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FOLDING MAPS ON SPACELIKE AND TIMELIKE SURFACES AND DUALITY

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Abstract

We study the reflectional symmetry of a generically embedded 2-dimensional surface M in the hyperbolic or de Sitter 3-dimensional spaces. This symmetry is picked up by the singularities of folding maps that are defined and studied here. We also define the evolute and symmetry set of M and prove duality results that relate them to the bifurcation sets of the family of folding maps.

1. Introduction

The investigation in this paper is the analogue of that in [5, 32] for surfaces in the Euclidean space $\mathbb{R}^3$. In [5] is studied the reflectional symmetry of a smooth surface $M \subset \mathbb{R}^3$ in planes in $\mathbb{R}^3$. A surface M is reflectionally more symmetric across planes with normals a principal direction at $p \in M$ than any other plane through p. This reflectional symmetry is studied via the family of folding maps, which is a 3-parameter family of mappings obtained by conjugating the fold map $(x, y, z) \to (x, y^2, z)$ by Euclidean motions ([2, 5]). The following result, with important geometric consequences, is shown in [5]: the bifurcation set of the family of folding maps is dual to the union of the focal and symmetry sets of M. The focal set and the symmetry set also arise as the bifurcation sets of the family of distance squared functions restricted to M. Recall that the distance squared function measures the contact of the surface with spheres, so the focal set is the centre of osculating spheres and the symmetry set is the centre of bi-tangent spheres to the surface. The duality result in [5] provides a powerfull tool for studying the affine geometry of the focal set of M and in turn obtain geometric information about the surface M itself; see for example [3, 4, 5, 27, 28, 32] and [7, 8] for the plane curves case.

Here we consider a smooth surface M in the hyperbolic space $H^3_+(-1)$ or in the de Sitter space S^3_1. The hyperbolic and the de Sitter spaces sit in the Minkowski space $\mathbb{R}^4_1$ endowed with the Laurentz pseudo-scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, where $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_0, y_1, y_2, y_3)$. In Section 3 we deal with surfaces in $H^3_+(-1)$. For such surfaces we define the family of folding maps, which is a 3-parameter family of mappings from $H^3_+(-1)$ to $H^3_+(-1)$ obtained by conjugat-

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ing the fold map $\left(\sqrt{x_0^2+x_1^2+x_2^2+1}, x_1, x_2, x_3\right) \mapsto \left(\sqrt{x_0^2+x_1^4+x_2^2+1}, x_1, x_2^2, x_3\right)$ by hyperbolic motions (see Section 3 for details). The first analogous result to the Euclidean case is that the surface M is reflectionally more symmetric across hyperplanes with normals a principal direction at $p \in M$ than any other hyperplane through p. For the analogous duality result we require some ingredients for dealing with the extrinsic geometry of submanifolds in $\mathbb{R}_1^{n+1}$. These are the duality concepts introduced by the first author in [11, 12], and the concepts of evolute and symmetry set of surfaces in $H^3(-1)$. The concept of evolute is introduced in [19, 22] and the symmetry set is defined in this paper. With these ingredients at hand, we show that the bifurcation set of the family of folding maps is dual to the union of the evolute and symmetry set (Theorem 5.3). The evolute and symmetry set are the local and multi-local strata of the bifurcation set of the family of timelike and spacelike height functions. We draw geometric consequences about the geometry of M from the duality result.

We also deal in this paper with families of folding maps on spacelike and timelike surfaces in S_1^3 and prove similar results to those for surfaces in the hyperbolic space (§5.2 and §5.3). We need to define for theses cases the notion of evolute and symmetry set. We do this following the same approach in [19, 22] using the timelike and spacelike height functions. We observe that timelike surfaces present distinct geometric properties to those of spacelike surfaces. This is due to the presence of two lightlike directions on each tangent space of the surface.

2. Preliminaries

The *Minkowski* $(n+1)$*-space* $(\mathbb{R}_1^{n+1}, \langle\ ,\ \rangle)$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed with the *pseudo scalar product*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i,$$

for $\boldsymbol{x} = (x_0, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}_1^{n+1}$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}_1^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, *lightlike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$.

The norm of a vector $\boldsymbol{x} \in \mathbb{R}_1^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{v} \in \mathbb{R}_1^{n+1}$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We have the following three pseudo-spheres in $\mathbb{R}_1^{n+1}$:

$$\begin{aligned}
\textit{Hyperbolic n-space}&: H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},\\
\textit{de Sitter n-space}&: S_1^n = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},\\
(\textit{open})\ \textit{lightcone}&: LC^* = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.
\end{aligned}$$

The hyperbolic space has two connected components, $H^n_+(-1) = \{\boldsymbol{x} \in H^n(-1) \mid x_0 \geq 1\}$ and $H^n_-(-1) = \{\boldsymbol{x} \in H^n(-1) \mid x_0 \leq -1\}$. We only consider embedded surfaces in $H^n_+(-1)$ as the study is similar for those embedded in $H^n_-(-1)$.

The wedge product of n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^{n+1}_1$ is given by

$$\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{a}_i = (a_0^i, a_1^i, \ldots, a_n^i)$, $i = 1, \ldots, n$. One can check that $\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_n \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, so the vector $\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo orthogonal to all the vectors $\boldsymbol{a}_i$, $i = 1, \ldots, n$.

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper (for more details see for example [1]). Let N be a $(2n+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Such a field is locally defined by a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point on N. The pair (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α a *contact form*.

A submanifold $\boldsymbol{i}\colon L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $d\boldsymbol{i}_x(T_x L) \subset K_{\boldsymbol{i}(x)}$ at any $x \in L$. A smooth fibre bundle $\pi\colon E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibres of π are Legendrian submanifolds. Let $\pi\colon E \to M$ be a Legendrian fibration. For a Legendrian submanifold $\boldsymbol{i}\colon L \subset E$, $\pi \circ \boldsymbol{i}\colon L \to M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ \boldsymbol{i}$ is called a *wavefront set* of $\boldsymbol{i}$ and is denoted by $W(\boldsymbol{i})$.

In [11, 12, 22] are considered five double fibrations. We recall here only those that are needed in this paper (and keep the notation of [11, 12, 22]).

(1) (a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{11}\colon \Delta_1 \to H^n(-1)$, $\pi_{12}\colon \Delta_1 \to S^n_1$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(5) (a) $S^n_1 \times S^n_1 \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{51}\colon \Delta_5 \to S^n_1$, $\pi_{52}\colon \Delta_5 \to S^n_1$,
(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_5$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$ and $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$ for $i = 1, 5$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$. The 1-forms θ_{i1} and θ_{i2}, $i = 1, 5$, define the same tangent hyperplane field over Δ_i which is denoted by K_i.

Theorem 2.1 ([11, 12, 22]). *The pairs* (Δ_i, K_i), $i = 1, 5$, *are contact manifolds and* π_{i1} *and* π_{i2} *are Legendrian fibrations.*

REMARK 2.2. (1) Given a Legendrian submanifold $\boldsymbol{i}: L \to \Delta_i$, $i = 1, 5$, Theorem 2.1 states that $\pi_{i1}(\boldsymbol{i}(L))$ is dual to $\pi_{i2}(\boldsymbol{i}(L))$ and vice-versa. We shall call this duality Δ_i-duality.
(2) If $\pi_{11}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{11}(\boldsymbol{i}(L)) \subset H^n_+(-1)$ at $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$. Conversely, if $\pi_{12}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{12}(\boldsymbol{i}(L)) \subset S^n_1$. The same properties hold for the Δ_5-duality.

3. Gauss maps of surfaces in $H^3_+(-1)$ and S^3_1

A *spacelike surface* is a surface whose tangent plane at any point is a spacelike vector space (i.e., the tangent plane contains only spacelike vectors). A *timelike surface* is a surface whose tangent plane at any point is a timelike vector space (i.e., it contains both spacelike and timelike vectors). Thus, any surface in $H^3_+(-1)$ is a spacelike surface, but this is not the case for surfaces in S^3_1. We shall work with some Gauss maps of an embedded surface M^h in $H^3_+(-1)$ or M^d in S^3_1. The differential of these maps are self-adjoint operators on M. An important observation for a spacelike surface is that the restriction of the pseudo-scalar product in $\mathbb{R}^4_1$ to the surface is a scalar product. Therefore, the differential of the Gauss map has always real eigenvalues. However, this is not the case for a timelike surface M^d in S^3_1 as the restriction of the pseudo-scalar product to M^d is Lorentzian [29]. We deal separatly with spacelike and timelike surfaces.

