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ON THE UNIQUENESS OF MARKOVIAN SELF-ADJOINT EXTENSION OF DIFFUSION OPERATORS ON INFINITE DIMENSIONAL SPACES

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1. Introduction

Let $(\mathcal{S}(R^d), L^2(R^d), \mathcal{S}'(R^d))$ be a rigged Hilbert space, where $\mathcal{S}(R^d)$ is the Schwartz space of test functions and $\mathcal{S}'(R^d)$ is its dual space. Letting $\{e_i\}_{i=1}^\infty \subset \mathcal{S}(R^d)$ be a complete orthonormal basis of $L_2(R^d)$, we put $FC_0^\infty = \{f; f \text{ is a function on } \mathcal{S}'(R^d) \text{ of the form } f(\xi) = \tilde{f}(\langle \xi, e_{i_1} \rangle, \dots, \langle \xi, e_{i_n} \rangle) \text{ for some } n \text{ and a real } C_0^\infty(R^n)\text{-function } \tilde{f}\}$, where $\langle \cdot, \cdot \rangle$ is the dualization between $\mathcal{S}'(R^d)$ and $\mathcal{S}(R^d)$. Let ν be a quasi-invariant measure on $\mathcal{S}'(R^d)$ with respect to $\mathcal{S}(R^d)$. We call the measure ν *admissible* if the symmetric bilinear form $\mathcal{E}_\nu(u, v) = \frac{1}{2} (Du, Dv)_{L^2(R^d) \otimes L^2(\nu)}$, $u, v \in FC_0^\infty$, is closable. Its closed extension $(\mathcal{E}_\nu, \mathcal{F}_\nu)$ is said to be the *energy form* associated with the quasi-invariant admissible measure ν . Here, $Du = \sum_{i=1}^\infty e_i \otimes D_i u \in L^2(R^d) \otimes L^2(\nu)$ and D_i is a derivative in the direction of e_i . Furthermore, a self-adjoint operator H_ν representing the energy form $(\mathcal{E}_\nu, \mathcal{F}_\nu)$ is said to be a *diffusion operator*. For example, the probability measure μ_0 on $\mathcal{S}'(R^d)$ defined by the following formula is quasi-invariant and admissible:

$$\int_{\mathcal{S}'(R^d)} e^{i\langle \xi, \phi \rangle} d\mu_0(\xi) = e^{-1/4(\phi, (\Delta + m^2)^{-1/2} \phi)}, \quad \phi \in \mathcal{S}(R^d),$$

where (\cdot, \cdot) is the scalar product in $L^2(R^d)$.

Let μ_0^* be the Euclidian random field $\langle \xi^*, \psi \rangle$ over R^{d+1} , defined by

$$\int_{\mathcal{S}'(R^{d+1})} e^{i\langle \xi^*, \psi \rangle} d\mu_0^*(\xi^*) = e^{-1/2(\psi, (\Delta + m^2)^{-1} \psi)}, \quad \psi \in \mathcal{S}(R^{d+1}).$$

The random field $\langle \xi^*, \psi \rangle$ can be regarded as the restriction to $\mathcal{S}(R^{d+1})$ of the generalized random field indexed by the Sobolev space H_{-1} the completion of $\mathcal{S}(R^{d+1})$ with respect to the norm $\|(-\Delta + m^2)^{-1/2} \psi\|$. We denote by Σ_0 the σ -field generated by random variable $\{\langle \xi^*, \delta_0 \otimes \phi \rangle; \phi \in \mathcal{S}(R^d)\}$, and regard the restriction of μ_0^* to Σ_0 as the measure on $\mathcal{S}'(R^d)$ by the natural identifica-

tion of $\langle \xi, \phi \rangle$ with $\langle \xi^*, \delta_0 \otimes \phi \rangle$. Then, it coincides with μ_0 and the diffusion operator H_{μ_0} corresponding to μ_0 is nothing but the energy operator H of free Euclidean field model μ_0^* . To see this, it is enough to show that H_{μ_0} and H are the same operators on FC_0^∞ and that the symmetric operator $S = H_{\mu_0} \upharpoonright FC_0^\infty$ has a unique Markovian self-adjoint extension, where the notation $H_{\mu_0} \upharpoonright FC_0^\infty$ indicates the restriction of H_{μ_0} to FC_0^∞ . In fact, the operator S is known to be an essentially self-adjoint operator.

