2)} = \frac{2\epsilon r_0^2 D_0}{m(r)}$$

and

$$\mathrm{Cap}^1(B(r)) \le \frac{C_2}{m(r)}$$

with $C_2 = (2\epsilon r_0^2 + (1-\epsilon)^{-1})D_0 \le 5D_0 + 8r_0^2$.

Using this and (4.38) in (4.37) gives

$$\mathbb{P}_{x_0}(B_n) \le C_1 C_2 \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_n))}. \tag{4.43}$$

Consider the decreasing sequence (t_n) defined by $t_0 = r_0/4$ and

$$m(\sqrt{t_{n+1}}) - m(\sqrt{t_n}) = \frac{D_0}{4} m(\sqrt{t_n}).$$

By (4.41), we must have

$$t_{n+1} \le \frac{t_n}{2}. \tag{4.44}$$

Moreover,

$$m(\sqrt{t_n}) - m(\sqrt{t_{n-1}}) = \frac{D_0}{4} m(\sqrt{t_{n-1}}) = \frac{D_0}{4+D_0} m(\sqrt{t_n}) = \frac{4D_0}{(4+D_0)^2} m(\sqrt{t_{n+1}}).$$

Thus, we have

$$\begin{aligned}
\int_0^{r_0^2} \frac{1}{m(\phi(s))V(\sqrt{s})}\,ds &\ge \sum_n \int_{t_n}^{t_{n-1}} \frac{1}{m(\phi(s))V(\sqrt{s})}\,ds \\
&\ge \sum_n \frac{1}{m(\phi(t_n))} \int_{t_n}^{t_{n-1}} \frac{1}{V(\sqrt{s})}\,ds \\
&= 2\sum_n \frac{1}{m(\phi(t_n))} \left(m(\sqrt{t_n}) - m(\sqrt{t_{n-1}})\right) \\
&\ge \frac{8D_0}{(4+D_0)^2} \sum_n \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_n))}.
\end{aligned}$$

It follows that the hypothesis (4.35) implies

$$\sum_n \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_n))} < \infty.$$

Together with (4.43) and (4.44), this shows that the series

$$\sum t_n, \qquad \sum \mathbb{P}_{x_0}(B_n)$$

converge, as desired. □

5. Two-sided Gaussian bounds and some consequences

5.1. Iterated logarithm lower bound. This section shows that, under some suitable assumptions,

$$\limsup_{t\to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} \geq 1, \qquad \mathbb{P}_x \text{almost surely.}$$

Proposition 5.1. *Fix a ball B_0 and assume that the doubling inequality holds around B_0. Assume further that the heat kernel satisfies the following lower bound for some $\kappa > 0$. For any $\epsilon \in (0, 1)$, there exists $c_\epsilon > 0$ such that for all $t > 0$ and all $x, y \in M$ such that $B(x, \sqrt{t}), B(y, \sqrt{t}) \subset 2B_0$,*

$$h(t, x, y) \geq \frac{c_\epsilon}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{(1+\epsilon)\, d(x, y)^2}{\kappa t}\right).$$

Then, for any starting point $x \in (1/4)B_0$, $\mathbb{P}_x$-almost surely,

$$\limsup_{t\to 0} \frac{d(x, X_t)}{\sqrt{\kappa t \log\log(1/t)}} \geq 1.$$

Together with Proposition 4.1, this gives the following statement.

Corollary 5.2. *Assume that the parabolic Harnack inequality* (3.19) *holds around the ball B_0. Then there exists a constant $c > 0$ such that for all $x \in (1/4)B_0$,*

$$c \leq \limsup_{t\to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} \leq 1, \qquad \mathbb{P}_x \text{ almost surely.}$$

In some cases we are able to prove that $c = 1$ but, in general, this seems to be a difficult problem. See Section 6.

The proof of Proposition 5.1 given below differs slightly from classical arguments and has the advantage to be independent of any corresponding upper bound. We need the following simple lemma.

Lemma 5.3. *Under the hypotheses of* Proposition 5.1, *for any $\epsilon \in (0, 1)$, there exists a constant $c_\epsilon > 0$ such that for any x, t, r with $0 < t < r^2$ and $B(x, r) \subset$*

$(1/4)B_0$, *we have*

$$\mathbb{P}_x(d(x, X_t) \geq r) \geq c_\epsilon \exp\left(-(1+\epsilon)\frac{r^2}{\kappa t}\right).$$

Proof. Fix $B(x, r) \subset (1/4)B_0$ and $\epsilon \in (0, 1)$. Let ξ be a point such that $d(x, \xi) = (1 + \epsilon/2)r$. Observe that for any $y \in B_1 = B(\xi, \epsilon r/2)$ we have $d(x, y) \leq (1 + \epsilon)r$. Moreover the ball $B(y, r)$ is contained in $2B_0$ so that we can apply the heat kernel lower bound at (t, x, y) with $t \in (0, r^2)$, $y \in B_1$. This gives

$$\begin{aligned}
\mathbb{P}_x(d(x, X_t) \geq r) &= \int_{d(x,y)>r} h(t, x, y)\, dy \geq \int_{B_1} h(t, x, y)\, dy \\
&\geq c_\epsilon \frac{\mu(B(\xi, \epsilon r/2))}{\mu(B(x, \sqrt{t}))} \exp\left(-(1+\epsilon)^3 \frac{r^2}{\kappa t}\right) \\
&\geq c'_\epsilon \exp\left(-(1+\epsilon)^3 \frac{r^2}{\kappa t}\right).
\end{aligned}$$

The desired conclusion follows. □

Fix $\eta \in (0, 1)$. For $t > 0$, set $r_t = \eta\sqrt{\kappa t \log\log(1/t)}$ and

$$A_t = \{d(x, X_t) \geq r_t\}, \qquad A_t^c = \{d(x, X_t) < r_t\}.$$

We let $\mathcal{F}_t$ be the σ-algebra generated by $\{X_\tau : 0 \leq \tau \leq t\}$.

Lemma 5.4. *Under the hypotheses of* Proposition 5.1, *for any* $\epsilon \in (0, 1)$ *there exists* $c_\epsilon > 0$ *such that*

$$\mathbf{1}_{A_s^c}\mathbb{E}^x(\mathbf{1}_{A_t}|\mathcal{F}_s) \geq c_\epsilon \left(\log\frac{1}{t}\right)^{-(1+\epsilon)\eta^2} \mathbf{1}_{A_s^c}$$

for all $t > s > 0$ *and* $t, s/t$ *small enough.*

Proof. Using the Markov property, the triangle inequality and Lemma 5.3, we obtain

$$\begin{aligned}
\mathbf{1}_{A_s^c}\mathbb{E}^x(\mathbf{1}_{A_t}|\mathcal{F}_s) &= \mathbf{1}_{A_s^c}\mathbb{P}_{X_s}(d(x, X_{t-s}) \geq r_t) \\
&\geq \mathbf{1}_{A_s^c}\mathbb{P}_{X_s}(d(X_0, X_{t-s}) \geq r_t + r_s) \\
&\geq \left(\inf_{z:\, d(x,z)<r_s} \mathbb{P}_z(d(z, X_{t-s}) \geq r_t + r_s)\right)\mathbf{1}_{A_s^c} \\
&\geq \left(c'_\epsilon \exp\left(-(1+\epsilon)\frac{(r_t + r_s)^2}{\kappa(t-s)}\right)\right)\mathbf{1}_{A_s^c}.
\end{aligned}$$

The desired result follows because $r_s = o(r_t)$ as t and s/t tend to 0. $\square$

Proof of Proposition 5.1. Set $t_n = e^{-n\log n}$ and

$$A_n = A_{t_n} = \{d(x, X_{t_n}) \geq r_{t_n}\}, \qquad r_t = \eta\sqrt{\kappa t \log\log\frac{1}{t}}$$

with $\eta \in (0,1)$. We have

$$\mathbb{P}_x([A_n \text{ i.o.}]^c) = \lim_{m\to\infty}\mathbb{P}_x\left(\bigcap_{k\geq m} A_k^c\right).$$

Thus it suffices to show that $\mathbb{P}_x\left(\bigcap_{k\geq m} A_k^c\right) = 0$. For any $n \geq m$, we have

$$\begin{aligned}\mathbb{P}_x\left(\bigcap_{k=m}^n A_k^c\right) &= \mathbb{E}^x\left[\mathbf{1}_{\bigcap_{m+1}^n A_k^c}\mathbf{1}_{A_{m+1}^c}\mathbb{E}^x(\mathbf{1}_{A_m^c}|\mathcal{F}_{t_{m+1}})\right]\\ &= \mathbb{E}^x\left[\mathbf{1}_{\bigcap_{m+1}^n A_k^c}\left(1-\mathbf{1}_{A_{m+1}^c}\mathbb{E}^x(\mathbf{1}_{A_m}|\mathcal{F}_{t_{m+1}})\right)\right].\end{aligned}$$