3.1. Spacelike surfaces in $H^3_+(-1)$ and S^3_1. The extrinsic geometry of hypersurfaces in the hyperbolic space is studied in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23]. We deal with local properties of mappings, so we consider an embedding $\boldsymbol{x}^h: U \to H^3_+(-1)$, where U is an open subset of $\mathbb{R}^2$, and write $M^h = \boldsymbol{x}^h(U)$. Since $\langle \boldsymbol{x}^h, \boldsymbol{x}^h \rangle \equiv -1$, we have $\langle \boldsymbol{x}^h_{u_i}, \boldsymbol{x}^h \rangle \equiv 0$, for $i = 1, 2$, where $u = (u_1, u_2) \in U$ and $\boldsymbol{x}^h_{u_i} = \partial \boldsymbol{x}^h / \partial u_i$. We define the spacelike unit normal vector $\boldsymbol{x}^d(u)$ to the surface at $\boldsymbol{x}^h(u)$ by

$$\boldsymbol{x}^d(u) = \frac{\boldsymbol{x}^h(u) \wedge \boldsymbol{x}^h_{u_1}(u) \wedge \boldsymbol{x}^h_{u_2}(u)}{\|\boldsymbol{x}^h(u) \wedge \boldsymbol{x}^h_{u_1}(u) \wedge \boldsymbol{x}^h_{u_2}(u)\|}.$$

We call the mapping $\boldsymbol{x}^d: U \to S^3_1$ the *de Sitter Gauss indicatrix* of M^h ([17]). For any $p = \boldsymbol{x}^h(u_0) \in M^h$, one can show that $\boldsymbol{x}^d_{u_i} \in T_p M^h$. The linear transformation $A^d_p = -d\boldsymbol{x}^d(u_0)$, called the *de Sitter shape operator*, is a self-adjoint operator. Because the restriction of the pseudo-scalar product in $\mathbb{R}^4_1$ to M^h is a scalar product, A^d_p has an orthogonal basis formed by its eigenvectors when its eigenvalues are distinct. Its eigenvalues $(\kappa_d)_i$, $i = 1, 2$, are called the (*de Sitter*) *principal curvature* and the corresponding eigenvectors $\boldsymbol{p}^d_i$, $i = 1, 2$, are called the (*de Sitter*) *principal directions*. We say that a point

$p = \boldsymbol{x}(u_0) \in M^h$ is an *umbilic* point if $A^d_p = k_d(p)id_{T_pM^h}$. We also say that M is *totally umbilic* if all points of M are umbilic.

DEFINITION 3.1. A surface given by the intersection of $H^3_+(-1)$ with a spacelike, timelike or lightlike hyperplane is called respectively *sphere*, *equidistant surface* or *horosphere*. The intersection of the surface with timelike hyperplane through the origin is called a *hyperbolic plane* (*plane* for short).

We call a vector $\boldsymbol{v}$ in, respectively, $H^3(-1)$, S^3_1 or LC^* which is orthogonal to a given sphere, equidistant surface or horosphere, the centre of the sphere, equidistant surface or horosphere.

Proposition 3.2 ([6, 17, 19]). *Suppose that $M^h = \boldsymbol{x}^h(U)$ is totally umbilic. Then $\kappa_d(p)$ is a constant κ_d for all $p \in M^h$. Under this condition, we have the following classification.*
(1) *If $\kappa_d^2 > 1$, then M^h is part of a sphere.*
(2) *If $\kappa_d^2 = 1$, then M^h is part of a horosphere.*
(3) *If $\kappa_d^2 < 1$, then M^h is part of an equidistant surface. In particular, if $\kappa_d = 0$, then M^h is a part of a plane.*

The mapping $\mathcal{L}_1 \colon U \to \Delta_1$ defined by $\mathcal{L}_1(u) = (\boldsymbol{x}^h(u), \boldsymbol{x}^d(u))$ is a Legendrian embedding. If we start with a spacelike embedding $\boldsymbol{x}^d \colon U \to S^3_1$, we can construct a mapping $\boldsymbol{x}^h \colon U \to H^3_+(-1)$ in exactly the same way as above and still obtain the Legendrian embedding $\mathcal{L}_1$.

By definition, $\mathcal{L}_1$ is a Legendrian embedding if and only if $\boldsymbol{x}^h_{u_i}$ is tangent to the spacelike surface $M^d = \boldsymbol{x}^d(U)$. The map $\boldsymbol{x}^h \colon U \to H^3_+(-1)$ is called the *hyperbolic Gauss indicatrix* of M^d. Therefore, we have a linear transformation $A^h_p = -d\boldsymbol{x}^h \colon T_pM^d \to T_pM^d$, called the *hyperbolic shape operator* of M^d at p, which is a self-adjoint operator. Because the restriction of the pseudo-scalar product in $\mathbb{R}^4_1$ to M^d is a scalar product (M^d is spacelike), A^h_p has an orthogonal basis formed by its eigenvectors when its eigenvalues are distinct. Its eigenvalues $(\kappa_h)_i$, $i = 1, 2$, are called the *hyperbolic principal curvature* and the corresponding eigenvectors $\boldsymbol{p}^h_i$, $i = 1, 2$, are called the *hyperbolic principal directions*. We say that a point $p = \boldsymbol{x}^d(u_0) \in M^d$ is an *umbilic* point if $A^h_p = k_h(p)id_{T_pM^d}$. We also say that M^d is *totally umbilic* if all points of M^d are umbilic.

DEFINITION 3.3. A surface given by the intersection of S^3_1 and a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane is respectively called a *hyperbolic*, an *elliptic* or a *parabolic de Sitter quadric*. In particular, we call an elliptic (resp. hyperbolic) de Sitter quadric through the origin a *flat elliptic* (*resp. hyperbolic*) *de Sitter quadric*.

We call a vector $\boldsymbol{v}$ in, respectively, $H^3(-1)$, S^3_1 or LC^* which is orthogonal to a given hyperbolic, elliptic or parabolic de Sitter quadric, the centre of the hyperbolic, elliptic or parabolic de Sitter quadric.

The following classification of totally umbilic spacelike surfaces in the de Sitter space follows in the same way as that of surfaces in hyperbolic space.

Proposition 3.4. *Suppose that $M^d = \boldsymbol{x}^d(U)$ is a totally umbilic spacelike surface in S_1^3. Then $\kappa_h(p)$ is constant κ_h. Under this condition, we have the following classification.*
(1) *If $\kappa_h^2 > 1$, then M^d is part of a hyperbolic de Sitter quadric.*
(2) *If $\kappa_h^2 = 1$, then M^d is part of a parabolic de Sitter quadric.*
(3) *If $\kappa_h^2 < 1$, then M^d is part of an elliptic de Sitter quadric. In particular, if $\kappa_h = 0$, then M^d is a part of a flat elliptic de Sitter quadric.*

3.2. Timelike surfaces in S_1^3. Some aspect of the extrinsic differential geometry of timelike hypersurfaces in S_1^n from the view point of singularity theory are studied in [11]. The tangent space at each point on a timelike surface in S_1^3 is timelike, so it contains two lightlike directions. This makes such surfaces behave in a distinct way to the spacelike ones.

Let $\boldsymbol{x}\colon U \to S_1^3$ denote an embedding of a timelike surface, where U is an open subset of $\mathbb{R}^2$. For any $u \in U$, we have $\langle \boldsymbol{x}(u), \boldsymbol{x}(u)\rangle = 1$, so $\langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}(u)\rangle = 0$, $i = 1, 2$. We also have a unit normal vector $\boldsymbol{x}^*(u)$ to the surface at $p = \boldsymbol{x}(u)$ given by

$$\boldsymbol{x}^*(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)\|}.$$

The vector $\boldsymbol{x}^*(u)$ is spacelike. We call $\boldsymbol{x}^*\colon U \to S_1^3$ the *de Sitter Gauss map* of $M^d = \boldsymbol{x}(U)$. One can show that for any $p = \boldsymbol{x}(u_0) \in M^d$, $\boldsymbol{x}^*_{u_i}(u_0) \in T_pM^d$ $(i = 1, 2)$. Therefore, we have a linear transformation $A_p = -d\boldsymbol{x}^*\colon T_pM^d \to T_pM^d$, which is a self-adjoint operator. Because the restriction of the pseudo-scalar product in $\mathbb{R}_1^4$ to M^d is still a pseudo-scalar product (M^d is timelike), A_p does not always have real eigenvalues. When A_p has two distinct eigenvalues κ_i, $i = 1, 2$, we call them the *principal curvature* of the surface at p, and the corresponding eigenvectors $\boldsymbol{p}_i$, $i = 1, 2$, are called the *principal directions*. The set of points where the eigenvalues coincide is of interest and we label it the *lightlike principal locus*.