Albeverio and Høegh-Krohn have raised a question in [3] whether the diffusion operator associated with $\mu^* \upharpoonright \Sigma_0$ is identical with energy operator of the Euclidean field model μ^* with trigonometric (or exponential) interaction and have shown that these operators are the same on FC_0^∞ when $d=1$. Thus, we now cope with the question: what kind of quasi-invariant admissible measure ν induces the symmetric operator $S_\nu = H_\nu \upharpoonright FC_0^\infty$ with a unique Markovian self-adjoint extension?

In this paper, we consider this problem in a simpler case that ν is an absolutely continuous measure with respect to the Wiener measure on the abstract Wiener space (H, B, μ) . We conclude the uniqueness of Markovian self-adjoint extension of S_ν under the condition that the Radon-Nikodym derivative ρ^2 is strictly positive and belongs to the space $D_\infty (= \bigcap_{p \geq 1, r \in \mathbb{R}^1} D_p^r)$, where D_p^r is the Sobolev space of order r and degree p on the Wiener space. In the proof, we use the Malliavin's calculus and in particular the hypoellipticity of the Ornstein-Uhlenbeck generator. We note at the end of this paper that, if ρ happens to be a tame function, then Wielens' method [8] applies and S_ν becomes essentially self-adjoint.

2. Notations and the closability of a symmetric form

Let (H, B, μ) be an abstract Wiener space and $\{e_i\}_{i=1}^\infty \subset B^*$ (dual space of B) be a complete orthonormal basis of H . We set $FC_0^\infty = \{f; f \text{ is a function on } B \text{ of the form } f(x) = \tilde{f}(\langle e_{i_1}, x \rangle, \dots, \langle e_{i_n}, x \rangle) \text{ for some } n \text{ and } \tilde{f} \in C_0^\infty(\mathbb{R}^n)\}$ and $FC_0^\infty(H) = \{F; F \text{ is a } H\text{-valued function on } B \text{ which is of form } F(x) = \sum_{i=1}^n e_i \otimes f_i(x) \text{ for some } n, f_i \in FC_0^\infty\}$. We denote by D_p^r the completion of FC_0^∞ with respect to the norm $\|f\|_p^r = \|f\|_p + \|D^r f\|_p$, where $D^r f, f \in FC_0^\infty$, is the r -times iteration of the Fréchet derivative which is an element in $L_p(B \rightarrow \overbrace{H \otimes \cdots \otimes H}^r)$. Note that

$$\|D^r f\|_p = \|\{\sum_{(n_1, \dots, n_r) \in N^r} (D_{n_1}(D_{n_2} \cdots (D_{n_r} f)))^2\}^{1/2}\|_p$$

where D_i is the derivative in the direction of e_i . It is convenient to use two different expressions of D_p^1 ($1 < p < \infty$) according to Sugita [7] and Kusuoka [6]:

$$(2.1) \quad D_p^1 = \left\{ \begin{array}{l} \text{There exists some } g \in L_p(B \rightarrow H) \\ u \in L_p(\mu); \text{ such that } (u, D^*v) = (g, v), \text{ for} \\ \text{any } v \in FC_0^\infty(H) \end{array} \right\}$$

and

$$(2.2) \quad D_p^1 = \left\{ \begin{array}{l} u \text{ is stochastic } H \text{ Gateaux differentiable} \\ \text{(SGD) with respect to } \mu, \text{ ray absolutely} \\ u \in L_p(\mu); \text{ continuous (RAC) and the stochastic} \\ \text{Gateaux derivative } Du \text{ of } u \text{ satisfies that} \\ \|Du(x)\|_H \in L_p(\mu) \end{array} \right\}$$

