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TIME CHANGES IN DIRICHLET SPACE THEORY

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1. Introcution

Let $\mathcal{M}=(\Omega, \mathcal{F}_t, X_t, \{P_x\}_{x\in X})$ be an m-symmetric Hunt process on state space X, where X is a locally compact separable metric space and m is a Radon measure on X which is strictly positive on each non-empty open set. We assume that the Dirichlet space $(\mathcal{E}, \mathcal{F})$ associated with $\mathcal{M}$ is C_0-regular and irreducible. In this situation M. Fukushima [5] developed the stochastic calculus for additive functionals of $\mathcal{M}$. Using this stochastic calculus, we can investigate the time change for $\mathcal{M}$ in relation to the Dirichlet space.

Let A_t be a positive continuous additive functional of $\mathcal{M}$ whose Revuz measure is a positive Radon measure μ charging no set of zero capacity. Denote by $\tilde{Y}$ the set $\{x\in X; P_x(A_t>0 \text{ for any } t>0)=1\}$ which is called the fine support of A_t. The time changed process of $\mathcal{M}$ by A_t is given by $\mathcal{M}^t=(\Omega, \mathcal{F}_{\tau_t}, X_{\tau_t}, \{P_x\}_{x\in\tilde{Y}})$, where $\tau_t=\inf\{s>0; A_s>t\}$. It is known that $\mathcal{M}^t$ is a normal, right continuous strong Markov process (M. Sharpe [17]), which is also symmetric with respect to μ. When $\mathcal{M}$ is transient, M. Fukushima [5] characterized the extended Dirichlet space associated with $\mathcal{M}^t$ in the framework of the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ (M. Silverstein [18]). Y. Ōshima [15, 16] obtained analogous results for recurrent cases. P.J. Fitzsimmons [3] extended those characterizations to a general symmetric Borel right process without C_0-regularity by making a reduction to the transient case. However none of the above mentioned articles treated an important question whether the C_0-regularity of the Dirichlet space is preserved under the time change. Only recently, M. Fukushima-Y. Ōshima [8] gave an affirmative answer to this question under the condition that $X-\tilde{Y}$ is of zero capacity.

In this paper we show the C_0-regularity of the Dirichlet space associated with $\mathcal{M}^t$ in the present generality. Denote by Y the support of the measure μ. It is known that Y includes $\tilde{Y}$ except for an exceptional set and $\mu(Y-\tilde{Y})=0$ ([5]). In Section 3 we present a simple and direct way of characterizing the Dirichlet space $(\mathcal{E}^\mu_Y, \mathcal{F}^\mu_Y)$ on $L^2(Y;\mu)$ associated with $\mathcal{M}^t$ and prove its C_0-regularity. Similarly as in Fitzsimmons [3], the subprocess of $\mathcal{M}$ by the multiplicative functional e^{-A_t} plays an important role in our approach. The C_0-regularity of

$(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ enables us to show in Section 4 that $\mathcal{M}^t$ is actually a Hunt process after a modification on an exceptional set.

M. Fukushima [5] raised a question in his book whether $Y-\tilde{Y}$ is of zero capacity or not in general. In the last section we give an example that $Y-\tilde{Y}$ is not of zero capacity in the class of birth and death processes.

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2. Potential theoretic properties related to the Feynman-Kac formula

In this section we investigate some properties of the subprocess of a symmetric Hunt process. We use some notations as in Fukushima [5]. Let $(\mathcal{E}, \mathcal{F})$ be a C_0-regular Dirichlet space on $L^2(X; m)$. Then we can consider the associated m-symmetric Hunt process $\mathcal{M}=(\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, P_x)$ on the canonical path space Ω. The family of transition kernels of $\mathcal{M}$ is denoted by $\{p_t, t>0\}$. In this paper we use following notation. For a Borel measure γ on X and Borel functions f and g on X, $(f,g)_\gamma=\int_X f(x)g(x)\gamma(dx)$. We assume $(\mathcal{E}, \mathcal{F})$ is irreducible, namely a Borel set $A\subset X$ satisfies either $m(A)=0$ or $m(X-A)=0$ whenever $p_t(I_A u)=I_A p_t u$, m-$a.e.$ for all $t>0$ and $u\in \boldsymbol{B}_b^+(X)$, where I_A is the indicator function of a set A and $\boldsymbol{B}_b^+(X)$ denotes the family of all bounded non-negative Borel functions on X. The capacity associated with $(\mathcal{E}, \mathcal{F})$ will be called the $\mathcal{E}_1$-capacity; for any open set G,

$$\mathcal{E}_1\text{-Cap}(G) = \inf\{\mathcal{E}_1(u,u);\ u\in\mathcal{F},\ u\geq 1\ m\text{-}a.e. \text{ on } G\} \tag{2.1}$$

and, for any set $A\subset X$,

$$\mathcal{E}_1\text{-Cap}(A) = \inf\{\mathcal{E}_1\text{-Cap}(G);\ A\subset G,\ open\}. \tag{2.2}$$

A statement Γ depending on $x\in A$ is said to hold q.e. on A if there exists a set N of zero $\mathcal{E}_1$-capacity such that Γ is true for $x\in A-N$. The quasi-continuous function with respect to $\mathcal{E}_1$-capacity is called $\mathcal{E}_1$-quasi-continuous.

Fix a non-trivial positive Radon measure μ on X charging no set of zero $\mathcal{E}_1$-capacity. Then μ belongs to the class S of all smooth measures and there exists a unique positive continuous additive functional (abbreviated to PCAF) A_t characterized by

$$\langle\mu, f\rangle = \lim_{t\to 0}\frac{1}{t}E_m\left[\int_0^t f(X_s)\,dA_s\right],\quad f\in\boldsymbol{B}^+(X), \tag{2.3}$$

where $\boldsymbol{B}^+(X)$ denotes the family of all non-negative Borel functions on X and $\langle\mu, f\rangle$ denotes $\int_X f(x)\mu(dx)$. E_γ denotes integration by $P_\gamma(d\omega)=\int_X P_x(d\omega)\gamma(dx)$ for a Borel measure γ on X. The measure μ is called Revuz measure of

A_t (Fukushima [5]).

For each $\alpha>0$, we let $\mathcal{M}^{\alpha\mu}=(\tilde{\Omega}, Y_t, Q_x)$ be the subprocess of $\mathcal{M}$ transformed by the multiplicative functional $e^{-\alpha A_t}$; namely, the transition function $p_t^{\alpha\mu}$ of $\mathcal{M}^{\alpha\mu}$ is given by

$$p_t^{\alpha\mu} f(x) = \int_{\tilde{\Omega}} f(Y_t)\, dQ_x = E_x[e^{-\alpha A_t} f(X_t)]\,, \quad f\in \boldsymbol{B}^+(X)\,. \tag{2.4}$$

Theorem 2.1. (*Ōshima* [16]) *$\mathcal{M}^{\alpha\mu}$ is m-symmetric and the associated Dirichlet space on $L^2(X; m)$ is given by*

$$\begin{cases} \mathcal{F}^\mu = \mathcal{F}\cap L^2(X;\mu) \\ \mathcal{E}^{\alpha\mu}(u,v) = \mathcal{E}(u,v)+\alpha(u,v)_\mu\,, \ \text{for } u, v\in\mathcal{F}^\mu\,. \end{cases} \tag{2.5}$$

Furthermore $(\mathcal{E}^{\alpha\mu}, \mathcal{F}^\mu)$ is C_0-regular. Here $u\in\mathcal{F}\cap L^2(X;\mu)$ means that its $\mathcal{E}_1$-quasi-continuous version $\tilde{u}$ belongs to $L^2(X;\mu)$.

Proposition 2.2. *$(\mathcal{E}^{\alpha\mu}, \mathcal{F}^\mu)$ is irreducible and transient.*

Proof. Suppose $A\in\boldsymbol{B}(X)$ is $p_t^{\alpha\mu}$-invariant, then for a fixed $B\in\boldsymbol{B}(X)$ and $t>0$, $p_t^{\alpha\mu} I_{A\cap B}=I_A p_t^{\alpha\mu} I_B$ *m-a.e.* Here $\boldsymbol{B}(X)$ is the Borel σ-algebra of X. Since $P_x(A_t<\infty, t<\zeta)=1$ q.e. $x\in X$ (Fukushima [5]), it holds that $p_t I_{A\cap B}=0$ *m-a.e.* on $X-A$. This statement is true with A replaced by $X-A$. Hence $p_t I_{A\cap B}\leq I_A p_t I_B$ *m-a.e.* and $p_t I_{(X-A)\cap B}\leq I_{(X-A)} p_t I_A$ *m-a.e.* Therefore $p_t I_{A\cap B}=I_A p_t I_B$ *m-a.e.* for any $B\in\boldsymbol{B}(X)$ and $t>0$ and consequently A is p_t-invariant. The irreducibility of $\mathcal{M}^{\mu\mu}$ is proved. Next suppose $(\mathcal{E}^{\alpha\mu}, \mathcal{F}^\mu)$ is non-transient, then it is conservative by the irreducibility, $p_t^{\alpha\mu}1(x)=E_x[e^{-\alpha A_t}]=1$ *m-a.e.* $x\in X, t>0$ (Ōshima [16]). Hence $P_x(A_t=0$ for any $t>0)=1$ *m-a.e.* $x\in X$, contradicting to the non-triviality of μ. The proof is complete.

By these properties, the extended Dirichlet space $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^{\alpha\mu})$ of $(\mathcal{E}^{\alpha\mu}, \mathcal{F}^\mu)$ is well-defined as the completion of $\mathcal{F}^\mu$ by the $\mathcal{E}^{\alpha\mu}$-norm. Since an $\mathcal{E}^{\alpha\mu}$-Cauchy sequence is an $L^2(X;\mu)$-Cauchy sequence, $\mathcal{F}_e^{\alpha\mu}$ is a subspace of $L^2(X;\mu)$. Since $\mathcal{F}_e^{\alpha\mu}$ is independent of $\alpha>0$, we denote $\mathcal{F}_e^\mu$ instead of $\mathcal{F}_e^{\alpha\mu}$.

