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INTERSECTION NUMBERS FOR LOGARITHMIC *K*-FORMS

KEIJI MATSUMOTO

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1. Introduction

Let $L_j : \ell_j = 0$ $(0 \le j \le n+1)$ be hyperplanes in the k-dimensional projective space $\mathbf{P}^k$ in general position. For

$$\alpha = (\alpha_0, \dots, \alpha_{n+1}), \quad \alpha_j \in \mathbf{C} \setminus \mathbf{Z}, \quad \sum_{j=0}^{n+1} \alpha_j = 0,$$

we consider the logarithmic 1-from

$$\omega = \sum_{j=0}^{n+1} \alpha_j \frac{d\ell_j}{\ell_j}$$

and the covariant derivation $\nabla_\omega = d + \omega\wedge$ with respect to ω on $X = \mathbf{P}^k \setminus \cup_{j=0}^{n+1} L_j$. The k-th twisted de Rham cohomology group on X with respect to ω and that with compact support are defined as

$$\begin{aligned} H^k(X, \nabla_\omega) &= \{\xi \in \mathcal{E}^k(X) \mid \nabla_\omega \xi = 0\}/\nabla_\omega \mathcal{E}^{k-1}(X), \\ H_c^k(X, \nabla_\omega) &= \{\xi \in \mathcal{E}_c^k(X) \mid \nabla_\omega \xi = 0\}/\nabla_\omega \mathcal{E}_c^{k-1}(X), \end{aligned}$$

where $\mathcal{E}^m(X)$ is the space of smooth m-forms on X and $\mathcal{E}_c^m(X)$ is the space of s-mooth m-forms on X with compact support. It is known that the inclusion of $\mathcal{E}_c^m(X)$ in $\mathcal{E}^m(X)$ induces a natural isomorphism of $H_c^k(X, \nabla_\omega)$ onto $H^k(X, \nabla_\omega)$ and that $H^k(X, \nabla_\omega)$ is spanned by

$$\begin{aligned} \varphi_I &= d\log\left(\frac{\ell_{i_0}}{\ell_{i_1}}\right) \wedge d\log\left(\frac{\ell_{i_1}}{\ell_{i_2}}\right) \wedge \dots \wedge d\log\left(\frac{\ell_{i_{k-1}}}{\ell_{i_k}}\right), \\ I &= (i_0, i_1, \dots, i_k), \quad 0 \le i_0 < i_1 < \dots < i_k \le n+1. \end{aligned}$$

The groups $H_c^k(X, \nabla_\omega)$ and $H^k(X, \nabla_{-\omega})$ for $-\alpha = (-\alpha_0, \dots, -\alpha_{n+1})$ are dual to each other under the pairing

$$\int_X \xi \wedge \eta,$$

where ξ and η are representatives of $H_c^k(X, \nabla_\omega)$ and $H^k(X, \nabla_{-\omega})$, respectively. The inverse $\iota_\omega^k : H^k(X, \nabla_\omega) \to H_c^k(X, \nabla_\omega)$ of this natural isomorphism induces the pairing $\langle\ ,\ \rangle_\omega$ on the space spanned by the φ_I's as follows:

$$\langle \varphi_I, \varphi_J \rangle_\omega = \int_X \iota_\omega^k(\varphi_I) \wedge \varphi_J.$$

We call $\langle \varphi_I, \varphi_J \rangle_\omega$ *the intersection number* of the logarithmic k-forms φ_I and φ_J with respect to ω. Our main theorem 2.1 evaluates explicitly the intersection numbers.

It seems that Professor K. Cho has obtained the same result in his private note [3] by using algebraic geometrical tools: spectral sequences, hypercohomology, the Serre duality, etc. In this paper, we prove the theorem by using only the Stokes theorem and the residue theorem. Since our proof bases on elementary tools, we can apply the technique in the proof to other computations for intersection numbers of forms, refer to [8] and [9].

Let us here recall the role of the intersection numbers of logarithmic k-forms in the theory of twisted period relations for hypergeometric functions. Let $H_k(X, \mathcal{L}_{-\omega})$ be the k-th twisted homology group with respect to the locally constant sheaf $\mathcal{L}_{-\omega}$ defined by holomorphic functions ψ on $U \subset X$ satisfying $\nabla_{-\omega}\psi = 0$. Note that the ψ are branches of $u^\alpha = \prod_{j=0}^{n+1} \ell^{\alpha_j}$ and that any element γ of $H_k(X, \mathcal{L}_{-\omega})$ is represented by a finite sum of pairs of a k-th topological chain ρ_i and a branch $u_{\rho_i}^\alpha$ of u^α along ρ_i. The group $H^k(X, \nabla_\omega)$ and $H_k(X, \mathcal{L}_{-\omega})$ are dual to each other under the pairing

$$[\varphi, \gamma] = \sum_i \int_{\rho_i} u_{\rho_i}^\alpha \varphi,$$

where φ is a representative of $H^k(X, \nabla_\omega)$. The so-called Euler integral representations for different hypergeometric functions can be interpreted as this pairing.

The isomorphism ι_ω^k and the duality between $H_c^k(X, \nabla_\omega)$ and $H^k(X, \nabla_{-\omega})$ induce an isomorphism of $H^k(X, \nabla_\omega)$ onto $H_k(X, \mathcal{L}_\omega)$ and the duality between $H_k(X, \mathcal{L}_{-\omega})$ and $H_k(X, \mathcal{L}_\omega)$. The value $\langle \gamma, \gamma' \rangle$ for $\gamma \in H_k(X, \mathcal{L}_{-\omega})$ and $\gamma' \in H_k(X, \mathcal{L}_\omega)$ is called the intersection number of γ and γ' with respect to ω. The intersection theory for representatives of $H_k(X, \mathcal{L}_{\mp\omega})$ is established in [6]. Choose bases of four groups $H^k(X, \nabla_{\pm\omega})$ and $H_k(X, \mathcal{L}_{\mp\omega})$ as

$$\xi_i^+ \in H^k(X, \nabla_{+\omega}), \quad \eta_i^- \in H^k(X, \nabla_{-\omega}),$$
$$\sigma_i^+ \in H_k(X, \mathcal{L}_{-\omega}), \quad \tau_i^- \in H_k(X, \mathcal{L}_{+\omega}),$$

and make four $\binom{n}{k} \times \binom{n}{k}$ matrices:

$$I_{ch} = \left(\langle \xi_i^+, \eta_j^- \rangle\right)_{ij}, \quad I_h = \left(\langle \sigma_i^+, \tau_j^- \rangle\right)_{ij},$$

$$\Pi^{+\alpha} = \left([\xi_i^+, \sigma_j^+]\right)_{ij}, \quad \Pi^{-\alpha} = \left([\eta_i^-, \tau_j^-]\right)_{ij}.$$

The naturality of the four pairings leads to *the twisted period relations*:

$$\Pi^{+\alpha}\, {}^tI_h^{-1}\, {}^t\Pi^{-\alpha} = I_{ch}, \quad \text{i.e.,} \quad {}^t\Pi^{-\alpha}\, I_{ch}^{-1} \Pi^{+\alpha} = {}^tI_h.$$

In this way, the evaluation of intersection numbers of logarithmic k-forms together with results in [6] yields quadratic identities for hypergeometric functions. Refer to [5] and [7] for other applications of the twisted period relations.

