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Theorems of the Phragmén-Lindelöf Type on an Open Riemann Surface

by Tadashi KURODA^{*)}

Introduction

1. In the theory of analytic functions of a complex variable, the maximum principle for regular functions plays important roles. Especially, in the investigation of the behaviour of a single-valued analytic function with a general existence domain, maximum principles of the Lindelöf type and theorems of the Phragmén-Lindelöf type are very important.

In this paper, we shall prove some theorems of the Phragmén-Lindelöf type and state some applications of them. The Iversen property of a covering surface spread over the complex plane is essentially deduced from the fact that a theorem of the Phragmén-Lindelöf type holds for a region on the covering surface.

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§ 1.

2. Let F be an open Riemann surface and let $\{F_n\}$ ($n=0, 1, 2, \dots$) be an exhaustion of F such that, for each n , the boundary Γ_n of F_n consists of a finite number of analytic closed curves and such that F_n is contained in F_{n+1} with its boundary Γ_n and further such that each component of $F - F_n$ is non-compact. We denote by $u_n(p)$ the harmonic function in $F_n - F_{n-1}$ ($n \geq 1$) which is equal to zero on Γ_{n-1} and to $\log \sigma_n$ on Γ_n and whose conjugate function $v_n(p)$ has the variation 2π on Γ_{n-1} , i.e.,

$$\int_{\Gamma_{n-1}} dv_n = 2\pi,$$

where the integral is taken in the positive sense with respect to F_{n-1} . The quantity $\log \sigma_n$ is the so-called harmonic modulus of the open set $F_n - \bar{F}_{n-1}$. If we choose an additive constant of $v_n(p)$ suitably, the

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regular function $u_n(p) + iv_n(p)$ maps $F_n - F_{n-1}$ with a finite number of suitable slits onto a slit-rectangle $0 \leq u_n < \log \sigma_n$, $0 < v_n < 2\pi$ in a one to one conformal manner. Hence the function $u(p) + iv(p)$ defined by $u_n(p) + iv_n(p) + \sum_{i=1}^{n-1} \log \sigma_i$ for each $F_n - F_{n-1}$ ($n \geq 1$) maps $F - F_0$ with at most an enumerable number of suitable slits onto a strip domain $0 \leq u < \sum_{i=1}^{\infty} \log \sigma_i$, $0 < v < 2\pi$ with at most an enumerable number of slits one to one conformally. This strip domain is the graph associated with the exhaustion $\{F_n\}$ in the sense of Noshiro [6]. We put

$$R = \sum_{i=1}^{\infty} \log \sigma_i.$$

By Sario-Noshiro's theorem [8], [6], there exists an exhaustion $\{F_n\}$ ($n=0, 1, 2, \dots$) of F satisfying $R = \infty$ if and only if F has a null boundary.

3. Let G be a non-compact domain on an open Riemann surface F whose relative boundary C consists of at most an enumerable number of analytic curves being compact or non-compact and clustering nowhere in F . For the sake of convenience, we shall call such a domain G a non-compact region on F . If a non-compact region on F is prolongable analytically over an open Riemann surface F^* , we shall say that G is imbedded conformally into F^* .

Here we shall give a condition for G to be able to be imbedded conformally into an open Riemann surface with null boundary.

If we denote by γ_r the niveau curve $u(p) = r$ ($0 < r < R$) on F , γ_r consists of a finite number of analytic closed curves and separates the ideal boundary of F from F_0 . Denoting by θ_r the part of γ_r contained in G and putting

$$\int_{\theta_r} dv = \theta(r),$$

we have the following

Theorem 1. *The non-compact region G on F can be imbedded conformally into an open Riemann surface with null boundary, if and only if there exists an exhaustion of F such that the integral*

$$(1) \quad \int \frac{dr}{\theta(r)}$$

is divergent.

Proof. First we shall prove the necessity of the condition. For the purpose, we may suppose that F has a null boundary. As stated

above, there exists an exhaustion $\{F_n\}$ ($n=0, 1, 2, \dots$) such that $R=\infty$. Since $\theta(r) \leq 2\pi$, the integral (1) is divergent for this exhaustion.

Next we shall give the proof of sufficiency. By the usual process of symmetrization, we can construct an open Riemann surface \hat{G} . There is given an indirectly conformal mapping of \hat{G} on itself which leaves every point on C fixed, where C is the relative boundary of G with respect to F . It is sufficient to prove that \hat{G} has a null boundary under our condition.