By using $\mathcal{E}^{\alpha\mu}$ instead of $\mathcal{E}_1$, we can define the $\mathcal{E}^{\alpha\mu}$-capacity as in (2.1) and (2.2).

Lemma 2.3. (i) *For each $\alpha>0$ and a subset N of X, $\mathcal{E}_1$-*Cap$(N)=0$ *if and only if $\mathcal{E}^{\alpha\mu}$-*Cap$(N)=0$.

(ii) *For each $\alpha>0$ and a function u on X. u is $\mathcal{E}_1$-quasi-continuous if and only if u is $\mathcal{E}^{\alpha\mu}$-quasi-continuous.*

Proof. The proof is the same as in Lemma 3.1.6, Theorem 3.1.5 of Fukushima [5].

Since each $u\in\mathcal{F}_e^\mu$ has an $\mathcal{E}^{\alpha\mu}$-quasi-continuous modification $\tilde{u}$, we may and we shall assume that all elements of $\mathcal{F}_e^\mu$ are $\mathcal{E}_1$-quasi-continuous by this lemma.

Theorem 2.4. *For $\nu\in S$ and PCAF B_t of $\mathcal{M}$, the following conditions are equivalent to each other. If B_t is associated with ν by the Revuz correspondence* (2.3), *then one of (and hence all of) the following conditions are satisfied.*

(i) $E_x[\int_0^\infty e^{-ps-\alpha A_s}f(X_s)\,dB_s]$ *is an $\mathcal{E}_1$-quasi-continuous modification of $U_p^{\alpha\mu}(f\nu)$ for any $p>0$ and $f\in \boldsymbol{B}^+(X)$, such that $f\nu\in S_0(\mathcal{E}_1^{\alpha\mu})$, where $S_0(\mathcal{E}_1^{\alpha\mu})$ is the family of all positive Radon measures of finite energy integrals with respect to $\mathcal{E}_1^{\alpha\mu}$ and $U_p^{\alpha\mu}(f\nu)$ denotes the p-potential of $f\nu$*

(ii) $E_{hm}[\int_0^\infty e^{-ps-\alpha A_s}f(X_s)\,dB_s]=\langle f\nu,\,R_p^{\alpha\mu}h\rangle,\,p>0,\,f,\,h\in\boldsymbol{B}^+(X)$, *where $R_p^{\alpha\mu}$ is thd resolvent kernel of $\mathcal{M}^{\alpha\mu}$.*

(iii) $E_{hm}[\int_0^t e^{-\alpha A_s}f(X_s)\,dB_s]=\int_0^t\langle f\nu,\,p_s^{\alpha\mu}h\rangle\,ds,\,t>0,\,f,\,h\in\boldsymbol{B}^+(X).$

(iv) $\lim_{t\to 0}\frac{1}{t}E_{hm}[\int_0^t e^{-\alpha A_s}f(X_s)\,dB_s]=\langle f\nu,\,h\rangle$ *for any p-excessive function h of $\mathcal{M}^{\alpha\mu}$ $(p\geqq 0)$ and $f\in\boldsymbol{B}^+(X)$.*

(v) $\uparrow\lim_{t\to 0}\frac{1}{t}E_{hm}[\int_0^t e^{-ps-\alpha A_s}f(X_s)\,dB_s]=\langle f\nu,\,h\rangle$ *for any p-excessive function h of $\mathcal{M}^{\alpha\mu}$ $(p\geqq 0)$ and $f\in\boldsymbol{B}^+(X)$.*

(vi) $\uparrow\lim_{q\to\infty}qE_{hm}[\int_0^\infty e^{-(p+q)s-\alpha A_s}f(X_s)\,dB_s]=\langle f\nu,\,h\rangle$ *for any p-excessive function h of $\mathcal{M}^{\alpha\mu}$ $(p\geqq 0)$ and $f\in\boldsymbol{B}^+(X)$.*

We prepare two lemmas to prove the above theorem.

Lemma 2.5. *For any $\nu\in S$, there exists a sequence K_n of increasing compact sets such that $I_{K_n}\nu\in S_0$ with $U_1(I_{K_n}\nu)\in L^\infty(X;m)$ and $\lim_{n\to\infty}\mathcal{E}_1\text{-}Cap(K-K_n)=0$ for any compact set K. In particular $\mathcal{E}_1\text{-}Cap(X-\bigcup_{n=1}^\infty K_n)=0$. Here S_0 is the space of all positive Radon measures of finite energy integrals with respect to $\mathcal{E}_1$ and $U_1\gamma$ denotes its 1-potential of $\gamma\in S_0$.*

Proof. First we prove in case $\nu\in S_0$. Then there exists a nest $\{F_k\}$ on X such that $\widetilde{U_1(\nu)}\in C(\{F_k\})$. Choose compact sets E_n increasing to X such that $E_n\subset \operatorname{Int} E_{n+1}$ and put $K_n=F_n\cap E_n$, where $\operatorname{Int} E_{n+1}$ is the largest open set included in E_{n+1}. Then we have $\mathcal{E}_1\text{-Cap}(K-K_n)\leqq\mathcal{E}_1\text{-Cap}(K-F_n)+\mathcal{E}_1\text{-Cap}(K-E_n)\to 0$, $n\to\infty$, because $K\subset E_n$ for large n. Since $\widetilde{U_1(\nu)}$ is bounded on K_n, $\|U_1(\widetilde{I_{K_n}}\nu)\|_\infty$ is bounded by the same constant in view of Lemma 3.2.3 of Fukushima [5]. Next we prove in case $\nu\in S$. By Theorem 3.2.3 of Fukushima [5], there exists a sequence $\{\tilde{K}_n\}$ of increasing compact sets such that $I_{\tilde{K}_n}\nu\in S_0$ and $\lim_{n\to\infty}\mathcal{E}_1\text{-Cap}\,(K-\tilde{K}_n)=0$ for any compact set K. For each $I_{\tilde{K}_n}\nu$ there exists increasing pact set K_n^l such that $U_1(I_{K_n^l\cap \tilde{K}_n}\nu)\in L^\infty(X;m)$ and $\lim_{l\to\infty}\mathcal{E}_1\text{-Cap}(K-K_n^l)=0$ for any compact set K. We put $K_n=\bigcup_{i=1}^n K_i^n\cap\tilde{K}_i$. Then K_n satisfies the desired assertion. The latter assertion is clear from $\mathcal{E}_1\text{-Cap}(E_n-\bigcup_{l=1}^\infty K_l)\leqq\mathcal{E}_1\text{-Cap}(E_n-K_l)\to 0$, as $l\to\infty$. The proof is complete.

Lemma 2.6. *Let B_t be the PCAF of $\mathcal{M}$ associated with $\nu\in S_0$ with $U_1\nu\in L^\infty(X;m)$ and let C_t be the PCAF of $\mathcal{M}$ associated with $\gamma\in S_0$.*

$$\lim_{t\to 0}\frac{1}{t}E_{hm}\Big[\int_0^t e^{-C_s}dB_s\Big]=\langle\nu,\tilde{h}\rangle\,,\ \textit{for any}\ h\in\boldsymbol{B}^+(X)\cap\mathcal{F}\,.$$

Proof. By Lemma 5.1.4 and Theorem 5.1.1 of Fukushima [5],

$$\lim_{t\to 0}\frac{1}{t}E_{hm}[B_t]=\langle\nu,\tilde{h}\rangle,\ \ h\in\boldsymbol{B}^+(X)\cap\mathcal{F}\,.$$

It suffices to show that

$$\lim_{t\to 0}\frac{1}{t}E_{hm}\Big[\int_0^t(1-e^{-C_s})\,dB_s\Big]=0,\ h\in\boldsymbol{B}^+(X)\cap\mathcal{F}\,. \tag{2.6}$$

Put $c_t(x)=E_x[C_t]$ and $b_t(x)=E_x[B_t]$. Since $E_{hm}[C_t]=(h,c_t)_m<\infty$ by (5.1.15) of Fukushima [5] and $||b_t||_\infty\leqq e^t\,||U_1\nu||_\infty$, we have

$$\begin{aligned}
E_{hm}\Big[\int_0^t(1-e^{-C_s})\,dB_s\Big]&\leqq E_{hm}\Big[\int_0^t C_s\,dB_s\Big]=E_{hm}\Big[\int_0^t(B_t-B_s)\,dC_s\Big]\\
&\leqq E_{hm}\Big[\int_0^t B_t(\theta_s)\,dC_s\Big]\\
&\leqq\varliminf_{n\to\infty}\sum_{k=0}^{n-1}E_{hm}[B_t(\theta_{(k+1/n)t})\,(C_{(k+1/n)t}-C_{(k/n)t})]\\
&\leqq\varliminf_{n\to\infty}\sum_{k=0}^{n-1}E_{hm}[E_x[B_t(\theta_{(k+1/n)t})\,|\,\mathcal{F}_{(k+1/n)t}]\,(C_{(k+1/n)t}-C_{(k/n)t})]\\
&=\varliminf_{n\to\infty}\sum_{k=0}^{n-1}E_{hm}[b_t(X_{(k+1/n)t})\,(C_{(k+1/n)t}-C_{(k/n)t})]\\
&=E_{hm}[\varliminf_{n\to\infty}\sum_{k=0}^{n-1}b_t(X_{(k+1/n)t})\,(C_{(k+1/n)t}-C_{(k/n)t})]\,.
\end{aligned}$$