Here, a function u is called *SGD*, if there exists a measurable map $Du; B \rightarrow H$ such that for any $k \in B^*$, the convergence $\frac{1}{t}[u(x+tk) - u(x) - t(Du(x), k)_H] \rightarrow 0$, $t \rightarrow 0$, take place in probability with respect to μ , and u is called *RAC*, if for any $k \in B^*$, there exists a measurable function u_k such that

- 1) $\tilde{u}_k(x) = u(x)$ for μ -a.e.
- 2) $\tilde{u}_k(x+tk)$ is absolutely continuous in t for each $x \in B$ (See [6; Definition 1,1 and Definition 1,2]). Then, we have for $u \in D_p^1$, $\|Du(x)\|_H = \sqrt{\sum_{i=1}^{\infty} (Du(x), e_i)^2} = \sqrt{\sum_{i=1}^{\infty} (D_i u(x))^2}$, where $D_i u(x) = \lim_{t \downarrow 0} \frac{1}{t} (\tilde{u}_{e_i}(x+te_i) - \tilde{u}_{e_i}(x))$.

We fix a function ρ on B satisfying

$$(2.3) \quad \text{i) } \rho > 0 \quad \text{ii) } \rho \in D_\infty$$

where $D_\infty = \bigcap_{\substack{p \geq 1 \\ r \in \mathbb{R}^1}} D_p^r$. We define the symmetric bilinear form $(\mathcal{E}_\rho, FC_0^\infty)$ by

$$(2.4) \quad \mathcal{E}_\rho(u, v) = \frac{1}{2} \int_B (Du(x), Dv(x))_H \rho^2(x) d\mu, \quad u, v \in FC_0^\infty.$$

Lemma 1. $(\mathcal{E}_\rho, FC_0^\infty)$ is closable on $L_2(\rho^2 \mu)$.

Proof. We follow the argument of [1; Theorem 2.3]. Since $D_i^* \rho^2 = -2\rho D_i \rho + \langle e_i, x \rangle \rho^2(x)$, we have for $g \in FC_0^\infty$,

$$\begin{aligned} (Dg, e_i \otimes 1)_{H \otimes L^2(\rho^2 \mu)} &= (D_i g, \rho^2)_{L^2(\mu)} \\ &= (g, -2\rho D_i \rho + \langle e_i, x \rangle \rho^2)_{L^2(\mu)} \\ &= (g, -2 \frac{D_i \rho}{\rho} + \langle e_i, x \rangle)_{L^2(\rho^2 \mu)}. \end{aligned}$$

By noting that $\int \left(\frac{D_i \rho}{\rho} \right)^2 \rho^2 d\mu \leq \|\rho\|_{\frac{1}{2}}^2 < \infty$, we see that $-2 \frac{D_i \rho}{\rho} + \langle e_i, x \rangle \in L_2(\rho^2 \mu)$

and $e_i \otimes 1 \in [D_p^*]$, where D_p^* denote the adjoint operator of D which is an operator from $L_2(\rho^2\mu)$ to $H \otimes L_2(\rho^2\mu)$. Put $\beta(e_i) = D_p^*(e_i \otimes 1)$. Then, we see that for $g, f \in FC_0^\infty$

$$\begin{aligned} (Dg, e_i \otimes f)_{H \otimes L_2(\rho^2\mu)} &= (D_i g \cdot f, 1)_{\rho^2\mu} \\ &= (D_i (g \cdot f) - g D_i f, 1)_{\rho^2\mu} \\ &= (g, \beta(e_i) f - D_i f)_{\rho^2\mu}. \end{aligned}$$

Because the function $\beta(e_i) f - D_i f$ belongs to $L_2(\rho^2\mu)$, it holds that $e_i \otimes f \in \mathcal{D}[D_p^*]$ and consequently $FC_0^\infty(H)$ is contained in $\mathcal{D}[D_p^*]$. Since $FC_0^\infty(H)$ is dense in $H \otimes L_2(\rho^2\mu)$, the closure $\bar{D} = (D_p^*)^*$ is well defined and hence $(\mathcal{E}_p, FC_0^\infty)$ is closable. q.e.d.

