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Theorems of the Phragmén-Lindelöf Type on an Open Riemann Surface

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Introduction

1. In the theory of analytic functions of a complex variable, the maximum principle for regular functions plays important roles. Especially, in the investigation of the behaviour of a single-valued analytic function with a general existence domain, maximum principles of the Lindelöf type and theorems of the Phragmén-Lindelöf type are very important.

In this paper, we shall prove some theorems of the Phragmén-Lindelöf type and state some applications of them. The Iversen property of a covering surface spread over the complex plane is essentially deduced from the fact that a theorem of the Phragmén-Lindelöf type holds for a region on the covering surface.

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§ 1.

2. Let F be an open Riemann surface and let $\{F_n\}$ $(n=0,1,2,\cdots)$ be an exhaustion of F such that, for each n, the boundary Γ_n of F_n consists of a finite number of analytic closed curves and such that F_n is contained in F_{n+1} with its boundary Γ_n and further such that each component of F-F_n is non-compact. We denote by $u_n(p)$ the harmonic function in F_n-F_{n-1} $(n\geqq 1)$ which is equal to zero on Γ_{n-1} and to $\log\sigma_n$ on Γ_n and whose conjugate function $v_n(p)$ has the variation 2π on Γ_{n-1}, i.e.,

$$\int_{\Gamma_{n-1}} dv_n = 2\pi ,$$

where the integral is taken in the positive sense with respect to F_{n-1}. The quantity $\log\sigma_n$ is the so-called harmonic modulus of the open set $F_n-\bar{F}_{n-1}$. If we choose an additive constant of $v_n(p)$ suitably, the

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regular function $u_n(p)+iv_n(p)$ maps F_n-F_{n-1} with a finite number of suitable slits onto a slit-rectangle $0\leqq u_n<\log\sigma_n$, $0<v_n<2\pi$ in a one to one conformal manner. Hence the function $u(p)+iv(p)$ defined by $u_n(p)+iv_n(p)+\sum_{i=1}^{n-1}\log\sigma_i$ for each F_n-F_{n-1} $(n\geqq 1)$ maps $F-F_0$ with at most an enumerable number of suitable slits onto a strip domain $0\leqq u<\sum_{i=1}^{\infty}\log\sigma_i$, $0<v<2\pi$ with at most an enumerable number of slits one to one conformally. This strip domain is the graph associated with the exhaustion $\{F_n\}$ in the sense of Noshiro [6]. We put

$$R=\sum_{i=1}^{\infty}\log\sigma_i\,.$$

By Sario-Noshiro's theorem [8], [6], there exists an exhaustion $\{F_n\}$ $(n=0,1,2,\cdots)$ of F satisfying $R=\infty$ if and only if F has a null boundary.

3. Let G be a non-compact domain on an open Riemann surface F whose relative boundary C consists of at most an enumerable number of analytic curves being compact or non-compact and clustering nowhere in F. For the sake of convenience, we shall call such a domain G a non-compact region on F. If a non-compact region on F is prolongable analytically over an open Riemann surface F^*, we shall say that G is imbedded conformally into F^*.

Here we shall give a condition for G to be able to be imbedded conformally into an open Riemann surface with null boundary.

If we denote by γ_r the niveau curve $u(p)=r(0<r<R)$ on F, γ_r consists of a finite number of analytic closed curves and separates the ideal boundary of F from F_0. Denoting by θ_r the part of γ_r contained in G and putting

$$\int_{\theta_r}dv=\theta(r)\,,$$

we have the following

Theorem 1. *The non-compact region G on F can be imbedded conformally into an open Riemann surface with null boundary, if and only if there exists an exhaustion of F such that the integral*

$$(1)\qquad\int^{R}\frac{dr}{\theta(r)}$$

is divergent.

Proof. First we shall prove the necessity of the condition. For the purpose, we may suppose that F has a null boundary. As stated above, there exists an exhaustion $\{F_n\}$ $(n=0, 1, 2, \cdots)$ such that $R=\infty$. Since $\theta(r) \leqq 2\pi$, the integral (1) is divergent for this exhaustion.