This is equal to

$$E_{hm}\Big[\int_0^t b_t(X_s)\,dC_s\Big]=\int_0^t\langle b_t\,\gamma,p_s\,\tilde{h}\rangle\,ds\,,$$

because b_t is $\mathcal{E}_1$-quasi-continuous. Hence we have, for sufficiently small $t>0$

$$\begin{aligned}
\frac{1}{t}E_{hm}\Big[\int_0^t(1-e^{-C_s})\,dB_s\Big]&\leqq\frac{1}{t}\int_0^t\langle b_t,\gamma,p_s\,\tilde{h}\rangle\,ds\\
&\leqq\langle b_t\,\gamma,\tilde{h}\rangle+||b_t||_\infty\langle\gamma,\,|\frac{S_t}{t}\tilde{h}-\tilde{h}|\rangle
\end{aligned}$$

$$\leqq\langle b_t\,\mu,\tilde{h}\rangle+e\|U_1\,\nu\|_\infty\langle\gamma,\,|\frac{S_t}{t}\tilde{h}-\tilde{h}|\rangle\,,$$

where $S_t\tilde{h}$ denotes $\int_0^t p_s\tilde{h}ds$. Since

$$\lim_{t\to 0}\langle\gamma,\,|\frac{S_t}{t}\tilde{h}-\tilde{h}|\rangle\leqq\lim_{t\to 0}\sqrt{\mathcal{E}_1(U_1\gamma,\,U_1\gamma)}\sqrt{\mathcal{E}_1(\frac{S_t}{t}\tilde{h}-\tilde{h},\frac{S_t}{t}\tilde{h}-\tilde{h})}=0\,,$$

we arrive at (2.6). The proof is complete.

Proof of Theorem 2.4. The equivalence of (i) and (ii) is easy. The implication (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) and (v)$\Rightarrow$(vi)$\Rightarrow$(ii) is also clear (Kim [13]). We first show the implication (iv)$\Rightarrow$(v). Suppose that (iv) is satisfied. We may assume that the right hand side of (v) is finite by Lemma 2.5. We put

$$g_t(x)=E_x[\int_0^t e^{-ps-\alpha A_s}f(X_s)\,dB_s]\,,\quad \phi_s(t)=e^{-ps}(p_s^{\alpha\mu}h,g_t)_m\,.$$

Then $\phi_s(t)$ is a subadditive function on $[0,\infty)$. We get

$$\lim_{t\to 0}\frac{\phi_s(t)}{t}=\sup_{t>0}\frac{\phi_s(t)}{t}=\langle f\nu,\,e^{-ps}\,p_s^{\alpha\mu}\,h\rangle<\infty\,.$$

Hence we have

$$\begin{aligned}\int_0^t\langle f\nu,\,e^{-ps}\,p_s^{\alpha\mu}\,h\rangle\,ds&=\int_0^t\lim_{u\to 0}\frac{\phi_s(u)}{u}\,ds=\lim_{u\to 0}\frac{1}{u}\int_0^t\phi_s(u)\,ds\\&=\lim_{u\to 0}\frac{1}{u}\int_0^t(h,\,e^{-ps}\,p_s^{\alpha\mu}\,g_u)_m\,ds\\&=\lim_{u\to 0}\frac{1}{u}\int_0^t(h,g_{u+s}-g_s)_m\,ds\\&=\lim_{u\to 0}\frac{1}{u}\int_t^{t+u}(h,g_s)_m\,ds-\lim_{u\to 0}\frac{1}{u}\int_0^u(h,g_s)_m\,ds\\&=(h,g_t)_m=E_{hm}[\int_0^t e^{-ps-\alpha A_s}f(X_s)\,dB_s]\,.\end{aligned}\tag{2.7}$$

Therefore

$$\begin{aligned}\frac{1}{t}E_{hm}[\int_0^t e^{-ps-\alpha A_s}f(X_s)\,dB_s]&=\frac{1}{t}\int_0^t\langle f\nu,\,e^{-ps}\,p_s^{\alpha\mu}\,h\rangle\,ds\\&=\int_0^1\langle f\nu,\,e^{-pts}\,p_{ts}^{\alpha\mu}\,h\rangle\,ds\nearrow\langle f\nu,h\rangle,\,t\searrow 0\,.\end{aligned}$$

Next we prove (iii) by assuming that B_t is associated with ν. By the uniqueness of the Revuz correspondence and Lemma 2.5, we may assume that $f=1$, $\nu\in S_0$ with $U_1\,\nu\in L^\infty(X;m)$. Using Lemma 2.6 and similar computation of (2.7), we have

$$E_{hm}[\int_0^t e^{-\alpha A_s}\,dB_s] = \int_0^t \langle \nu, p_s^{\alpha\mu}\tilde{h}\rangle\,ds\,, \quad h\in \boldsymbol{B}^+(X)\cap \mathcal{F}\,.$$

Hence we get (iii) for $h\in\boldsymbol{B}^+(X)\cap L^2(X; m)$, because $E_x[\int_0^t e^{-\alpha A_s}\,dB_s]\in L^2(X; m)$. Approximating $h\in\boldsymbol{B}^+(X)$ by $h_n\in\boldsymbol{B}^+(X)\cap L^2(X; m)$ with $h_n\nearrow h$. We can prove (iii). The proof is complete.

We define the kernels $\tilde{R}_\alpha^p$ by

$$\tilde{R}_\alpha^p f(x) = E_x[\int_0^\infty e^{-pt-\alpha A_t} f(X_t)\,dA_t]\,, \quad f\in\boldsymbol{B}^+(X)\,. \tag{2.8}$$

$\tilde{R}_\alpha^0$ is denoted by $\tilde{R}_\alpha$.

Corollary 2.7. *For each $f\in\boldsymbol{B}^+(X)\cap L^2(X; \mu)$, the Radon measure $f\mu$ on X is of finite 0-order energy integral with respect to $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$ and for each $p\geqq 0$, $\tilde{R}_\alpha^p f$ is $\mathcal{E}_1$-quasi-continuous modification of $U_p^{\alpha\mu}(f\mu)$. In particular the following duality relation holds*

$$(\tilde{R}_\alpha^p f, g)_m = (f, R_p^{\alpha\mu} g)_\mu \quad f, g\in\boldsymbol{B}^+(X)\,. \tag{2.9}$$

Proof. The first assertion is clear from

$$\langle f\mu, |v|\rangle \leqq \frac{1}{\sqrt{\alpha}}\,||f||_{L^2(\mu)}\sqrt{\mathcal{E}^{\alpha\mu}(|v|, |v|)} \quad v\in\mathcal{F}\cap C_0(X)\,.$$

In case that $p>0$, the second assertion follows from by Theorem 2.4 (ii). We show this in the case $p=0$. For $f, g\in\boldsymbol{B}^+(X)\cap L^2(X; \mu)$, we have

$$\begin{aligned}(\tilde{R}_\alpha f, g)_\mu &= \lim_{p\to 0}(\tilde{R}_\alpha^p f, g)_\mu = \lim_{p\to 0}\langle g\mu, \widetilde{U_p^{\alpha\mu}}(f\mu)\rangle \\ &= \lim_{p\to 0}\mathcal{E}_p^{\alpha\mu}(U_p^{\alpha\mu}(f\mu), U_p^{\alpha\mu}(g\mu)) = \lim_{p\to 0}\langle f\mu, \widetilde{U_p^{\alpha\mu}}(g\mu)\rangle \\ &= \lim_{p\to 0}(f, \tilde{R}_\alpha^p g)_\mu = (f, \tilde{R}_\alpha g)_\mu\end{aligned}$$

and

$$||\tilde{R}_\alpha f||_{L^2(\mu)} = \lim_{p\to 0}||\tilde{R}_\alpha^p f||_{L^2(\mu)} \leqq \frac{1}{\alpha}\,||f||_{L^2(\mu)}\,.$$

Hence $\tilde{R}_\alpha$ can be extended to a symmetric contractive resolvent operator $\tilde{G}_\alpha$ on $L^2(X; \mu)$ which is strongly continuous and Markovian. Especially $\tilde{R}_\alpha f$ belongs to $L^2(X; \mu)$ for any $f\in\boldsymbol{B}^+(X)\cap L^2(X; \mu)$. For $q>p>0$ and $f\in\boldsymbol{B}^+(X)\cap L^2(X; \mu)$,

$$\begin{aligned}&\mathcal{E}^{\alpha\mu}(\tilde{R}_\alpha^p f-\tilde{R}_\alpha^q f, \tilde{R}_\alpha^p f-\tilde{R}_\alpha^q f)\leqq \mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f-\tilde{R}_\alpha^q f, \tilde{R}_\alpha^p f-\tilde{R}_\alpha^q f) \\ &\quad\leqq \mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f, \tilde{R}_\alpha^p f)-2\mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f, \tilde{R}_\alpha^q f)+\mathcal{E}_q^{\alpha\mu}(\tilde{R}_\alpha^q f, \tilde{R}_\alpha^q f)-(q-p)\,(\tilde{R}_\alpha^q f, \tilde{R}_\alpha^q f)_m \\ &\quad\leqq \langle f\mu, \tilde{R}_\alpha^p f\rangle-2\langle f\mu, \tilde{R}_\alpha^q f\rangle+\langle f\mu, \tilde{R}_\alpha^q f\rangle \\ &\quad= \langle f\mu, \tilde{R}_\alpha^p f\rangle-\langle f\mu, \tilde{R}_\alpha^q f\rangle\,.\end{aligned}$$