Applying Lemma 5.4 with ϵ chosen so that $(1+\epsilon)\eta^2 = 1$, we obtain

$$\mathbb{P}_x\left(\bigcap_{k=m}^n A_k^c\right) \leq \mathbb{P}_x\left(\bigcap_{k=m+1}^n A_k^c\right)\left(1 - \frac{c}{m\log m}\right).$$

By induction, it follows that

$$\mathbb{P}_x\left(\bigcap_{k=m}^n A_k^c\right) \leq \prod_m^n\left(1 - \frac{c}{k\log k}\right) \leq e^{-\sum_m^n c/(k\log k)}.$$

As the series $\sum 1/(k\log k)$ diverges, this proves that $\mathbb{P}_x(\bigcap_{k\geq m} A_k^c) = 0$. As $\eta \in (0,1)$ is arbitrary, this finishes the proof of Proposition 5.1. $\square$

5.2. Lower bound for the Levy modulus of continuity. This section shows that, under some suitable assumptions,

$$\lim_{\epsilon\to 0}\sup_{\substack{0<s<t<1\\ t-s<\epsilon}}\frac{d(X_s, X_t)}{\sqrt{(t-s)\log(1/(t-s))}} \geq 1.$$

We start with the following result.

Proposition 5.5. *Fix $r_0 > 0$ and assume that the doubling inequality holds up to scale r_0. Assume further that the heat kernel satisfies the following lower bound for* *some $\kappa > 0$. For any $\epsilon \in (0,1)$, there exists $c_\epsilon > 0$ such that for all $t \in (0, r_0^2)$ and all $x, y \in M$,*

$$h(t,x,y) \geq \frac{c_\epsilon}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{(1+\epsilon)\, d(x,y)^2}{\kappa t}\right).$$

Then, for any starting point $x \in M$, $\mathbb{P}_x$-almost surely,

$$\lim_{\epsilon \to 0} \sup_{\substack{0<s<t<1 \\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{\kappa(t-s)\log(1/(t-s))}} \geq 1.$$

Before giving a proof of this proposition, we state the following theorem which is an immediate corollary of Theorems 3.5, 3.6 and Propositions 4.2, 5.5.

Corollary 5.6. *Assume that the parabolic Harnack inequality* (3.19) *holds up to scale r_0. Then there exists a constant $c > 0$ such that*

$$c \leq \lim_{\epsilon \to 0} \sup_{\substack{0<s<t<1 \\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{4(t-s)\log(1/(t-s))}} \leq 1.$$

Note that, in general, it seems difficult to show that one can take $c = 1$ in the above statement. However, for many special cases discussed below in Section 6, it is indeed possible to show that $c = 1$.

Proof of Proposition 5.5. The proof follows very classical lines. Fix $k \leq n$ and consider

$$A_n^k = \left\{ \sup_{0 \leq j < 2^k} d(X_{j2^{-n}}, X_{(j+1)2^{-n}}) \leq h(2^{-n}) \right\}$$

where $h(s) = \sqrt{\eta \kappa t \log(1/t)}$ with $\eta \in (0,1)$ to be chosen later (do not confuse this real function h with the heat kernel $h(t,x,y)$). Here κ is the constant appearing in Proposition 5.5. We claim that

$$\mathbb{P}_x(A_n^k) \leq \left(\sup_{z \in M} \mathbb{P}_z(A_n^0) \right)^k. \tag{5.1}$$

Indeed, by the Markov property, we have $\mathbb{P}_x(A_n^k) = \mathbb{E}_x(\mathbf{1}_{A_n^{k-1}} \mathbb{P}_{X_{k2^{-n}}}(A_n^0))$. The claim follows by induction.

Next, let $\epsilon \in (0,1)$ and $z \in M$ be arbitrary. Pick a point ξ such that $d(z,\xi) = (1+\epsilon)h(2^{-n})$. Then write

$$\mathbb{P}_z(A_n^0) = 1 - \mathbb{P}_z(d(z, X_{2^{-n}}) > h(2^{-n})) = 1 - \int_{d(z,y)>h(2^{-n})} h(2^{-n}, z, y)\, d\mu(y)$$

$$\leq 1 - \int_{d(\xi,y)<\epsilon h(2^{-n})} h(2^{-n}, z, y)\, d\mu(y)$$
$$\leq 1 - \frac{c_\epsilon \mu(B(\xi, \epsilon h(2^{-n}))}{\mu(B(z, 2^{-n/2}))} \exp(-(1+2\epsilon)^2 \eta \log 2^n).$$

For the last inequality we have used the heat kernel lower bound assumed in Proposition 5.5 and the fact that $d(z, y) \leq (1+2\epsilon)h(2^{-n})$ if $y \in B(\xi, \epsilon h(2^{-n}))$. By the assumed doubling property and (3.4), we have

$$\frac{\mu(B(\xi, \epsilon h(2^{-n})))}{\mu(B(z, 2^{-n/2}))} \geq D_1^{-1}(\epsilon^2 \eta \kappa)^{\nu/2}(\log 2^n)^{-\nu/2}.$$

Hence, if we pick $\eta = \eta_\epsilon = (1-\epsilon)/(1+3\epsilon)^2$, we have

$$\mathbb{P}_z(A_n^0) \leq 1 - c'_\epsilon 2^{-n(1-\epsilon)} \tag{5.2}$$

for $n > n_\epsilon$, n_ϵ large enough. Together with (5.1), (5.2) gives

$$\mathbb{P}_x(A_n^n) \leq \left(1 - c'_\epsilon 2^{-n(1-\epsilon)}\right)^{2^n} \leq \exp\left(-c'_\epsilon 2^{\epsilon n}\right).$$

Hence the series $\sum_n \mathbb{P}_x(A_n^n)$ converges. By the Borel-Cantelli lemma, it follows that for any $\epsilon \in (0, 1)$,

$$\limsup_{n\to\infty} \sup_{0\leq k<n} \frac{d(X_{k2^{-n}}, X_{(k+1)2^{-n}})}{\sqrt{\eta_\epsilon \kappa 2^{-n} \log 2^n}} \geq 1.$$

When ϵ tend to zero η_ϵ tend to 1 and we obtain the desired result. □

5.3. Upper bound for the rate of escape. The aim of this section is to prove a converse to Proposition 4.4.

Proposition 5.7. *Fix a ball $B_0 = B(x_0, r_0) \subset M$. Assume that the parabolic Harnack inequality* (3.19) *holds around B_0. Set*

$$V(r) = \mu(B(x_0, r)), \qquad m(r) = \int_r^{2r_0} \frac{s\, ds}{V(s)}.$$

Assume that m tends to ∞ at 0 *and let $\phi(t) \leq \sqrt{t}$ be an increasing positive function on* $(0, r_0^2)$ *such that*

$$\int_0^{r_0^2} \frac{1}{m(\phi(s))V(\sqrt{s})}\, ds = \infty. \tag{5.3}$$

Then, $\mathbb{P}_{x_0}$-almost surely,

$$\liminf_{t\to 0} \frac{d(x_0, X_t)}{\phi(t)} = 0.$$

REMARK. The proof of Proposition 5.7 is quite technical and is the most difficult part of this paper. Observe that the condition that $\phi(t) \le \sqrt{t}$ is quite harmless because $\phi(t) = \sqrt{t}$ does satisfy (5.3). Note also that the remark made after Proposition 4.4 applies to Proposition 5.7 as well so that it suffices to show that

$$\liminf_{t\to 0} \frac{d(x_0, X_t)}{\phi(t)} \le 1 \qquad \mathbb{P}_{x_0}\text{-almost surely.}$$

Together, Propositions 4.4, 5.7 and Theorems 3.5, 3.6 yield the following statement.