Next we shall give the proof of sufficiency. By the usual process of symmetrization, we can construct an open Riemann surface $\hat{G}$. There is given an indirectly conformal mapping of $\hat{G}$ on itself which leaves every point on C fixed, where C is the relative boundary of G with respect to F. It is sufficient to prove that $\hat{G}$ has a null boundary under our condition.

Let Δ be a simply connected domain in G such that the boundary of Δ is an analytic closed curve and such that the closure $\bar{\Delta}$ of Δ is contained in G. Denote by $\tilde{\Delta}$ and $\bar{\tilde{\Delta}}$ the images of Δ and $\bar{\Delta}$, respectively, under the indirectly conformal mapping of $\hat{G}$ on itself. We choose an exhaustion $\{\hat{G}_n\}$ $(n=1, 2, \cdots)$ of $\hat{G}$ such that, for each n, $\hat{G}_n$ contains $\tilde{\Delta}$ and $\bar{\tilde{\Delta}}$ and is symmetric with respect to C. If we construct the harmonic measure $\omega_n(p)$ $(p \in \hat{G}_n - (\bar{\Delta} \cup \bar{\tilde{\Delta}}))$ of the boundary of $\hat{G}_n$ with respect to the domain $\hat{G}_n - (\bar{\Delta} \cup \bar{\tilde{\Delta}})$, we get a sequence $\{\omega_n(p)\}$ $(n=1, 2, \cdots)$ of uniformly bounded harmonic functions. It is easily seen from the configuration of $\hat{G}_n$ that $\omega_n(p) = \omega_n(\tilde{p})$, where $\tilde{p}$ is the image of the point p under the indirectly conformal mapping of $\hat{G}$ on itself. Since $0 < \omega_n(p) < 1$ for each n, we can select a subsequence of $\{\omega_n(p)\}$ which is uniformly convergent on $G - (\bar{\Delta} \cup \bar{\tilde{\Delta}})$ in the wider sense and which has a uniquely determined limiting function $\omega(p)$. This function $\omega(p)$ is harmonic in $G - (\bar{\Delta} \cup \bar{\tilde{\Delta}})$ and equals zero on the boundary of Δ and $\tilde{\Delta}$. From the fact that $\omega_n(p) = \omega_n(\tilde{p})$ for each n, we can see that the normal derivative $\frac{\partial \omega}{\partial \nu}$ vanishes at every point on C. $\hat{G}$ has a null boundary if and only if the function $\omega(p)$ is identically equal to zero. Hence we shall prove that $\omega(p)$ vanishes throughout $G - \bar{\Delta}$ under our condition.

Now we construct a graph $0 \leqq u < R$, $0 < v < 2\pi$ associated with an exhaustion $\{F_n\}$ $(n=0, 1, 2, \cdots)$ for which the integral (1) is divergent. Without loss of generality, we may assume that F_0 is identical to Δ. Let us denote by G_r the open subset of $G - \bar{\Delta}$ consisting of points, each of which satisfies the condition $0 < u(p) < r$ $(0 < r < R)$. The boundary of G_r consists of θ_r, a part of C and the boundary of Δ. It is evident that G_r is not empty for any $r > 0$. Denoting by $D(r)$ the Dirichlet integral of $\omega(p)$ taken over G_r, we have

$$D(r)=\int_{\theta_r}\omega\frac{\partial\omega}{\partial u}dv\,,$$

because $\omega(p)$ equals zero on the boundary of Δ and the normal derivative $\frac{\partial\omega}{\partial\nu}$ vanishes at every point on C. By the Schwarz inequality, we get

$$(D(r))^2\leqq\int_{\theta_r}dv\int_{\theta_r}\left(\frac{\partial\omega}{\partial u}\right)^2 dv$$
$$\leqq\theta(r)\frac{dD(r)}{dr}\,,$$

whence follows that

$$\frac{dr}{\theta(r)}\leqq\frac{dD(r)}{(D(r))^2}\,.$$

Integrating both sides, we obtain

$$\int_{r_0}^{r}\frac{dr}{\theta(r)}\leqq\frac{1}{D(r_0)}-\frac{1}{D(r)}\leqq\frac{1}{D(r_0)}\,,$$

where r_0 is a positive number fixed arbitrarily. Since the integral of the left hand side is divergent as $r\to R$, the Dirichlet integral $D(r_0)$ of $\omega(p)$ taken over the non-empty open set G_{r_0} must be equal to zero and hence the function $\omega(p)$ must reduce to the constant zero. Thus our proof is complete.