The last term tends to zero as $q, p\to 0$. Hence $\{\tilde{R}_\alpha^p f\}_{p>0}$ is an $\mathcal{E}^{\alpha\mu}$-Cauchy sequence and $\tilde{R}_\alpha^p f$ increase to $\tilde{R}_\alpha f$ as $p\searrow 0$. Therefore $\tilde{R}_\alpha f$ belongs to the extended Dirichlet space $\mathcal{F}_e^\mu$ and $\tilde{R}_\alpha^p f$ converges to $\tilde{R}_\alpha f$ in $\mathcal{E}^{\alpha\mu}$. Hence we get for each $v\in\mathcal{F}\cap C_0(X)$,

$$\begin{aligned}\mathcal{E}^{\alpha\mu}(\tilde{R}_\alpha f, v) &= \lim_{p\to 0}\mathcal{E}^{\alpha\mu}(\tilde{R}_\alpha^p f, v)\\ &= \lim_{p\to 0}\mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f, v)-\lim_{p\to 0} p(\tilde{R}_\alpha^p f, v)_m\\ &= \langle f\mu, v\rangle = \mathcal{E}^{\alpha\mu}(U^{\alpha\mu}(f\mu), v)\,,\end{aligned}$$

because

$$\begin{aligned}p(\tilde{R}_\alpha^p f, v) &= p\mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f, R_p^{\alpha\mu} v)\\ &\leqq p\sqrt{\mathcal{E}_p^{\alpha\mu}(\tilde{R}_\alpha^p f, \tilde{R}_\alpha^p f)}\sqrt{\mathcal{E}_p^{\alpha\mu}(R_p^{\alpha\mu} v, R_p^{\alpha\mu} v)}\\ &= p\sqrt{\langle f\mu, \tilde{R}_\alpha^p f\rangle}\sqrt{(v, R_p^{\alpha\mu} v)_m}\\ &\leqq \sqrt{p}\sqrt{\langle f\mu, \tilde{R}_\alpha f\rangle}\sqrt{(v, v)_m}\,.\end{aligned}$$

Thus we have $\tilde{R}_\alpha f = U^{\alpha\mu}(f\mu)$ m-a.e., because $\mathcal{F}\cap C_0(X)$ is dense in $\mathcal{F}_e^\mu$. Since $\tilde{R}_\alpha f$ is an excessive function with respect to $\mathcal{M}^{\alpha\mu}$, $\tilde{R}_\alpha f$ is finely continuous q.e. with respect to $\mathcal{M}^{\alpha\mu}$. By Theorem 4.3.2 of Fukushima [5] and Lemma 2.3, $\tilde{U}^{\alpha\mu}(f\mu)$ is also finely continuous q.e. with respect to $\mathcal{M}^{\alpha\mu}$. Since $\tilde{R}_\alpha f = U^{\alpha\mu}(f\mu)$ m-a.e., we get $\tilde{R}_\alpha f = \tilde{U}^{\alpha\mu}(f\mu)$ $\mathcal{E}_1^{\alpha\mu}$-q.e. by Lemma 4.2.5 of Fukushima [5]. We have that $\tilde{R}_\alpha f$ is $\mathcal{E}_1$-quasi-continuous. The proof is complete.

3. Time changed regular Dirichlet space

In this section, we shall construct a C_0-regular Dirichlet space on $L^2(Y;\mu)$ associated with the time changed process $\mathcal{M}^t$ of $\mathcal{M}$ on $\tilde{Y}$, where Y is the support of μ and $\tilde{Y}$ is the fine support of A_t; $\tilde{Y}=\{x\in X-N; P_x(A_t>0,\text{ for any } t>0)=1\}$, N being an exceptional set for A_t.

We let $\mathcal{F}_{eX-\tilde{Y}}^\mu=\{u\in\mathcal{F}_e^\mu; u=0 \text{ q.e. on } \tilde{Y}\}$. This is a closed subspace of $\mathcal{F}_e^\mu$ and the Hilbert space $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$ admits the orthogonal decomposition

$$\mathcal{F}_e^\mu = \mathcal{F}_{eX-\tilde{Y}}^\mu \oplus \mathcal{H}_{\tilde{Y}}^{\alpha\mu}\,, \tag{3.1}$$

where $\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$ is the orthogonal complement of $\mathcal{F}_{eX-\tilde{Y}}^\mu$ with respect to $\mathcal{E}^{\alpha\mu}$. Denote by $\mathcal{P}^{\alpha\mu}$ the orthogonal projection on $\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$. For $f\in\mathcal{F}_e^\mu$, $u=\mathcal{P}^{\alpha\mu}f$ if and only if $u\in\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$ and $u=f$ q.e. on $\tilde{Y}$. Note that the space $\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$ is independent of $\alpha>0$. Indeed for any $u\in\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$ and $\beta>0$,

$$\mathcal{E}^{\beta\mu}(u, v) = \mathcal{E}^{\alpha\mu}(u, v)+(\beta-\alpha)(u, v)_\mu = 0\,,\quad v\in\mathcal{F}_{eX-\tilde{Y}}^\mu\,,$$

because $\mu(X-\tilde{Y})=0$ (Fukushima [5]). Hence $u\in\mathcal{H}_{\tilde{Y}}^{\beta\mu}$. Consequently $\mathcal{P}^{\alpha\mu}$ is also independent of $\alpha>0$. Hereafter $\mathcal{H}_{\tilde{Y}}^{\alpha\mu}$ (resp. $\mathcal{P}^{\alpha\mu}$) will be denoted by $\mathcal{H}_{\tilde{Y}}^\mu$ (resp. $\mathcal{P}^\mu$). We notice that, if $u, v\in\mathcal{F}_e^\mu$ and $u=v$ q.e. on $\tilde{Y}$, then $\mathcal{P}^\mu u=\mathcal{P}^\mu v$.

Lemma 3.1. *If $u, v \in \mathcal{F}_e^\mu$ and $u=v$ μ-a.e on Y, then $u=v$ q.e. on $\tilde{Y}$ and consequently $\mathcal{P}^\mu u=\mathcal{P}^\mu v$.*

Proof. We put $w_n=|u-v|\wedge n$, then $w_n=0$ μ-a.e. By virtue of the duality relation (2.9), we get

$$\langle m, \tilde{R}_\alpha w_n\rangle=\langle w_n \mu, R_0^{\alpha\mu} 1\rangle = 0 .$$

Thus we have that $\tilde{R}_\alpha w_n=0$ m-a.e. Since $\tilde{R}_\alpha w_n$ is an excessive function of $\mathcal{M}^{\alpha\mu}$, it is finely continuous q.e. with respect to $\mathcal{M}^{\alpha\mu}$. By Lemma 4.2.5 of Fukushima [5], we get $\tilde{R}_\alpha w_n=0$ q.e. By $\mathcal{E}_1$-quasi-continuity of w_n,

$$w_n(x) = \lim_{\alpha\to\infty} E_x[\int_0^\infty \alpha e^{-\alpha A_t} w_n(X_t)\, dA_t] = 0,\ q.e.\ x\in\tilde{Y} ,$$

which implies $u=v$ *q.e.* on $\tilde{Y}$. The proof is complete.

Define a symmetric bilinear form on $L^2(Y;\mu)$ by

$$(3.2)\qquad \begin{cases} \mathcal{F}_Y^\mu = \{u\in L^2(Y;\mu);\, u=v|_Y\ \mu\text{-a.e. on } Y \text{ for some } v\in\mathcal{F}_e^\mu\} \\ \mathcal{E}_Y^\mu(u,u) = \mathcal{E}(\mathcal{P}^\mu v, \mathcal{P}^\mu v), \quad \text{for } u\in\mathcal{F}_Y^\mu,\ v\in\mathcal{F}_e^\mu \text{ s.t. } u = v|_Y\ \mu\text{-a.e.,} \end{cases}$$

where $v|_Y$ is the restriction of function v to Y.

By Lemma 3.1 this is well-defined. Furthermore, $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is a closed symmetric form on $L^2(Y;\mu)$. Indeed, suppose that $\{u_n\}\subset\mathcal{F}_Y^\mu$ is an $\mathcal{E}_{Y,\alpha}^\mu$-Cauchy sequence, then there exists $v_n\in\mathcal{F}_e^\mu$ such that $u_n=v_n|_Y$ μ-a.e. and $\{\mathcal{P}^\mu v_n\}$ is an $\mathcal{E}^{\alpha\mu}$-Cauchy sequence in $\mathcal{H}_{\tilde{Y}}^\mu$. Since $\mathcal{H}_{\tilde{Y}}^\mu$ is a closed subspace of $\mathcal{F}_e^\mu$, there exists $v\in\mathcal{F}_e^\mu$ such that $\mathcal{P}^\mu v_n$ converge to $\mathcal{P}^\mu v$ in as $n\to\infty$. We put $u=v|_Y$, then $u\in\mathcal{F}_Y^\mu$. We get

$$\lim_{n\to\infty} \mathcal{E}_{Y\alpha}^\mu(u-u_n, u-u_n) = \lim_{n\to\infty} \mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu(v-v_n), \mathcal{P}^\mu(v-v_n)) = 0 ,$$

which implies the closedness of $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$.