We denote by $(\mathcal{E}_p, \mathcal{F})$ the closed extension of $(\mathcal{E}_p, FC_0^\infty)$.

3. The uniqueness of Markovian self-adjoint extension

Let H_p be a self-adjoint operator associated with the closed form $(\mathcal{E}_p, \mathcal{F})$ and S be a symmetric operator defined by $S = H_p \upharpoonright FC_0^\infty$. S can be represented as

$$(3.1) \quad Su = \frac{1}{2} \mathcal{L}u + \frac{1}{\rho} \langle D\rho, Du \rangle_H, \quad u \in FC_0^\infty,$$

where \mathcal{L} is a Ornstein-Uhlenbeck generator. We denote by $\mathcal{A}_M(S)$ the totality of Markovian self-adjoint extensions: $A \in \mathcal{A}_M(S)$ means that A is a self-adjoint extension of S which generates a strongly continuous contraction Markovian semi-group on $L_2(\rho^2\mu)$. H_p is called the Friedrichs extension of S and is an element of $\mathcal{A}_M(S)$. Then, the following theorem holds.

Theorem 1. *Under the condition (2.3), $\mathcal{A}_M(S)$ has only one element H_p , namely, S has a unique Markovian self-adjoint extension.*

For any $A \in \mathcal{A}_M(S)$, the form domain $\mathcal{D}[\sqrt{-A}]$ is orthogonally decomposed with respect to $\mathcal{E}_{A,\alpha} (= (\sqrt{-A} \cdot, \sqrt{-A} \cdot)_{\rho^2\mu} + \alpha(\cdot, \cdot)_{\rho^2\mu})$ as

$$(3.2) \quad \mathcal{D}[\sqrt{-A}] = \mathcal{F} \oplus (\mathcal{N}_\alpha \cap \mathcal{D}[\sqrt{-A}]),$$

where $\mathcal{N}_\alpha = \{u \in L_2(\rho^2\mu); (\alpha I - S^*)u = 0\}$ ([4; Theorem 2.3.2]). Hence, for the proof of Theorem 1 we must show that

$$(3.3) \quad \mathcal{N}_\alpha \cap \mathcal{D}[\sqrt{-A}] \subset \mathcal{F}.$$

In order to prove (3.3) we introduce the intermediate space \mathcal{H} by (3.4) and prove that $\mathcal{N}_\alpha \cap \mathcal{D}[\sqrt{-A}] \subset \mathcal{H} \subset \mathcal{F}$ (Lemma 2 and Lemma 4).

Let $\{a_l(t)\}_{l=1}^\infty$ be a sequence of $C_0^\infty(R^1)$ -functions satisfying that

$$\begin{aligned} \text{i) } 0 \leq a_l(t) \leq 1 \quad \text{ii) } a_l(t) &= \begin{cases} 1 & \text{on } \frac{1}{2^l} < t < 2^l \\ 0 & \text{on } t \leq \frac{1}{2^{l+1}}, t \geq 2^{l+1} \end{cases} \\ \text{iii) } |a'_l(t)| &\leq \begin{cases} c 2^{l+1} & \text{on } t \leq \frac{1}{2^l} \\ c' & \text{otherwise} \end{cases} \text{ for some constants } c \text{ and } c'. \end{aligned}$$

We put $\phi_l(x) = a_l \circ \rho(x)$. If a function u satisfies $\phi_l \cdot u \in \bigcup_{1 < \rho < 2} D_\rho^1$, for any l , then we have $D(\phi_{l+1} \cdot u) = D(\phi_l \cdot u)$ μ -a.e. on $\mathcal{M}_l = \left\{ \frac{1}{2^l} < \rho < 2^l \right\}$, because $D(\phi_l \cdot u) = D(\phi_l \cdot \phi_{l+1} u) = \phi_l \cdot D(\phi_{l+1} \cdot u) + \phi_{l+1} u \cdot D\phi_l$. Therefore, we can well define Du by

$$Du = D(\phi_l \cdot u) \quad \text{on } \mathcal{M}_l.$$

Let we consider the function space

$$(3.4) \quad \mathcal{H} = \left\{ u \in L_2(\rho^2 \mu); \quad \begin{aligned} &\phi_l \cdot u \in \bigcup_{1 < \rho < 2} D_\rho^1 \text{ for any } l \text{ and} \\ &\int \langle Du, Du \rangle_H \rho^2 d\mu < \infty \end{aligned} \right\}$$

Then, we have the following lemma.