This theorem is the same as the result essentially which was obtained by Noshiro (Cf. [3]). Further, the following is easily obtained from the proof of the above theorem.

Corollary (KURAMOCHI [2]). *Suppose that G is a non-compact region on an open Riemann surface with null boundary. Then the double $\hat{G}$, which is obtained from G by the process of symmetrization, has also a null boundary.*

§ 2.

4. Here we shall state some theorems of the Phragmén-Lindelöf type. Let F be an open Riemann surface and let G be a non-compact region on F with the relative boundary C. In the following, we choose an exhaustion $\{F_n\}$ $(n=0,1,2,\cdots)$ of F satisfying the condition $F_0\cap G=0$ and associate the graph $0\leqq u<R$, $0<v<2\pi$ with F which corresponds to this exhaustion and we denote by γ_r the niveau curve

$u(p)=r$ on F as in §1. We shall prove the following

Theorem 2. *Suppose that a function $f(p)$ regular in G is continuous on $G\cup C$ and that $|f(p)|$ is single-valued on $G\cup C$ and satisfies the condition $|f(p)|\leqq 1$ on C. If there exists a point p_0 in G such that $|f(p_0)|>1$, then*

$$\varliminf_{r\to R}\frac{(\log M(r))^2}{\int_{r_0}^{r}\frac{dr}{\theta(r)}}>0\,,$$

where $M(r)$ is the maximum of $|f(p)|$ on $\theta_r(=\gamma_r\cap G)$ and $u(p_0)=r_0$ and further, $\theta(r)=\int_{\theta_r}dv$.

Proof. We put $h(p)=\overset{+}{\log}|f(p)|$, where, for any real number x, $\overset{+}{\log}\,x$ is the maximum of zero and $\log x$. Let us denote by G_r the open subset of G which consists of points of G satisfying $u(p)<r$. If $u(p_0)=r_0$, $h(p)$ is non-constant in G_r for any number $r\geqq r_0$. Denoting by $D(r)$ the Dirichlet integral of $h(p)$ taken over G_r, we have

$$D(r)=\int_{\theta_r}h\frac{\partial h}{\partial u}dv\,,$$

for, $h(p)$ is non-constant in G_r and harmonic at every point p satisfying $h(p)=\log|f(p)|>0$ and reduces to the constant zero elsewhere. It is obvious that $D(r)$ is positive for any $r\geqq r_0$. By the Schwarz inequality, we get

$$\begin{aligned}(D(r))^2&\leqq\int_{\theta_r}h^2dv\int_{\theta_r}\left(\frac{\partial h}{\partial u}\right)^2dv\\&\leqq\theta(r)\,(\log M(r))^2\frac{dD(r)}{dr}\,,\end{aligned}$$

or

$$\frac{dr}{\theta(r)}\leqq(\log M(r))^2\frac{dD(r)}{(D(r))^2}\,.$$

Integrating both sides, we obtain

$$\begin{aligned}\int_{r_0}^{r}\frac{dr}{\theta(r)}&\leqq(\log M(r))^2\left[\frac{1}{D(r_0)}-\frac{1}{D(r)}\right]\\&\leqq(\log M(r))^2\frac{1}{D(r_0)}\,,\end{aligned}$$

because $M(r)$ is a monotonically increasing function of r. Hence it follows that, for any $r>r_0$,

$$0 < D(r_0) \leqq \frac{(\log M(r))^2}{\int_{r_0}^{r} \frac{dr}{\theta(x)}},$$

which proves our theorem.

This theorem implies the following which contains Kusunoki's result [4].

Theorem 3. *Under the same conditions in Theorem* 2,

$$\varliminf_{r \to R} \frac{\log M(r)}{\sqrt{r}} > 0 .$$

5. In the preceding section we dealt with the regular function with uniform modulus. Here we shall consider the single-valued regular function.