2. Main Theorem

For a fixed $(k+1)\times(n+2)$ matrix $x = (x_{ij})_{0\le i\le k, 0\le j\le n+1}$ such that no $(k+1)$-minor vanishes, put

$$\ell_j = \sum_{i=0}^{k} t_i x_{ij}, \quad L_j = \{t \in \mathbf{P}^k \mid \ell_j = 0\}, \quad 0 \le j \le n+1,$$

where $t = (t_0, \ldots, t_k)$ is a coordinate system of the k-dimensional complex projective space $\mathbf{P}^k$. Note that any k hyperplanes $L_{j_1}, \ldots, L_{j_k}$ intersect at one point $L_{j_1,\ldots,j_k}$, through which passes no other hyperplanes L_j. For

$$\alpha = (\alpha_0, \ldots, \alpha_{n+1}), \quad \alpha_j \in \mathbf{C}\setminus\mathbf{Z}, \quad \sum_{j=0}^{n+1} \alpha_j = 0,$$

we consider the logarithmic 1-form

$$\omega = \sum_{j=0}^{n+1} \alpha_j \frac{d\ell_j}{\ell_j}$$

on $\mathbf{P}^k$ and the covariant derivation $\nabla_\omega = d + \omega\wedge$ with respect to ω on $X = \mathbf{P}^k - \cup_{j=0}^{n+1} L_j$. Note that

$$\nabla_\omega \circ \nabla_\omega = 0.$$

Let $\mathcal{E}^m(X)$ be the space of smooth m-forms on X and $\mathcal{E}_c^m(X)$ the space of s-mooth m-forms on X with compact support. The m-th twisted de Rham cohomology groups on X with respect to ω and those with compact support are defined as

$$H^m(X, \nabla_\omega) = \{\xi \in \mathcal{E}^m(X) \mid \nabla_\omega \xi = 0\}/\nabla_\omega \mathcal{E}^{m-1}(X),$$
$$H_c^m(X, \nabla_\omega) = \{\xi \in \mathcal{E}_c^m(X) \mid \nabla_\omega \xi = 0\}/\nabla_\omega \mathcal{E}_c^{m-1}(X).$$

It is known that the natural inclusion of $\mathcal{E}_c^m(X)$ in $\mathcal{E}^m(X)$ induces the isomorphism between $H^m(X,\nabla_\omega)$ and $H_c^m(X,\nabla_\omega)$ under the condition $\alpha_j \in \mathbf{C}\setminus\mathbf{Z}$ for any j; let

$$\iota_\omega^m : H^m(X,\nabla_\omega) \to H_c^m(X,\nabla_\omega)$$

be the inverse of this isomorphism.

Let $\mathcal{F}^m(X)$ be the $\mathbf{C}$ vector space spanned by

$$\begin{aligned}\varphi_I &= d\log\left(\frac{\ell_{i_0}}{\ell_{i_1}}\right)\wedge d\log\left(\frac{\ell_{i_1}}{\ell_{i_2}}\right)\wedge\ldots\wedge d\log\left(\frac{\ell_{i_{m-1}}}{\ell_{i_m}}\right),\\ I &= (i_0,i_1,\ldots,i_m),\quad 0\le i_0<i_1<\ldots<i_m\le n+1,\end{aligned}$$

where we regard $\mathcal{F}^0(X)$ and $\mathcal{F}^{-1}(X)$ as $\mathbf{C}$ and $\{0\}$, respectively. It is shown in [1] that the natural inclusion of $\mathcal{F}^m(X)$ into $\mathcal{E}^m(X)$ induces the isomorphism from $\mathcal{F}^m(X)/(\omega\wedge\mathcal{F}^{m-1}(X))$ to $H^m(X,\nabla_\omega)$. This isomorphism implies that $H^m(X,\nabla_\omega)$ is 0 for $m<k$ and that the φ_I with multi-index $I=(i_0,i_1,\ldots,i_k)$ $i_0=0,\ i_k\le n$, span $H^k(X,\nabla_\omega)$; especially the rank of $H^k(X,\nabla_\omega)$ is $\binom{n}{k}$.

For $-\alpha=(-\alpha_0,\ldots,-\alpha_{n+1})$, we consider $H^k(X,\nabla_{-\omega})$ and $H_c^k(X,\nabla_{-\omega})$. The groups $H_c^k(X,\nabla_\omega)$ and $H^k(X,\nabla_{-\omega})$ are dual to each other under the pairing

$$\int_X \xi\wedge\eta,$$

where ξ and η are representatives of $H_c^k(X,\nabla_\omega)$ and $H^k(X,\nabla_{-\omega})$, respectively. The isomorphism ι_ω^k induces the pairing $\langle\ ,\ \rangle_\omega$ on $\mathcal{F}^k(X)$, which descents to that on $\mathcal{F}^k(X)/(\omega\wedge\mathcal{F}^{k-1}(X))$, as follows:

$$\langle\varphi,\varphi'\rangle_\omega = \int_X \iota_\omega^k(\varphi)\wedge\varphi',$$

where $\varphi,\varphi'\in\mathcal{F}^k(X)$. We call $\langle\varphi,\varphi'\rangle_\omega$ the intersection number of the logarithmic k-forms φ and φ' with respect to ω. The following is our main theorem.

Theorem 2.1. *For multi-indices*

$$\begin{aligned}I &= (i_0,i_1,\ldots,i_k),\quad 0\le i_0<i_1<\ldots<i_k\le n+1,\\ J &= (j_0,j_1,\ldots,j_k),\quad 0\le j_0<j_1<\ldots<j_k\le n+1,\end{aligned}$$

we have

$$\langle \varphi_I, \varphi_J \rangle_\omega = \begin{cases} (2\pi\sqrt{-1})^k \dfrac{\sum_{i\in I}\alpha_i}{\prod_{i\in I}\alpha_i} & \textit{if } I = J, \\ (2\pi\sqrt{-1})^k \dfrac{(-1)^{\mu+\nu}}{\prod_{i\in I\cap J}\alpha_i} & \textit{if } \#(I\cap J) = k, \\ 0 & \textit{otherwise}, \end{cases}$$

where in case $\#(I \cap J) = k$, μ and ν are determined as

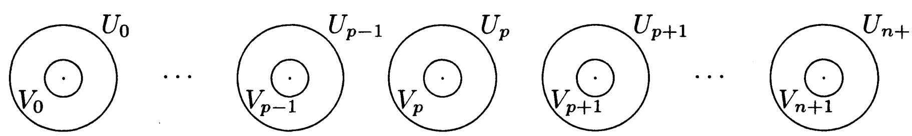
$$\{i_\mu\} = I \setminus (I \cap J), \quad \{j_\nu\} = J \setminus (I \cap J).$$

3. Preliminaries

We prepare some notations. Let $\mathbf{Z}_{\leq 0}$ be the set $\{0, -1, -2, \ldots\}$. For multi-indices $P_{m+1} = (p_0, \ldots, p_m)$, $p_0 < p_1 < \ldots < p_m$ of cardinality $m+1$ and $P_m = (q_1, \ldots, q_m)$ of cardinality m, set

$$\delta(P_m; P_{m+1}) = \begin{cases} (-1)^\mu & \text{if } P_m \subset P_{m+1}, \\ 0 & \text{if } P_m \not\subset P_{m+1}, \end{cases}$$

where μ is determined as $\{p_\mu\} = P_{m+1} \setminus P_m$ in case $P_m \subset P_{m+1}$. For every $p = 0, \ldots, n+1$, we take sufficiently small tubular neighborhoods U_p and V_p of L_p satisfying $V_p \subset U_p$. Put $D_p = U_p \setminus V_p$; the case $k = 1$, see the following.