Let Δ be a simply connected domain in G such that the boundary of Δ is an analytic closed curve and such that the closure $\bar{\Delta}$ of Δ is contained in G . Denote by $\tilde{\Delta}$ and $\bar{\tilde{\Delta}}$ the images of Δ and $\bar{\Delta}$, respectively, under the indirectly conformal mapping of \hat{G} on itself. We choose an exhaustion $\{\hat{G}_n\}$ ($n=1, 2, \dots$) of \hat{G} such that, for each n , \hat{G}_n contains $\tilde{\Delta}$ and $\bar{\tilde{\Delta}}$ and is symmetric with respect to C . If we construct the harmonic measure $\omega_n(p)$ ($p \in \hat{G}_n - (\bar{\tilde{\Delta}} \cup \tilde{\tilde{\Delta}})$) of the boundary of \hat{G}_n with respect to the domain $\hat{G}_n - (\bar{\tilde{\Delta}} \cup \tilde{\tilde{\Delta}})$, we get a sequence $\{\omega_n(p)\}$ ($n=1, 2, \dots$) of uniformly bounded harmonic functions. It is easily seen from the configuration of \hat{G}_n that $\omega_n(p) = \omega_n(\tilde{p})$, where \tilde{p} is the image of the point p under the indirectly conformal mapping of \hat{G} on itself. Since $0 < \omega_n(p) < 1$ for each n , we can select a subsequence of $\{\omega_n(p)\}$ which is uniformly convergent on $G - (\bar{\tilde{\Delta}} \cup \tilde{\tilde{\Delta}})$ in the wider sense and which has a uniquely determined limiting function $\omega(p)$. This function $\omega(p)$ is harmonic in $G - (\bar{\tilde{\Delta}} \cup \tilde{\tilde{\Delta}})$ and equals zero on the boundary of Δ and $\tilde{\Delta}$. From the fact that $\omega_n(p) = \omega_n(\tilde{p})$ for each n , we can see that the normal derivative $\frac{\partial \omega}{\partial \nu}$ vanishes at every point on C . \hat{G} has a null boundary if and only if the function $\omega(p)$ is identically equal to zero. Hence we shall prove that $\omega(p)$ vanishes throughout $G - \bar{\Delta}$ under our condition.

Now we construct a graph $0 \leq u < R$, $0 < v < 2\pi$ associated with an exhaustion $\{F_n\}$ ($n=0, 1, 2, \dots$) for which the integral (1) is divergent. Without loss of generality, we may assume that F_0 is identical to Δ . Let us denote by G_r the open subset of $G - \bar{\Delta}$ consisting of points, each of which satisfies the condition $0 < u(p) < r$ ($0 < r < R$). The boundary of G_r consists of θ_r , a part of C and the boundary of Δ . It is evident that G_r is not empty for any $r > 0$. Denoting by $D(r)$ the Dirichlet integral of $\omega(p)$ taken over G_r , we have

$$D(r) = \int_{\theta_r} \omega \frac{\partial \omega}{\partial u} dv,$$

because $\omega(p)$ equals zero on the boundary of Δ and the normal derivative $\frac{\partial \omega}{\partial \nu}$ vanishes at every point on C . By the Schwarz inequality, we get

$$\begin{aligned} (D(r))^2 &\leq \int_{\theta_r} dv \int_{\theta_r} \left(\frac{\partial \omega}{\partial u} \right)^2 dv \\ &\leq \theta(r) \frac{dD(r)}{dr}, \end{aligned}$$

whence follows that

$$\frac{dr}{\theta(r)} \leq \frac{dD(r)}{(D(r))^2}.$$

Integrating both sides, we obtain

$$\int_{r_0}^r \frac{dr}{\theta(r)} \leq \frac{1}{D(r_0)} - \frac{1}{D(r)} \leq \frac{1}{D(r_0)},$$

where r_0 is a positive number fixed arbitrarily. Since the integral of the left hand side is divergent as $r \rightarrow R$, the Dirichlet integral $D(r_0)$ of $\omega(p)$ taken over the non-empty open set G_{r_0} must be equal to zero and hence the function $\omega(p)$ must reduce to the constant zero. Thus our proof is complete.

This theorem is the same as the result essentially which was obtained by Noshiro (Cf. [3]). Further, the following is easily obtained from the proof of the above theorem.