Corollary 5.8. *Fix a ball $B_0 = B(x_0, r_0) \subset M$. Assume that the parabolic Harnack inequality* (3.19) *holds around B_0. Set*

$$V(r) = \mu(B(x_0, r)), \qquad m(r) = \int_r^{2r_0} \frac{s\,ds}{V(s)}.$$

Assume that m tends to ∞ at 0 and let $\phi(t) \le \sqrt{t}$ be an increasing positive function on $(0, r_0^2)$. Then

$$\liminf_{t\to 0} \frac{d(x_0, X_t)}{\phi(t)} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_{x_0}\text{-a.s. iff the integral } \int_0^{r_0^2} \frac{1}{m(\phi(s))V(\sqrt{s})}\,ds \begin{cases} \text{diverges} \\ \text{converges.} \end{cases}$$

Before entering the proof of Proposition 5.7, we need to introduce some notation. Let $\tau = \inf\{t > 0 : X_t \in M \setminus B_0\}$ (i.e., the first exit time from B_0). As $\mathbb{P}_{x_0}$-almost surely $\tau > 0$, we can prove Proposition 5.7 by looking at the process killed at time τ. This will provide us with a useful localization. Let $h^0(t, x, y)$ denote the Dirichlet heat kernel in B_0 and set

$$g^0(x, y) = \int_0^\infty h^0(t, x, y)\,dt.$$

Thus $g^0(x, y)$ is the Green function for the Laplacian with Dirichlet boundary condition in B_0. By shrinking B_0 if necessary, we can assume that for $t \ge r_0^2$

$$\forall z, y \in B_0, \qquad h^0(t, z, y) \le C_3 e^{-c_3 t/r_0^2}. \tag{5.4}$$

This is equivalent to say that the lowest Dirichlet eigenvalue in B_0 is bounded below by $c_3 r_0^{-2}$. See [26, Theorem 2.5]. We will need the following inequalities concerning $h^0(t, x, y)$. For $y, z \in (1/2)B_0$ and $t \le r_0^2$, we have

$$\frac{c_1}{\mu(B(y, \sqrt{t}))} \exp\left(-\frac{C_1\, d(y, z)^2}{t}\right) \le h^0(t, y, z) \le \frac{C_2}{\mu(B(y, \sqrt{t}))} \exp\left(-c_2 \frac{d(y, z)^2}{t}\right). \tag{5.5}$$

In fact, the upper bound holds for all $y, z \in B_0$, $0 < t \le r_0^2$. See [26, Lemma 3.9].

Note that these estimates are available because of the assumption that the parabolic Harnack inequality (3.19) holds around B_0. See Theorem 3.6. The estimates (5.4) and (5.5) imply easily that for all $y, z \in (1/2)B_0$, we have

$$c \int_{d(y,z)^2}^{4r_0^2} \frac{ds}{\mu(B(y,\sqrt{s}))} \le g^0(y,z) \le C \int_{d(y,z)^2}^{4r_0^2} \frac{ds}{\mu(B(y,\sqrt{s}))}. \tag{5.6}$$

We need estimates of the function

$$u_r(z) = \mathbb{P}_z(X_s \in B(x_0, r) \text{ for some } s < \tau).$$

For two non-negative functions f, g, we write $f \approx g$ to indicate that there exist finite positive constants c, C such that $cg \le f \le Cg$ on the relevant domain.

Lemma 5.9. *Under the assumptions of* Proposition 5.7 *and assuming that* (5.4) *holds, we have*

$$u_r(z) \approx \min\left\{1, \frac{m(d(x_0,z))}{m(r)}\right\}$$

for all $z \in (1/2)B_0$ *and* $r \le r_0/4$.

Proof. It is well known that $u_r(z)$ admits the representation (see, e.g., [8])

$$u_r(z) = \int_{\partial B(x_0,r)} g^0(z,y)\, d\nu_{B(x_0,r)}(y)$$

where $\nu_{B(x_0,r)}$ is the equilibrium measure for $B(x_0, r)$. From this and (5.6) it follows that

$$u_r(x_0) = 1 = \int_{\partial B(x_0,r)} g^0(x_0,y)\, d\nu_{B(x_0,r)}(y) \approx m(r) \int_{\partial B(x_0,r)} d\nu_{B(x_0,r)}(y).$$

Thus, using (5.6) again, if $d(x_0, z) \ge 2r$ and $z \in (1/2)B_0$, we have

$$u_r(z) \approx \frac{m(d(x_0,z))}{m(r)}.$$

Moreover, u_r is bounded above by 1 and, if $d(x_0, z) \le 2r$, $u_r(z) \ge cm(3r)/m(r) \ge c' > 0$ as desired. □

Our next task is to bound

$$v_r(z,t) = \mathbb{P}_z(X_s \in B(x_0, r) \text{ for some } s \text{ with } t < s < \tau).$$

By the strong Markov property, we have

$$v_r(z,t) = \int h^0(t,z,y) u_r(y)\, d\mu(y).$$

Lemma 5.10. *Under the assumptions of* Proposition 5.7 *and assuming that* (5.4) *holds, there exist constants* $C, c > 0$ *such that*

$$c\frac{m(\sqrt{t})}{m(r)} \le v_r(x_0, t) \le C\frac{m(\sqrt{t})}{m(r)} \tag{5.7}$$

and

$$v_r(z, t) \le C\frac{m(\sqrt{t})}{m(r)}$$

for all $z \in B_0$, $r \le r_0/8$ *and* $t \in [(r/2)^2, (r_0/8)^2]$.

Proof. If t is of order r^2, the results are clear so we assume that $t > r^2$. Let x be a point such that $d(x_0, x) = 2\sqrt{t}$. Note that the ball $B(x, \sqrt{t})$ is contained in $(1/2)B_0$ and write

$$v_r(x_0, t) \ge \int_{B(x,\sqrt{t})} h^0(t, x_0, y)u_r(y)\, d\mu(y) \ge c\frac{m(\sqrt{t})}{m(r)}$$

where the last inequality follows from Lemma 5.9, (5.5) and (3.4). This proves the desired lower bound for $v_r(x_0, t)$.

We are left with the task of proving the upper bound

$$v_r(z, t) \le C\frac{m(\sqrt{t})}{m(r)}.$$

Write

$$\begin{aligned} v_r(z, t) &= \int_{B_0} h^0(t, z, y)u_r(y)\, dy \\ &= \left(\int_{B_0\setminus B(x_0,\sqrt{t})} + \int_{B(x_0,\sqrt{t})\setminus B(x_0,r)} + \int_{B(x_0,r)}\right) h^0(t, z, y)u_r(y)\, d\mu(y). \end{aligned}$$

In the first integral $d(x_0, y) \ge \sqrt{t}$, hence $u_r(y) \le Cm(\sqrt{t})/m(r)$. Moreover

$$\int h^0(t, z, y)\, d\mu(y) \le 1.$$

In the second and third integrals, use $h^0(t, z, y) \le C/V(\sqrt{t})$ (this follows from (5.5), the doubling property and the fact that $y \in B(x_0, \sqrt{t})$) and Lemma 5.9. This gives

$$v_r(z, t) \le C\left(\frac{m(\sqrt{t})}{m(r)} + \frac{1}{V(\sqrt{t})m(r)}\int_{B(x_0,\sqrt{t})\setminus B(x_0,r)} m(d(x_0, y))\, d\mu(y) + \frac{V(r)}{V(\sqrt{t})}\right).$$

Next, use integration by parts and the fact that $m'(s) = -s/V(s)$ to estimate

$$\begin{aligned}\int_{B(x_0,\sqrt{t})\setminus B(x_0,r)} m(d(x_0,y))\,d\mu(y) &= \int_r^{\sqrt{t}} m(s)\,dV(s)\\ &\le m(\sqrt{t})V(\sqrt{t}) + t.\end{aligned}$$

This yields

$$\begin{aligned}v_r(z,t) &\le C\left(\frac{m(\sqrt{t})}{m(r)} + \frac{t}{V(\sqrt{t})m(r)} + \frac{V(r)}{V(\sqrt{t})}\right)\\ &\le C\left(1 + \frac{t}{V(\sqrt{t})m(\sqrt{t})} + \frac{V(r)m(r)}{V(\sqrt{t})m(\sqrt{t})}\right)\frac{m(\sqrt{t})}{m(r)}.\end{aligned}$$

We need to show that the factor in brackets is bounded. We have

$$\frac{t}{V(\sqrt{t})} \le C\int_{\sqrt{t}}^{2\sqrt{t}} \frac{s\,ds}{V(s)} \le Cm(\sqrt{t}).$$

To bound

$$\frac{V(r)m(r)}{V(\sqrt{t})m(\sqrt{t})}$$

observe that for any $0 < r < R < r_0$, we have

$$\begin{aligned}V(R)m(R) - V(r)m(r) &= [V(R) - V(r)]m(R) - V(r)[m(r) - m(R)]\\ &\ge -V(r)\int_r^R \frac{s\,ds}{V(s)} \ge -R^2\end{aligned}$$

and, using the volume doubling property,

$$V(R)m(R) = V(R)\int_R^{2r_0} \frac{s\,ds}{V(s)} \ge V(R)\int_R^{2R} \frac{s\,ds}{V(s)} \ge cR^2.$$

Thus, $V(r)m(r) \le CV(R)m(R)$. This proves that

$$v_r(x,t) \le C\frac{m(\sqrt{t})}{m(r)}$$

and ends the proof of Lemma 5.10. □

Let us now consider the quantity

$$v_r(x,s,t) = \mathbb{P}_x(X_a \in B(x_0,r) \text{ for some } a \in (s,t),\ \ a < \tau).$$

We need the following definition.