Put

$$\begin{aligned} L_{p_1,\ldots,p_m} &= L_{p_1} \cap \ldots \cap L_{p_m}, & U_{p_1,\ldots,p_m} &= U_{p_1} \cap \ldots \cap U_{p_m}, \\ V_{p_1,\ldots,p_m} &= V_{p_1} \cap \ldots \cap V_{p_m}, & D_{p_1,\ldots,p_m} &= D_{p_1} \cap \ldots \cap D_{p_m}. \end{aligned}$$

Note that if $m > k$ then $U_{p_1,\ldots,p_m}$ is empty. Let h_p be a smooth function satisfying

$$\begin{aligned} h_p(t) = 1 \quad & t \in V_p, \\ 0 \leq h_p(t) \leq 1 \quad & t \in D_p, \\ h_p(t) = 0 \quad & t \in U_p^c. \end{aligned}$$

Note that

$$\begin{aligned}\nabla_\omega(\xi\wedge\eta) &= d(\xi\wedge\eta)+\omega\wedge\xi\wedge\eta\\ &= d\xi\wedge\eta+(-1)^m\xi\wedge d\eta+\omega\wedge\xi\wedge\eta\\ &= (\nabla_\omega\xi)\wedge\eta+(-1)^m\xi\wedge d\eta\\ &= (d\xi)\wedge\eta+(-1)^m\xi\wedge\nabla_\omega\eta,\end{aligned}$$

for a smooth m-form ξ and a smooth form η on X.

4. Proof of Theorem for $k=1$

Theorem for $k=1$ is proved in [2]. In this section, we give another proof of Theorem for $k=1$, since this will be a model of our proof of Theorem for general k. The key point for computation of the value $\langle\varphi_I,\varphi_J\rangle_\omega$ is to find the image $\iota_\omega^1(\varphi_I)$ explicitly. Since $\iota_\omega^1(\varphi_I)$ is with compact support, it vanishes on small neighborhoods around L_p. Thus we need local solutions of the differential equation $\nabla_\omega\psi=\varphi_I$ on U_p. After finding $\iota_\omega^1(\varphi_I)$, the value $\int_X\iota_\omega^1(\varphi_I)\wedge\varphi_J$ can be computed by the Stokes theorem and the residue theorem. We will give the local solutions of $\nabla_\omega\psi=\varphi_I$ and the image $\iota_\omega^1(\varphi_I)$ in Lemma 4.1 and Lemma 4.2, respectively.

Lemma 4.1. *If $\alpha_p\notin\mathbf{Z}_{\le 0}$, for any $\varphi_I=d\log(\ell_{i_0}/\ell_{i_1})\in\mathcal{F}^1(X)$ there exists a unique holomorphic function ψ^p on U_p such that*

$$\nabla_\omega\psi^p=\varphi_I,\quad \text{on } U_p\setminus\{L_p\}.$$

The value of ψ^p at L_p is the ratio of the residues of φ_I and ω at L_p:

$$\psi^p(L_p)=\frac{\delta_{p,i_0}-\delta_{p,i_1}}{\alpha_p}=\frac{\delta(p;I)}{\alpha_p},$$

where $\delta_{i,j}$ is Kronecker's symbol.

Proof. In terms of a local coordinate z around L_p, ω and φ_I can be expressed as

$$\omega=\left(\frac{a_{-1}}{z}+a_0+a_1z+\cdots\right)dz,\quad \varphi_I=\left(\frac{b_{-1}}{z}+b_0+b_1z+\cdots\right)dz;$$

note that

$$a_{-1}=\alpha_p,\quad b_{-1}=\delta_{p,i_0}-\delta_{p,i_1}=\delta(p;I).$$

Put

$$\psi^p=\sum_{m=0}^{\infty}c_mz^m.$$

Then $\nabla_\omega \psi^p = \varphi_I$ reduces to

$$\sum_{m=0}^{\infty}\left(mc_m + \sum_{q=-1}^{m-1} a_q c_{m-q-1}\right) z^{m-1} = \sum_{m=0}^{\infty} b_m z^{m-1}.$$

This equation implies that $c_0 = b_{-1}/a_{-1}$ and that c_m are expressed in terms of the a_m's and the b_m's by the assumption $\alpha_p \notin \mathbf{Z}_{\leq 0}$. Since ω and φ_I are meromorphic around L_p, the series $\sum_{m=0}^{\infty} c_m z^m$ converges on the small neighborhood U_p of L_p. □

Note that the function $h_p\psi^p$ and the form $\psi^p dh_p$ are regarded as defined on X.

Lemma 4.2. *The form*

$$\tilde{\varphi}_I = \varphi_I - \sum_{p=0}^{n+1} \nabla_\omega (h_p \psi^p)$$

is with compact support on X.

Proof. Since

$$\nabla_\omega (h_p\psi^p) = \psi^p dh_p + h_p \nabla_\omega \psi^p = \psi^p dh_p + h_p \varphi_I,$$

the form $\tilde{\varphi}_I$ vanishes on each V_p. □

Proof of Theorem for $k = 1$. Lemma 4.2 asserts that $\tilde{\varphi}_I$ represents $\iota_\omega^1(\varphi_I) \in H_c^1(X, \nabla_\omega)$. Since

$$\tilde{\varphi}_I \wedge \varphi_J = -\sum_{p=0}^{n+1} \psi^p dh_p \wedge \varphi_J = -\sum_{p=0}^{n+1} d(h_p \psi^p \varphi_J)$$

for $\varphi_J \in \mathcal{F}^1(X)$, we have

$$\begin{aligned} \langle \varphi_I, \varphi_J \rangle_\omega &= \int_X \iota_\omega^1(\varphi_I) \wedge \varphi_J \\ &= -\sum_{p=0}^{n+1} \int_{D_p} \psi^p dh_p \wedge \varphi_J = -\sum_{p=0}^{n+1} \int_{\partial D_p} h_p \psi^p \varphi_J. \end{aligned}$$

Since $\partial D_p = \partial U_p - \partial V_p$,

$$h_p = \begin{cases} 0 & \text{on } \partial U_p, \\ 1 & \text{on } \partial V_p; \end{cases}$$

we have

$$-\int_{\partial D_p} h_p\psi^p\varphi_J = 2\pi\sqrt{-1}\mathrm{Res}_{L_p}(\psi^p\varphi_J)$$
$$= 2\pi\sqrt{-1}\frac{\delta(p;I)\delta(p;J)}{\alpha_p}.$$

This proves Theorem for $k=1$. □

5. Proof of Theorem for $k=2$

In this section, we give a proof of Theorem for $k=2$, since it will help the reader understand the proof for general k. The key point is to find the image $\iota_\omega^2(\varphi_I)$ explicitly. We need not only 1-forms $\tilde{\psi}^p$ on U_p such that $\nabla_\omega\tilde{\psi}^p = \varphi_I$ but also functions $\tilde{\psi}^{p,q}$ on $U_{p,q}$ such that $\nabla_\omega\tilde{\psi}^{p,q} = \tilde{\psi}^p - \tilde{\psi}^q$. In order to construct $\tilde{\psi}^p$ and $\tilde{\psi}^{p,q}$, we prepare three lemmas. We will give the image $\iota_\omega^2(\varphi_I)$ in Lemma 5.4. Once $\iota_\omega^2(\varphi_I)$ is found explicitly, we have only to apply the Stokes theorem and the residue theorem repeatedly.

For a fixed multi-index $P=(p,q)$, let (z_1,z_2) be local coordinates around L_P such that L_p and L_q are expressed by $z_1=0$ and $z_2=0$, respectively.