Lemma 2. *It holds that*

$$(3.5) \quad \mathcal{H} \subset \mathcal{F}.$$

Proof. For any $u \in \mathcal{H}$, we see that $u_{(N)} = (-N \vee u) \wedge N \in \mathcal{H}$ since $\phi_l \cdot u_{(N)} = ((-N \phi_l) \vee \phi_l u) \wedge N \phi_l \in D_\rho^1$ by (2.2), $u_{(N)}$ converges to u in $\mathcal{E}_{\rho,1}$. Furthermore $u_{(N)}$ can be approximated by $\phi_l \cdot u_{(N)} \in \mathcal{H}$. In fact, we have

$$(3.6) \quad \begin{aligned} &\int \|Du_{(N)} - D(\phi_l \cdot u_{(N)})\|_H^2 \rho^2 d\mu \\ &\leq 2 \left[\int |1 - \phi_l|^2 \|Du_{(N)}\|_H^2 \rho^2 d\mu + \int u_{(N)}^2 \|D\phi_l\|_H^2 \rho^2 d\mu \right]. \end{aligned}$$

The second term of the right hand side is equal to $\int u_{(N)}^2 (a'_l(\rho))^2 \|D\rho\|_H^2 \rho^2 d\mu$ and is not greater than $\int_{\{\rho \leq 1/2^l\}} u_{(N)}^2 \cdot (c2^{l+1})^2 \|D\rho\|_H^2 \rho^2 d\mu + \int_{\{\rho \geq 2^l\}} u_{(N)}^2 c'^2 \|D\rho\|_H^2 \rho^2 d\mu \leq 4c^2 N^2 \int_{\{\rho \leq 1/2^l\}} \|D\rho\|_H^2 d\mu + c'^2 N^2 \int_{\{\rho \geq 2^l\}} \|D\rho\|_H^2 \rho^2 d\mu$, which tends to zero as $l \rightarrow \infty$ by the assumption (2.3). Hence the left hand side of (3.6) tends to zero as $l \rightarrow \infty$.

Next we show that there exists a sequence $\{f_m\}_{m=1}^\infty \subset FC_0^\infty$ such that

$$(3.7) \quad \phi_{l+1} f_m \rightarrow \phi_l u_{(N)} \quad (m \rightarrow \infty) \text{ in } \mathcal{E}_{\rho,1}.$$

Since we see $\phi_l u_{(N)} \in D_{2,b}^1$ by

$$\begin{aligned} \int \|D(\phi_l u_{(N)})\|_H^2 d\mu &\leq 2 \left[\int \phi_l^2 \|Du_{(N)}\|_H^2 d\mu + \int u_{(N)}^2 \|D\phi_l\|_H^2 d\mu \right] \\ &\leq 2 \left[\int_{\{1/2^{l+1} < \rho < 2^{l+1}\}} \phi_l^2 \|Du_{(N)}\|_H^2 d\mu + N^2 \int \|D\phi_l\|_H^2 d\mu \right] \\ &\leq 2^{2l+3} \int \|Du_{(N)}\|_H^2 \rho^2 d\mu + 2N^2 \int \|D\phi_l\|_H^2 d\mu \\ &< \infty, \end{aligned}$$

there exists a sequence $\{f_m\}_{m=1}^\infty \subset FC_0^\infty$ such that 1) $|f_m| \leq N$, 2) $f_m \rightarrow \phi_l u_{(N)}$, μ -a.e., 3) $f_m \rightarrow \phi_l u_{(N)}$ in D_2^1 . Then, (3.7) follows because