DEFINITION 5.1. Let $C > c > 0$ be such that (5.7) holds true. For $t \in [0, (r_0/8)^2]$, define $\theta(t)$ to be such that

$$m(\sqrt{t}) = \frac{2C}{c} m\left(\sqrt{\theta(t)}\right).$$

Note that θ is an increasing function and $s < \theta(s)$.

Lemma 5.11. *Under the assumptions of* Proposition 5.7 *and assuming that* (5.4) *holds, we have*:

1. *For all* $(r/2)^2 < s < t < (r_0/8)^2$, $x \in B_0$,

$$v_r(x, s, t) \leq K \frac{m(\sqrt{s})}{m(r)}.$$

2. *For all* $(r/2)^2 < s < \theta(s) < t < (r_0/8)^2$,

$$v_r(x_0, s, t) \approx \frac{m(\sqrt{s})}{m(r)}.$$

3. *For all* $(r/2)^2 < s < t < (r_0/8)^2$, $x \in B(x_0, \sqrt{s})$,

$$|v_r(x, s, t) - v_r(x_0, s, t)| \leq K \left(\frac{d(x_0, x)}{\sqrt{s}}\right)^{\alpha} \frac{m(\sqrt{s})}{m(r)}.$$

Proof. The first statement follows from Lemma 5.10 since $v_r(x, s, t) \leq v_r(x, s)$. To prove the second statement, write

$$v_r(x_0, s, t) \geq v_r(x_0, s) - v_r(x_0, t).$$

The desired result then follows from (5.7) and Definition 5.1.

To prove the third statement, observe that, by the strong Markov property, for any $0 < s < t$, we have

$$v_r(x, s, t) = \int v_r\left(z, \frac{s}{2}, t - \frac{s}{2}\right) h^0\left(\frac{s}{2}, x, z\right) d\mu(z).$$

Hence,

$$|v_r(x, s, t) - v_r(x_0, s, t)| \leq \int v_r\left(z, \frac{s}{2}, t - \frac{s}{2}\right) \left|h^0\left(\frac{s}{2}, x, z\right) - h^0\left(\frac{s}{2}, x_0, z\right)\right| d\mu(z).$$

Assume that $x \in B(x_0, \sqrt{s})$. By (3.20), for $x \in B(x_0, \sqrt{s})$, we have

$$\left|h^0\left(\frac{s}{2}, x, z\right) - h^0\left(\frac{s}{2}, x_0, z\right)\right| \leq K \left(\frac{d(x_0, x)}{\sqrt{s}}\right)^{\alpha} \frac{1}{V(\sqrt{s})} \exp\left(-\epsilon \frac{d(x_0, z)^2}{s}\right).$$

Hence, after some computation,

$$|v_r(x,s,t) - v_r(x_0,s,t)| \le K\left(\frac{d(x_0,x)}{\sqrt{s}}\right)^{\alpha}\frac{m(\sqrt{s})}{m(r)}. \qquad \square$$

Proof of Proposition 5.7. Fix $t_0 = (r_0/8)^2$ and define t_n inductively by

$$t_{n+1} = \theta^{-1}\left(\frac{t_n}{2}\right) \tag{5.8}$$

with θ as in Definition 5.1. Note that this definition implies that, for all integers $n < m$,

$$t_m \le 2^{-(m-n)} t_n. \tag{5.9}$$

Set

$$A_n = \{d(x_0, X_s) < r_n \text{ for some } s \in (t_{n+1}, t_n),\ \ s < \tau\} \qquad \text{where } r_n = \phi(t_{n+1}).$$

Then, by Lemma 5.11 (2), there exists $c_0 > 0$ such that

$$\mathbb{P}_{x_0}(A_n) = v_{r_n}(x_0, t_{n+1}, t_n) \ge c_0 \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_{n+1}))}.$$

We claim that the series

$$\sum \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_{n+1}))} \tag{5.10}$$

diverges. Indeed, for any decreasing sequence t_n such that $m(\sqrt{t_n}) \ge \epsilon m(\sqrt{t_{n+1}})$ for some $\epsilon > 0$, we have

$$\begin{aligned}
\int_{t_{N+1}}^{t_0} \frac{ds}{m(\phi(s))V(\sqrt{s})} &= \sum_0^N \int_{t_{n+1}}^{t_n} \frac{ds}{m(\phi(s))V(\sqrt{s})} \\
&\le \sum_0^N \frac{1}{m(\phi(t_n))} \int_{t_{n+1}}^{t_n} \frac{ds}{V(\sqrt{s})} \\
&\le 2\sum_0^N \frac{m(\sqrt{t_{n+1}})}{m(\phi(t_n))} \le \frac{2}{\epsilon}\sum_0^N \frac{m(\sqrt{t_n})}{m(\phi(t_n))}.
\end{aligned}$$

By the definition of the function θ, the sequence t_n at (5.8) satisfies the condition $m(\sqrt{t_n}) \ge \epsilon m(\sqrt{t_{n+1}})$ (for some $\epsilon > 0$). Hence, by (5.3), the series at (5.10) diverges as claimed.

If the events A_n were independent, we could apply the Borel-Cantelli lemma to conclude the proof of Proposition 5.7 but, obviously, the A_n are not independent. We will need a more sophisticated version of the Borel-Cantelli lemma which deals with asymptotically independent events. First, we use the following elementary observation: for any sequence of non-negative numbers a_n such that $\sum a_n = \infty$ we can find an increasing sequence m_i of integers such that

$$\sum_i a_{m_i} = \infty, \qquad \lim_{i\to\infty}(m_{i+1} - m_i) = \infty \quad \text{and} \quad a_{m_i} > m_i^{-2}.$$

Let m_i be such a sequence for the series $\sum \mathbb{P}_{x_0}(A_n)$ and set

$$A'_i = A_{m_i}.$$

To show that

$$\liminf_{t\to 0} \frac{d(x_0, X_t)}{\phi(t)} \le 1 \qquad \mathbb{P}_{x_0}\text{-a.s.}$$

it suffices to show that

$$\mathbb{P}_{x_0}(A'_i \text{ i.o.}) = 1.$$

This will follow from the divergence of $\sum_i \mathbb{P}_{x_0}(A'_i)$ if we can show that the events A'_i are asymptotically independent, that is,

$$\lim_{\substack{i,j\to\infty \\ i<j}} \frac{\mathbb{P}_{x_0}(A'_i \cap A'_j)}{\mathbb{P}_{x_0}(A'_i)\mathbb{P}(A'_j)} = 1. \tag{5.11}$$

We claim that, indeed, (5.11) holds. To prove this claim we will show that

$$\lim_{\substack{i,j\to\infty \\ i<j}} \left| \frac{\mathbb{P}_{x_0}(A'_i | A'_j)}{v_{r'_i}(x_0, \sigma_i, \theta_i)} - 1 \right| = 0 \tag{5.12}$$

and

$$\lim_{i\to\infty} \left| \frac{\mathbb{P}_{x_0}(A'_i)}{v_{r'_i}(x_0, \sigma_i, \theta_i)} - 1 \right| = 0 \tag{5.13}$$

where

$$r'_i = r_{m_i} = \phi(t_{m_i+1}), \qquad \sigma_i = t_{m_i+1} - t_{m_j}, \qquad \theta_i = t_{m_i} - t_{m_j}, \qquad i < j.$$

Observe that these parameters satisfy

$$\sigma_i \in \left(\frac{t_{m_i+1}}{2}, t_{m_i+1} \right), \qquad \theta_i \in \left(\frac{t_{m_i}}{2}, t_{m_i} \right), \qquad (r'_i)^2 \le 2\sigma_i. \tag{5.14}$$

It follows that

$$\sigma_i < \theta(\sigma_i) < \theta_i.$$

This will allow us to use Lemma 5.11 in what follows.