Lemma 5.1. *If $\alpha_p,\alpha_q\notin \mathbf{Z}_{\le 0}$, for any $\varphi_I\in\mathcal{F}^2(X)$, $I=(i_0,i_1,i_2)$, there exist 1-forms ψ_P^p and ψ_P^q on $U_P\setminus(L_p\cup L_q)$ such that*

$$\nabla_\omega\psi_P^p = \nabla_\omega\psi_P^q = \varphi_I, \quad \text{on } U_P\setminus(L_p\cup L_q),$$
$$\psi_P^p = f_P^p(z_1,z_2)\frac{dz_2}{z_2}, \quad \psi_P^q = f_P^q(z_1,z_2)\frac{dz_1}{z_1},$$

where f_P^p and f_P^q are holomorphic on U_P satisfying

$$f_P^p(L_P) = \frac{\delta(P;I)}{\alpha_p}, \quad f_P^q(L_P) = -\frac{\delta(P;I)}{\alpha_q}.$$

Moreover, we can extend ψ_P^p and ψ_P^q to single-valued holomorphic 1-forms on $U_p\cap X$ and $U_q\cap X$, respectively.

Proof. Note that in terms of the local coordinates (z_1,z_2), the forms ω and φ_I are expressed as

$$\omega = \left(\frac{\alpha_p}{z_1}+\cdots\right)dz_1 + \left(\frac{\alpha_q}{z_2}+\cdots\right)dz_2, \quad \varphi_I = (\delta(P;I)+\cdots)\frac{dz_1\wedge dz_2}{z_1z_2}.$$

We can regard

$$\nabla_\omega\left(f_P^p(z_1,z_2)\frac{dz_2}{z_2}\right)=\varphi_I$$

as an ordinary differential equation with independent variable z_1 (z_2 being a parameter) and

$$\nabla_\omega\left(f_P^q(z_1,z_2)\frac{dz_1}{z_1}\right)=\varphi_I$$

with independent variable z_2 (z_1 being a parameter). Follow the argument in the proof of Lemma 4.1. □

Lemma 5.2. *There exists a unique holomorphic function ψ_P^P on U_P such that*

$$\nabla_\omega\psi_P^P=\psi_P^p-\psi_P^q,\quad \text{on } U_P\setminus(L_p\cup L_q)$$

for $P=(p,q),\ p<q$. The value of ψ_P^P at L_P is

$$\psi_P^P(L_P)=\frac{\delta(P;I)}{\alpha_p\alpha_q}.$$

Proof. For a smooth function f, we have

$$\nabla_\omega f=\left(\frac{\partial}{\partial z_1}f+\left(\frac{\alpha_p}{z_1}+\cdots\right)f\right)dz_1+\left(\frac{\partial}{\partial z_2}f+\left(\frac{\alpha_q}{z_2}+\cdots\right)f\right)dz_2$$

We can find ψ_P^P satisfying

$$\left(\frac{\partial}{\partial z_2}\psi_P^P+\left(\frac{\alpha_q}{z_2}+\cdots\right)\psi_P^P\right)dz_2=\psi_P^p=f_P^p(z_1,z_2)\frac{dz_2}{z_2},$$

by regarding the above as an ordinary differential equation with variable z_2 (z_1 being a parameter) and by following the argument in the proof of Lemma 4.1. Since $\nabla_\omega\circ\nabla_\omega=0$ and $\nabla_\omega\psi_P^p=\nabla_\omega\psi_P^q=\varphi_I$, we have

$$\begin{aligned}0&=\nabla_\omega\{\nabla_\omega\psi_P^P-(\psi_P^p-\psi_P^q)\}\\&=\nabla_\omega\left\{\left(\frac{\partial}{\partial z_1}\psi_P^P+\left(\frac{\alpha_p}{z_1}+\cdots\right)\psi_P^P\right)dz_1+\psi_P^q\right\}.\end{aligned}$$

By regarding this as an ordinary differential equation with variable z_2 (z_1 being a parameter), the uniqueness of the solution in Lemma 4.1 implies

$$\left(\frac{\partial}{\partial z_1}\psi_P^P+\left(\frac{\alpha_p}{z_1}+\cdots\right)\psi_P^P\right)dz_1+\psi_P^q=0,$$

which shows

$$\nabla_\omega \psi_P^P = \psi_P^p - \psi_P^q. \qquad \square$$

Lemma 5.3. *Let S and S' be multi-indices of cardinality two including the index p. There exists a unique holomorphic function ${}_{S'}\psi_S^p$ on $U_p \setminus \cup_{q\neq p} L_q$ such that*

$$\nabla_\omega {}_{S'}\psi_S^p = \psi_{S'}^p - \psi_S^p, \quad \text{on } U_p \cap X.$$

The function ${}_{S'}\psi_S^p$ vanishes along $L_p \setminus \cup_{q\neq p} L_q$.

Proof. Follow a similar argument in the proof of Lemma 5.2 for local coordinates (z_1, z_2) so that L_p is expressed by $z_2 = 0$. Since $\psi_{S'}^p - \psi_S^p$ is holomorphic on $U_p \setminus \cup_{q\neq p} L_q$ and L_p is a component of the pole divisor of ω, ${}_{S'}\psi_S^p$ must vanish along $L_p \setminus \cup_{q\neq p} L_q$. $\square$

For each p, put

$$\tilde{\psi}^p = \psi_{P\langle p\rangle}^p,$$

where $P\langle p\rangle$ is the smallest multi-index of cardinality two including p in the lexicographic order; note that on $U_p \cap X$,

$$\nabla_\omega \tilde{\psi}^p = \varphi_I.$$

For each multi-index $P = (p, q)$, put

$$\tilde{\psi}^P = \psi_P^P + {}_{P\langle p\rangle}\psi_P^p - {}_{P\langle q\rangle}\psi_P^q;$$

note that on $U_P \cap X$,

$$\begin{aligned}\nabla_\omega \tilde{\psi}^P &= (\psi_P^p - \psi_P^q) + (\psi_{P\langle p\rangle}^p - \psi_P^p) - (\psi_{P\langle q\rangle}^q - \psi_P^q) \\ &= \psi_{P\langle p\rangle}^p - \psi_{P\langle q\rangle}^q = \tilde{\psi}^p - \tilde{\psi}^q.\end{aligned}$$

We can regard $h_p \tilde{\psi}^p$ and $h_q \tilde{\psi}^P dh_p$ as defined on X.

Lemma 5.4. *The form*

$$\tilde{\varphi}_I = \varphi_I - \nabla_\omega \left(\sum_{p=0}^{n+1} g^p h_p \tilde{\psi}^p \right) - \nabla_\omega \left(\sum_{P=(p,q)} h_q \tilde{\psi}^P dh_p \right)$$

is with compact support on X, *where*

$$g^0 = 1, \quad g^p = \prod_{r=0}^{p-1}(1-h_r), \ p \geq 1.$$

Proof. We show that $\tilde{\varphi}_I$ vanishes on V_r. Suppose $P = (p,q)$ with $p < q$ and $r \in \{p,q\}$. Then on U_P, $\tilde{\varphi}_I$ reduces to

$$\begin{aligned}
&\varphi_I - \nabla_\omega\left(h_p\tilde{\psi}^p + (1-h_p)h_q\tilde{\psi}^q\right) - \nabla_\omega\left((h_q\tilde{\psi}^P)dh_p\right) \\
&= (1-h_p)(1-h_q)\varphi_I - (1-h_q)dh_p \wedge \tilde{\psi}^p - (1-h_p)dh_q \wedge \tilde{\psi}^q \\
&\quad + \tilde{\psi}^P dh_p \wedge dh_q.
\end{aligned}$$

On $U_r \setminus \cup_{r\in P} U_P$, $\tilde{\varphi}_I$ reduces to

$$\varphi_I - \nabla_\omega(h_r\tilde{\psi}^r) = (1-h_r)\varphi_I - dh_r \wedge \tilde{\psi}^r.$$