Proposition 3.5. (1) *For a generic timelike surface M^d in the de Sitter space, the lightlike principal locus is a curve on M^d. It can be characterised as the set of points on M^d where the two principal directions coincide and become a lightlike direction.*
(2) *The lightlike principal locus divides the surfaces into two regions. In one of them there are no principal directions and in the other there are two distinct principal directions at each point. In the later case, the principal directions are orthogonal and one is spacelike while the other is timelike.*

Proof. (1) The computations here are similar to the case of scalar product. Denote by

$$E = \langle \boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_1} \rangle, \quad F = \langle \boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2} \rangle, \quad G = \langle \boldsymbol{x}_{u_2}, \boldsymbol{x}_{u_2} \rangle$$

the coefficients of the (pseudo) first fundamental form and by

$$\begin{aligned} l &= \langle A_p(\boldsymbol{x}_{u_1}), \boldsymbol{x}_{u_1} \rangle = \langle \boldsymbol{x}^*, \boldsymbol{x}_{u_1 u_1} \rangle, \\ n &= \langle A_p(\boldsymbol{x}_{u_1}), \boldsymbol{x}_{u_2} \rangle = \langle \boldsymbol{x}^*, \boldsymbol{x}_{u_1 u_2} \rangle, \\ m &= \langle A_p(\boldsymbol{x}_{u_2}), \boldsymbol{x}_{u_2} \rangle = \langle \boldsymbol{x}^*, \boldsymbol{x}_{u_2 u_2} \rangle \end{aligned}$$

those of the (pseudo) second fundamental form. Then the matrix of A_p with respect to the basis $\{\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2}\}$ is given by the usual formula

$$\frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

It follows that the equation of the principal direction is also given by the usual formula

$$(Gm - Fn)\, du_2^2 + (Gl - En)\, du_1\, du_2 + (Fl - Em)\, du_1^2 = 0,$$

equivalently by,

$$\begin{vmatrix} du_2^2 & -du_1\, du_2 & du_1^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0.$$

The discriminant of the above quadratic differential equation is

$$\delta(u_1, u_2) = ((Gl - En)^2 - 4(Gm - Fn)(Fl - Em))(u_1, u_2).$$

The set $\delta^{-1}(0)$ (the lightlike principal locus) is either empty or is a curve on generic surfaces M^d. (Recall that on generic two dimensional Riemannian surfaces, the set $\delta^{-1}(0)$ consists of isolated umbilic points; see for example [31].)

A principal direction $\boldsymbol{p} = du_1 \boldsymbol{x}_{u_1} + du_2 \boldsymbol{x}_{u_2}$ in $T_p M^d$ is lightlike if and only if

$$\langle \boldsymbol{p}, \boldsymbol{p} \rangle = G\, du_2^2 + 2F\, du_1\, du_2 + E\, du_1^2 = 0.$$

The resultant of this equation with that of the principal directions is

$$(EG - F^2)^2((Gl - En)^2 - 4(Gm - Fn)(Fl - Em)).$$

As $EG - F^2 \neq 0$, it follows that a principal direction is lightlike at a point p if and only if p is on the lightlike principal locus.

(2) In the region $\delta > 0$ the equation of the principal directions has two distinct solutions. It has no solutions in the region where $\delta < 0$. The two principal directions at points in the region where $\delta > 0$ are orthogonal (this follows from the fact that $\kappa_1\langle \boldsymbol{p}_1, \boldsymbol{p}_2\rangle = \langle A_p(\boldsymbol{p}_1), \boldsymbol{p}_2\rangle = \langle \boldsymbol{p}_1, A_p(\boldsymbol{p}_2)\rangle = \kappa_2\langle \boldsymbol{p}_1, \boldsymbol{p}_2\rangle$, and $\kappa_1 \neq \kappa_2$). As neither of them are lightlike, one has to be timelike and the other spacelike (see for example Theorem 3.1.4 in [30]). □

REMARK 3.6. The generic configurations of the lines of principal curvature on a timelike surface in S_1^3 (i.e., the pair of foliations defined by the principal directions) are studied in [24].

We can interpret the de Sitter Gauss map $\boldsymbol{x}^*$ by the Legendrian duality. We have a Legendrian embedding $\mathcal{L}_5\colon U \to \Delta_5$ defined by $\mathcal{L}_5(u) = (\boldsymbol{x}(u), \boldsymbol{x}^*(u))$. Therefore, $\boldsymbol{x}(U)$ and $\boldsymbol{x}^*(U)$ are Δ_5-dual to each other.

We can also define the notion of umbilic points and have a classification of totally umbilic timelike surfaces in S_1^3 (see [11]). The arguments are similar to the spacelike case and are omitted.

4. Evolute and symmetry set

In this section we introduce the notion of evolutes and symmetry sets for surfaces in $H_+^3(-1)$ or S_1^3. We distinguish, as before, the cases when the surface is spacelike or timelike.

4.1. Spacelike surfaces in $H_+^3(-1)$ and S_1^3. In [19] (see also [18] for the curves case) is introduced the notion of evolute (or focal surface) of a hypersurface in a hyperbolic space. For a surface $\boldsymbol{x}^h\colon U \to H_+^3(-1)$, the *total evolute* (evolute for short) of $\boldsymbol{x}^h(U) = M^h$ is defined by

$$\mathrm{TE}^{\pm}_{M^h} = \bigcup_{i=1}^{2}\left\{ \pm \frac{1}{\sqrt{|(\kappa_d)_i^2(u) - 1|}}((\kappa_d)_i(u)\boldsymbol{x}^h(u) + \boldsymbol{x}^d(u)),\ u \in U \right\},$$

where $(\kappa_d)_i(u)$, $i = 1, 2$, are the de Sitter principal curvature at $\boldsymbol{x}^h(u)$. Observe that $\mathrm{TE}^-_{M^h}$ is the reflection of $\mathrm{TE}^+_{M^h}$ with respect to the origin (so we have two copies of the total evolute). We assume here that $p = \boldsymbol{x}^h(u)$ is not a horoparabolic point, that is, $(\kappa_d)_i^2(u) \neq 1$ for $i = 1, 2$. The evolute has the following decomposition

$$\mathrm{TE}^{\pm}_{M^h} = \mathrm{HE}^{\pm}_{M^h} \cup \mathrm{SE}^{\pm}_{M^h},$$

where $\mathrm{HE}^{\pm}_{M^h}$ is the hyperbolic space component of the evolute and corresponds to points u where $(\kappa_d)_i^2(u) > 1$, and $\mathrm{SE}^{\pm}_{M^h}$ is the de Sitter component of the evolute and corresponds to points u where $(\kappa_d)_i^2(u) < 1$.

The evolute has some interesting geometric properties. Let

$$H_h^T : U \times H^3(-1) \to \mathbb{R}$$

denote the hyperbolic timelike height function given by $H_h^T(u, \boldsymbol{v}) = \langle \boldsymbol{x}^h(u), \boldsymbol{v} \rangle$, and

$$H_h^S : U \times S_1^3 \to \mathbb{R}$$

denote the hyperbolic spacelike height function given by $H_h^S(u, \boldsymbol{v}) = \langle \boldsymbol{x}^h(u), \boldsymbol{v} \rangle$. The function H_h^T measures the contact of the surface with spheres and H_h^S measures its contact with equidistant surfaces (see Definition 3.1). One can show that the evolute is the union of the "local" strata of the bifurcation sets $\mathrm{LBif}(H_h^T)$ and $\mathrm{LBif}(H_h^S)$ of the families H_h^T and H_h^S respectively, [19]. The local (resp. multi-local) stratum of, say the family H_h^T, is the set of parameters $\boldsymbol{v} \in H^3(-1)$ for which $H_{h\,\boldsymbol{v}}^T = H_h^T(\cdot\,, \boldsymbol{v})$ has a unstable local (resp. multi-local) singularity. We have,

$$\mathrm{LBif}(H_h^T) = \mathrm{HE}_{M^h}^+ \cup \mathrm{HE}_{M^h}^-,$$
$$\mathrm{LBif}(H_h^S) = \mathrm{SE}_{M^h}^+ \cup \mathrm{SE}_{M^h}^-.$$

Therefore, the evolute parametrises the centres of spheres or equidistant surfaces that have degenerate contact with M^h (i.e., parametrises the set of $\boldsymbol{v}$ for which $H_{h\,\boldsymbol{v}}^T = H_h^T(\cdot\,, \boldsymbol{v})$ or $H_{h\,\boldsymbol{v}}^S = H_h^S(\cdot\,, \boldsymbol{v})$ has a singularity of type A_2 or worse). Observe that if u is a degenerate singularity of $H_{h\,\boldsymbol{v}}^T$ (resp. $H_{h\,\boldsymbol{v}}^S$) then it is also a degenerate singularity of $H_{h\,-\boldsymbol{v}}^T$ (resp. $H_{h\,-\boldsymbol{v}}^S$). This is why we have two copies $\mathrm{TE}_{M^h}^+$ and $\mathrm{TE}_{M^h}^-$ of the evolute. The evolute can also be characterised as a caustic, and therefore has generic Lagrangian singularities [19, 22].

We have the following observation needed for the duality result in this paper.