$$\begin{aligned} \int \|D(\phi_{l+1} f_m) - D(\phi_l u_{(N)})\|_H^2 \rho^2 d\mu &= \int \|D(\phi_{l+1}(f_m - \phi_l u_{(N)}))\|_H^2 \rho^2 d\mu \\ &\leq 2 \left[\int \phi_{l+1}^2 \|Df_m - D(\phi_l u_{(N)})\|_H^2 \rho^2 d\mu + \int (f_m - \phi_l u_{(N)})^2 \|D\phi_{l+1}\|_H^2 \rho^2 d\mu \right] \\ &\leq 2^{2l+5} \int \|Df_m - D(\phi_l u_{(N)})\|_H^2 d\mu + 2 \int (f_m - \phi_l u_{(N)})^2 \|D\phi_{l+1}\|_H^2 \rho^2 d\mu \\ &\rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

Finally we take a sequence $\{g_n\}_{n=1}^\infty \subset FC_0^\infty$ satisfying that $g_n \rightarrow \phi_{l+1} f_m$ in D_4^1 . Then, we see that

$$(3.8) \quad g_n \rightarrow \phi_{l+1} f_m \quad \text{in } \mathcal{E}_{p,1},$$

since $\int \|D(\phi_{l+1} f_m) - Dg_n\|_H^2 \rho^2 d\mu \leq (\int \|D(\phi_{l+1} f_m) - Dg_n\|_H^4 d\mu)^{1/2} \cdot (\int \rho^4 d\mu)^{1/2}$. q.e.d.

Denote by $\bar{S}^{(p)}$, $1 < p$, the closure of S in $L_p(\rho^2 \mu)$. We need the following lemma in the proof of Lemma 4.

Lemma 3. *If $w \in D_p^2$, $p > 1$, then for any l , $\phi_l w \in \bigcap_{\substack{1 < p' < p \\ p' \leq 2}} \mathcal{D}[\bar{S}^{(p')}]$ and*

$$(3.9) \quad \bar{S}^{(p')}(\phi_l w) = \frac{1}{2} \mathcal{L}(\phi_l w) + \frac{1}{\rho} \langle D\rho, D(\phi_l w) \rangle_H.$$

Proof. First of all we show that $\phi_l \psi \in \mathcal{D}[\bar{S}^{(2)}]$, for $\psi \in FC_0^\infty$. Take a sequence $\{g_k\}_{k=1}^\infty \subset FC_0^\infty$ such that g_k converges to ϕ_l with respect to $\|\cdot\|_2^2$. Then, we obtain

$$\begin{aligned} (3.10) \quad S(g_k \psi) &= \frac{1}{2} \mathcal{L} g_k \psi + \frac{1}{2} g_k \mathcal{L} \psi + \frac{1}{2} \langle Dg_k, D\psi \rangle_H + \frac{g_k}{\rho} \langle D\rho, D\psi \rangle_H \\ &\quad + \frac{\psi}{\rho} \langle D\rho, Dg_k \rangle_H \\ &\xrightarrow{k \rightarrow \infty} \frac{1}{2} \mathcal{L} \phi_l \psi + \frac{1}{2} \phi_l \mathcal{L} \psi + \frac{1}{2} \langle D\phi_l, D\psi \rangle_H + \frac{\phi_l}{\rho} \langle D\rho, D\psi \rangle_H \\ &\quad + \frac{\psi}{\rho} \langle D\rho, D\phi_l \rangle_H \end{aligned}$$

$$= \frac{1}{2} \mathcal{L}(\phi \psi) + \frac{1}{\rho} \langle D\rho, D(\phi \psi) \rangle_H,$$

the convergence being in $L_2(\rho^2\mu)$. In fact, by Schwartz inequality we have

$$\int |\mathcal{L}g_k\psi - \mathcal{L}\phi\psi|^2 \rho^2 d\mu \leq \left(\int |\mathcal{L}g_k - \mathcal{L}\phi|^4 d\mu \right)^{1/2} \left(\int (\psi\rho)^4 d\mu \right)^{1/2} \rightarrow 0 \quad (k \rightarrow \infty),$$

and in the same way we can show the convergence of other terms of (3.10).