Let G be a non-compact region on F with the relative boundary C and let $f(p) = U(p) + iV(p)$ be a single-valued regular function in G being continuous on $G \cup C$. Denote by G_r the open subset of G, every point of which satisfies the condition $u(p) < r$.

Suppose that the real part $U(p)$ of $f(p)$ equals zero on C. The part θ_r of the niveau curve $\gamma_r : u(p) = r$ contained in G consists of at most a finite number of components θ_r^i $(i = 1, 2, \cdots, n = n(r))$. If we denote by $D(r)$ the Dirichlet integral of $f(p)$ taken over G_r, then we get

$$D(r) = \sum_{i=1}^{n(r)} \int_{\theta_r^i} U dV = \sum_{i=1}^{n(r)} \int_{\theta_r^i} U \frac{\partial U}{\partial u} dv .$$

In the case of θ_r^i which is a cross-cut of G, since by Wirtinger's inequality

$$\int_{\theta_r^i} U^2 dv \leqq \frac{(\theta_i(r))^2}{\pi^2} \int_{\theta_r^i} \left(\frac{\partial U}{\partial v}\right)^2 dv ,$$

where $\theta_i(r) = \int_{\theta_r^i} dv$, we have

$$\begin{aligned} \left(\int_{\theta_r^i} U \frac{\partial U}{\partial u} dv \right)^2 &\leqq \int_{\theta_r^i} U^2 dv \int_{\theta_r^i} \left(\frac{\partial U}{\partial u}\right)^2 dv \\ &\leqq \frac{(\theta_i(r))^2}{\pi^2} \int_{\theta_r^i} \left(\frac{\partial U}{\partial v}\right)^2 dv \int_{\theta_r^i} \left(\frac{\partial U}{\partial u}\right)^2 dv , \end{aligned}$$

and hence we obtain

$$\int_{\theta_r^i} U \frac{\partial U}{\partial u} dv \leqq \frac{\theta_i(r)}{2\pi} \int_{\theta_r^i} \left[\left(\frac{\partial U}{\partial u}\right)^2 + \left(\frac{\partial U}{\partial v}\right)^2 \right] dv .$$

Next we consider the case of θ_r^j being a loop-cut of G. We can choose a constant m_j such that $\int_{\theta_r^j}(U-m_j)dv=0$. By Wirtinger's inequality, we have

$$\int_{\theta_r^j}(U-m_j)^2dv \leqq \frac{(\theta_j(r))^2}{4\pi^2}\int_{\theta_r^j}\left(\frac{\partial U}{\partial v}\right)^2 dv\,.$$

On the other hand, since $f(p)$ is single-valued, it follows that

$$\int_{\theta_r^j} UdV = \int_{\theta_r^j}(U-m_j)dV\,,$$

whence we obtain

$$\begin{aligned}\left(\int_{\theta_r^j} U\frac{\partial U}{\partial u}dv\right)^2 &= \left(\int_{\theta_r^j}(U-m_j)\frac{\partial U}{\partial u}dv\right)^2\\ &\leqq \int_{\theta_r^j}(U-m_j)^2dv\int_{\theta_r^j}\left(\frac{\partial U}{\partial u}\right)^2dv\\ &\leqq \frac{(\theta_j(r))^2}{4\pi^2}\int_{\theta_r^j}\left(\frac{\partial U}{\partial v}\right)^2dv\int_{\theta_r^j}\left(\frac{\partial U}{\partial u}\right)^2dv\,.\end{aligned}$$

Thus we get

$$\int_{\theta_r^j} U\frac{\partial U}{\partial u}dv \leqq \frac{\theta_j(r)}{4\pi}\int_{\theta_r^j}\left[\left(\frac{\partial U}{\partial u}\right)^2+\left(\frac{\partial U}{\partial v}\right)^2\right]dv\,.$$