Proposition 4.1. *Let q be a smooth point on the evolute associated to the principal curvature $(\kappa_d)_i$, $i = 1$ or 2. Then the normal to the evolute at q is parallel to the principal direction $\boldsymbol{p}_i^d$ associated to $(\kappa_d)_i$.*

Proof. Let $c_i^{\pm} : U \to H^3(-1) \cup S_1^3$, $i = 1, 2$, given by

$$c_i^{\pm}(u) = \pm \frac{1}{\sqrt{|(\kappa_d)_i^2(u) - 1|}}((\kappa_d)_i(u)\boldsymbol{x}^h(u) + \boldsymbol{x}^d(u)),$$

be a local parametrisation of the evolute. Let p be the point on the surface corresponding to the point q on the evolute. As q is a smooth point on the evolute, the principal curvatures are distinct at p. We can then choose a local parametrisation $\boldsymbol{x}^h : U \to H_+^3(-1)$ of the surface at p so that $u_i = \text{constant}$, $i = 1, 2$, represent the lines of curvatures. The part of the evolute that is associated to a given principal curvature $(\kappa_d)_i$ is parametrised by $c_i^{\pm}(u) = \lambda^{\pm}(u)((\kappa_d)_i(u)\boldsymbol{x}^h(u) + \boldsymbol{x}^d(u))$ where $\lambda^{\pm}(u) = \pm 1 \Big/ \sqrt{|(\kappa_d)_i^2(u) - 1|}$. We have

$\langle c_i^{\pm}, \boldsymbol{p}_i^d \rangle = \langle \boldsymbol{x}^h, \boldsymbol{p}_i^d \rangle = 0$ as $\boldsymbol{p}_i^d$ is tangent to M^h. Because of the chosen parametrisation, we have $\langle \boldsymbol{x}_{u_j}^h, \boldsymbol{p}_i^d \rangle = 0$ for $j \neq i$. Also $\langle \boldsymbol{x}_{u_i}^d, \boldsymbol{p}_i^d \rangle = \langle d\boldsymbol{x}^d . \boldsymbol{x}_{u_i}^h, \boldsymbol{p}_i^d \rangle = \langle -\kappa_i \boldsymbol{x}_{u_i}^h, \boldsymbol{p}_i^d \rangle$ and $\langle \boldsymbol{x}_{u_j}^d, \boldsymbol{p}_i^d \rangle = -\kappa_j \langle \boldsymbol{x}_{u_j}^h, \boldsymbol{p}_i^d \rangle = 0$ for $j \neq i$. Therefore,

$$\left\langle \frac{\partial c_i^{\pm}}{\partial u_i}, \boldsymbol{p}_i^d \right\rangle = \left\langle \lambda_{u_i}^{\pm}((\kappa_d)_i \boldsymbol{x}^h + \boldsymbol{x}^d) + \lambda^{\pm}\left(\left(\frac{\partial}{\partial u_i}(\kappa_d)_i\right)\boldsymbol{x}^h + (\kappa_d)_i \boldsymbol{x}_{u_i}^h + \boldsymbol{x}_{u_i}^d\right), \boldsymbol{p}_i^d \right\rangle = 0$$

and for $j \neq i$,

$$\left\langle \frac{\partial c_i^{\pm}}{\partial u_j}, \boldsymbol{p}_i^d \right\rangle = \left\langle \lambda_{u_j}^{\pm}((\kappa_d)_i \boldsymbol{x}^h + \boldsymbol{x}^d) + \lambda^{\pm}\left(\left(\frac{\partial}{\partial u_j}(\kappa_d)_i\right)\boldsymbol{x}^h + (\kappa_d)_i \boldsymbol{x}_{u_j}^h + \boldsymbol{x}_{u_j}^d\right), \boldsymbol{p}_i^d \right\rangle = 0$$

which proves the assertion. □

We consider now the multi-local strata of the bifurcation sets of the spacelike and timelike height functions. (This is analogous to the study of the multi-local stratum of the distance squared function on surfaces in the Euclidean space $\mathbb{R}^3$.)

DEFINITION 4.2. The *symmetry set* of M^h, denoted by SS, is the closure of centres of spheres in $H^3(-1)$ or equidistant surfaces in S_1^3 that are tangent to M^h in at least two distinct points. It is the union of the closure of the multi-local strata of the bifurcation sets of the spacelike and timelike family of height functions H_h^S and H_h^T.

We denote by SS^T (resp. SS^S) the component of the symmetry set related to the timelike (resp. spacelike) family of height function.

Proposition 4.3. (1) *A point $q \in H^3(-1) \cup S_1^3$ is on the* SS *of a surface $M^h \subset H_+^3(-1)$ if and only if there exists two distinct points p_1 and p_2 on M^h such that the tangent planes $T_{p_1}M^h$ and $T_{p_2}M^h$ are symmetric with respect to the equidistant surface orthogonal to the geodesic joining p_1 and p_2 and passing through the midpoint of the segment p_1p_2.*
(2) *Let q be a smooth point on the* SS *corresponding to the bi-tangency of a sphere* (*resp. equidistant surface*) *to the surface M^h at two points p_1 and p_2. Then the normal to the* SS *at q is the normal to the equidistant surface in* (1).

Proof. (1) Let $\boldsymbol{x}_1^h : U_1 \to M^h$ and $\boldsymbol{x}_2^h : U_2 \to M^h$ be local coordinates on M^h around $\boldsymbol{x}_1^h(0, 0) = p_1$ and $\boldsymbol{x}_2^h(0, 0) = p_2$. By a hyperbolic motion, we can suppose that the equidistant surface orthogonal to the geodesic joining p_1 and p_2 and passing through the midpoint of the segment p_1p_2 is given by $x_2 = 0$. If $\boldsymbol{v}_0 = (0, 0, 1, 0)$, then $p_2 = p_1 - 2\langle p_1, \boldsymbol{v}_0 \rangle \boldsymbol{v}_0$.

The height function $H_{h\,\boldsymbol{v}}^T$ (resp. $H_{h\,\boldsymbol{v}}^S$) has two singularities at p_1 and p_2 at the same level if and only if $\boldsymbol{v} = \lambda p_1 + \mu \boldsymbol{e}_1 = \alpha p_2 + \beta \boldsymbol{e}_2$ with $-\lambda^2 + \mu^2 = -1$ and $-\alpha^2 + \beta^2 = -1$ (resp. $-\lambda^2 + \mu^2 = 1$ and $-\alpha^2 + \beta^2 = 1$) and $\langle p_1, \boldsymbol{v} \rangle = \langle p_2, \boldsymbol{v} \rangle$. Here $\boldsymbol{e}_1$

and $\boldsymbol{e}_2$ are the normal vectors to the surface at p_1 and p_2 respectively. Since $\langle p_i, p_i\rangle = -1$ and $\langle p_i, \boldsymbol{e}_i\rangle = 0$ for $i = 1, 2$, it follows that

$$\langle p_1, \boldsymbol{v}\rangle = -\lambda = \alpha\langle p_1, p_2\rangle + \beta\langle p_1, \boldsymbol{e}_2\rangle. \tag{1}$$

We have $\langle p_1 - p_2, \boldsymbol{v}\rangle = 0$. Therefore $\langle p_1 - p_2, \alpha p_2 + \beta\boldsymbol{e}_2\rangle = 0$, equivalently,

$$\alpha + \alpha\langle p_1, p_2\rangle + \beta\langle p_1, \boldsymbol{e}_2\rangle = 0. \tag{2}$$

It follows from equations (1) and (2) that $\lambda = \alpha$ and hence $\mu = \pm\beta$. We can assume that $\mu = \beta$ by changing the orientation of the surface at p_2 if necessary (by taking $-\boldsymbol{e}_2$ as the normal vector at p_2). Now $\lambda p_1 + \mu\boldsymbol{e}_1 = \alpha p_2 + \beta\boldsymbol{e}_2$, so $\boldsymbol{e}_1 - \boldsymbol{e}_2$ is parallel to $p_1 - p_2$, and hence is parallel to $\boldsymbol{v}_0$. This implies that $\boldsymbol{e}_2$ is symmetric to $\boldsymbol{e}_1$ with respect to the plane $x_2 = 0$ and hence the normal plane $N_{p_2}M^h$ (generated by p_2 and $\boldsymbol{e}_2$) is symmetric to the normal plane $N_{p_1}M^h$ (generated by p_1 and $\boldsymbol{e}_1$) with respect to $x_2 = 0$. Consequently, $T_{p_2}M^h$ is symmetric to $T_{p_1}M^h$ with respect to $x_2 = 0$.