Next, if $\{h_m\}_{m=1}^\infty \subset FC_0^\infty$ converges to w with respect to $\|\cdot\|_p^2$, we have

$$\begin{aligned} (3.11) \quad \bar{S}^{(2)}(\phi, h_m) &= \frac{1}{2} \mathcal{L}\phi, h_m + \frac{1}{2} \phi, \mathcal{L}h_m + \frac{1}{2} \langle D\phi, Dh_m \rangle_H + \frac{1}{2} \phi, \mathcal{L}h_m \\ &\quad + \frac{1}{2} \langle D\phi, Dh_m \rangle_H \\ &\xrightarrow{m \rightarrow \infty} \frac{1}{2} \mathcal{L}\phi, w + \frac{1}{2} \phi, \mathcal{L}w + \frac{1}{2} \langle D\phi, Dw \rangle_H + \frac{w}{\rho} \langle D\rho, D\phi \rangle_H \\ &\quad + \frac{\phi}{\rho} \langle D\rho, Dw \rangle_H \\ &= \frac{1}{2} \mathcal{L}(\phi, w) + \frac{1}{\rho} \langle D\rho, D(\phi, w) \rangle_H, \end{aligned}$$

the convergence being in $L_{p'}(\rho^2\mu)$. In fact, by Hölder inequality, we get

$$\begin{aligned} \int |\mathcal{L}\phi, w - \mathcal{L}\phi, h_m|^{p'} \rho^2 d\mu &\leq \left(\int |w - h_m|^p d\mu \right)^{p'/p} \left(\int |\mathcal{L}\phi|^{p'} \rho^2 d\mu \right)^{p-p'/p} \\ &\rightarrow 0 \quad (m \rightarrow \infty), \end{aligned}$$

and the second and the third terms of (3.11) also converge to the corresponding terms in $L_{p'}(\rho^2\mu)$. Furthermore,

$$\begin{aligned} \int \left| \frac{w}{\rho} \langle D\rho, D\phi \rangle_H - \frac{h_m}{\rho} \langle D\rho, D\phi \rangle_H \right|^{p'} \rho^2 d\mu &= \int |w - h_m|^{p'} |\langle D\rho, D\phi \rangle_H|^{p'} \rho^{2-p'} d\mu \\ &\leq \left(\int |w - h_m|^p d\mu \right)^{p'/p} \left(\int |\langle D\rho, D\phi \rangle_H|^{p'} \rho^{2-p'} d\mu \right)^{p-p'/p} \\ &\rightarrow 0 \quad (m \rightarrow \infty), \end{aligned}$$

and the last term in (3.11) also tends to $\frac{\phi}{\rho} \langle D\rho, Dw \rangle_H$. q.e.d.

Take any element $A \in \mathcal{A}_M(S)$ and let $\{T_t\}_{t \geq 0}$ be a semi-group on $L_2(\rho^2\mu)$ corresponding to A . Then, by the contractivity and symmetry, we can extend $\{T_t\}_{t \geq 0}$ to a strongly continuous semi-group $\{T_t^{(p)}\}_{t \geq 0}$ on $L_p(\rho^2\mu)$. We denote by $\{G_\alpha^{(p)}\}_{\alpha > 0}$ the corresponding resolvent.

Lemma 4. *It hold that*

$$(3.12) \quad \mathcal{N}_\alpha \cap \mathcal{D}[\sqrt{-A}] \subset \mathcal{H} \quad \text{for } A \in \mathcal{A}_M(S).$$

Proof. Take any element $v \in \mathcal{N}_\alpha \cap \mathcal{D}[\sqrt{-A}]$. We first show that $\phi, v \in$

$\bigcap_{1 < p < 2} D_p^1$ for any l . Let $w = \frac{\phi_l}{\rho^2} \in D_\infty$. Then, by Lemma 3, we see $w\psi$
 $\left(= \phi_{l+1} \frac{\phi_l \psi}{\rho^2} \right) \in \mathcal{D}[\bar{S}^{(2)}]$, for any $\psi \in FC_0^\infty$. Then, by the definition

$$(v\rho^2, \alpha\phi - \bar{S}^{(2)}\phi)_\mu = 0, \quad \text{for } \phi \in \mathcal{D}[\bar{S}^{(2)}].$$