Therefore, it holds for any number i that

$$\int_{\theta_r^i} U\frac{\partial U}{\partial u}dv \leqq \frac{\Theta(r)}{2\pi}\int_{\theta_r^i}\left[\left(\frac{\partial U}{\partial u}\right)^2+\left(\frac{\partial U}{\partial v}\right)^2\right]dv\,,$$

where $\Theta(r)=\underset{1\leqq i\leqq n(r)}{\text{Max}}\,\theta_i(r)$. Summing up these inequalities for $i=1, 2, \cdots, n(r)$, we have

$$D(r)\leqq\frac{\Theta(r)}{2\pi}\frac{dD(r)}{dr}\,,$$

or

$$2\pi\frac{dr}{\Theta(r)}\leqq\frac{dD(r)}{D(r)}\,.$$

Integrating both sides, we obtain

$$2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}\leqq\log\frac{D(r)}{D(r_0)}\,,$$

where r_0 is a suitable number such that there exists a point p_0 of G satisfying $u(p_0)=r_0$. Hence it follows that

$$D(r_0)e^{2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}} \leqq D(r)\,. \tag{2}$$

On the other hand, since

$$\frac{d}{dr}\left(\int_{\theta_r} U^2 dv\right) = 2\int_{\theta_r} U\frac{\partial U}{\partial u}dv = 2D(r)\,,$$

it is easy to see that

$$\int_{r_0}^{r} D(r)dr = \frac{1}{2}\left(\int_{\theta_r} U^2 dv - \int_{\theta_{r_0}} U^2 dv\right)$$
$$\leqq \frac{1}{2}\int_{\theta_r} U^2 dv \leqq \pi(M^*(r))^2\,,$$

where $M^*(r)$ is the maximum of $|U(p)|$ on θ_r. From this and (2), we get

$$\frac{(M^*(r))^2}{\int_{r_0}^{r} e^{2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}}dr} \geqq \frac{D(r_0)}{\pi}\,.$$

If the function is non-constant, $D(r_0)$ is positive. Thus we have the following

Theorem 4. *Suppose that $f(p)$ is a single-valued regular function in a non-compact region G on an open Riemann surface and that the real part of $f(p)$ is equal to zero on the relative boundary of G. Denote by $M^*(r)$ the maximum of the absolute values of the real part of $f(p)$ on θ_r. If*

$$\varliminf_{r\to R} \frac{(M^*(r))^2}{\int_{r_0}^{r} e^{2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}}dr} = 0\,,$$

then $f(p)$ reduces to a constant.

The argument of the above proof is due to Pfluger [7].

This theorem is applicable to investigate the behaviour of functions on an open Riemann surface satisfying the condition similar to that of Pfluger.

§ 3.

6. Let F be an open Riemann surface and let $w=f(p)$ be a non-constant single-valued analytic function defined on F. The space formed by elements $q=[p, f(p)]$ defines a covering surface Φ spread

over the w-plane and the point $q=[p, f(p)]$ has the projection $w=f(p)$. The correspondence $p \leftrightarrow q$ gives a topological and conformal mapping between F and Φ.

Let Φ_Δ be any connected piece of Φ lying on the disc (c_ρ), where (c_ρ) is the disc $|w-w_0|<\rho$ for any finite point $w=w_0$ and for any positive number ρ or is the disc $|w|>\frac{1}{\rho}$ for any positive number ρ. We shall denote by Δ the domain of F corresponding to Φ_Δ by $p \leftrightarrow q$. If Δ is non-empty for a disc (c_ρ) and if either there exists a point p in Δ such that $w^*=f(p)$ or there exists a path in Δ tending to the ideal boundary of F such that $\lim f(p)=w^*$ along the path, where w^* is the centre of (c_ρ), then we shall say that Φ has the Iversen property.

Mori [5] proved that Φ has the Iversen property if F belongs to the class O_{HB} which is the class of Riemann surfaces not allowing the existence of the non-constant single-valued bounded harmonic function. In the case of F with null boundary, Stoïlow [10] proved this result.

7. Let $\{F_n\}$ $(n=0, 1, 2, \cdots)$ be an exhaustion of F and let the strip domain $0 \leqq u < R$, $0 < v < 2\pi$ be the graph of F associated with the exhaustion $\{F_n\}$. The niveau curve $\gamma_r : u(p)=r$ consists of a finite number of closed analytic curves γ_r^i $(i=1, \cdots, m=m(r))$. Put

$$\Lambda(r) = \operatorname*{Max}_{1 \leqq i \leqq m(r)} \int_{\gamma_r^i} dv .$$

Then the following was proved by Pfluger [7].