Theorem 3.2. *$(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is the Dirichlet space on $L^2(Y;\mu)$ associated with the time changed process $\mathcal{M}^t$ of $\mathcal{M}$. $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is C_0-regular.*

Proof. The resolvent operator $\tilde{G}_\alpha$ in the proof of Corollary 2.7 can be regarded to be defined on $L^2(Y;\mu)$ because $\mu(X-Y)=0$. $\tilde{G}_\alpha$ is the L^2-resolvent of the μ-symmetric Markov process $\mathcal{M}^t$. For the first statement it is enough to show that, for $u\in L^2(Y;\mu)$ and $v\in\mathcal{F}_Y^\mu$,

$$(3.3)\qquad \begin{cases} \tilde{G}_\alpha u\in\mathcal{F}_Y^\mu \\ \mathcal{E}_{Y\alpha}^\mu(\tilde{G}_\alpha u, v) = (u,v)_\mu . \end{cases}$$

We may assume $u\in \boldsymbol{B}^+(Y)\cap L^2(Y;\mu)$. For any Borel extention $\bar{u}$ of u on X, $\tilde{G}_\alpha u=\tilde{R}_\alpha \bar{u}|_Y$ μ-a.e. By Corollary 2.7 and the definition of $\mathcal{F}_Y^\mu$, $\tilde{G}_\alpha u$ belongs to

$\mathcal{F}_Y^\mu$. Let $\bar{v}$ be an element of $\mathcal{F}_e^\mu$ such that $v=\bar{v}|_Y$ μ-a.e. Noting that $\mathcal{P}^\mu f=f$ μ-a.e. for each $f\in\mathcal{F}_e^\mu$,

$$\begin{aligned}\mathcal{E}_{Y\alpha}^\mu(\tilde{G}_\alpha u, v) &= \mathcal{E}_Y^\mu(\tilde{G}_\alpha u, v)+\alpha(\tilde{G}_\alpha u, v)_\mu \\ &= \mathcal{E}(\mathcal{P}^\mu \tilde{R}_\alpha \bar{u}, \mathcal{P}^\mu \bar{v})+\alpha(\tilde{R}_\alpha \bar{u}, \bar{v})_\mu \\ &= \mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu \tilde{R}_\alpha \bar{u}, \mathcal{P}^\mu \bar{v}) = \mathcal{E}^{\alpha\mu}(\tilde{R}_\alpha \bar{u}, \mathcal{P}^\mu \bar{v}) \\ &= (\bar{u}, \mathcal{P}^\mu \bar{v})_\mu = (u, v)_\mu.\end{aligned}$$

Next we shall show that $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is C_0-regular. For each $u\in C_0(Y)$, there exists $v\in C_0(X)$ such that $u=v|_Y$ by virtue of Tietze's extention theorem. Since $(\mathcal{E}, \mathcal{F})$ is C_0-regular, there exists $\{v_n\}\subset\mathcal{F}\cap C_0(X)$ which converge to v uniformly on X. By definition of $\mathcal{F}_Y^\mu$, $u_n=v_n|_Y$ belongs to not only $\mathcal{F}_Y^\mu$ but $C_0(Y)$, because Y is closed. Hence we have that u is approximated by elements of $\mathcal{F}_Y^\mu\cap C_0(Y)$ uniformly on Y. Next for each $u\in\mathcal{F}_Y^\mu$, there exists $v\in\mathcal{F}_e^\mu$ such that $u=v|_Y$ μ-a.e. By virtue of C_0-regularity of $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$, we have that for some $\{v_n\}\subset\mathcal{F}\cap C_0(X)$, $\lim_{n\to\infty}\mathcal{E}^{\alpha\mu}(v-v_n, v-v_n)=0$. Then $u_n=v_n|_Y$ belongs to $\mathcal{F}_Y^\mu\cap C_0(Y)$ by the same reason as above. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\mathcal{E}_{Y\alpha}^\mu(u-u_n, u-u_n) &= \lim_{n\to\infty}\mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu(v-v_n), \mathcal{P}^\mu(v-v_n)) \\ &\leqq \lim_{n\to\infty}\mathcal{E}^{\alpha\mu}(v-v_n, v-v_n) = 0\,,\end{aligned}$$

which means that $\mathcal{F}_Y^\mu\cap C_0(Y)$ is dense in $\mathcal{F}_Y^\mu$. The proof is complete.

4. Time changed Hunt process

On account of the C_0-regularity of $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$, we can consider a μ-symmetric Hunt process $\mathcal{M}_Y^\mu=(\hat{\Omega}, \hat{\mathcal{F}}_\infty, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{P}_x)$ on the state space Y associated with $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ (Theorem 6.2.1 of Fukushima [5]). In this section we investigate a relation between $\mathcal{M}_Y^\mu$ and the time changed process $\mathcal{M}^t$.

Lemma 4.1. *For a Radon measure ν on X such that $\nu(X-\tilde{Y})=0$, ν is of 0-order finite energy integral with respect to $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$ if and only if $\nu|_Y$ is of α-order finite energy integral with respect to $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$, where $\nu|_Y$ is the restriction of ν to Y.*

Proof. Suppose that ν is of 0-order finite energy integral with respect to $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$. For each $u\in\mathcal{F}_Y^\mu\cap C_0(Y)$, there exist $v\in\mathcal{F}_e^\mu$ and $w\in C_0(X)$ such that $u=v|_Y$ μ-a.e. and $u=w|_Y$. By Lemma 3.1, $v=w$ q.e. on $\tilde{Y}$, that is, $u=\mathcal{P}^\mu v$ q.e. on $\tilde{Y}$. We get

$$\begin{aligned}\int_Y |u(x)|\,\nu|_Y(dx) = \int_{\tilde{Y}} |\mathcal{P}^\mu v(x)|\,\nu(dx) &\leqq \text{const}\sqrt{\mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu v, \mathcal{P}^\mu v)} \\ &= \text{const}\sqrt{\mathcal{E}_{Y\alpha}^\mu(u, u)}\;.\end{aligned}$$

Conversely, suppose that $\nu|_Y$ is of α-order finite energy integral with respect to $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$. For each $v \in \mathcal{F} \cap C_c(X)$, $u = v|_Y$ belongs to $\mathcal{F}_Y^\mu \cap C_0(Y)$. We have

$$\int_X |v(x)|\nu(dx) = \int_Y |u(x)|\nu|_Y(dx) \leqq \text{const}\sqrt{\mathcal{E}_{Y\alpha}^\mu(u, u)}$$
$$= \text{const}\sqrt{\mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu v, \mathcal{P}^\mu v)} \leqq \text{const}\sqrt{\mathcal{E}^{\alpha\mu}(v, v)}\,.$$

The proof is complete.

Similary as in Section 1 we can define the notion of $\mathcal{E}_{Y1}^\mu$-capacity on Y, $\mathcal{E}_{Y1}^\mu$-q.e. and $\mathcal{E}_{Y1}^\mu$-quasi-continuous functions.

Theorem 4.2. *For a Borel set $B \subset Y$,*

$$\mathcal{E}_{Y\alpha}^\mu\text{-Cap}(B \cap \tilde{Y}) = 0 \quad \textit{if and only if} \quad \mathcal{E}^{\alpha\mu}\text{-Cap}(B \cap \tilde{Y}) = 0\,.$$

Proof. $\mathcal{E}_{Y\alpha}^\mu$-Cap$(B \cap \tilde{Y}) = 0$ is equivalent to $\nu(B \cap \tilde{Y}) = 0$ for any $\nu \in S_0(\mathcal{E}_{Y\alpha}^\mu)$, where $\nu \in S_0(\mathcal{E}_{Y\alpha}^\mu)$ is the family of all positive Radon measures on Y of α-order finite energy integral with respect to $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$. By the above lemma, this is equivalent to $\tilde{\nu}(B \cap \tilde{Y}) = 0$ for any $\tilde{\nu} \in S_0(\mathcal{E}^{\alpha\mu})$, where $S_0(\mathcal{E}^{\alpha\mu})$ is the family of all positive Radon measures on X of 0-order finite energy integral with respect to $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$. This is equivalent to $\mathcal{E}^{\alpha\mu}$-Cap$(B \cap \tilde{Y}) = 0$. The proof is complete.

Lemma 4.3. *If u is an $\mathcal{E}_1$-quasi-continuous function on X, then $u|_Y$ is $\mathcal{E}_{Y\alpha}^\mu$-quasi-continuous on Y.*

Proof. Suppose that u is $\mathcal{E}_1$-quasi-continuous. By Lemma 2.3 u is $\mathcal{E}^{\alpha\mu}$-quasi-continuous. Hence there exists an increasing sequence $\{F_n\}$ of closed sets such that $u|_{F_n}$ is continuous and $\lim_{n\to\infty} \mathcal{E}^{\alpha\mu}$-Cap$(X - F_n) = 0$. Let $\tilde{e}_n^\alpha$ be an $\mathcal{E}_1$-quasi-continuous version of an equilibrium potential of $X - F_n$ with respect to $(\mathcal{E}^{\alpha\mu}, \mathcal{F}_e^\mu)$. Since $\tilde{e}_n^\alpha|_Y = 1$ q.e. on $Y - Y \cap F_n$, we have

$$\mathcal{E}_{Y\alpha}^\mu\text{-Cap}(Y - Y \cap F_n) \leqq \mathcal{E}_{Y\alpha}^\mu(\tilde{e}_n^\alpha|_Y, \tilde{e}_n^\alpha|_Y) = \mathcal{E}^{\alpha\mu}(\mathcal{P}^\mu \tilde{e}_n^\alpha, \mathcal{P}^\mu \tilde{e}_n^\alpha)$$
$$\leqq \mathcal{E}^{\alpha\mu}(\tilde{e}_n^\alpha, \tilde{e}_n^\alpha) = \mathcal{E}^{\alpha\mu}\text{-Cap}(X - F_n)\,,$$

which implies $u|_Y$ is $\mathcal{E}_{Y\alpha}^\mu$-quasi-continuous on Y. The proof is complete.

Theorem 4.4. $\mathcal{E}_{Y1}^\mu$-Cap$(Y - \tilde{Y}) = 0$. *In particular $Y - \tilde{Y}$ is an exceptional set of $\mathcal{M}_Y^\mu$.*

Proof. Let $\hat{R}_\alpha$ be the resolvent kernel of $\mathcal{M}_Y^\mu$. Then for $f \in \boldsymbol{B}^+(Y) \cap L^2(Y; \mu)$

$$\hat{R}_\alpha f = \tilde{R}_\alpha f \ \mu\text{-a.e. on } Y \quad \text{for each } \alpha > 0\,,$$

by definition of $\tilde{G}_\alpha$. By Corollary 2.7 and Lemma 4.3 $\tilde{R}_\alpha f$ is $\mathcal{E}_{Y1}^\mu$-quasi-continuous on Y. We get

(4.1) $\hat{R}_\alpha f=\tilde{R}_\alpha f$, for any $\alpha>0$, $\mathcal{E}^\mu_{Y1}$-q.e. on $Y, f\in\boldsymbol{B}^+(Y)\cap L^2(Y;\mu)$.