(2) We consider the setting in (1) and deal with the multi-local singularities of the timelike height function. The case of the spacelike height function follows in the same way. Consider the map $\Phi^T\colon U_1 \times U_2 \times H^3_+(-1) \to \mathbb{R}^5$ given by

$$(u, v, \boldsymbol{v}) \mapsto \left(\langle \boldsymbol{x}^h_1(u), \boldsymbol{v}\rangle - \langle \boldsymbol{x}^h_2(v), \boldsymbol{v}\rangle, \left\langle \boldsymbol{x}^h_{1_{u_1}}(u), \boldsymbol{v}\right\rangle, \left\langle \boldsymbol{x}^h_{1_{u_2}}(u), \boldsymbol{v}\right\rangle, \left\langle \boldsymbol{x}^h_{2_{v_1}}(v), \boldsymbol{v}\right\rangle, \left\langle \boldsymbol{x}^h_{2_{v_2}}(v), \boldsymbol{v}\right\rangle\right)$$

with $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then $\mathrm{SS}^T = \pi_3((\Phi^T)^{-1}(0))$, where π_3 is the canonical projection to the third component. To prove the statement it is enough to show that $\langle \boldsymbol{v}_0, d\boldsymbol{v}\rangle = 0$ at q, where $\boldsymbol{v} \in \mathrm{SS}^T$. Since $(u, v, \boldsymbol{v}) \in (\Phi^T)^{-1}(0)$, we have $\langle \boldsymbol{x}^h_1(u) - \boldsymbol{x}^h_2(v), \boldsymbol{v}\rangle = 0$. By differentiating, we have $\langle \boldsymbol{x}^h_1(u) - \boldsymbol{x}^h_2(v), d\boldsymbol{v}\rangle = 0$, and the assertion follows from the fact that $p_1 - p_2$ is parallel to $\boldsymbol{v}_0$. □

We now introduce the notion of evolute of a spacelike surface in de Sitter space. For a spacelike surface $\boldsymbol{x}^d\colon U \to S^3_1$, we define the *total evolute* of $\boldsymbol{x}^d(U) = M^d$ by

$$\mathrm{TE}^{\pm}_{M^d} = \bigcup_{i=1}^{2}\left\{ \pm\frac{1}{\sqrt{|(\kappa_h)^2_i(u) - 1|}}(\boldsymbol{x}^h(u) + (\kappa_h)_i(u)\boldsymbol{x}^d(u)),\ u \in U \right\},$$

where $(\kappa_h)_i(u)$, $i = 1, 2$, are the hyperbolic principal curvature at $\boldsymbol{x}^d(u)$. We assume here that $(\kappa_h)^2_i(u) \neq 1$ for $i = 1, 2$. The total evolute has the following decomposition

$$\mathrm{TE}^{\pm}_{M^d} = \mathrm{HE}^{\pm}_{M^d} \cup \mathrm{SE}^{\pm}_{M^d},$$

where $\mathrm{HE}^{\pm}_{M^d}$ denotes the hyperbolic part of the total evolute and corresponds to point u where $(\kappa_h)^2_i(u) < 1$ and $\mathrm{SE}^{\pm}_{M^d}$ denotes the de Sitter part of the total evolute and corresponds to point u where $(\kappa_h)^2_i(u) > 1$. Let

$$H^T_d\colon U \times H^3(-1) \to \mathbb{R}$$

denote the de Sitter timelike height function given by $H_d^T(u, \boldsymbol{v}) = \langle \boldsymbol{x}^d(u), \boldsymbol{v} \rangle$, and

$$H_d^S \colon U \times S_1^3 \to \mathbb{R}$$

denote the de Sitter spacelike height function given by $H_d^S(u, \boldsymbol{v}) = \langle \boldsymbol{x}^d(u), \boldsymbol{v} \rangle$. The function H_d^T measures the contact of the surface with hyperbolic de Sitter quadrics and H_d^S measures its contact with elliptic de Sitter quadrics (see Definition 3.3). One can show that the evolute is the union of the local strata of the bifurcation sets $\mathrm{LBif}(H_d^T)$ and $\mathrm{LBif}(H_d^S)$ of the families H_d^T and H_d^S respectively. More precisely,

$$\mathrm{LBif}(H_d^T) = \mathrm{HE}_{M^d}^+ \cup \mathrm{HE}_{M^d}^-,$$
$$\mathrm{LBif}(H_d^S) = \mathrm{SE}_{M^d}^+ \cup \mathrm{SE}_{M^d}^-.$$

We consider now the multi-local singularities of the spacelike and timelike height functions.

DEFINITION 4.4. The *symmetry set* of M^d, denoted by SS, is defined to be the closure of the set of centres of elliptic and hyperbolic de Sitter quadrics that are tangent to M^d in at least two distinct points. It is the union of the closure of the multi-local strata of the bifurcation sets of the spacelike and timelike height functions H_d^S and H_d^T respectively.

REMARK 4.5. Suppose that both surfaces $\boldsymbol{x}^h(U) = M^h$ and $\boldsymbol{x}^d(U) = M^d$ are smooth for the Legendrian embedding $\mathcal{L}_1 \colon U \to \Delta_1$ given by $\mathcal{L}_1(u) = (\boldsymbol{x}^h(u), \boldsymbol{x}^d(u))$. Then the principal curvatures satisfy $\kappa_h(u)\kappa_d(u) = 1$ by Δ_1-duality. It follows that the total evolutes of M^h and M^d coincide. In [22] is given a unified interpretation of these concepts as caustics of a certain Lagrangian submanifold in the symplectification $\Delta_1 \times \mathbb{R}_+$ of (Δ_1, K_1).

4.2. Timelike surfaces in S_1^3. We define the *de Sitter evolute* of a parametrised timelike surface $\boldsymbol{x} \colon U \to S_1^3$ to be the set

$$\mathrm{SE}_{M^d}^{\pm} = \bigcup_{i=1}^{2} \left\{ \pm \frac{1}{\sqrt{\kappa_i^2(u) + 1}} (\kappa_i(u)\boldsymbol{x}(u) + \boldsymbol{x}^*(u)),\ u \in U \right\},$$

where $\kappa_i(u)$, $i = 1, 2$ are the principal curvature at $\boldsymbol{x}(u)$. The evolute is related to the family of spacelike height functions

$$H^S \colon U \times S_1^3 \to \mathbb{R}$$

given by $H^S(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v} \rangle$. The function H^S measures the contact of the surface with elliptic de Sitter quadrics (see Definition 3.3). Let $H_{\boldsymbol{v}}^S(u) = H^S(u, \boldsymbol{v})$. One can easily show the following.

Proposition 4.6. *The spacelike height function H_v^S is singular at u if and only if there exist real numbers λ, μ such that $\boldsymbol{v} = \lambda \boldsymbol{x}(u) + \mu \boldsymbol{e}(u)$ and $\lambda^2 + \mu^2 = 1$.*

By Proposition 4.6, the discriminant (or catastrophe set) of H^S is given by

$$C(H^S) = \{(u, \boldsymbol{v}) \in U \times S_1^3 \mid \boldsymbol{v} = \lambda \boldsymbol{x}(u) + \mu \boldsymbol{e}(u),\ \lambda^2 + \mu^2 = 1\}.$$

We also have

$$\frac{\partial^2 H^S}{\partial u_i \partial u_j}(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i u_j}(u), \boldsymbol{v} \rangle = -\lambda g_{ij} + \mu h_{ij}$$

on $C(H^S)$, where $g_{11} = E$, $g_{12} = g_{21} = F$ and $g_{22} = G$. If $\mu = 0$, then $\boldsymbol{v} = \pm \boldsymbol{x}$ and $\det(\mathcal{H}(H_v^S)(u)) = \det(g_{ij}) \neq 0$, where $\mathcal{H}$ denotes the Hessian of H_v^S. So, $\det(\mathcal{H}(h_v^S)(u)) = 0$ if and only if λ/μ is a principal curvature. It follows that the local bifurcation set, $\mathrm{LBif}(H^S)$, of the family of the spacelike height functions is the evolute of M^d, that is, $\mathrm{LBif}(H^S) = \mathrm{SE}^+_{M^d} \cup \mathrm{SE}^-_{M^d}$.

REMARK 4.7. There is no hyperbolic component of the evolute of a timelike surface $\boldsymbol{x}\colon U \to S_1^3$. The timelike height function $H^T\colon U \times H^3(-1) \to \mathbb{R}$ is not singular at any point on $\boldsymbol{x}(U)$. The reason being that any hyperbolic de Sitter quadric (whose tangent spaces are spacelike) is always transverse to a timelike surface.

For the duality result in this paper, we require the normal to the evolute.

Proposition 4.8. *Let q be a smooth point on the de Sitter evolute of a timelike surface $M^d \subset S_1^3$ associated to a point $p \in M^d$ not on the lightlike principal locus of M^d. Then the normal to the evolute at q is along the principal direction $\boldsymbol{p}_i$ ($i = 1$ or 2), associated to the principal curvature κ_i defining q.*

The proof is similar to that of Proposition 4.1 and is omitted.

We consider now the multi-local singularities of the spacelike height function.

DEFINITION 4.9. The *symmetry set* of M^d, denoted by SS, is defined to be the closure of the centres of elliptic de Sitter quadrics that are tangent to M^d in at least two distinct points. It is the closure of the multi-local stratum of the bifurcation set of the spacelike height function H^S.