If the integral

$$\int_0^R e^{4\pi \int_0^r \frac{dr}{\Lambda(r)}} dr \tag{3}$$

is divergent, there exists no non-constant single-valued bounded analytic function on F.

Hence we can see that if the integral

$$\int_0^R e^{2\pi \int_0^r \frac{dr}{\Lambda(r)}} dr$$

is divergent, there exists no non-constant single-valued bounded analytic function on $F \in O_{AB}$. Further we can prove the following which was found by Z. Kuramochi.

Theorem 5. *If the integral* (3) *is divergent,* Φ *has the Iversen property.*

Proof. As mentioned above, there exists no non-constant single-valued bounded analytic function on F. Hence the set of values taken by $w=f(p)$ is everywhere dense in the w-plane. Therefore, for any disc (c_ρ), there exists at least a connected piece of Φ lying over (c_ρ). We choose such an arbitrary piece Φ_Δ and denote by Δ the domain on F corresponding to Φ_Δ by the mapping $p \leftrightarrow q$. It is easily seen that, by the mapping $p \leftrightarrow q$, the relative boundary of Δ corresponds to that of Φ_Δ lying over the circumference of (c_ρ).

If Δ is compact in F, it is easy to see that there exists a point p_0 such that $f(p_0)=w^*$, where the point w^* is the centre of (c_ρ). Hence we suppose that Δ is non-compact. Then Δ is a non-compact region on F.

Let (c) be any concentric circular disc of (c_ρ) contained in (c_ρ) and let E be the set of points which lie in the closure $\overline{(c)}$ of (c) and are not covered by Φ_Δ. As is easily seen, for our purpose it is sufficient to prove that E is the set of class $N_{\mathfrak{B}}$ in the sense of Ahlfors-Beurling [1]. Since Φ_Δ is connected, the complementary set of E with respect to the whole w-plane is connected.

Let δ be the domain in the w-plane which is a complementary domain of E with respect to the whole w-plane and contains the circumference of (c_ρ). Suppose that E is not the set on the class $N_{\mathfrak{B}}$. Then, by Sario's theorem [8], [9], there exists a non-constant single-valued bounded regular function $g(w)$ in $\delta \cap (c_\rho)$ whose real part equals zero on the circumference of (c_ρ). Noticing the fact that the complementary domain of E with respect to the whole w-plane is connected and putting $\psi(p)=g(f(p))$, we can see that $\psi(p)$ is a non-constant single-valued bounded regular function in Δ and the real part of $\psi(p)$ is equal to zero on the relative boundary of Δ. Denote by θ_r^i $(i=1, \cdots, n=n(r))$ the components of the common part of γ_r and Δ. Putting $\Theta(r)=\max_{1\leq i\leq n(r)} \int_{\theta_r^i} dv$ and denoting by $M^*(r)$ the maximum of the absolute values of the real part of $\psi(p)$ on $\bigcup_{i=1}^{n(r)} \theta_r^i$, we have from Theorem 4

$$\varliminf_{r\to R} \frac{(M^*(r))^2}{\int_{r_0}^{r} e^{2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}} dr} > 0 .$$

On the other hand, since $\psi(p)$ and so $M^*(r)$ is bounded, we see by our assumption that

$$\varliminf_{r\to R} \frac{(M^*(r))^2}{\int_{r_0}^{r} e^{2\pi\int_{r_0}^{r}\frac{dr}{\Theta(r)}} dr} \leq \varliminf_{r\to R} \frac{(M^*(r))^2}{\int_{r_0}^{r} e^{2\pi\int_{r_0}^{r}\frac{dr}{\Lambda(r)}} dr} = 0 ,$$

which is a contradiction.

Hence the set E belongs to the class $N_{\mathfrak{B}}$. Thus our theorem is proved.

Remark. This implies Stoïlow's theorem stated above. For, if F has a null boundary, then we can choose a graph such that $R=\infty$ and we can see by Theorem 1 that for such a graph the integral

$$\int^{\infty} \frac{dr}{\Lambda(r)}$$

is divergent.

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