Let $\sigma_{\tilde{Y}}(\omega)=\inf\{t>0;\ X_t\in\tilde{Y}\}$ be the first hitting time of $\tilde{Y}$. Then we have for $f\in C_0^+(Y)$

$$f(x)=\lim_{n\to\infty}n\hat{R}_n f(x)=\lim_{n\to\infty}n\tilde{R}_n f(x)=E_x[f(X_{\sigma_{\tilde{Y}}})]\ \mathcal{E}^\mu_{Y1}\text{-q.e. } x\in Y,$$

because $\sigma_{\tilde{Y}}(\omega)=R(\omega)=\inf\{t>0;\ A_t(\omega)>0\}$ (Ōshima [16]). We put

$$\mathcal{H}=\{A\in\boldsymbol{B}(Y);\ I_B(x)=E_x[I_A(X_{\sigma_{\tilde{Y}}})]\ \mathcal{E}^\mu_{Y1}\text{-q.e. } x\in Y\},$$

then $\mathcal{H}$ is a Dynkin class which contains all open sets of Y. W ehave $\mathcal{H}=\boldsymbol{B}(Y)$. Owing to the finely closedness of $\tilde{Y}$ (Ōshima [16]),

$$I_{Y-\tilde{Y}}(x)=E_x[I_{Y-\tilde{Y}}(X_{\sigma_{\tilde{Y}}})]=0\ \mathcal{E}^\mu_{Y1}\text{-q.e. } x\in Y,$$

which implies $\mathcal{E}^\mu_{Y1}$-Cap$(Y-\tilde{Y})=0$. The proof is complete.

Next we show that the time changed process $\mathcal{M}^t=(\Omega, \mathcal{F}_{\tau_t}, X_{\tau_t}, \{P_x\}_{x\in\tilde{Y}})$ can be realized as a Hunt process if its state $\tilde{Y}$ is modified. Let Δ be an extra point such that Y_Δ is a one point compactification of Y. When Y is already compact, Δ is adjoined as an isolated point. We call a Borel set $\tilde{B}\subset\tilde{Y}$ is $\mathcal{M}^t$-invariant if $P_x(X_{\tau_t}\in\tilde{B}_\Delta$ for any $t\geqq 0)=1$ for any $x\in\tilde{B}$ and a Borel set $B\subset Y$ is $\mathcal{M}^\mu_Y$-invariant if $P_x(\hat{X}_t\in B_\Delta$ for any $t\geqq 0$, $\hat{X}_{t-}\in B_\Delta$ for any $t>0)=1$ for any $x\in B$.

Lemma 4.5. *For any set $N\subset\tilde{Y}$ with $\mathcal{E}^\mu_{Y\alpha}$-Cap$(N)=0$, there exists a Borel set $\tilde{N}$ such that $N\cap(Y-\tilde{Y})\subset\tilde{N}\subset Y$, $\mu(\tilde{N})=0$ and $\tilde{Y}-\tilde{N}=Y-\tilde{N}$ is not only $\mathcal{M}^\mu_Y$-invariant but also $\mathcal{M}^t$-invariant. In particular $\mathcal{E}^\mu_{Y\alpha}$-Cap$(\tilde{N})=0$.*

Proof. By Theorem 4.4 and Theorem 4.2.1 of Fukushima [5] we can first find a Borel set N_1 such that $N\cap(Y-\tilde{Y})\subset N_1\subset Y$, $\mu(N_1)=0$ and $\tilde{Y}-N_1=Y-N_1$ is $\mathcal{M}^\mu_Y$-invariant. By Theorem 4.2 we can find a properly exceptional set $\tilde{N}_2$ of $\mathcal{M}$ such that $N_1\cap\tilde{Y}\subset\tilde{N}_2$. Put $N_2=\tilde{N}_2\cap\tilde{Y}$. Then $\tilde{Y}-N_2$ is an $\mathcal{M}^t$-invariant set and $\tilde{Y}-N_2\subset\tilde{Y}-N_1$. Similarly we can find a Borel set N_3 such that $N_2\cup(Y-\tilde{Y})\subset N_3$, $\mu(N_3)=0$ and $\tilde{Y}-N_3=Y-N_3$ is $\mathcal{M}^\mu_Y$-invariant and $\tilde{Y}-N_3\subset\tilde{Y}-N_2$. Hence we have a sequence $\{N_k\}$ of Borel sets such that

$$\begin{cases} N_0=N, \\ N_{2k}\cup(Y-\tilde{Y})\subset N_{2k+1}\subset Y,\ \tilde{Y}-N_{2k+1}\subset\tilde{Y}-N_{2k} \quad (k\geqq 0), \\ N_{2k-1}\cap\tilde{Y}\subset N_{2k}\subset\tilde{Y},\ \tilde{Y}-N_{2k}\subset\tilde{Y}-N_{2k-1} \quad (k\geqq 1) \end{cases}$$

and $\tilde{Y}-N_{2k-1}$ (resp. $\tilde{Y}-N_{2k}$) is $\mathcal{M}^\mu_Y$-invariant (resp. $\mathcal{M}^t$-invariant) for each

$k \geqq 1$ (resp. $k \geqq 0$). Put $\tilde{N} = \bigcup_{k=0}^{\infty} N_k$. Then $\tilde{N}$ satisfies the desired assertion, because countable intersection of $\mathcal{M}_Y^{\mu}$-invariant set (resp. $\mathcal{M}^t$-invariant set) is $\mathcal{M}_Y^{\mu}$-invariant (resp. $\mathcal{M}^t$-invariant). The proof is complete.

We put

$$\begin{aligned}\tilde{\Omega} = \{\tilde{\omega} \in Y_{\Delta}^{[0,\infty]};\ &\tilde{\omega}(\cdot) \text{ is right continuous on } [0, \infty)\,, \\ &\tilde{\omega}(\infty) = \Delta,\ \tilde{\omega}(s) = \Delta \text{ implies } \tilde{\omega}(t) = \Delta\,, \text{ for any } t \geqq s\}\,,\end{aligned}$$

$\tilde{X}_t(\tilde{\omega}) = \tilde{\omega}(t)$, $\tilde{\omega} \in \tilde{\Omega}$, $\tilde{\mathcal{F}}_{\infty}^0 = \sigma\{\tilde{X}_s;\ s \in [0, \infty]\}$, $\tilde{\mathcal{F}}_t^0 = \sigma\{\tilde{X}_s;\ s \in [0, t]\}$. Define maps $\Pi_1 : \hat{\Omega} \to \tilde{\Omega}$ and $\Pi_2 : \Omega \to \tilde{\Omega}$ by

$$\begin{aligned}\Pi_1(\hat{\omega})(t) &= \hat{X}_t(\omega),\ \hat{\omega} \in \hat{\Omega},\ t \in [0, \infty]\,, \\ \Pi_2(\omega)(t) &= X_{\tau_t(\omega)}(\omega),\ \omega \in \Omega,\ t \in [0, \infty]\,.\end{aligned}$$

Then we get $\Pi_1^{-1} \tilde{\mathcal{F}}_t^0 \subset \hat{\mathcal{F}}_t$, $\Pi_1^{-1} \tilde{\mathcal{F}}_{\infty}^0 \subset \hat{\mathcal{F}}_{\infty}$ and $\Pi_2^{-1} \tilde{\mathcal{F}}_t^0 \subset \mathcal{F}_{\tau_t}$, $\Pi_2^{-1} \tilde{\mathcal{F}}_{\infty}^0 \subset \mathcal{F}_{\infty}$. In particular $\Pi_1^{-1}\{\tilde{X}_t \in B\} = \{\hat{X}_t \in B\}$, $\Pi_2^{-1}\{\tilde{X}_t \in B\} = \{X_{\tau_t} \in B\}$ for a Borel set $B \subset Y$. Therefore we can define probability measures $\tilde{P}_x^{(i)}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_{\infty}^0)$ $(i=1, 2)$ by

$$\tilde{P}_x^{(1)}(\tilde{\Lambda}) = \begin{cases} \hat{P}_x(\Pi_1^{-1}(\tilde{\Lambda}))\,, & x \in \tilde{Y}_{\Delta},\ \tilde{\Lambda} \in \tilde{\mathcal{F}}_{\infty}^0 \\ \delta\{\tilde{\omega}_x\}(\tilde{\Lambda})\,, & x \in Y - \tilde{Y},\ \tilde{\Lambda} \in \tilde{\mathcal{F}}_{\infty}^0\,, \end{cases}$$

$$\tilde{P}_x^{(2)}(\tilde{\Lambda}) = \begin{cases} P_x(\Pi_2^{-1}(\tilde{\Lambda}))\,, & x \in \tilde{Y}_{\Delta},\ \tilde{\Lambda} \in \tilde{\mathcal{F}}_{\infty}^0 \\ \delta\{\tilde{\omega}_x\}(\tilde{\Lambda})\,, & x \in Y - \tilde{Y},\ \tilde{\Lambda} \in \tilde{\mathcal{F}}_{\infty}^0\,, \end{cases}$$

where $\tilde{\omega}_x(t) = x$ for any $t \in [0, \infty)$.