We have the following result analogous to Proposition 4.3.

Proposition 4.10. (1) *A point $q \in S_1^3$ is on the* SS *of a timelike surface $M^d \subset S_1^3$ if and only if there exists two distinct points p_1 and p_2 on M^d such that the tangent planes $T_{p_1}M^d$ and $T_{p_2}M^d$ are symmetric with respect to the sphere orthogonal to the geodesic joining p_1 and p_2 and passing through the midpoint of the segment p_1p_2.*

(2) *Let q be a smooth point on the* SS *corresponding to the bi-tangency of an elliptic de Sitter quadric to the surface M^d at two points p_1 and p_2. Then the normal to the* SS *at q is the normal to the sphere in* (1).

Proof. The proof is similar to that of Proposition 4.3. We consider, by Lorentzian motion, the sphere to be the intersection of the spacelike hyperplane $x_0 = 0$ with S_1^3 and follow the same steps in the proof of Proposition 4.3. □

5. The folding family

5.1. Surfaces in $H_+^3(-1)$. We shall restrict our study to 2-dimensional surfaces in $H_+^3(-1)$. However, the construction of the family of folding maps we give here is valid in $H_+^n(-1)$, $n \geq 3$, and for any embedded submanifold in $H_+^n(-1)$. For the surface case in $H_+^3(-1)$, the folding maps can be represented locally by a map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. A classification of the singularities of such mappings are well known (see for example [25]) and one can deduce interesting geometrical properties of the surface from the singularities of the folding maps.

In the Euclidean case, given a plane $P \subset \mathbb{R}^3$, the folding map in $\mathbb{R}^3$ with respect to P identifies points with the same distance to P. If we want to follow this construction for surfaces embedded in the hyperbolic space $H_+^3(-1)$, we need to identify points with the same distance to some "flat" object. Planes are surfaces with de Sitter principal curvatures vanishing at all points ([6, 19]) and horospheres are surfaces with lightcone principal curvatures vanishing at all points ([17]). As we are aiming to pick up the principal directions of the surface M^h and the fact that these are the same for the de Sitter and lightcone shape operators, it is enough to consider folding with respect to planes. We observe that a folding with respect to an equidistant surface can be brought, by a hyperbolic motion, to a folding with respect to a plane.

Following the construction in the Euclidean case, folding with respect to a plane in $H_+^3(-1)$ means taking two distinct points on the same geodesic that are at the same distance d from the plane and mapping them to the point on this geodesic that is at a distance d^2 to the plane. This map is slightly messy to work with, and as we are only interested in its $\mathcal{A}$-singularities, where $\mathcal{A}$ denotes the Mather left-right group, we shall construct an $\mathcal{A}$-equivalent map as follows. (This new map still sends symmetric points with respect to a fixed plane to the same image.)

The planes of interest above are timelike as they are normal to a geodesic which has a spacelike tangent vector. Consider folding with respect to the timelike hyperplane $x_2 = 0$. Thus, we seek a fold map that identifies any two points (x_0, x_1, x_2, x_3) and $(x_0, x_1, -x_2, x_3)$ in $H_+^3(-1)$. As

$$H_+^3(-1) = \left\{ \left(\sqrt{x_1^2 + x_2^2 + x_3^2 + 1}, x_1, x_2, x_3 \right) \;\middle|\; (x_1, x_2, x_3) \in \mathbb{R}^3 \right\},$$

we define the folding map with respect to the timelike hyperplane $x_2 = 0$ as the map

$$f_1 \colon H^3_+(-1) \to H^3_+(-1)$$

given by $f_1\Big(\sqrt{x_1^2 + x_2^2 + x_3^2 + 1}, x_1, x_2, x_3\Big) = \Big(\sqrt{x_1^2 + x_2^4 + x_3^2 + 1}, x_1, x_2^2, x_3\Big)$.

Let $p \in H^3_+(-1)$ and C be the geodesic through p and orthogonal to $H^3_+(-1) \cap HP(\boldsymbol{e}_2, 0)$ at some point q. Then C is parametrised by $c(\theta) = \cosh(\theta)q + \sinh(\theta)\boldsymbol{e}_2$. Thus $p = c(\theta_0)$, for some θ_0, and the symmetric point of p on C with respect to $H^3_+(-1) \cap HP(\boldsymbol{e}_2, 0)$ is the point $\bar{p} = c(-\theta_0)$. It is clear that $f_1(p) = f_1(\bar{p})$. This means that f_1 send the symmetric points with respect to the plane $H^3_+(-1) \cap HP(\boldsymbol{e}_2, 0)$ to the same image. It follows that this property is invariant under the Lorentzian isometry. Therefore we can proceed as in [2, 5]. The timelike hyperplane $x_2 = 0$ is of course arbitrary. If we are interested in studying the reflectional symmetry of the surface M^h with respect to all timelike hyperplanes, we need to consider the family of folding maps parametrised by these hyperplanes. Let $\mathrm{SO}_0(1, 3)$ denotes the positive Lorentzian group. We define

$$\bar{F} \colon H^3_+(-1) \times \mathrm{SO}_0(1, 3) \to H^3_+(-1)$$

by $\bar{F}(p, A) = (A^{-1} \circ f_1 \circ A)(p)$. This is a 6-parameter family of folding maps. However, there are some redundant parameters that can be eliminated by considering the quotient of $\mathrm{SO}_0(1, 3)$ by the subgroup H_2 of motions that preserve $x_2 = 0$ (i.e., $HP(\boldsymbol{e}_2, 0)$). We then obtain a family

$$F \colon H^3_+(-1) \times \mathrm{SO}_0(1, 3)/H_2 \to H^3_+(-1).$$

We shall now show that $\mathrm{SO}_0(1, 3)/H_2 \cong S^3_1$. We consider the action of $\mathrm{SO}_0(1, 3)$ on S^3_1 defined by $\boldsymbol{v}A$ for any $(A, \boldsymbol{v}) \in \mathrm{SO}_0(1, 3) \times S^3_1$. It is well known (cf., [10]) that this action is transitive (of course, one can also show by direct linear algebra arguments that this fact holds). Consider the two isotropic subgroups of $\mathrm{SO}_0(1, 3)$ defined by

$$H_i = \{A \in \mathrm{SO}_0(1, 3) \mid \boldsymbol{e}_i A = \boldsymbol{e}_i\}, \quad i = 2, 3.$$

Let

$$P_{(3,4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SO}_0(1, 3)$$

so that $\boldsymbol{e}_2 P_{(3,4)} = \boldsymbol{e}_3$. One can show that if $A \in H_3$ then $P_{(3,4)} A P^{-1}_{(3,4)} \in H_2$, so that we have a diffeomorphism

$$\Psi \colon \mathrm{SO}_0(1, 3)/H_3 \to \mathrm{SO}_0(1, 3)/H_2$$

between homogeneous spaces defined by $\Psi([A]) = [P_{(3,4)}AP_{(3,4)}^{-1}]$. Since

$$H_3 = \left\{ \begin{pmatrix} B & {}^t\mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \;\middle|\; B \in \mathrm{SO}_0(1,2) \right\},$$

we have the canonical diffeomorphisms

$$\mathrm{SO}_0(1,3)/H_2 \cong \mathrm{SO}_0(1,3)/\mathrm{SO}_0(1,2) \cong S_1^3.$$

Therefore the family of folding maps can be considered as a family

$$F\colon H_+^3(-1) \times S_1^3 \to H_+^3(-1).$$

Given an embedding $\boldsymbol{x}\colon M^h \to H_+^3(-1)$, we obtain a family

$$F_{\boldsymbol{x}}\colon M^h \times S_1^3 \to H_+^3(-1)$$

by restriction to $M^h \times S_1^3$. We have the following result where the term generic is defined in terms of transversality to submanifolds of multi-jet spaces (see for example [9]).

Theorem 5.1. *For a residual set of embeddings $\boldsymbol{x}\colon M^h \to H_+^3(-1)$, the family $F_{\boldsymbol{x}}$ is a generic family of mappings.*

Proof. The map f_1 defined above is a fold map, so it is an $\mathcal{A}$-stable map. Therefore, the corresponding 3-dimensional family F is an $\mathcal{A}$-versal family of mappings in the sense of Montaldi [26]. The assertion follows now from Montaldi's theorem in [26]. □

For a given $\boldsymbol{v} \in S_1^3$ and a point $p_0 \in M^h$, one can choose local coordinates so that $F_{\boldsymbol{x}}^{\boldsymbol{v}}(p) = F_{\boldsymbol{x}}(p, \boldsymbol{v})$ can be considered locally as a map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. It follows from Theorem 5.1 that for generic embeddings of the surface, only singularities of $\mathcal{A}_e$-codimension ≤ 3 can occur in the members of the family of folding maps (3 being the dimension of the parameter space S_1^3). Therefore, we have the following result.