Note that dh_r and $(1-h_r)$ vanish on V_r. □

Proof of Theorem for $k = 2$. Lemma 5.4 asserts that $\tilde{\varphi}_I$ represents $\iota^2_\omega(\varphi_I) \in H^2_c(X, \nabla_\omega)$. Note that $\tilde{\varphi}_I \wedge \varphi_J$ vanishes on $(\cup_{i=0}^{n+1} D_p)^c$ for $\varphi_J \in \mathcal{F}^2(X)$. By the expressions of $\tilde{\varphi}_I$ on U_P and on U_r in the proof of Lemma 5.4, we have

$$\begin{aligned}
\langle \varphi_I, \varphi_J \rangle_\omega &= \int_X \iota^2_\omega(\varphi_I) \wedge \varphi_J \\
&= \sum_{P=(p,q)} \int_{D_P} \tilde{\psi}^P dh_p \wedge dh_q \wedge \varphi_J.
\end{aligned}$$

Express D_P and φ_J as

$$D_P = \{(z_1, z_2) \mid \varepsilon_1 \leq |z_1|, |z_2| \leq \varepsilon_2\}, \quad \varphi_J = \varphi_J(z_1, z_2)dz_1 \wedge dz_2$$

in terms of the local coordinates (z_1, z_2) around L_P, and use the Stokes theorem and the residue theorem, then

$$\begin{aligned}
&\int_{D_P} \tilde{\psi}^P dh_p \wedge dh_q \wedge \varphi_J = \int_{\partial D_P} h_p\tilde{\psi}^P dh_q \wedge \varphi_J \\
&= -\int_{|z_1|=\varepsilon_1, \varepsilon_1 \leq |z_2| \leq \varepsilon_2} \tilde{\psi}^P dh_q(z_2) \wedge \varphi_J(z_1, z_2)dz_1 \wedge dz_2 \\
&= -2\pi\sqrt{-1} \int_{\varepsilon_1 \leq |z_2| \leq \varepsilon_2} dh_q(z_2) \wedge \left(\mathrm{Res}_{z_1=0}\tilde{\psi}^P \varphi_J(z_1, z_2)\right) dz_2
\end{aligned}$$

$$= 2\pi\sqrt{-1}\int_{|z_2|=\varepsilon_1}\left(\mathrm{Res}_{z_1=0}\tilde{\psi}_P^P\varphi_J(z_1,z_2)\right)dz_2$$
$$= (2\pi\sqrt{-1})^2\mathrm{Res}_{z_2=0}\left(\mathrm{Res}_{z_1=0}\tilde{\psi}_P^P\varphi_J(z_1,z_2)\right).$$

By Lemma 5.2, Lemma 5.3 and

$$\lim_{(z_1,z_2)\to(0,0)} z_1z_2\varphi_J(z_1,z_2)=\delta(P;J),$$

we have

$$\mathrm{Res}_{z_2=0}\left(\mathrm{Res}_{z_1=0}\tilde{\psi}_P^P\varphi_J(z_1,z_2)\right)=\frac{\delta(P;I)\delta(P;J)}{\alpha_p\alpha_q},$$

which yields Theorem for $k=2$. □

6. Proof of Theorem for general k

The key point of our proof of Theorem is to find the image $\iota_\omega^k(\varphi_I)$ explicitly. We need systems of $(k-m)$-forms $\tilde{\psi}^{P_m}$ on U_{P_m} such that

$$\nabla_\omega\tilde{\psi}^{P_1} = \varphi_I,$$
$$\nabla_\omega\tilde{\psi}^{P_m} = \sum_{P_{m-1}\subset P_m}\delta(P_{m-1};P_m)\tilde{\psi}_P^{P_{m-1}},\quad 2\le m\le k,$$

where P_m's are multi-indices of cardinality m. To construct such systems we prepare some lemmas. Once $\iota_\omega^k(\varphi_I)$ is found, we have only to apply the Stokes theorem and the residue theorem repeatedly.

For a fixed multi-index $P=(p_1,\dots,p_k)$, let $(z_1,\dots,z_k)$ be coordinates around L_P such that each L_{p_μ} is expressed by $z_\mu=0$.

Lemma 6.1. *For $\varphi_I\in\mathcal{F}^k(X)$ and a multi-index P of cardinality k, if $\alpha_p\notin\mathbf{Z}_{\le 0}$, $p\in P$, there uniquely exists a system of $(k-m)$-forms $\psi_P^{P_m}$ on $U_P\cap X$ with multi-index $P_m\subset P$ of cardinality m such that*

$$\nabla_\omega\psi_P^{P_1} = \varphi_I,$$
$$\nabla_\omega\psi_P^{P_m} = \sum_{P_{m-1}\subset P_m}\delta(P_{m-1};P_m)\psi_P^{P_{m-1}},\quad 2\le m\le k,$$
$$\psi_P^{P_m} = f_P^{P_m}\bigwedge_{1\le\nu\le k}^{p_\nu\notin P_m}\frac{dz_\nu}{z_\nu},$$

where $f_P^{P_m}$ is holomorphic on U_P satisfying

$$f_P^{P_m}(L_P) = (-1)^{m(m+1)/2}\delta(P;I)\prod_{1\leq\lambda\leq k}^{p_\lambda\in P_m}\frac{(-1)^\lambda}{\alpha_{p_\lambda}}.$$

Moreover, we can extend the form $\psi_P^{P_m}$ on $U_{P_m}\setminus\cup_{p\notin P_m}L_p$ holomorphically.

Proof. Follow the arguments in the proofs of Lemma 5.1 and Lemma 5.2. □

Let $P\langle P_m\rangle$ be the smallest multi-index of cardinality k including P_m in the lexicographic order. Put

$$\tilde\psi^p = \psi^p_{P\langle p\rangle}.$$

Since we have

$$\nabla_\omega(\psi^p_{S'} - \psi^p_S) = \varphi_I - \varphi_I = 0$$

for multi-indices S and S' including p, the argument in the proof of Lemma 5.3 implies that there uniquely exists holomorphic $(k-2)$-form ${}_{S'}\psi^p_S$ on $U_p\cap X$ vanishing along $L_p\setminus\cup_{q\neq p}L_q$ such that the expression of ${}_{S'}\psi^p_S$ in terms of the local coordinates $(z_1,\dots,z_k)$ around L_S consists of terms not including dz_ν, where ν is determined by $s_\nu = p$, and that

$$\nabla_\omega\, {}_{S'}\psi^p_S = \psi^p_{S'} - \psi^p_S.$$

For a multi-index S of cardinality k including p and q, we have

$$\begin{aligned}\tilde\psi^p - \tilde\psi^q &= (\psi^p_{P\langle p\rangle} - \psi^p_S) - (\psi^q_{P\langle q\rangle} - \psi^q_S) + (\psi^p_S - \psi^q_S)\\ &= \nabla_\omega({}_{P\langle p\rangle}\psi^p_S - {}_{P\langle q\rangle}\psi^q_S + \psi^{pq}_S).\end{aligned}$$

Put

$$\begin{aligned}\tilde\psi^{pq}_S &= {}_{P\langle p\rangle}\psi^p_S - {}_{P\langle q\rangle}\psi^q_S + \psi^{pq}_S,\\ \tilde\psi^{pq} &= \tilde\psi^{pq}_{P\langle pq\rangle}.\end{aligned}$$

Note that

$$\nabla_\omega(\tilde\psi^{pq}_{S'} - \tilde\psi^{pq}_S) = (\tilde\psi^p - \tilde\psi^q) - (\tilde\psi^p - \tilde\psi^q) = 0.$$

for multi-indices S and S' of cardinality k including p and q.