We start by proving (5.13) which is simpler. By the strong Markov property, we have

$$\mathbb{P}_{x_0}(A_i') = \int v_{r_i'}(z, t_{m_i+1} - t_{m_j}, t_{m_i} - t_{m_j}) h^0(t_{m_j}, x_0, z)\, d\mu(z).$$

Hence

$$\begin{aligned} |\mathbb{P}_{x_0}(A_i') - v_{r_i'}(x_0, \sigma_i, \theta_i)| \le \int & |v_{r_i'}(z, \sigma_i, \theta_i) - v_{r_i'}(x_0, \sigma_i, \theta_i)| h^0(t_{m_j}, x_0, z)\, d\mu(z) \\ & + v_{r_i'}(x_0, \sigma_i, \theta_i)\mathbb{P}_{x_0}(\tau < t_{m_j}). \end{aligned} \tag{5.15}$$

By (3.21) and the hypotheses of Proposition 5.7,

$$\mathbb{P}_{x_0}(\tau < t_{m_j}) \le K \exp\left(-\frac{r_0^2}{4t_{m_j}}\right).$$

This shows that the last term in (5.15) is harmless. To bound

$$\int |v_{r_i'}(z, \sigma_i, \theta_i) - v_{r_i'}(x_0, \sigma_i, \theta_i)| h^0(t_{m_j}, x_0, z)\, d\mu(z)$$

we integrate over $B(x_0, \sqrt{\sigma_i})$ and $B_0 \setminus B(x_0, \sqrt{\sigma_i})$ separately. Using Lemma 5.11 (3), the integral over $B(x_0, \sqrt{\sigma_i})$ is bounded by

$$K \frac{m(\sqrt{\sigma_i})}{m(r_i')} \int_{B(x_0, \sqrt{\sigma_i})} \left(\frac{d(x_0, x)}{\sqrt{\sigma_i}}\right)^\alpha h^0(t_{m_j}, x_0, x)\, d\mu(x).$$

By the upper bound in (5.5) and the doubling property around B_0, we have (for a different constant K)

$$\int_{B(x_0, \sqrt{\sigma_i})} \left(\frac{d(x_0, x)}{\sqrt{\sigma_i}}\right)^\alpha h^0(t_{m_j}, x_0, x)\, d\mu(x) \le K \left(\frac{t_{m_j}}{\sigma_i}\right)^{\alpha/2}.$$

Thus this part of the integral is bounded by

$$K \frac{m(\sqrt{\sigma_i})}{m(r_i')} \left(\frac{t_{m_j}}{\sigma_i}\right)^{\alpha/2}.$$

For the part of the integral over $B_0 \setminus B(x_0, \sqrt{\sigma_i})$, Lemma 5.11 (1), the upper bound in (5.5) and the doubling property around B_0 give the bound

$$K \frac{m(\sqrt{\sigma_i})}{m(r_i')} \exp\left(-\frac{\epsilon \sigma_i}{t_{m_j}}\right)$$

for some $\epsilon > 0$. By Lemma 5.11 (2), we conclude that

$$\begin{aligned}
&\int |v_{r_i'}(z,\sigma_i,\theta_i) - v_{r_i'}(x_0,\sigma_i,\theta_i)| h^0(t_{m_j},x_0,z)\,d\mu(z) \\
&\le K \frac{m(\sqrt{\sigma_i})}{m(r_i')} \left(\frac{t_{m_j}}{\sigma_i}\right)^{\alpha/2} \\
&\le K \left(\frac{t_{m_j}}{\sigma_i}\right)^{\alpha/2} v_{r_i'}(x_0,\sigma_i,\theta_i).
\end{aligned}$$

Hence

$$|\mathbb{P}_{x_0}(A_i') - v_{r_i'}(x_0,\sigma_i,\theta_i)| \le K\left(\left(\frac{t_{m_j}}{\sigma_i}\right)^{\alpha/2} + \exp\left(-\frac{r_0^2}{4t_{m_j}}\right)\right) v_{r_i'}(x_0,\sigma_i,\theta_i)$$

By (5.14) and (5.9), $t_{m_j}/\sigma_i \le 2t_{m_j}/t_{m_i+1} \le 2^{-(m_j-m_i)} \le 2^{-(m_{i+1}-m_i)}$. As $m_{i+1} - m_i$ tends to infinity with i, this proves (5.13).

We now turn to the proof of (5.12). Consider the conditional measure λ defined for any Borel set $E \subset M$ by

$$\lambda(E) = \mathbb{P}_{x_0}(X_{t_{m_j}} \in E,\ t_{m_j} < \tau | A_j').$$

By the strong Markov property, we have

$$\mathbb{P}_{x_0}(A_i'|A_j') = \int v_{r_i'}(z, t_{m_i+1} - t_{m_j}, t_{m_i} - t_{m_j})\,d\lambda(z).$$

Hence

$$\begin{aligned}
|\mathbb{P}_{x_0}(A_i'|A_j') - v_{r_i'}(x_0,\sigma_i,\theta_i)| \le &\int |v_{r_i'}(z,\sigma_i,\theta_i) - v_{r_i'}(x_0,\sigma_i,\theta_i)|\,d\lambda(z) \\
&+ v_{r_i'}(x_0,\sigma_i,\theta_i)\mathbb{P}_{x_0}(\tau < t_{m_j}|A_j').
\end{aligned}$$

By the choice of the sequence m_j, we have $\mathbb{P}_{x_0}(A_j') > m_j^{-2}$. Hence

$$\mathbb{P}_{x_0}(\tau < t_{m_j}|A_j') \le m_j^2 \mathbb{P}_{x_0}(\tau < t_{m_j}) \le K m_j^2 \exp\left(-\frac{r_0^2}{4t_{m_j}}\right).$$

Next, set $B_i = B(x_0, \sqrt{\sigma_i})$, $i = 1, 2, \ldots$, and use Lemma 5.11 to write

$$\begin{aligned}
&\int \left|v_{r_i'}(z,\sigma_i,\theta_i) - v_{r_i'}(x_0,\sigma_i,\theta_i)\right| d\lambda(z) \\
&= \int_{B_i} \left|v_{r_i'}(z,\sigma_i,\theta_i) - v_{r_i'}(x_0,\sigma_i,\theta_i)\right| d\lambda(z) \\
&\quad + \int_{B_0\setminus B_i} \left|v_{r_i'}(z,\sigma_i,\theta_i) - v_{r_i'}(x_0,\sigma_i,\theta_i)\right| d\lambda(z)
\end{aligned}$$

$$\leq K v_{r_i'}(x_0, \sigma_i, \theta_i) \left(\int_{B_i} \left(\frac{d(x_0, x)}{\sqrt{\sigma_i}} \right)^{\alpha} d\lambda(x) + \int_{B_0 \setminus B_i} d\lambda(x) \right)$$

Set

$$\zeta = \inf\{t > t_{m_j+1} : d(x_0, X_t) < r_{m_j}\}.$$

Then $\mathbb{P}_{x_0}(A_j') = \mathbb{P}_{x_0}(\zeta < t_{m_j}, \zeta < \tau)$ and by the strong Markov property, for any measurable function $f \geq 0$, we have

$$\int f(x)\, d\lambda(x) = \frac{1}{\mathbb{P}_{x_0}(A_j')} \mathbb{E}_{x_0} \left(\mathbf{1}_{\{\zeta < t_{m_j}, \zeta < \tau\}} \int h^0(t_{m_j} - \zeta, X_\zeta, x) f(x)\, d\mu(x) \right)$$
$$\leq \sup \left\{ \int h^0(s, z, x) f(x)\, d\mu(x) : 0 < s < t_{m_j},\ d(x_0, z) < r_{m_j} \right\}.$$

By the upper bound in (5.5) and the doubling property around B_0, for all $s \in (0, t_{m_j})$ and $z \in B(x_0, r_{m_j})$, we have

$$\int_{B_i} \left(\frac{d(x_0, x)}{\sqrt{\sigma_i}} \right)^{\alpha} h^0(s, z, x)\, d\mu(x) \leq K \left(\frac{r_{m_j} + \sqrt{s}}{\sqrt{\sigma_i}} \right)^{\alpha}$$

and

$$\int_{B_0 \setminus B_i} h^0(s, z, x)\, d\mu(x) \leq K \exp\left(-\frac{\epsilon \sigma_i}{s} \right).$$

To obtain these inequalities, note that $\sqrt{\sigma_i} \pm r_{m_j} \approx \sqrt{\sigma_i}$ for $i \neq j$ large enough and use this to move the center x_0 of B_i to z so that these integrals can be bounded exactly as in the proof of (5.13). As $r_{m_j} \leq \sqrt{t_{m_j}}$, it follows that

$$\int \left| v_{r_i'}(z, \sigma_i, \theta_i) - v_{r_i'}(x_0, \sigma_i, \theta_i) \right| d\lambda(z) \leq K v_{r_i'}(x_0, \sigma_i, \theta_i) \left(\frac{t_{m_j}}{\sigma_i} \right)^{\alpha/2}.$$

Thus

$$\left| \mathbb{P}_{x_0}(A_i' | A_j') - v_{r_i'}(x_0, \sigma_i, \theta_i) \right| \leq K v_{r_i'}(x_0, \sigma_i, \theta_i) \left(\left(\frac{t_{m_j}}{\sigma_i} \right)^{\alpha/2} + m_j^2 \exp\left(-\frac{r_0^2}{4 t_{m_j}} \right) \right).$$

We have $t_{m_j} \leq 2^{-m_j} t_0$ and $t_{m_j}/\sigma_i \leq 2^{-(m_{i+1} - m_i)}$. As $m_{i+1} - m_i$ tends to infinity with i, this proves (5.12). $\square$

6. Examples

The authors started to write this paper in order to collect results concerning the regularity of paths of diffusions in finite dimensional type settings. There are excellent well known references for Brownian motion on $\mathbb{R}$ such as [30, 33, 40] but references treating more general setups are harder to find. The best studied of the three problems discussed here is probably the law of iterated logarithm and its more advanced version due to Strassen (see, e.g., [14, 45]). The work [3] is very much in the spirit of the present paper. There is much less literature on Dvoretzky-Erdös rate of escape [15]. For random walks in $\mathbb{R}^n$, see [18, 39].