Corollary (KURAMOCHI [2]). *Suppose that G is a non-compact region on an open Riemann surface with null boundary. Then the double \hat{G} , which is obtained from G by the process of symmetrization, has also a null boundary.*

§ 2.

4. Here we shall state some theorems of the Phragmén-Lindelöf type. Let F be an open Riemann surface and let G be a non-compact region on F with the relative boundary C . In the following, we choose an exhaustion $\{F_n\}$ ($n=0, 1, 2, \dots$) of F satisfying the condition $F_0 \cap G = 0$ and associate the graph $0 \leq u < R$, $0 < v < 2\pi$ with F which corresponds to this exhaustion and we denote by γ_r the niveau curve

$u(p) = r$ on F as in §1. We shall prove the following

Theorem 2. *Suppose that a function $f(p)$ regular in G is continuous on $G \cup C$ and that $|f(p)|$ is single-valued on $G \cup C$ and satisfies the condition $|f(p)| \leq 1$ on C . If there exists a point p_0 in G such that $|f(p_0)| > 1$, then*

$$\lim_{r \rightarrow R} \frac{(\log M(r))^2}{\int_{r_0}^r \frac{dr}{\theta(r)}} > 0,$$

where $M(r)$ is the maximum of $|f(p)|$ on $\theta_r (= \gamma_r \cap G)$ and $u(p_0) = r_0$ and further, $\theta(r) = \int_{\theta_r} dv$.

Proof. We put $h(p) = \log^+ |f(p)|$, where, for any real number x , $\log^+ x$ is the maximum of zero and $\log x$. Let us denote by G_r the open subset of G which consists of points of G satisfying $u(p) < r$. If $u(p_0) = r_0$, $h(p)$ is non-constant in G_r for any number $r \geq r_0$. Denoting by $D(r)$ the Dirichlet integral of $h(p)$ taken over G_r , we have

$$D(r) = \int_{\theta_r} h \frac{\partial h}{\partial u} dv,$$

for, $h(p)$ is non-constant in G_r and harmonic at every point p satisfying $h(p) = \log |f(p)| > 0$ and reduces to the constant zero elsewhere. It is obvious that $D(r)$ is positive for any $r \geq r_0$. By the Schwarz inequality, we get

$$\begin{aligned} (D(r))^2 &\leq \int_{\theta_r} h^2 dv \int_{\theta_r} \left(\frac{\partial h}{\partial u}\right)^2 dv \\ &\leq \theta(r) (\log M(r))^2 \frac{dD(r)}{dr}, \end{aligned}$$

or

$$\frac{dr}{\theta(r)} \leq (\log M(r))^2 \frac{dD(r)}{(D(r))^2}.$$

Integrating both sides, we obtain

$$\begin{aligned} \int_{r_0}^r \frac{dr}{\theta(r)} &\leq (\log M(r))^2 \left[\frac{1}{D(r_0)} - \frac{1}{D(r)} \right] \\ &\leq (\log M(r))^2 \frac{1}{D(r_0)}, \end{aligned}$$

because $M(r)$ is a monotonically increasing function of r . Hence it follows that, for any $r > r_0$,

$$0 < D(r_0) \leq \frac{(\log M(r))^2}{\int_{r_0}^r \frac{dr}{\theta(x)}},$$

which proves our theorem.

This theorem implies the following which contains Kusunoki's result [4].

Theorem 3. *Under the same conditions in Theorem 2,*

$$\lim_{r \rightarrow R} \frac{\log M(r)}{\sqrt{r}} > 0.$$

5. In the preceding section we dealt with the regular function with uniform modulus. Here we shall consider the single-valued regular function.

Let G be a non-compact region on F with the relative boundary C and let $f(p) = U(p) + iV(p)$ be a single-valued regular function in G being continuous on $G \cup C$. Denote by G_r the open subset of G , every point of which satisfies the condition $u(p) < r$.

Suppose that the real part $U(p)$ of $f(p)$ equals zero on C . The part θ_r of the niveau curve $\gamma_r: u(p) = r$ contained in G consists of at most a finite number of components θ_r^i ($i = 1, 2, \dots, n = n(r)$). If we denote by $D(r)$ the Dirichlet integral of $f(p)$ taken over G_r , then we get

$$D(r) = \sum_{i=1}^{n(r)} \int_{\theta_r^i} U dV = \sum_{i=1}^{n(r)} \int_{\theta_r^i} U \frac{\partial U}{\partial u} dv.$$