Lemma 6.2. *There uniquely exists holomorphic $(k-3)$-form ${}_S\tilde\psi^{pq}_{S'}$ on $U_{pq}\cap X$ satisfying the following conditions*:

(i) *the expression of* ${}_{S'}\tilde{\psi}^{pq}_{S}$ *in terms of the local coordinates* $(z_1, \dots, z_k)$ *around* L_S *consists of terms not including* dz_ν, *where* $s_\nu \in \{p, q\}$,

(ii) *it decomposes into a sum of a holomorphic form on* $U_{pq} \setminus \cup_{r\neq p} L_r$ *vanishing along* L_p *and that on* $U_{pq} \setminus \cup_{r \neq q} L_r$ *vanishing along* L_q,

(iii) $\nabla_{\omega\, S'}\tilde{\psi}^{pq}_{S} = \tilde{\psi}^{pq}_{S'} - \tilde{\psi}^{pq}_{S}$.

Proof. It is sufficient to show the existence and uniqueness of ${}_{S}\tilde{\psi}^{pq}_{S'}$ satisfying the condition (ii) and (iii) for $k = 3$, $(p, q) = (1, 2)$ and $S = (1, 2, 3)$, $S' = (0, 1, 2)$. By the expression of $\tilde{\psi}^{12}_{S}$ and $\tilde{\psi}^{12}_{S'}$, in terms of the local coordinate (z_1, z_2, z_3) around L_S, $\tilde{\psi}^{pq}_{S'} - \tilde{\psi}^{pq}_{S}$ is expressed as

$$(f^1_1 + f^2_1)dz_1 + (f^1_2 + f^2_2)dz_2 + f_3 dz_3,$$

where f^i_j is holomorphic in the variable z_i and f^1_2 and f^2_1 vanish along $z_1 = 0$ and $z_2 = 0$, respectively. Since f^1_1 and f^2_2 are holomorphic in the variable z_1 and z_2, respectively, there exist F_1 and F_2 vanishing along $z_1 = 0$ and $z_2 = 0$, respectively, such that

$$\frac{\partial F_1}{\partial z_1} + \omega_1 F_1 = f^1_1, \quad \frac{\partial F_2}{\partial z_2} + \omega_2 F_2 = f^1_2,$$

where $\omega = \omega_1 dz_1 + \omega_2 dz_2 + \omega_3 dz_3$. Since

$$\nabla_\omega(\tilde{\psi}^{12}_{S'} - \tilde{\psi}^{12}_{S}) = 0, \quad \nabla_\omega \circ \nabla_\omega (F_1 + F_2) = 0,$$

we have

$$\left(\frac{\partial}{z_2} + \omega_2\right)\left\{f^2_1 - \left(\frac{\partial F_2}{\partial z_1} + F_2\omega_2\right)\right\} = \left(\frac{\partial}{z_1} + \omega_1\right)\left\{f^1_2 - \left(\frac{\partial F_1}{\partial z_2} + F_1\omega_1\right)\right\}.$$

Since the left hand side of the above is holomorphic in z_2 and the right hand side of the above is holomorphic in z_1, both sides are holomorphic in z_1 and z_2, which implies that

$$f^2_1 - \left(\frac{\partial F_2}{\partial z_1} + F_2\omega_2\right), \quad f^1_2 - \left(\frac{\partial F_1}{\partial z_2} + F_1\omega_1\right)$$

are holomorphic in z_1 and z_2. By following the argument in the proof of Lemma 5.2, there exists function F vanishing along $z_1 = 0$ and $z_2 = 0$ such that

$$\left(\frac{\partial}{z_1} + \omega_1\right) F = f^2_1 - \left(\frac{\partial F_2}{\partial z_1} + F_2\omega_2\right), \quad \left(\frac{\partial}{z_2} + \omega_2\right) F = f^1_2 - \left(\frac{\partial F_1}{\partial z_2} + F_1\omega_1\right).$$

Put

$${}_{S}\tilde{\psi}^{pq}_{S'} = (F + F_1) + F_2,$$

then the coefficients of dz_1 and dz_2 of $\nabla_{\omega S}\tilde{\psi}^{pq}_{S'}$ coincides with those of $\tilde{\psi}^{pq}_{S'} - \tilde{\psi}^{pq}_{S}$. The argument in the proof of Lemma 5.2 implies the coincidence of the coefficients of dz_3 and the uniqueness of ${}_S\tilde{\psi}^{pq}_{S'}$. □

For a multi-index S of cardinality k including p, q and r, we have

$$\begin{aligned}
&\tilde{\psi}^{pq} - \tilde{\psi}^{pr} + \tilde{\psi}^{qr} \\
&= (\tilde{\psi}^{pq}_{P\langle pq\rangle} - \tilde{\psi}^{pq}_{S}) - (\tilde{\psi}^{pr}_{P\langle pr\rangle} - \tilde{\psi}^{pr}_{S}) + (\tilde{\psi}^{qr}_{P\langle qr\rangle} - \tilde{\psi}^{qr}_{S}) + (\tilde{\psi}^{pq}_{S} - \tilde{\psi}^{pr}_{S} + \tilde{\psi}^{qr}_{S}) \\
&= \nabla_\omega({}_{P\langle pq\rangle}\tilde{\psi}^{pq}_{S} - {}_{P\langle pr\rangle}\tilde{\psi}^{pr}_{S} + {}_{P\langle qr\rangle}\tilde{\psi}^{qr}_{S}) + ({}_{P\langle p\rangle}\psi^{p}_{S} - {}_{P\langle q\rangle}\psi^{q}_{S} + \psi^{pq}_{S}) \\
&\quad -({}_{P\langle p\rangle}\psi^{p}_{S} - {}_{P\langle r\rangle}\psi^{r}_{S} + \psi^{pr}_{S}) + ({}_{P\langle q\rangle}\psi^{q}_{S} - {}_{P\langle r\rangle}\psi^{r}_{S} + \psi^{qr}_{S}) \\
&= \nabla_\omega({}_{P\langle pq\rangle}\tilde{\psi}^{pq}_{S} - {}_{P\langle pr\rangle}\tilde{\psi}^{pr}_{S} + {}_{P\langle qr\rangle}\tilde{\psi}^{qr}_{S} + \psi^{pqr}_{S}).
\end{aligned}$$

Put

$$\begin{aligned}
\tilde{\psi}^{pqr}_{S} &= {}_{P\langle pq\rangle}\tilde{\psi}^{pq}_{S} - {}_{P\langle pr\rangle}\tilde{\psi}^{pr}_{S} + {}_{P\langle qr\rangle}\tilde{\psi}^{qr}_{S} + \psi^{pqr}_{S} \\
\tilde{\psi}^{pqr} &= \tilde{\psi}^{pqr}_{P\langle pqr\rangle}.
\end{aligned}$$

In general, for multi-indices P_m of cardinality m and S of cardinality k including P_m, we define $(k-m)$-forms on U_{P_m}

$$\begin{aligned}
\tilde{\psi}^{P_m}_{S} &= \sum_{P_{m-1}\subset P_m} \delta(P_{m-1}; P_m)\, {}_{P\langle P_{m-1}\rangle}\tilde{\psi}^{P_m}_{S} + \psi^{P_m}_{S}, \\
\tilde{\psi}^{P_m} &= \tilde{\psi}^{P_m}_{P\langle P_m\rangle}.
\end{aligned}$$

For multi-indices P_m of cardinality m and S, S' of cardinality k including P_m, there uniquely exists holomorphic $(k-m-1)$-form ${}_S\tilde{\psi}^{P_m}_{S'}$ on $U_{P_m} \cap X$ satisfying the following conditions:

(i) the expression of ${}_{S'}\tilde{\psi}^{P_m}_{S}$ in terms of the local coordinates $(z_1, \dots, z_k)$ around L_S consists of terms not including dz_ν, where $s_\nu \in P_m$,

(ii) it decomposes into a sum of holomorphic forms on $U_{P_m} \setminus \cup_{r\neq p_\mu} L_r$ vanishing along L_{p_μ} for certain $p_\mu \in P_m$,

(iii) $\nabla_{\omega S'}\tilde{\psi}^{P_m}_{S} = \tilde{\psi}^{P_m}_{S'} - \tilde{\psi}^{P_m}_{S}$.