Below we present applications of the results proved in the previous sections to different natural settings such as Riemannian manifolds and sub-elliptic symmetric diffusions. There are overlaps between our different examples but we hope that this presentation will make it easy for the reader to find the statements concerning her preferred example.

6.1. Brownian motion on Riemannian manifolds. As expected for an asymptotic result, the law of iterated logarithm holds on any smooth complete Riemannian manifold M.

Theorem 6.1. *Let M be a smooth complete Riemannian manifold without boundary. Let $(X_t, \mathbb{P}_x)$ denote Brownian motion on M, i.e., the diffusion process driven by the Laplace-Beltrami operator* $\Delta = -\operatorname{div}\operatorname{grad}$. *Then, for any $x \in M$, $\mathbb{P}_x$-almost surely,*

$$\limsup_{t\to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} = 1.$$

Proof. This easily follows from Sections 4.1 and 5.1. The needed Gaussian heat kernel bounds can be extracted form [34] (see also [43, 54]). □

REMARK. This Theorem extends without changes to manifolds with boundary as long as we consider "interior" starting point x (i.e., points that are not on the boundary). In fact, the same result should hold even if the starting point x is on the boundary. For instance, in the half space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0, \infty)$, the law of iterated logarithm holds from any starting point including points of the form $(x_1, \dots, x_n, 0)$. The difficulty here is to obtain the needed Gaussian estimates up to the boundary. The paper [57] gives the needed upper bound as well as some lower bound for compact manifolds but fails to provide the sharp lower bound involving $\exp(-d(x, y)^2/(4(1 + \epsilon)t))$ for arbitrary small ϵ. Further work is needed to obtain the sharp form of the law of iterated logarithm by the approach of this paper when the starting point is on the boundary.

Turning to Lévy's result concerning the modulus of continuity of Brownian paths, let us start by noting that it would be very unreasonable to expect such a result to hold in the generality of Theorem 6.1. This is because Lévy's result concerns the uniform regularity of Brownian paths on the time interval $[0, 1]$. At the very least, such a control requires non-explosion (explosion occurs when Brownian motion goes to infinity in finite time). For background on the explosion phenomenon on Riemannian manifolds, see the excellent article [22]. Even without explosion, it is likely that any Lévy type result requires some uniform assumption on the local geometry of M. The following result appeals to the convenient setting of manifolds with Ricci curvature bounded below.

Theorem 6.2. *Assume that* (M, g) *is a smooth complete Riemannian manifold* (*without boundary*) *whose Ricci curvature tensor satisfies* $\mathrm{Ric} \geq -Kg$ *for some* $K \geq 0$. *Let* $(X_t, \mathbb{P}_x)$ *denotes Brownian motion on* M. *Then, for any* $x \in M$ *and* $\mathbb{P}_x$*-almost surely, we have*

$$\lim_{\epsilon \to 0} \sup_{\substack{0<s<t<1 \\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{4(t-s)\log(1/(t-s))}} = 1.$$

This follows from Sections 4.2 and 5.2. The needed heat kernel estimates can be found in [34], [54].

For complete Riemannian manifolds, the local version of the result of Dvoretzky and Erdös holds as in $\mathbb{R}^n$.

Theorem 6.3. *Let* (M, g) *be a smooth complete Riemannian manifold* (*without boundary*) *of dimension* n. *Let* $(X_t, \mathbb{P}_x)$ *denotes Brownian motion on* M. *If* $n \geq 3$, $x \in M$, *and* ψ *is a positive increasing function, then*

$$\liminf_{t \to 0} \frac{d(x, X_t)}{\psi(t)\sqrt{t}} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x\text{-}a.s. \iff \sum_k \psi(2^{-k})^{n-2} \begin{cases} \textit{diverges} \\ \textit{converges.} \end{cases}$$

This follows form Sections 4.3, 5.3 and the heat kernel estimates of [34, 43]. Using the result of [57], the same statement holds for compact manifolds with boundary (including the case where the starting point x is on the boundary). For the case where $n = 2$, see Section 6.3 below.

6.2. Left invariant diffusions on groups. Let G be a connected real Lie group equipped with a Haar measure μ (this measure may be either left or right invariant, it will not matter in what follows). Consider a set $\xi = \{\xi_1, \dots, \xi_k\}$ of left invariant vector fields such that the Lie algebra generated by ξ equals the Lie algebra of G. In such cases, we say that ξ satisfies the Hörmander condition (see, e.g., [28, 38, 56]). Consider the Dirichlet form $\mathcal{E}$ on $L^2(G, \mu)$ obtained as the least extension of

$$f \mapsto \int_G \sum_1^k |\xi_i f|^2 \, d\mu, \qquad f \in \mathcal{C}_0^\infty(G).$$

We let $(X_t, \mathbb{P}_x)$ be the associated diffusion. If μ is a right Haar measure, then the associated infinitesimal generator $-L$ is given by $-L = \sum_1^k \xi_i^2$. If μ is a left Haar measure then $-L = \sum_1^k \xi_i^2 - \sum_1^k \lambda_i \xi_i$ where $\lambda_i = m^{-1}\xi_i m$ and m is the modular function (as m is multiplicative, the λ_i's are constant). See, e.g., [43, 56]. The hypothesis that ξ satisfies the Hörmander condition implies that the intrinsic distance $d = d_\xi$ is finite, continuous and defines the topology of G (e.g., [38, 56]). Independently of whether μ is a left or right Haar measure, the volume is uniformly doubling up to scale 1 and there exists an integer $N = N_\xi$ such that $\mu(B(x,r)) \approx c(x)r^N$ (uniformly) for all $x \in G$ and all $r \in (0,1)$. Here $c(x) = 1$ if μ is a left Haar measure whereas $c(x) = m(x)$ if μ is a right Haar measure. See [56] for how to compute the integer N_ξ. This integer is greater or equal to the topological dimension with equality if and only if ξ contains a linear basis of the Lie algebra of G.

Theorem 6.4. *Referring to the setup above, assume that ξ satisfies the Hörmander condition. Then, for any $x \in G$, we have*

$$\lim_{t\to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} = 1, \quad \mathbb{P}_x\text{-almost surely;}$$

$$\lim_{\epsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{4(t-s)\log(1/(t-s))}} = 1, \quad \mathbb{P}_x\text{-almost surely;}$$

and, assuming $N = N_\xi > 2$,

$$\liminf_{t\to 0} \frac{d(x, X_t)}{\psi(t)\sqrt{t}} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x\text{-a.s.} \iff \sum_k \psi(2^{-k})^{N-2} \begin{cases} \text{diverges} \\ \text{converges} \end{cases}$$

for any increasing positive function ψ.

The proof follows from the general results of the present paper and the theory developed in [56] which provides the necessary parabolic Harnack inequality up to scale 1 (this in fact goes back to Bony [9]). The precise Gaussian heat kernel lower bound needed to obtain the sharp form of the law of iterated logarithm and the modulus of continuity are taken from [55].

6.3. Sub-elliptic diffusions. Let Ω be a non-empty open set in $\mathbb{R}^n$. Let L be a second order differential operator of the form

$$L = -v^{-1} \sum_{i,j} \partial_i (v a_{i,j} \partial_j)$$

where v, $a_{i,j} = a_{j,i}$ are smooth functions and $v > 0$, $\sum_{i,j} a_{i,j}(x) y_i y_j \geq 0$ for all $x \in \Omega$, $(y_i)_1^n \in \mathbb{R}^n$. We say that this operator L is sub-elliptic if there exists $s > 0$ and $C > 0$

such that

$$(6.16) \qquad \forall u \in \mathcal{C}_0^\infty(\Omega), \qquad \|u\|_{W_s^2} \leq C \int (|u|^2 + uLu)\, d\mu$$

where $d\mu(x) = v(x)\, dx$ and W_s^2 is the classical Sobolev space in $\mathbb{R}^n$. If $\Omega = \mathbb{R}^n$, we say that L is uniformly sub-elliptic if (6.16) holds, and the functions $v, v^{-1}, a_{i,j}$ and their partial derivatives of any order are bounded functions. We refer the reader to [32] for an excellent exposition concerning such operators and for further references. Note that sums of squares of Hörmander vector fields are sub-elliptic (see [28, 32]).