In the case of θ_r^i which is a cross-cut of G , since by Wirtinger's inequality

$$\int_{\theta_r^i} U^2 dv \leq \frac{(\theta_i(r))^2}{\pi^2} \int_{\theta_r^i} \left(\frac{\partial U}{\partial v} \right)^2 dv,$$

where $\theta_i(r) = \int_{\theta_r^i} dv$, we have

$$\begin{aligned} \left(\int_{\theta_r^i} U \frac{\partial U}{\partial u} dv \right)^2 &\leq \int_{\theta_r^i} U^2 dv \int_{\theta_r^i} \left(\frac{\partial U}{\partial u} \right)^2 dv \\ &\leq \frac{(\theta_i(r))^2}{\pi^2} \int_{\theta_r^i} \left(\frac{\partial U}{\partial v} \right)^2 dv \int_{\theta_r^i} \left(\frac{\partial U}{\partial u} \right)^2 dv, \end{aligned}$$

and hence we obtain

$$\int_{\theta_r^i} U \frac{\partial U}{\partial u} dv \leq \frac{\theta_i(r)}{2\pi} \int_{\theta_r^i} \left[\left(\frac{\partial U}{\partial u} \right)^2 + \left(\frac{\partial U}{\partial v} \right)^2 \right] dv.$$

Next we consider the case of θ_r^j being a loop-cut of G . We can choose a constant m_j such that $\int_{\theta_r^j} (U - m_j) dv = 0$. By Wirtinger's inequality, we have

$$\int_{\theta_r^j} (U - m_j)^2 dv \leq \frac{(\theta_j(r))^2}{4\pi^2} \int_{\theta_r^j} \left(\frac{\partial U}{\partial v}\right)^2 dv.$$

On the other hand, since $f(p)$ is single-valued, it follows that

$$\int_{\theta_r^j} U dV = \int_{\theta_r^j} (U - m_j) dV,$$

whence we obtain

$$\begin{aligned} \left(\int_{\theta_r^j} U \frac{\partial U}{\partial u} dv\right)^2 &= \left(\int_{\theta_r^j} (U - m_j) \frac{\partial U}{\partial u} dv\right)^2 \\ &\leq \int_{\theta_r^j} (U - m_j)^2 dv \int_{\theta_r^j} \left(\frac{\partial U}{\partial u}\right)^2 dv \\ &\leq \frac{(\theta_j(r))^2}{4\pi^2} \int_{\theta_r^j} \left(\frac{\partial U}{\partial v}\right)^2 dv \int_{\theta_r^j} \left(\frac{\partial U}{\partial u}\right)^2 dv. \end{aligned}$$

Thus we get

$$\int_{\theta_r^j} U \frac{\partial U}{\partial u} dv \leq \frac{\theta_j(r)}{4\pi} \int_{\theta_r^j} \left[\left(\frac{\partial U}{\partial u}\right)^2 + \left(\frac{\partial U}{\partial v}\right)^2\right] dv.$$

Therefore, it holds for any number i that

$$\int_{\theta_r^i} U \frac{\partial U}{\partial u} dv \leq \frac{\Theta(r)}{2\pi} \int_{\theta_r^i} \left[\left(\frac{\partial U}{\partial u}\right)^2 + \left(\frac{\partial U}{\partial v}\right)^2\right] dv,$$

where $\Theta(r) = \text{Max}_{1 \leq i \leq n(r)} \theta_i(r)$. Summing up these inequalities for $i = 1, 2, \dots, n(r)$, we have

$$D(r) \leq \frac{\Theta(r)}{2\pi} \frac{dD(r)}{dr},$$

or

$$2\pi \frac{dr}{\Theta(r)} \leq \frac{dD(r)}{D(r)}.$$

Integrating both sides, we obtain

$$2\pi \int_{r_0}^r \frac{dr}{\Theta(r)} \leq \log \frac{D(r)}{D(r_0)},$$

where r_0 is a suitable number such that there exists a point p_0 of G satisfying $u(p_0) = r_0$. Hence it follows that

$$(2) \quad D(r_0)e^{2\pi \int_{r_0}^r \frac{dr}{\theta(r)}} \leq D(r).$$

On the other hand, since

$$\frac{d}{dr} \left(\int_{\theta_r} U^2 dv \right) = 2 \int_{\theta_r} U \frac{\partial U}{\partial u} dv = 2D(r),$$

it is easy to see that

$$\begin{aligned} \int_{r_0}^r D(r) dr &= \frac{1}{2} \left(\int_{\theta_r} U^2 dv - \int_{\theta_{r_0}} U^2 dv \right) \\ &\leq \frac{1}{2} \int_{\theta_r} U^2 dv \leq \pi (M^*(r))^2, \end{aligned}$$

where $M^*(r)$ is the maximum of $|U(p)|$ on θ_r . From this and (2), we get

$$\frac{(M^*(r))^2}{\int_{r_0}^r e^{2\pi \int_{r_0}^r \frac{dr}{\theta(r)}} dr} \geq \frac{D(r_0)}{\pi}.$$