Thus we have the following lemma.

Lemma 6.3. *The forms $\tilde{\psi}^{P_m}$ satisfy*

$$\begin{aligned}
\nabla_\omega \tilde{\psi}^{P_1} &= \varphi_I, \\
\nabla_\omega \tilde{\psi}^{P_m} &= \sum_{P_{m-1}\subset P_m} \delta(P_{m-1}; P_m)\tilde{\psi}^{P_{m-1}}, \quad 2 \le m \le k,
\end{aligned}$$

on $U_{P_m} \setminus \cup_{p\in P_m} L_p$ for any multi-index P_m.

For multi-index $P = (p_1, \dots, p_k)$, by using $\tilde{\psi}^{P_m}$ and h_p ($P_m \subset P$, $p \in P$), we construct a smooth k-form such that it is cohomologous to φ_I and that it vanishes identically on V_P. We fix $P = (1, \dots, k)$ for simplicity. Put

$$g^{\mu,\nu} = \prod_{\lambda=\mu}^{\nu} (1 - h_\lambda) = (1 - h_\mu)(1 - h_{\mu+1}) \cdots (1 - h_\nu),$$

where $1 \le \mu \le \nu \le k$. Note that

$$\begin{aligned} & 1 - \sum_{\lambda=\mu}^{\nu} g^{\mu,\lambda-1} h_\lambda \\ &= 1 - h_\mu - (1 - h_\mu)h_{\mu+1} - \dots - (1 - h_\mu) \cdots (1 - h_{\nu-1})h_\nu = g^{\mu,\nu}, \end{aligned}$$

and that

$$dg^{\mu,\nu} = -\sum_{\lambda=\mu}^{\nu} g^{\mu,\lambda-1}\, dh_\lambda\, g^{\lambda+1,\nu},$$

where we regard $g^{\mu,\mu-1}$ as 1. For a multi-index $P_m = (\mu(1), \dots, \mu(m)) \subset P$, put

$$\begin{aligned} \eta^\lambda(P_m) &= g^{\mu(\lambda-1)+1,\mu(\lambda)-1} dh_{\mu(\lambda)}, \\ H(P_m) &= \eta^1(P_m) \wedge \dots \wedge \eta^{m-1}(P_m) \cdot g^{\mu(m-1)+1,\mu(m)-1} h_{\mu(m)}, \\ G(P_m) &= \eta^1(P_m) \wedge \dots \wedge \eta^{m-1}(P_m) \wedge \eta^m(P_m), \end{aligned}$$

where we regard $\mu(0)$ as 0. Note that $H(P_m)$ is $(m-1)$-form and $G(P_m)$ is m-form and that

$$G(P_m) = H(P_m) \wedge d\log h_{\mu(m)}.$$

Put

$$\Psi^m = \sum_{P_m \subset P} H(P_m) \wedge \tilde{\psi}^{P_m}.$$

Lemma 6.4.

$$\begin{aligned} \nabla_\omega \Psi^1 &= (1 - g^{1,k})\varphi_I + \sum_{P_2 \subset P} H(P_2) \wedge \nabla_\omega \tilde{\psi}^{P_2} \\ &\quad + \sum_{P_1 \subset P} G(P_1) g^{\mu(1)+1,k} \wedge \tilde{\psi}^{P_1}, \\ (-1)^{m-1} \nabla_\omega \Psi^m &= \sum_{P_m \subset P} H(P_m) \wedge \nabla_\omega \tilde{\psi}^{P_m} + \sum_{P_{m+1} \subset P} H(P_{m+1}) \wedge \nabla_\omega \tilde{\psi}^{P_{m+1}} \end{aligned}$$

$$+\sum_{P_m\subset P} G(P_m)g^{\mu(m)+1,k}\wedge\tilde{\psi}^{P_m},\quad 2\le m\le k-1,$$

$$(-1)^{k-1}\nabla_\omega\Psi^k = H(P_k)\wedge\nabla_\omega\tilde{\psi}^P + dh_1\wedge\ldots\wedge dh_k\cdot\tilde{\psi}^P,$$

where $P_m=(\mu(1),\ldots,\mu(m))\subset P=(1,\ldots,k)$. *The form*

$$\varphi_I-\sum_{m=1}^k\nabla_\omega\Psi^m = g^{1,k}\varphi+\sum_{m=1}^k\sum_{P_m\subset P}(-1)^m G(P_m)g^{\mu(m)+1,k}\wedge\tilde{\psi}^{P_m}$$

vanishes on $\cup_{p\in P}V_p$.

Proof. Since we have

$$\begin{aligned}\nabla_\omega\Psi^m &= \sum_{P_m\subset P}\nabla_\omega\{H(P_m)\wedge\tilde{\psi}^{P_m}\}\\ &= \sum_{P_m\subset P}(dH(P_m)\wedge\tilde{\psi}^{P_m}+(-1)^{m-1}H(P_m)\wedge\nabla_\omega\tilde{\psi}^{P_m}),\end{aligned}$$

we show that

$$\begin{aligned}\sum_{P_m\subset P} dH(P_m)\wedge\tilde{\psi}^{P_m} &= (-1)^{m+1}\sum_{P_{m+1}\subset P}H(P_{m+1})\wedge\nabla_\omega\tilde{\psi}^{P_{m+1}}\\ &\quad+(-1)^{m-1}\sum_{P_m\subset P}G(P_m)g^{\mu(m)+1,k}\wedge\tilde{\psi}^{P_m}.\end{aligned}$$

Since

$$\begin{aligned}dg^{\mu(\lambda-1)+1,\mu(\lambda)-1} &= -\sum_{\nu=\mu(\lambda-1)+1}^{\mu(\lambda)-1} g^{\mu(\lambda-1)+1,\nu-1}\,dh_\nu\, g^{\nu+1,\mu(\lambda)-1}\\ &= -\sum_{\nu=\mu(\lambda-1)+1}^{\mu(\lambda)-1}\eta^\lambda(P_m^\lambda(\nu))\cdot g^{\nu+1,\mu(\lambda)-1},\end{aligned}$$

where $P_m^\lambda(\nu)$ is the multi-index of cardinality $m+1$ as

$$P_m^\lambda(\nu)=(\mu(1),\ldots,\mu(\lambda-1),\nu,\mu(\lambda),\ldots,\mu(m)),\quad \mu(\lambda-1)<\nu<\mu(\lambda),$$

we have

$$\begin{aligned}&dH(P_m)\\ &= -\sum_{\nu=1}^{\mu(1)-1}\eta^1(P_m^1(\nu))\wedge\ldots\wedge\eta^m(P_m^1(\nu))\cdot g^{\mu(m-1)+1,\mu(m)-1}h_{\mu(m)}\end{aligned}$$