Consider a regular strictly local Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(M, \mu))$ where M is a manifold and $\mathcal{C}_0^\infty(M) \subset \mathcal{D}$. We say that $(\mathcal{E}, \mathcal{D}, L^2(M, \mu))$ is sub-elliptic if in any small enough coordinate chart U the measure μ has the form $d\mu(x) = v(x)\, dx$ for some smooth positive function v and the associated infinitesimal generator restricted to $\mathcal{C}_0^\infty(U)$ is a sub-elliptic operator as defined above. Under this assumption, the intrinsic distance d is continuous and defines the topology of M. For simplicity, we include in our definition of a sub-elliptic Dirichlet space the additional global hypothesis that (M, d) is complete. Two simple examples of interest are the case when $M = \mathbb{R}^n$ and the case when M is a compact manifold (without boundary).

Theorem 6.5. *Assume that $(\mathcal{E}, \mathcal{D}, L^2(M, \mu))$ is a sub-elliptic Dirichlet space as defined above. Let $(X_t, \mathbb{P}_x)$ be the associated diffusion. Then for any $x \in G$, we have*

$$\lim_{t \to 0} \frac{d(x, X_t)}{\sqrt{4t \log\log(1/t)}} = 1, \qquad \mathbb{P}_x\text{-almost surely.}$$

The local Gaussian heat kernel estimates needed to apply Propositions 4.1, 5.1 are given in [54, (0.5)] and [55, Theorem 2]. For background concerning the relevant and very non-trivial local dilation structure due to Fefferman and Phong, the doubling property and Poincaré inequalities, see [32].

Theorem 6.6. *Assume that $(\mathcal{E}, \mathcal{D}, L^2(M, \mu))$ is a sub-elliptic Dirichlet space as defined above. Assume further that either M is a compact manifold (without boundary) or $M = \mathbb{R}^n$ and L is uniformly sub-elliptic. Let $(X_t, \mathbb{P}_x)$ be the associated diffusion. Then for any $x \in G$, we have*

$$\lim_{\epsilon \to 0} \sup_{\substack{0<s<t<1 \\ t-s<\epsilon}} \frac{d(X_s, X_t)}{\sqrt{4(t-s)\log(1/(t-s))}} = 1, \qquad \mathbb{P}_x\text{-almost surely.}$$

This follows from Propositions 4.2, 5.5 and [54, 55]. Note that some kind of uniformity is needed here.

Concerning the rate of escape, although Propositions 4.4, 5.7 apply to this situation (because the parabolic Harnack inequality (3.19) holds), it appears to be difficult to give an explicit sharp result in the present generality. The difficulty concerns the precise behavior of the volume function $r \mapsto \mu(B(x,r))$ for a fixed x as r tends to 0. What is known (and, in a sense, follows immediately from the fact this function is doubling) is that there are two positive reals α, β such that

$$c(x)r^{\alpha} \leq \mu(B(x,r)) \leq C(x)r^{\beta}$$

for small r. In fact, we always can take $\alpha \geq n$ where n is the topological dimension of M (see [16, p.255]).

However, if we restrict ourself to the easier case when L can be written as a sum of squares of smooth vector fields (Hörmander type operators [28]) a sharp local rate of escape can be obtained. Thus assume that

$$L = -\sum_{1}^{k} \xi_i^2 + \xi_0 \tag{6.17}$$

where ξ_i, $0 \leq i \leq k$ are smooth vector fields. Because we assume that L is self-adjoint the vector field ξ_0 belongs to the span of $\{\xi_1, \dots, \xi_k\}$ and does not play an important role here. Such an operator satisfies (6.16) in an open set $\Omega \subset \mathbb{R}^n$ (i.e., is sub-elliptic) if and only if there is an integer N such that the vector fields $\xi_1, \dots, \xi_k$ and all their brackets of order less than $N+1$ span $\mathbb{R}^n$ at any point in Ω. Moreover, in such a case, the volume growth function centered at point x satisfies

$$c(x)r^{N_x} \leq \mu(B(x,r)) \leq C(x)r^{N_x} \tag{6.18}$$

for all r small enough. Here N_x is an integer that is a lower semi-continuous function of x (see [38, Theorem 1] and also [53, (2.9)]). With this information Propositions 4.4, 5.7 give the following statement.

Theorem 6.7. *Assume that* $(\mathcal{E}, \mathcal{D}, L^2(M,\mu))$ *is a sub-elliptic Dirichlet space as define above, that the generator* L *is locally of the form* (6.17) *and that the topological dimension of* M *is at least* 2. *Let* $(X_t, \mathbb{P}_x)$ *be the associated diffusion. For each* $x \in M$, *let* $N_x \geq 2$ *be such that* (6.18) *holds.*

- *If* $N_x > 2$ *then*

$$\liminf_{t\to 0} \frac{d(x,X_t)}{\psi(t)\sqrt{t}} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x \text{ a.s.} \iff \sum_k \psi(2^{-k})^{N_x-2} \begin{cases} \text{diverges} \\ \text{converges} \end{cases}$$

for any increasing positive function ψ.

- *If* $N_x = 2$ *then*

$$\liminf_{t\to 0} \frac{d(x,X_t)}{\phi(t)} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x \text{ a.s.} \iff \sum_k \left(\log \frac{1}{\phi(2^{-k})}\right)^{-1} \begin{cases} \text{diverges} \\ \text{converges} \end{cases}$$

for any increasing positive function ϕ.

It may be useful to illustrate the last result above with the following two explicit examples.

EXAMPLE (Brownian motion in the plane). Of course, Brownian motion in the plane is recurrent. However, locall escape still occurs albeit at a very slow rate. As mentioned in the introduction, Spitzer [44] proved the result in this case. Applying the last statement of Theorem 6.7 we see that the local rate of escape is slower than any power function. In fact, for planar Brownian motion and

$$\phi(t) = \exp\left(-\left(\log\frac{1}{t}\right)\left(\log\log\frac{1}{t}\right)^{\alpha}\right)$$

we have

$$\liminf_{t\to 0}\frac{d(x,X_t)}{\phi(t)} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x \text{ a.s.} \iff \alpha \begin{cases} > 1 \\ \leq 1 \end{cases}$$

By using the fact that X_t is equal in law to $tX_{t^{-1}}$, we see that, for ϕ as above,

$$\liminf_{t\to \infty}\frac{d(x,X_t)}{\phi(1/t)} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x \text{ a.s.} \iff \alpha \begin{cases} > 1 \\ \leq 1 \end{cases}$$

This tells us how close to its starting point Brownian motion can be found in the long run. Spitzer [44] notes that this disproves a conjecture of Lévy.

EXAMPLE (Grushin's diffusion). The simplest sub-elliptic operator is the Grushin operator L on $\mathbb{R}^2$ defined by

$$L = -(y^2\partial_x^2 + \partial_y^2).$$

See, e.g., [25, 32]. For this operator, the parabolic Harnack inequality (3.19) holds globally. The volume of small ball centred at a point p is quadratic (i.e., $N_p = 2$ in the notation of Theorem 6.7) if $p \neq (0,0)$ and cubic (i.e., $N_p = 3$) if $p = (0,0)$. The associated diffusion is transient. Thus at all points except $(0,0)$, the local rate of escape is slower than any power function. At $(0,0)$ it is $\sqrt{t}$ up to a logarithmic factor.

6.4. Long time results. In this paper, we have focused on short time path regularity. However both the law of iterated logarithm and the problem of the rate of escape have very natural long time version. For Brownian motion in Euclidean space, long and short time results are equivalent because, in law, $tX_{t^{-1}} = X_t$. In the general framework of this paper, short time and long time results are distinct but technically very similar. Following the proofs given in the previous sections, one easily obtains the following theorem which complements the results obtained in [21].

Theorem 6.8. *Assume that the Harnack inequality* (3.19) *holds at all scales* (*i.e.*, *up to scale* $r_0 = \infty$). *Then there exists a constant* $c > 0$ *such that, for any* $x \in M$,

$$c \leq \limsup_{t \to \infty} \frac{d(x, X_t)}{\sqrt{4t \log \log t}} \leq 1, \qquad \mathbb{P}_x\text{-}a.s.$$

Moreover, if $m(r) = \int_r^\infty s\, ds / V(s) < \infty$ *then, for any increasing positive function* ϕ, *we have*

$$\liminf_{t \to \infty} \frac{d(x, X_t)}{\phi(t)} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}_x\text{-}a.s. \iff \int_1^\infty \frac{ds}{m(\phi(s)) V(\sqrt{s})} \begin{cases} \text{diverges} \\ \text{converges} \end{cases}$$

This result applies for instance to Brownian motion on manifolds with non-negative Ricci curvature and to left invariant diffusions on Lie groups having polynomial volume growth. Note that, in general, one cannot take $c = 1$ in the lower bound concerning the law of iterated logarithm. However, in the case of Brownian motion on a Riemannian manifold with non-negtaive Ricci curvature and in the case of symmetric left invariant diffusions on nilpotent Lie groups one can show that $c = 1$. See the heat kernel lower estimates in [34, 55] and [56, p.62].