If the function is non-constant, $D(r_0)$ is positive. Thus we have the following

Theorem 4. *Suppose that $f(p)$ is a single-valued regular function in a non-compact region G on an open Riemann surface and that the real part of $f(p)$ is equal to zero on the relative boundary of G . Denote by $M^*(r)$ the maximum of the absolute values of the real part of $f(p)$ on θ_r . If*

$$\lim_{r \rightarrow R} \frac{(M^*(r))^2}{\int_{r_0}^r e^{2\pi \int_{r_0}^r \frac{dr}{\theta(r)}} dr} = 0,$$

then $f(p)$ reduces to a constant.

The argument of the above proof is due to Pfluger [7].

This theorem is applicable to investigate the behaviour of functions on an open Riemann surface satisfying the condition similar to that of Pfluger.

§ 3.

6. Let F be an open Riemann surface and let $w=f(p)$ be a non-constant single-valued analytic function defined on F . The space formed by elements $q=[p, f(p)]$ defines a covering surface Φ spread

over the w -plane and the point $q = [p, f(p)]$ has the projection $w = f(p)$. The correspondence $p \leftrightarrow q$ gives a topological and conformal mapping between F and Φ .

Let Φ_Δ be any connected piece of Φ lying on the disc (c_ρ) , where (c_ρ) is the disc $|w - w_0| < \rho$ for any finite point $w = w_0$ and for any positive number ρ or is the disc $|w| > \frac{1}{\rho}$ for any positive number ρ . We shall denote by Δ the domain of F corresponding to Φ_Δ by $p \leftrightarrow q$. If Δ is non-empty for a disc (c_ρ) and if either there exists a point p in Δ such that $w^* = f(p)$ or there exists a path in Δ tending to the ideal boundary of F such that $\lim f(p) = w^*$ along the path, where w^* is the centre of (c_ρ) , then we shall say that Φ has the Iversen property.

Mori [5] proved that Φ has the Iversen property if F belongs to the class O_{HB} which is the class of Riemann surfaces not allowing the existence of the non-constant single-valued bounded harmonic function. In the case of F with null boundary, Stoilow [10] proved this result.

7. Let $\{F_n\}$ ($n = 0, 1, 2, \dots$) be an exhaustion of F and let the strip domain $0 \leq u < R, 0 < v < 2\pi$ be the graph of F associated with the exhaustion $\{F_n\}$. The niveau curve $\gamma_r: u(p) = r$ consists of a finite number of closed analytic curves γ_r^i ($i = 1, \dots, m = m(r)$). Put

$$\Lambda(r) = \text{Max}_{1 \leq i \leq m(r)} \int_{\gamma_r^i} dv.$$

Then the following was proved by Pfluger [7].

If the integral

$$(3) \quad \int_0^R e^{4\pi \int_0^r \frac{dr}{\Lambda(r)}} dr$$

is divergent, there exists no non-constant single-valued bounded analytic function on F .

Hence we can see that if the integral

$$\int_0^R e^{2\pi \int_0^r \frac{dr}{\Lambda(r)}} dr$$

is divergent, there exists no non-constant single-valued bounded analytic function on $F \in O_{AB}$. Further we can prove the following which was found by Z. Kuramochi.

Theorem 5. *If the integral (3) is divergent, Φ has the Iversen property.*

Proof. As mentioned above, there exists no non-constant single-

valued bounded analytic function on F . Hence the set of values taken by $w=f(p)$ is everywhere dense in the w -plane. Therefore, for any disc (c_p) , there exists at least a connected piece of Φ lying over (c_p) . We choose such an arbitrary piece Φ_Δ and denote by Δ the domain on F corresponding to Φ_Δ by the mapping $p \leftrightarrow q$. It is easily seen that, by the mapping $p \leftrightarrow q$, the relative boundary of Δ corresponds to that of Φ_Δ lying over the circumference of (c_p) .