Proposition 5.2. *For a residual set of embeddings $\boldsymbol{x}\colon M^h \to H_+^3(-1)$, the folding maps $F_{\boldsymbol{x}}^{\boldsymbol{v}}\colon M^h \to H_+^3(-1)$ in the family $F_{\boldsymbol{x}}$ have local singularities $\mathcal{A}$-equivalent to one in* Table 1. *Moreover, these singularities are versally unfolded by the family $F_{\boldsymbol{x}}$.*

For a fixed embedding $\boldsymbol{x}\colon M^h \to H_+^3(-1)$, we define a mapping $\Psi_{\boldsymbol{x}}\colon S_1^3 \to C^\infty(M^h, H_+^3(-1))$ by $\Psi_{\boldsymbol{x}}(\boldsymbol{v}) = F_{\boldsymbol{x}}^{\boldsymbol{v}}$. We remark that $\Psi_{\boldsymbol{x}}$ is a continuous mapping with respect to the Whitney C^∞-topology of $C^\infty(M^h, H_+^3(-1))$. Since the set of stable mappings is open in $C^\infty(M^h, H_+^3(-1))$, the set of $\boldsymbol{v} \in S_1^3$ such that $F_{\boldsymbol{x}}^{\boldsymbol{v}}$ is a stable mapping is an open subset of S_1^3. It also follows from Theorem 5.1 that for a generic

Table 1. $\mathcal{A}_e$-codimension ≤ 3 singularities of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ ([25]).

Normal form	Name	$\mathcal{A}_e$-codimension
$(x, y, 0)$	Immersion	0
(x, y^2, xy)	Cross-cap	0
$(x, y^2, x^2y \pm y^{2k+1})$, $k = 1, 2, 3$	B_k	k
$(x, y^2, y^3 \pm x^{k+1}y)$, $k = 2, 3$	S_k	k
$(x, y^2, xy^3 \pm x^k y)$, $k = 3$	C_k	k

embedding $\boldsymbol{x}\colon M^h \to H^3_+(-1)$ and for $\boldsymbol{v}$ in an open and dense subset of S^3_1, the map $F^{\boldsymbol{v}}_{\boldsymbol{x}}\colon M^h \to H^3_+(-1)$ is stable, i.e., is locally an immersion, a cross-cap or a pair of transverse planes. The set of vectors $\boldsymbol{v} \in S^3_1$ for which $F^{\boldsymbol{v}}_{\boldsymbol{x}}$ is not $\mathcal{A}$-stable is the *bifurcation set*, $\mathrm{Bif}(F_{\boldsymbol{x}})$, of $F_{\boldsymbol{x}}$. This set consists of vectors $\boldsymbol{v}$ for which $F^{\boldsymbol{v}}_{\boldsymbol{x}}$ has a singularity more degenerate than a cross-cap (generically one of the B_k, S_k, C_k in Proposition 5.2) or the image has a multi-local singularity of type self tangency or worse. We have the following duality result, analogous to the one in [5] for the Euclidean case, where duality here refers to Δ_1-duality when the evolute/symmetry set lies in the hyperbolic space and Δ_5-duality when it is in the de Sitter space (see Theorem 2.1 and Remark 2.2).

Theorem 5.3. *The bifurcation set* $\mathrm{Bif}(F_{\boldsymbol{x}})$ *of the family of folding maps on a surface* $M^h \subset H^3_+(-1)$ *is the dual of the evolute and the symmetry set of* M^h. *More precisely, the local stratum of* $\mathrm{Bif}(F_{\boldsymbol{x}})$ *is the dual of the evolute and the multi-local stratum of* $\mathrm{Bif}(F_{\boldsymbol{x}})$ *is the dual of the symmetry set.*

Proof. We take the surface M^h, without loss of generality, in the hyperbolic Monge form (see [17])

$$\boldsymbol{x}(u_1, u_2) = \left(\sqrt{g^2(u_1, u_2) + u_1^2 + u_2^2 + 1},\, g(u_1, u_2),\, u_1,\, u_2\right)$$

at the origin, with g and its first derivatives vanishing at the origin. We write $j^2g(u_1, u_2) = a_{20}u_1^2 + a_{21}u_1u_2 + a_{22}u_2^2$. The restriction of the folding map f to M^h is given by

$$f_1(u_1, u_2) = \left(\sqrt{g^2(u_1, u_2) + u_1^4 + u_2^2 + 1},\, g(u_1, u_2),\, u_1^2,\, u_2\right).$$

If we project it to the tangent space of $H^3_+(-1)$ at $\boldsymbol{x}(0, 0)$ (i.e., to the space $x_0 = 0$) we obtain a map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ which is $\mathcal{A}$-equivalent to f_1 and is given by

$$\tilde{f}_1(u_1, u_2) = (g(u_1, u_2),\, u_1^2,\, u_2).$$

This map-germ has a singularity of type cross-cap at the origin if and only if $a_{21} \neq 0$, if and only if the normal to the hyperplane $x_2 = 0$ is not along a principal direction.

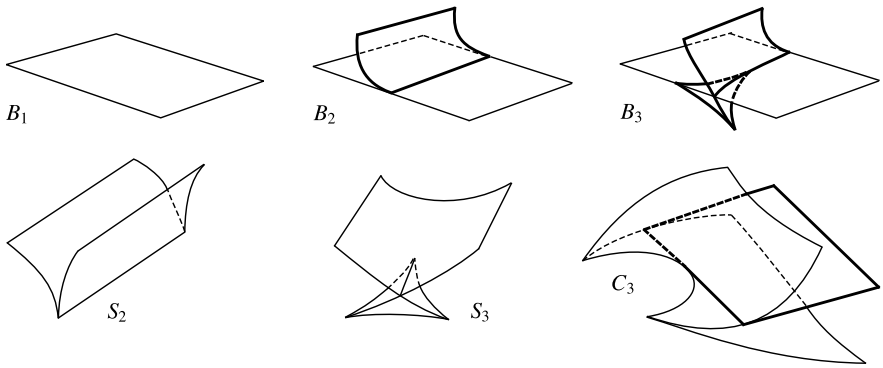

Fig. 1. Bifurcation sets (local strata in thin and multi-local strata in thick).

It follows then that the local stratum of the bifurcation set of F_x is the surface in S_1^3 traced by the (unit) principal directions of M^h. However, by Proposition 4.1, a principal direction is the normal to the evolute and by Theorem 2.1 (see also Remark 2.2), these normals trace the dual of the evolute. Here, duality refers to Δ_1-duality when the evolute lies in the hyperbolic space and Δ_5-duality when it is in the de Sitter space.

The duality for the multi-local stratum of the bifurcation set of the folding map follows from Proposition 4.3, Theorem 2.1 and Remark 2.2. □

Since the family F_x is an $\mathcal{A}$-versal unfolding of each of its singularities, we can deduce the model (up-to diffeomorphism) of its bifurcation set $\mathrm{Bif}(F_x)$, and hence of the dual of the evolute and symmetry set. The models for the local singularities are given in Fig. 1.

We can deduce from Theorem 5.3 and from the results in [19] the following geometric characterisations of the singularities of the folding maps:

B_1: General smooth point of the evolute.

S_2: De Sitter parabolic smooth point of the evolute.

S_3: Swallowtail of the de Sitter Gauss indicatrix at smooth point of the evolute.

These are also the points where the principal direction corresponding to the null principal curvature is tangent to the parabolic set of the evolute.

B_2: General cuspidal-edge point of the evolute.

B_3: Cuspidal-edge point of the evolute in the closure of the de Sitter parabolic curve on the symmetry set.

C_3: Intersection point of the cuspidal-edge and parabolic curves on the evolute.

Here, the *general smooth point* means a smooth point which is different from S_2 or S_3 and the *general cuspidal-edge point* means a cuspidal-edge point which is different from B_3 or C_3.

Following [5, 32], we shall call the pre-image on M^h of the de Sitter parabolic set of the evolute the *sub-parabolic curve* of M^h. In the Euclidean case, the sub-parabolic curve is the locus of points where lines of curvature have geodesic inflections. It is also the locus of points where one principal curvature has an extremal value along lines of the other principal curvature [27]. We have a similar characterisation for surfaces in the hyperbolic space. Recall that the restriction of the pseudo-scalar product to the hyperbolic space is a scalar product, so this space is a Riemannian manifold.

The lines of curvature (i.e., curves on M^h whose tangent at each point is a principal direction) are given, in the parameters space, by the usual equation

$$(Gm - Fn)\,du_2^2 + (Gl - En)\,du_2\,du_1 + (Fl - Em)\,du_1^2 = 0$$

(see for example [31]) where E, F, G are the coefficients of the first fundamental form and l, m, n are the coefficients of the (de Sitter) second fundamental form.