Theorem 4.6. *There exists a Borel set $\tilde{N} \subset Y$ such that $\mu(\tilde{N}) = 0$ and $\tilde{Y} - \tilde{N}$ is $\mathcal{M}_Y^{\mu}$-invariant and $\mathcal{M}^t$-invariant and*

$$\tilde{P}_x^{(1)} = \tilde{P}_x^{(2)} \text{ on } \tilde{\mathcal{F}}_{\infty}^0 \quad \text{for any} \quad x \in \tilde{Y} - \tilde{N}\,. \tag{4.2}$$

Proof. Denote by $\hat{p}_t$ and $\tilde{p}_t$ the transition kernel of $\mathcal{M}_Y^{\mu}$ and $\mathcal{M}^t$ respectively. By (4.1) and the uniqueness of Laplace transformation, we get for each $f \in C_0(Y)$,

$$\hat{p}_t f(x) = \tilde{p}_t f(x)\,, \text{ for any } t > 0,\ \mathcal{E}_{Y1}^{\mu}\text{-q.e. } x \in \tilde{Y}\,. \tag{4.3}$$

Using the separability of $C_0(Y)$, there exists a Borel set $N \subset Y$ with $\mathcal{E}_{Y1}^{\mu}$-Cap$(N) = 0$ such that

$$\hat{p}_t(x, B) = \tilde{p}_t(x, B)\,, \text{ for any } t > 0, \text{ Borel set } B \subset Y, \text{ and } x \in \tilde{Y} - N\,.$$

By Lemma 4.5 there exists a Borel set $\tilde{N} \subset \tilde{Y}$ such that $\mu(\tilde{N}) = 0$ and $\tilde{Y} - \tilde{N}$ is $\mathcal{M}_Y^{\mu}$-invariant and $\mathcal{M}^t$-invariant and

$$\hat{p}_t(x, B) = \tilde{p}_t(x, B)\,, \text{ for any } t>0, \text{ Borel set } B\subset Y, \text{ and } x\in\tilde{Y}-\tilde{N}\,.$$

Due to the Markov property of $\mathcal{M}_Y^\mu$ and $\mathcal{M}^t$, we then easily see that the finite dimentional distributions of $\tilde{P}_x^{(1)}$ rnd $\tilde{P}_x^{(2)}$ coincide for $x\in\tilde{Y}-\tilde{N}$. Therefore $\tilde{P}_x^{(1)}=\tilde{P}_x^{(2)}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_\infty^0)$, $x\in\tilde{Y}-\tilde{N}$, namely, $\mathcal{M}_Y^\mu|_{\tilde{Y}-\tilde{N}}$ and $\mathcal{M}_t|_{\tilde{Y}-\tilde{N}}$ induce the same law on $(\tilde{\Omega}, \tilde{\mathcal{F}}_\infty^0)$. On the other hand, $\mathcal{M}_Y^\mu|_{\tilde{Y}-\tilde{N}}$ is again a Hunt process because $\tilde{Y}-\tilde{N}$ is $\mathcal{M}_Y^\mu$-invariant. Hence we arrive at

Corollary 4.7. *$\mathcal{M}^t|_{\tilde{Y}-\tilde{N}}$ is a Hunt process on $\tilde{Y}-\tilde{N}$.*

5. Fine support of a PCAF

In this section we give an example related to birth and death processes where $\mathcal{E}_1$-capacity of the set $Y-\tilde{Y}$ is positive. By a birth and death process on the non-negative integers, we mean a time homogeneous Markov process with transition function $P_{ij}(t)$ such that

$$P_{ij}(t)\geqq 0,\ \sum_{k=0}^{\infty} P_{ik}(t)\leqq 1\,.$$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)\,P_{kj}(s)\,.$$

$$P_{ij}(0) = \delta_{ij} = \begin{cases} 1 & i=j\,, \\ 0 & i\neq j\,. \end{cases}$$

Moreover

$$\begin{cases} P_{ii+1}(t) = \lambda_i t+o(t) & (t\to 0) \\ P_{ii}(t) = 1-(\lambda_i+\mu_i)\,t+o(t) & (t\to 0) \\ P_{ii-1}(t) = \mu_i t+o(t) & (t\to 0)\,, \end{cases}$$

where $\lambda_i(i=0, 1, 2, 3, \cdots)$, $\mu_i(i=1, 2, 3, \cdots)$ are positive constants and $\mu_0=0$.

We let

$$\begin{cases} x_0 = 0\,, \\ x_1 = \dfrac{1}{\lambda_0}\,, \\ x_n = x_{n-1}+\dfrac{\mu_1\mu_2\cdots\mu_{n-1}}{\lambda_0\lambda_1\cdots\lambda_{n-1}}\,, & n\geqq 2\,, \\ m_0 = 1,\, m_n = \dfrac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,, & n\geqq 1\,. \end{cases}$$

We change the state space of the birth and death process from non-negative integers to $X=\{x_i\}_{i=0}^{\infty}$. The transition function is m-symmetric: $P_{ij}(t)\,m_i= P_{ji}(t)\,m_j$. We assume that

(A1) The boundary $x_\infty = \lim_{n\to\infty} x_n$ is regular: $x_\infty < \infty$, $\sum_{i=0}^{\infty} m_i < \infty$.

(A2) $$\sum_{i=0}^{\infty} m_i \sqrt{\lambda_i} < \infty,\ \mu_i < \lambda_i\,, \quad (i = 0, 1, 2, \cdots)\,.$$

We fix three positive numbers p_1, p_2, p_3 such that $p_1+p_2+p_3=1$ and we define the mass m_∞ on x_∞ by $m_\infty = p_3/p_2$, then the extended $\hat{m}$ is a positive Radon measure on the compact space $\hat{X} = X \cup \{x_\infty\}$ which is endowed with the relative topology of $\boldsymbol{R}$. Let us introduce a symmetric bilinear form on $L^2(\hat{X}; \hat{m})$ by

$$(5.1)\quad \begin{cases} \mathcal{F} = \{u \in C(\hat{X});\ \sum_{i=0}^{\infty} u^+(x_i)^2 (x_{i+1}-x_i) < \infty\} \\ \mathcal{E}(u, v) = \sum_{i=0}^{\infty} u^+(x_i)\, v^+(x_i)(x_{i+1}-x_i) + p_1/p_3\, u(x_\infty)\, v(x_\infty)\, m_\infty,\ u, v \in \mathcal{F}\,, \end{cases}$$

where $u^+(x_i) = \dfrac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i}$ $(i=0, 1, 2, \cdots)$.

Lemma 5.1. *$(\mathcal{E}, \mathcal{F})$ is an irreducible transient C_0-regular Dirichlet space on $L^2(\hat{X}; \hat{m})$.*

Proof. It is clear that every normal contraction operates on $(\mathcal{E}, \mathcal{F})$. To show the closedness, let $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F}$ be an $\mathcal{E}_1$-Cauchy sequence. Then $u_n^+ \in L^2(X; s)$ converges to some $f \in L^2(X; s)$ in $L^2(X; s)$ where s is a point measure on X such that $s(\{x_i\}) = x_{i+1}-x_i$. u_n converges to some $u \in L^2(X; m)$ in $L^2(X; m)$. From this and the inequality:

$$|u(x_i)-u(x_j)|^2 \leqq (x_i - x_j)\, \mathcal{E}(u, u) \quad u \in \mathcal{F}\,(0 \leqq i < j \leqq \infty)\,,$$

$u_n \in \mathcal{F}$ are equiuniformly-continuous and equibounded, Hence we conclude that there exists a subsequence $\{n_k\}$ such that u_{n_k} converges to a continuous function $\tilde{u}$ on $\hat{X}$ uniformly. Obviously $\tilde{u} = u$. Moreover

$$u^+(x_i) = \frac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i} = \lim_{k\to\infty} \frac{u_{n_k}(x_{i+1})-u_{n_k}(x_i)}{x_{i+1}-x_i} = \lim_{k\to\infty} u_{n_k}^+(x_i) = f(x_i)\,,$$

which implies that $u^+ \in L^2(X; s)$. Hence $u \in \mathcal{F}$ and u_n is $\mathcal{E}_1$-convergent to u.

Next we prove that $(\mathcal{E}, \mathcal{F})$ is C_0-regular. Since $\mathcal{F}$ is contained in $C(\hat{X})$, we have only to show that $\mathcal{F}$ is uniformly dense in $C(\hat{X})$. For any $u \in C^+(\hat{X})$, we put $u_n = I_{\{x_0, x_1, \cdots, x_n\}}\, u$, then u_n belongs to $\mathcal{F}$. By Dini's theorem, we have that u_n converges to u uniformly on $\hat{X}$. To show the irreducibility, consider a non-empty Borel set B of $\hat{X}$ such that $B^c = \hat{X} - B \neq \phi$ and $p_t\, I_B u = I_B p_t u$ for any $t > 0$ and $u \in \boldsymbol{B}_b^+(\hat{X})$. We may assume that $x_\infty \in B$. Then B should contain some other point than x_∞, because otherwise $\mathcal{F}$ contains the function $p_t\, I_{\{x_\infty\}} = I_{\{x_\infty\}}\, p_t\, I_{\{x_\infty\}}$ which is not continuous for small $t > 0$. Therefore there exists $x_{i+1} \in B$ such that $x_i \in \hat{X} - B$. We have

$$p_t(x_i, \{x_{i+1}\}) = p_t I_{\{x_{i+1}\}}(x_i) = p_t(I_{B^c} I_{\{x_{i+1}\}})(x_i) = I_{B^c}(x_i) p_t I_{x_{i+1}\}}(x_i) = 0$$

and

$$\mathcal{E}(I_{\{x_i\}}, I_{\{x_{i+1}\}}) = \lim_{t\to 0} \frac{1}{t} (I_{\{x_i\}}, I_{\{x_{i+1}\}} - P_t I_{\{x_{i+1}\}})_m = 0 ,$$

which contradicts $\mathcal{E}(I_{\{x_i\}}, I_{\{x_{i+1}\}}) = \lambda_i > 0$.

Suppose $(\mathcal{E}, \mathcal{F})$ is non-transient, then it is recurrent by irreducibility. Hence it is conservative, namely $p_t 1 = 1$ $\hat{m}$-a.e. (Ōshima [16]). Since 1 belongs to $L^2(\hat{X}; \hat{m})$, we have $\mathcal{E}(1, 1) = \lim_{t\to 0} (1 - p_t 1, 1)_{\hat{m}} = 0$, which contradicts that $\mathcal{E}(1, 1) = \frac{p_1}{p_3} m_\infty = \frac{p_1}{p_2} > 0$. The proof is complete.