If Δ is compact in F , it is easy to see that there exists a point p_0 such that $f(p_0)=w^*$, where the point w^* is the centre of (c_p) . Hence we suppose that Δ is non-compact. Then Δ is a non-compact region on F .

Let (c) be any concentric circular disc of (c_p) contained in (c_p) and let E be the set of points which lie in the closure $\overline{(c)}$ of (c) and are not covered by Φ_Δ . As is easily seen, for our purpose it is sufficient to prove that E is the set of class $N_{\mathfrak{B}}$ in the sense of Ahlfors-Beurling [1]. Since Φ_Δ is connected, the complementary set of E with respect to the whole w -plane is connected.

Let δ be the domain in the w -plane which is a complementary domain of E with respect to the whole w -plane and contains the circumference of (c_p) . Suppose that E is not the set on the class $N_{\mathfrak{B}}$. Then, by Sario's theorem [8], [9], there exists a non-constant single-valued bounded regular function $g(w)$ in $\delta \cap (c_p)$ whose real part equals zero on the circumference of (c_p) . Noticing the fact that the complementary domain of E with respect to the whole w -plane is connected and putting $\psi(p)=g(f(p))$, we can see that $\psi(p)$ is a non-constant single-valued bounded regular function in Δ and the real part of $\psi(p)$ is equal to zero on the relative boundary of Δ . Denote by θ_r^i ($i=1, \dots, n=n(r)$) the components of the common part of γ_r and Δ . Putting $\Theta(r) = \text{Max}_{1 \leq i \leq n(r)} \int_{\theta_r^i} dv$ and denoting by $M^*(r)$ the maximum of the absolute values of the real part of $\psi(p)$ on $\bigcup_{i=1}^{n(r)} \theta_r^i$, we have from Theorem 4

$$\lim_{r \rightarrow R} \frac{(M^*(r))^2}{\int_{r_0}^r e^{2\pi \int_{r_0}^r \Theta(r) dr} dr} > 0.$$

On the other hand, since $\psi(p)$ and so $M^*(r)$ is bounded, we see by our assumption that

$$\lim_{r \rightarrow R} \frac{(M^*(r))^2}{\int_{r_0}^r e^{2\pi \int_{r_0}^r \Theta(r) dr} dr} \leq \lim_{r \rightarrow R} \frac{(M^*(r))^2}{\int_{r_0}^r e^{2\pi \int_{r_0}^r \Lambda(r) dr} dr} = 0,$$

which is a contradiction.

Hence the set E belongs to the class $N_{\mathfrak{B}}$. Thus our theorem is proved.

Remark. This implies Stoilow's theorem stated above. For, if F has a null boundary, then we can choose a graph such that $R = \infty$ and we can see by Theorem 1 that for such a graph the integral

$$\int^{\infty} \frac{dr}{\Lambda(r)}$$

is divergent.

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References

- [1] L. Ahlfors-A. Beurling: Conformal invariants and function-theoretic null sets, *Acta Math.* **83** (1950), 101-129.
- [2] Z. Kuramochi: On covering surfaces, *Osaka Math. J.* **5** (1953), 155-201.
- [3] Z. Kuramochi-T. Kuroda: Note on the set of logarithmic capacity zero, *Proc. Jap. Acad.* **30** (1954), 566-569.
- [4] Y. Kusunoki: Maximum principle for analytic functions on open Riemann surfaces, *Mem. College of Sci., Univ. Kyoto*, **38** (1953), 61-66.
- [5] A. Mori: On Riemann surfaces on which no bounded harmonic function exists, *Journ. Jap. Math. Soc.* **3** (1951), 285-289.
- [6] K. Noshiro: Open Riemann surface with null boundary, *Nagoya Math. J.* **3** (1951), 73-79.
- [7] A. Pfluger: Über das Anwachsen eindeutiger analytischer Funktionen auf offenen Riemannschen Flächen, *Ann. Acad. Sci. Fenn., A. I.* **64** (1949).
- [8] L. Sario: Über Riemannsche Flächen mit hebbarem Rand, *Ann. Acad. Sci. Fenn., A. I.* **50** (1948).
- [9] L. Sario: Sur la classification des surfaces de Riemann, *Congr. Math. Scand., Trondheim* 1949, 229-238.
- [10] S. Stoilow: Sur les singularités des fonctions analytiques multiformes dont la surface de Riemann a sa frontière de mesure harmonique nulle, *Mathematica, Timisoara*, **19** (1943), 126-138.