ACKNOWLEDGEMENTS. The authors thank T. Kumagai for his help and comments.

References

[1] M. Barlow and R. Bass: *Brownian motion and harmonic analysis on Sierpinski carpets*, Canad. J. Math. **51** (1999), 673–744.

[2] M. Barlow and E. Perkins: *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields **79** (1988), 543–623.

[3] R. Bass and T. Kumagai: *Laws of the iterated logarithm for some symmetric diffusion processes*, Osaka J. Math. **37** (2000), 625–650.

[4] A. Bendikov and L. Saloff-Coste: *On- and off-diagonal heat kernel behaviors on certain infinite dimensional Dirichlet spaces*, Amer. J. Math. **122** (2000), 1205–1263.

[5] A. Bendikov and L. Saloff-Coste: *On the sample paths of Brownian motions on compact infinite dimensional groups*, Ann. Probab. **31** (2003), 1464–1493.

[6] M. Biroli and U. Mosco: *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, C.R. Acad. Sci. Paris **313** (1991), 593–598.

[7] M. Biroli and U. Mosco: *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Annali Mat. Pura Appl. **169** (1995), 125–181.

[8] R. Blumental and R. Getoor: Markov Processes and Potential Theory, Academic Press, New York and London, 1968.

[9] J.M. Bony: *Opérateurs elliptiques dégénérés associés aux axiomatiques de la théorie du potentiel*; in Potential Theory, C.I.M.E. (Stresa, 1969), Rome (1970).

[10] D. Burago, Y. Burago and S. Ivanov: A Course in Metric Geometry, Amer. Math. Soc. 2001.

[11] J. Cheeger and S-T.Yau: *A lower bound for the heat kernel*, Commun. Pure Appl. Math. **34** (1981), 465–480.

[12] E.B. Davies: Heat Kernels and Spectral Theory, Cambridge University Press, 1989.

[13] E.B. Davies: *Non-Gaussian aspects of heat kernel behaviour*, J. London Math. Soc. **55** (1997), 105–125.

[14] J-D. Deutschel and D. Stroock: Large Deviations, Academic Press, 1984.

[15] A. Dvoretzky and P. Erdös: *Some problems on random walk in space*; in Second Berkeley Symp., 1951, 353–367, University of California Press.

[16] C. Fefferman and A. Sanchez-Calle: *Fundamental solutions for second order subelliptic operators*, Annals of Math. **124** (1986), 247–272.

[17] M. Fukushima, Y. Oshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes. W. De Gruyter, 1994.

[18] P. Griffin: *An integral test for the rate of escape of d-dimensional random walk*, Ann. Probab. **11** (1983), 953–961.

[19] A. Grigor'yan: *The heat equation on non-compact Riemannian manifolds*, Matem. Sbornik, **182** (1991), 55–87. Engl. Transl. Math. USSR Sb. **72** (1992), 47–77.

[20] A. Grigor'yan: *Estimates of the heat kernels on Riemannian manifolds*; in Spectral Theory and Geometry, London Math. Soc. Lecture Note Series **273**, Eds. E.B Davies and Y. Safarov, 140–225, Cambridge University Press, 1999.

[21] A. Grigor'yan: *Escape rate of Brownian motion on Riemannian manifolds*, Applicable Anal, **71** (1999), 63–89.

[22] A. Grigor'yan: *Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[23] A. Grigor'yan and L. Saloff-Coste L: *Hitting probabilities for Brownian motion on Riemannian manifolds*, J. Math. Pures Appl. **81** (2002), 115–142.

[24] M. Gromov: Metric Structures for Riemannian and non-Riemannian Spaces, Birkhäuser, 1999.

[25] V. Grushin: *On a class of hypoelliptic pseudodifferential operators degenerate on a submanifold*, Math. USSR Sbornik **13** (1971), 155–185.

[26] W. Hebisch and L. Saloff-Coste: *Gaussian estimates for Markov chains and random walks on groups*, Ann. Prob. **21** (1993), 673–709.

[27] W. Hebisch and L. Saloff-Coste: *On the relation between elliptic and parabolic Harnack inequalities*, Ann. Inst. Fourier **51** (2001), 1437–1481.

[28] L. Hörmander: *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[29] G.A. Hunt: *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264–293.

[30] K. Ito and P. McKean: Diffusion Processes and Their Sample Paths. Springer, 1974.

[31] D. Jerison: *The Poincaré inequality for vector fields satisfying Hörmander condition*, Duke Math. J. **53** (1986), 503–523.

[32] D. Jerison and A. Sanchez-Calle: *Subelliptic second order differential operators*; in Complex Analysis III, Lect. Notes in Math. **1277** (1986), 47–77.

[33] I. Karatzas and S. Shreve: Brownian Motion and Stochastic Calculus, Springer, 1988.

[34] P. Li and S-T. Yau: *On the parabolic kernel of Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[35] T. Lyons: *Random thoughts on reversible potential theory*; in Summer School in Potential Theory, Joensuu 1990, Ed. Ilpo Laine, University of Joensuu, Publication in Sciences **26**, 71–114.

[36] P. Maheux and L. Saloff-Coste: *Analyse sur les boules d'un operateur sous-elliptique*, Math. Ann. **303** (1995), 713–740.

[37] J. Moser: *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **16** (1964), 101-134. Correction in **20** (1967), 231–236.

[38] A. Nagel, E. Stein and S. Wainger: *Balls and metrics defined by vector fields* I: *basic properties*, Acta Math. **55** (1985), 103–147.

[39] W. Pruitt: *The rate of escape of random walk*, Ann. Probab. **18** (1990), 1417–1461.

[40] D. Revuz and M. Yor: Continuous Martingales and Brownian Motion, Springer, 1991.

[41] L. Saloff-Coste: *A note on Poincaré, Sobolev and Harnack inequalities*, Internat. Math. Res. Notices **2** (1992), 27–38.

[42] L. Saloff-Coste: *Parabolic Harnack inequality for divergence-form second-order differential operators*; In Potential theory and degenerate partial differential operators (Parma). Potential Anal. **4** (1995), 429–467.
[43] L. Saloff-Coste: Aspects of Sobolev-Type Inequalities, Cambridge University Press, 2002.
[44] F. Spitzer: *Some theorems concerning* 2*-dimensional Brownian motion*, Trans. Amer. Math. Soc. **87** (1958), 187–197.
[45] D. Stroock: Probability Theory, An Anlytic View, Cambridge University Press, 1993.
[46] K-T. Sturm: *On the geometry defined by Dirichlet forms*; in Seminar on Stochastic Processes, Random Fields and Applications, Ascona, E. Bolthausen et al. Eds. Birkhäuser, Progress in Probability, **36** (1995), 231–242.
[47] K-T. Sturm: *Analysis on local Dirichlet spaces*—I, *Recurrence, conservativeness and* L^p*-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173–196.
[48] K-T. Sturm: *Sharp estimates for capacities and applications to symmetric diffusions*, Prob. Theory Relat. Fields **103** (1995), 73–89.
[49] K-T. Sturm: *Analysis on local Dirichlet spaces*—II, *Upper gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32** (1995), 275–312.
[50] K-T. Sturm: *Analysis on local Dirichlet spaces*—III, *The parabolic Harnack inequality*, J. Math. Pures Appl. **75** (1996), 273–297.
[51] K-T Sturm: *The geometric aspect of Dirichlet forms*, Studies in Advanced Math. **8** (1998), 233–277.
[52] M. Takeda: *On a martingale method for symmetric diffusion process and its application*, Osaka J. Math. **26** (1989), 605–623.
[53] N. Varopoulos: *Analysis on Lie groups*, J. Funct. Anal. **76** (1988), 346–410.
[54] N. Varopoulos: *Small time Gaussian estimates of heat diffusion kernels*. *Part* I: *The semigroup technique*, Bull. Sci. Math. **113** (1989), 253–277.
[55] N. Varopoulos: *Small time Gaussian estimates of heat diffusion kernels. Part* II. *The theory of large deviations*, J. Funct. Anal. **93** (1990), 1–33.
[56] N. Varopoulos, L. Saloff-Coste and Th. Coulhon: Analysis and Geometry on Groups, Cambridge University Press, 1993.
[57] J. Wang: *Global heat kernel estimates*, Pacific J. Math. **178** (1997), 377–398.

Alexander Bendikov
Instytut Matematyczny
Uniwersytetu Wrocławskiego
pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland

Laurent Saloff-Coste
Department of Mathematics
Malott Hall, Cornell University
Ithaca, NY 14853-4201, U.S.A.
e-mail: lsc@math.cornell.edu