REMAEK. Let A be the self-adjoint operator on $L^2(\hat{X}; \hat{m})$ corresponding to the Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\hat{X}; \hat{m})$. Then

$$\mathcal{D}(A) = \{u \in C(\hat{X}); \sum_{i=1}^{\infty} \left(\frac{u^+(x_i) - u^+(x_{i-1})}{m_i}\right)^2 m_i < \infty\} , \tag{5.2}$$

$$Au(x_i) = \begin{cases} \dfrac{u^+(x_0)}{m_0} & (i = 0) \\ \dfrac{u^+(x_i) - u^+(x_{i-1})}{m_i} & (1 \leqq i < \infty) \\ -\dfrac{p_1 u(x_\infty) + p_2 u^-(x_\infty)}{p_3} & (i = \infty) \quad (u^-(x_\infty) = \lim_{n\to\infty} u^+(x_n)) . \end{cases} \tag{5.3}$$

The last equation in (5.3) can be regarded as a boundary condition:

$$p_1 u(x_\infty) + p_2 u^-(x_\infty) + p_3 Au(x_\infty) = 0 \quad (p_1 + p_2 + p_3 = 1, p_i > 0, i = 1, 2, 3)$$

(Feller [2], Itô-McKean [11, 12]).

Let $\mathcal{M} = (\Omega, X_t, P_x)$ be the $\hat{m}$-symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$ on $\hat{X}$. Denote by $\mathcal{M} \otimes \mathcal{M} = (\tilde{\Omega}, \tilde{X}_t, \tilde{P}_x)$ the direct product process on $\hat{X} \times \hat{X}$ with its transition probability $\tilde{p}_t$. Then $\tilde{p}_t(f_1 \otimes f_2) = p_t f_1 \otimes p_t f_2$, $f_i \in \boldsymbol{B}^+(\hat{X})$. Hence $\tilde{p}_t$ is $\hat{m} \otimes \hat{m}$-symmetric. Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be the Dirichlet space on $L^2(\hat{X} \times \hat{X}, \hat{m} \otimes \hat{m})$ associated with the process $\mathcal{M} \otimes \mathcal{M}$. If $f_i \in \mathcal{F}$, $i = 1, 2$, then we have $f_1 \otimes f_2 \in \tilde{\mathcal{F}}$ and $\tilde{\mathcal{E}}(f_1 \otimes f_2, f_1 \otimes f_2) = \mathcal{E}(f_1, f_1)(f_2, f_2)_{\hat{m}} + \mathcal{E}(f_2, f_2)(f_1, f_1)_{\hat{m}}$. It is known that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is an irreducible transient regular Dirichlet space on $L^2(\hat{X} \times \hat{X}, \hat{m} \otimes \hat{m})$ (Ōshima [16]). Thus we have

$$\tilde{\mathcal{E}}\text{-Cap}(\{(x_k, x_k)\}) \leqq 4 m_k^2 \lambda_k \quad (k = 0, 1, 2, \cdots) . \tag{5.4}$$

In fact

$$\begin{aligned} \tilde{\mathcal{E}}\text{-Cap}(\{(x_k, x_k)\}) &= \inf \{\tilde{\mathcal{E}}(u, u); u \in \tilde{\mathcal{F}}_e, u \geqq 1 \ \hat{m} \otimes \hat{m}\text{-a.e. on } (x_k, x_k)\} \\ &\leqq \tilde{\mathcal{E}}(I_{\{(x_k, x_k)\}}, I_{\{(x_k, x_k)\}}) \end{aligned}$$

$$
\begin{aligned}
&= 2\tilde{\mathcal{E}}(I_{\{x_k\}}, I_{\{x_k\}})\,(I_{\{x_k\}}, I_{\{x_k\}})_{\hat{m}} \\
&= 2\Big(\sum_{i=0}^{\infty} (I^{+}_{\{x_k\}}(x_i))^2 (x_{i+1}-x_i) + \frac{p_1}{p_2}\,(I_{\{x_k\}}(x_\infty))^2\Big)\, m_k \\
&= 2m_k \sum_{i=0}^{\infty} \frac{(I_{\{x_k\}}(x_{i+1})-I_{\{x_k\}}(x_i))^2}{x_{i+1}-x_i} \\
&= \begin{cases} 2m_k\left(\dfrac{1}{x_k-x_{k-1}}+\dfrac{1}{x_{k+1}-x_k}\right) & (1\leqq k<\infty) \\[2ex] 2m_0\,\dfrac{1}{x_1-x_0} & (k=0)\,. \end{cases}
\end{aligned}
$$

The last term is dominated by the right hand side of (5.4) because $(x_k-x_{k-1})-(x_{k+1}-x_k)=\frac{\mu_1\mu_2\cdots\mu_{k-1}}{\lambda_0\lambda_1\cdots\lambda_{k-1}}\left(1-\frac{\mu_k}{\lambda_k}\right)$ which is positive by (A2).

Theorem 5.2. *Let Δ be the diagonal set of $\hat{X}\times\hat{X}$. Then (x_∞, x_∞) is irregular for $\Delta-\{(x_\infty, x_\infty)\}$ with respect to $\mathcal{M}\otimes\mathcal{M}$.*

Proof. It is enough to show that

$$
\sum_{i=0}^{\infty} \tilde{P}_{(x_\infty,x_\infty)}(\tilde{\sigma}_{\{(x_i,x_i)\}}<\infty)<\infty\,, \tag{5.5}
$$

where $\tilde{\sigma}_{\{(x_i,x_i)\}}=\inf\{t>0;\tilde{X}_t=(x_i, x_i)\}$, In fact, by Borel-Cantelli lemma, we have then a stronger assertion

$$
\begin{aligned}
\tilde{P}_{(x_\infty,x_\infty)}\,(\omega;\ &\text{there exists an integer } n(\omega) \text{ such that } X_t(\omega) \\
&\text{does not hit } (x_i, x_i) \text{ for any } i\geqq n(\omega)) = 1\,.
\end{aligned}
$$

Denote by $\tilde{g}(\tilde{x}, \tilde{y})=\int_0^\infty \tilde{p}_t(\tilde{x}, \tilde{y})dt$ the Green function of $\mathcal{M}\otimes\mathcal{M}$. Then

$$
\sum_{i=0}^{\infty} \tilde{P}_{(x_\infty,x_\infty)}(\tilde{\sigma}_{\{(x_i,x_i)\}}<\infty) = \sum_{i=0}^{\infty} \frac{\tilde{g}((x_\infty, x_\infty), (x_i, x_i))}{\tilde{g}((x_i, x_i), (x_i, x_i))}
$$

Since $\tilde{p}_t(\tilde{x}, \tilde{y})\leqq\sqrt{\tilde{p}_t(\tilde{x}, \tilde{x})}\,\sqrt{\tilde{p}_t(\tilde{y}, \tilde{y})}$, the right hand side is estimated by

$$
\begin{aligned}
\sum_{i=0}^{\infty}\sqrt{\frac{\tilde{g}((x_\infty, x_\infty), (x_\infty, x_\infty))}{\tilde{g}((x_i, x_i), (x_i, x_i))}} &= \frac{1}{\sqrt{\tilde{\mathcal{E}}\text{-Cap}(\{(x_\infty, x_\infty)\})}}\sum_{i=0}^{\infty}\sqrt{\tilde{\mathcal{E}}\text{-Cap}(\{(x_i, x_i)\})} \\
&\leqq \frac{1}{\sqrt{\tilde{\mathcal{E}}\text{-Cap}(\{(x_\infty, x_\infty)\})}}\sum_{i=0}^{\infty} 2m_i\sqrt{\lambda_i}<\infty\,.
\end{aligned}
$$

The proof is complete.

Now put $\mu=I_{\Delta-\{(x_\infty,x_\infty)\}}\hat{m}\otimes\hat{m}$. Then μ is a positive Radon measure on $\hat{X}\times\hat{X}$ charging no set of zero $\tilde{\mathcal{E}}_1$-capacity and its topological support is given by $Y=\mathrm{Supp}[\mu]=\Delta$ because (x_∞, x_∞) is an accumulation point of $\Delta-\{(x_\infty, x_\infty)\}$.

Let A_t be the associated PCAF with μ with respect to $\mathcal{M}\otimes\mathcal{M}$. Then $A_t=\int_0^t I_{\Delta-\{(x_\infty,x_\infty)\}}(\tilde{X}_s)ds$. By the last theorem, the fine support of A_t is given by $\tilde{Y}=\mathrm{Supp}\,[A_t]=\Delta-\{(x_\infty, x_\infty)\}$. Hence we have

$$\tilde{\mathcal{E}}_1\text{-Cap}(Y-\tilde{Y}) = \tilde{\mathcal{E}}_1\text{-Cap}(\{(x_\infty, x_\infty)\})\geqq m_\infty^2>0\,.$$

REMARKS. (i) By (5.6) and $p_t(L^2(\hat{X}, \hat{m}))\subset\mathcal{F}$, we know that $\mathcal{M}$ is a Feller process. Hence $\mathcal{M}\otimes\mathcal{M}$ is so. Therefore its Ray topology is equal to the original one by (9.27) of Sharpe [17] (Getoor [9]). Hence this also gives a counter example for $\mathcal{E}_1$-Cap$(Y^r-\tilde{Y})=0$, where Y^r is the Ray topological support of μ.

(ii) K. Th. Strum [19] obtained another example that $\mathcal{E}_1$-Cap$(Y-\tilde{Y})>0$ for d-dimentional Brownian motion $B_t(d\geqq2)$ by investigation of the fine topological structure of B_t.

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