Proposition 5.4. *The sub-parabolic curve of an embedded surface M^h in $H^3_+(-1)$ can be characterised as follows.*
(1) *It is the locus of points where one principal curvature has an extremal value along lines of the other principal curvature.*
(2) *It is the locus of points where the other lines of curvature have geodesic inflections.*

Proof. (1) We take the surface in hyperbolic Monge form as in the proof of Theorem 5.3 and write $j^3g(u_1, u_2) = a_{20}u_1^2 + a_{22}u_2^2 + a_{30}u_1^3 + a_{31}u_1^2u_2 + a_{32}u_1u_2^2 + a_{33}u_2^3$. Then folding along the hyperplane $x_2 = 0$ yields a singularity worse than a cross-cap. The folding map $\tilde{f}_1(u_1, u_2) = (g(u_1, u_2), u_1^2, u_2)$ has an S_2-singularity if and only if $a_{32} = 0$ (and $a_{30} \neq 0$). A calculation shows that the 1-jet of the principal curvature associated to the other principal direction (0, 0, 0, 1) at the origin (which is contained in the hyperplane $x_2 = 0$) is given by $j^1\kappa_2 = 2a_{22} + 2a_{32}u_1 + 6a_{33}u_2$. It has an extremal value along the line of principal curvature associated to (0, 0, 1, 0) if and only if $a_{32} = 0$, which proves statement (1).

(2) We get the initial term of the line of curvature tangent to $(0, 1)$ in the parameter space by solving the equation of the lines of curvature with the hyperbolic Monge form setting above. It is given by $(u_1(s), u_2(s)) = ((a_{32}/2(a_{20} - a_{22}))s^2 + \text{h.o.t.}, s)$. The principal curve $\boldsymbol{x}^h(u_1(s), u_2(s))$ has a geodesic inflection at the origin if and only if $a_{32} = 0$, if and only if $\boldsymbol{x}^h(0, 0)$ is a sub-parabolic point. □

5.2. Spacelike surfaces in S^3_1. As the surface is spacelike, we have everywhere defined principal directions (away from umbilic points) and these are spacelike. Therefore, we are interested in measuring the reflectional symmetry of the surface with respect to timelike hyperplanes. We proceed as in §5.1 and start by considering folding with respect to the hyperplane $x_2 = 0$. For the de Sitter space, unlike for the hyperbolic

space, one needs several charts to express it as the graph of a function. We define the fold map using a global parametrisation.

Let $g(u, \theta, \phi) = (x_0, x_1, x_2, x_3)(u, \theta, \phi)$ be a parametrisation of the de Sitter space S_1^3 given by

$$\begin{aligned} x_0 &= u, \\ x_1 &= \sqrt{1+u^2}\cos\theta\sin\phi, \\ x_2 &= \sqrt{1+u^2}\cos\phi, \\ x_3 &= \sqrt{1+u^2}\sin\theta\sin\phi, \end{aligned}$$

where $u \in \mathbb{R}$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. We define the folding map with respect to the hyperplane $x_2 = 0$ as the map $f_2 \colon S_1^3 \to S_1^3$ given by

$$f_2((x_0, x_1, x_2, x_3)(u, \theta, \phi)) = g(u, \theta, t(\phi))$$

where

$$t(\phi) = \frac{\pi}{2} - \frac{2}{\pi}\left(\phi - \frac{\pi}{2}\right)^2.$$

This is simply a folding map on each level sphere $x_0 =$ constant in S_1^3. We can then follow the same analysis in §5.1 and deduce the same duality result, where the evolute and symmetry set refer to the sets defined in §4.1. In practise, as we are considering local or multi-local properties of the surface, we can choose a different folding map f_2 defined on a chart where S_1^3 is given as a graph of a function. For example, we can work with the local chart $x_0 = \pm\sqrt{-1 + x_1^2 + x_2^2 + x_3^2}$, with $x_0 \neq 0$, and define the folding map as

$$f_2\left(\pm\sqrt{-1 + x_1^2 + x_2^2 + x_3^2}, x_1, x_2, x_3\right) = \left(\pm\sqrt{-1 + x_1^2 + x_2^4 + x_3^2}, x_1, x_2^2, x_3\right).$$

5.3. Timelike surfaces in S_1^3. The folding maps measure the reflectional symmetry of a surface with respect to hyperplanes. In the case of spacelike surfaces the hyperplanes of interest are those whose normals are principal directions. In the case of timelike surfaces, when the principal directions exist, one is timelike and the other is spacelike (Proposition 3.5). Thus, we need to consider two families of folding maps. One is with respect to timelike hyperplanes. This family is the same as that considered in §5.2. The duality result in §5.2 is valid here too (away from the lightlike principal locus), with duality meaning Δ_5-duality only (recall the there is no hyperbolic component of the evolute of a timelike surface in S_1^3). The second family, which we construct below, is the family of folding maps with respect to spacelike hyperplanes. We proceed

as in §5.1. Given the parametrisation $g(u, \theta, \phi)$ of S_1^3 in §5.2, we define the folding map with respect to the spacelike hyperplane $x_0 = 0$ as the map $f_3 \colon S_1^3 \to S_1^3$ given

$$f_3((x_0, x_1, x_2, x_3)(u, \theta, \phi)) = g(u^2, \theta, \phi).$$

In a local chart, say $x_1 = \pm\sqrt{1 + x_0^2 - x_2^2 - x_3^2}$, with $x_1 \neq 0$, the above folding map has the following expression

$$f_3\Big(x_0, \pm\sqrt{1 + x_0^2 - x_2^2 - x_3^2}, x_2, x_3\Big) = \Big(x_0^2, \pm\sqrt{1 + x_0^4 - x_2^2 - x_3^2}, x_2, x_3\Big).$$

We now proceed as in §5.1. The spacelike hyperplane $x_0 = 0$ is of course arbitrary. If we are interested in studying the reflectional symmetry of the surface M^d with respect to all spacelike hyperplanes, we need to consider the family of folding maps parametrised by these hyperplanes. We define

$$\bar{G} \colon S_1^3 \times \mathrm{SO}_0(1, 3) \to S_1^3$$

by $\bar{G}(p, A) = (A^{-1} \circ f_3 \circ A)(p)$. This is a 6-parameter family of folding maps. However, there are some redundant parameters and we need to consider the quotion of $\mathrm{SO}_0(1, 3)$ by the subgroup of Lorentzian motions that preserve $x_0 = 0$ (that is, $HP(\boldsymbol{e}_0, 0)$).

We consider the action of $\mathrm{SO}_0(1, 3)$ on $H_+^3(-1)$ defined by $\boldsymbol{v}A$ for any $(A, \boldsymbol{v}) \in \mathrm{SO}_0(1, 3) \times H_+^3(-1)$. It is well known that this action is transitive. Let

$$H_0 = \{A \in \mathrm{SO}_0(1, 3) \mid \boldsymbol{e}_0 A = \boldsymbol{e}_0\}$$

be an isotropic subgroup of $\mathrm{SO}_0(1, 3)$. Since

$$H_0 = \left\{ \begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \;\middle|\; B \in \mathrm{SO}(3) \right\},$$

we have the canonical diffeomorphisms

$$\mathrm{SO}_0(1, 3)/H_0 \cong \mathrm{SO}_0(1, 3)/\mathrm{SO}(3) \cong H_+^3(-1).$$

Therefore the map $\bar{G}$ gives rise to a 3-parameter family of folding maps

$$G \colon S_1^3 \times H_+^3(-1) \to S_1^3.$$

Given a timelike embedding $\boldsymbol{x} \colon M^d \to S_1^3$, we obtain a family

$$G_{\boldsymbol{x}} \colon M^d \times H_+^3(-1) \to S_1^3$$

by restriction to: $M^d \times H_+^3(-1)$. We obtain the following results following the same arguments as §5.1.

Theorem 5.5. *For a residual set of timelike embeddings* $\boldsymbol{x}\colon M^d \to S^3_1$*, the family* $G_{\boldsymbol{x}}$ *is a generic family of mappings.*

Proposition 5.6. *For a residual set of timelike embeddings* $\boldsymbol{x}\colon M^d \to S^3_1$*, the folding maps in the family* $G_{\boldsymbol{x}}$ *have local singularities* $\mathcal{A}$*-equivalent to one in* Table 1.

We consider now the map $G'_{\boldsymbol{x}}\colon M^d \setminus L \to S^3_1$, where L denotes the lightlike principal locus.

Theorem 5.7. *The bifurcation set* $\mathrm{Bif}(G'_{\boldsymbol{x}})$ *of the folding map on* $M^d \setminus L$ *is the* Δ_1*-dual of the de Sitter evolute and the symmetry set of* $M^d \setminus L$*. More precisely, the local stratum of the bifurcation set is the* Δ_1*-dual of the de Sitter evolute and the multi-local stratum is the* Δ_1*-dual of the symmetry set.*

Proof. The proof is similar to that of Theorem 5.3 and follows from Propositions 4.8 and 4.10. □

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