$$+(-1)^2 \sum_{\nu=\mu(1)+1}^{\mu(2)-1} \eta^1(P_m^2(\nu)) \wedge \ldots \wedge \eta^m(P_m^2(\nu)) \cdot g^{\mu(m-1)+1,\mu(m)-1} h_{\mu(m)}$$

$$\vdots$$

$$+(-1)^m \sum_{\nu=\mu(m-1)+1}^{\mu(m)-1} \eta^1(P_m^m(\nu)) \wedge \ldots \wedge \eta^m(P_m^m(\nu)) \cdot g^{\nu+1,\mu(m)-1} h_{\mu(m)}$$

$$+(-1)^{m-1} G(P_m)$$

$$= \sum_{\lambda=1}^{m} \sum_{\nu=\mu(\lambda-1)+1}^{\mu(\lambda)-1} (-1)^\lambda H(P_m^\lambda(\nu))$$

$$+(-1)^{m+1} \sum_{\nu=\mu(m)+1}^{k} G(P_m) g^{\mu(m)+1,\nu-1} h_\nu$$

$$+(-1)^{m-1} G(P_m) - (-1)^{m+1} \sum_{\nu=\mu(m)+1}^{k} G(P_m) g^{\mu(m)+1,\nu-1} h_\nu$$

$$= \sum_{\lambda=1}^{m+1} \sum_{\nu=\mu(\lambda-1)+1}^{\mu(\lambda)-1} (-1)^\lambda H(P_m^\lambda(\nu)) + (-1)^{m-1} G(P_m) g^{\mu(m)+1,k},$$

where we regard $\mu(m+1)$ as $k+1$. Thus

$$\sum_{P_m \subset P} dH(P_m) \wedge \tilde{\psi}^{P_m}$$

$$= \sum_{P_m \subset P} \Big\{ \sum_{\lambda=1}^{m+1} \sum_{\nu=\mu(\lambda-1)+1}^{\mu(\lambda)-1} (-1)^\lambda H(P_m^\lambda(\nu)) \wedge \tilde{\psi}^{P_m}$$

$$+(-1)^{m-1} G(P_m) g^{\mu(m)+1,k} \wedge \tilde{\psi}^{P_m} \Big\}$$

$$= (-1)^{m+1} \sum_{P_{m+1} \subset P} H(P_{m+1}) \wedge \nabla_\omega \tilde{\psi}^{P_{m+1}}$$

$$+(-1)^{m-1} \sum_{P_m \subset P} G(P_m) g^{\mu(m)+1,k} \wedge \tilde{\psi}^{P_m}.$$

Note that

$$\sum_{P_1 \subset P} H(P_1) \cdot \nabla_\omega \tilde{\psi}^{P_1} = \varphi_I - \left(1 - \sum_{\lambda=1}^{k} g^{1,\lambda-1} h_\lambda\right) \varphi_I$$

$$= (1 - g^{1,k}) \varphi_I.$$

for $m = 1$. Since each term of

$$\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi^m$$

includes either $(1-h_p)$ or dh_p for any index $p \in P$, it vanishes on $\cup_{p\in P} V_p$. □

For a general multi-index $P = (p_1, \ldots, p_k)$, put

$$\Psi_P^m = \sum_{P_m \subset P} H_P(P_m) \wedge \tilde{\psi}^{P_m},$$

where

$$\begin{aligned} g_P^{\mu,\nu} &= \prod_{\lambda=\mu}^{\nu} (1-h_{p_\lambda}), \quad 1 \le \mu \le \nu \le k, \\ \eta_P^\lambda(P_m) &= g_P^{\mu(\lambda-1)+1,\mu(\lambda)-1} dh_{p_{\mu(\lambda)}}, \\ H_P(P_m) &= \eta_P^1(P_m) \wedge \ldots \wedge \eta_P^{m-1}(P_m) \cdot g_P^{\mu(m-1)+1,\mu(m)-1} h_{p_{\mu(m)}}, \\ G_P(P_m) &= \eta_P^1(P_m) \wedge \ldots \wedge \eta_P^{m-1}(P_m) \wedge \eta_P^m(P_m). \end{aligned}$$

By an argument similar to the one in the proof of Lemma 6.4, we can show that

$$\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi_P^m = g_P^{1,k} \varphi + \sum_{m=1}^{k} \sum_{P_m \subset P} (-1)^m G_P(P_m) g_P^{\mu(m)+1,k} \wedge \tilde{\psi}^{P_m}$$

and that it vanishes on $\cup_{p\in P} V_p$. Moreover, Lemma 6.3 implies that the form $\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi_P^m$ coincides with $\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi_Q^m$ on $U_{P\cap Q}$ for any multi-indices P and Q of cardinality k. Hence we have a smooth k-form $\tilde{\varphi}_I$ such that it is cohomologous to φ_I and that it coincides with the forms $\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi_P^m$, $P_m \subset P$ on U_P for any P.

Proof of Theorem for general k. It is clear that $\tilde{\varphi}_I$ represents $\iota_\omega^k(\varphi_I) \in H_c^k(X, \nabla_\omega)$. By the expression of $\varphi_I - \sum_{m=1}^{k} \nabla_\omega \Psi_P^m$, we have

$$\begin{aligned} \langle \varphi_I, \varphi_J \rangle_\omega &= \int_X \iota_\omega^k(\varphi_I) \wedge \varphi_J \\ &= \sum_P \int_{D_P} \tilde{\psi}^P dh_{p_1} \wedge \ldots \wedge dh_{p_k} \wedge \varphi_J, \end{aligned}$$

where $P = (p_1, \ldots, p_k)$. Express D_P and φ_J as

$$\begin{aligned} D_P &= \{(z_1, \ldots, z_k) \mid \varepsilon_1 \le |z_1|, \ldots, |z_k| \le \varepsilon_2\}, \\ \varphi_J &= \varphi_J(z_1, \ldots, z_k) dz_1 \wedge \ldots \wedge dz_k, \end{aligned}$$

in terms of the local coordinates $(z_1, \dots, z_k)$ around L_P, and use the Stokes theorem and the residue theorem repeatedly, then

$$\int_{D_P} \tilde{\psi}^P dh_{p_1} \wedge \dots \wedge dh_{p_k} \wedge \varphi_J$$
$$= (2\pi\sqrt{-1})^k \mathrm{Res}_{z_k=0}\left(\mathrm{Res}_{z_{k-1}=0}\left(\cdots\left(\mathrm{Res}_{z_1=0}\tilde{\psi}^P_P \varphi_J(z_1, \dots, z_k)\right)\right)\right).$$

Note that

$$\lim_{(z_1,\dots,z_k)\to(0,\dots,0)} z_1 \cdots z_k \varphi_J(z_1, \dots, z_k) = \delta(P; J).$$

Since

$$\tilde{\psi}^P = \psi^P_P + \sum_{P_{k-1}\subset P} \delta(P_{k-1}; P)_{P\langle P_{k-1}\rangle}\tilde{\psi}^P_P,$$

and $_{P\langle P_{k-1}\rangle}\tilde{\psi}^P_P$ is a sum of holomorphic functions on $U_P \setminus \cup_{r\neq p_\mu} L_r$ vanishing along L_{p_μ} for certain $p_\mu \in P$, Lemma 6.1 implies

$$\mathrm{Res}_{z_k=0}\left(\mathrm{Res}_{z_{k-1}=0}\left(\cdots\left(\mathrm{Res}_{z_1=0}\tilde{\psi}^P_P \varphi_J(z_1, \dots, z_k)\right)\right)\right) = \frac{\delta(P; I)\delta(P; J)}{\alpha_{p_1} \cdots \alpha_{p_k}},$$

which completes our proof of Theorem. □

References

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Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima
739-8526, Japan