Hence, we obtain for $\psi \in FC_0^\infty$

$$(3.13) \quad (v\rho^2, \mathcal{L}(w\psi))_\mu = 2\alpha(v\rho^2, w\psi)_\mu - 2(v\rho, \langle D\rho, D(w\psi) \rangle_H)_\mu.$$

Now, we have for $\psi \in FC_0^\infty$

$$(\phi_l v, \mathcal{L}\psi)_\mu = (g, \psi)_\mu$$

where $g = 2\alpha v\rho^2 w - 2D^*(vw\rho D\rho) - 2\rho \langle D\rho, Dw \rangle_H - v\rho^2 \mathcal{L}w - D^*(v\rho^2 Dw)$. Now, we use the hypoellipticity of \mathcal{L} ([5]) as follows: since $vw\rho D\rho$ and $v\rho^2 Dw$ belong to $\bigcap_{1 < p < 2} L_p(B \rightarrow H)$, we have $g \in \bigcap_{1 < p < 2} D_p^{-1}$. By [5], $\phi_l v$ belongs to the domain of extended \mathcal{L} , $\mathcal{L}(\phi_l v) \in \bigcap_{1 < p < 2} D_p^{-1}$ and $\phi_l v = R(\mathcal{L}(\phi_l v)) \in \bigcap_{1 < p < 2} D_p^1$, where R is the resolvent of \mathcal{L} . Using this property of $\phi_l v$ and repeating the same procedure as above, we get $g \in \bigcap_{1 < p < 2} D_p^0$ and consequently $\phi_l v \in \bigcap_{1 < p < 2} D_p^2$ as was to be proved.

We next prove that $\int \langle Dv, Dv \rangle_H \rho^2 d\mu$ is finite. To this end, let $\{b_{(n)}(t)\}_{n=1}^\infty \subset C_b^\infty(R^1)$ be a sequence satisfying that

i) $b_{(n)}(t) = t$ on $-n \leq t \leq n$ ii) $b_{(n)}(t) - b_{(n)}(s) \leq t - s$, $t > s$ iii) $|b_{(n)}(t)| \leq n + 1$.
 Then, $v_{(n)} = b_{(n)}(v) \in \mathcal{D}[\sqrt{-A}]$ by virtue of the Markovian property of Dirichlet space $\mathcal{D}[\sqrt{-A}]$. According to [4; (2.3.24)], we get

$$\begin{aligned} \mathcal{E}_A(v_{(n)}, v_{(n)}) &= (\sqrt{-A}v_{(n)}, \sqrt{-A}v_{(n)}) \rho^2 \mu \\ &= \lim_{\beta \rightarrow \infty} \mathcal{E}_A^{(\beta)}(v_{(n)}, v_{(n)}) \\ &= \lim_{\beta \rightarrow \infty} \frac{1}{2} (f_\beta, 1) \rho^2 \mu, \end{aligned}$$

where $f_\beta = -\beta(v_{(n)}^2 - \beta G_\beta^{(2)} v_{(n)}^2) + 2\beta v_{(n)}(v_{(n)} - \beta G_\beta^{(2)} v_{(n)}) + v_{(n)}^2(1 - \beta G_\beta^{(2)} 1)$. First, we see that

$$\begin{aligned} (3.14) \quad \lim_{\beta \rightarrow \infty} -\beta(v_{(n)}^2 - \beta G_\beta^{(2)} v_{(n)}^2, \phi_l) \rho^2 \mu &= \lim_{\beta \rightarrow \infty} -\beta(v_{(n)}^2, \phi_l - \beta G_\beta^{(2)} \phi_l) \rho^2 \mu \\ &= \lim_{\beta \rightarrow \infty} -\beta(v_{(n)}^2, \beta G_\beta^{(2)} \bar{S}^{(2)} \phi_l) \rho^2 \mu \\ &= (v_{(n)}^2, \bar{S}^{(2)} \phi_l) \rho^2 \mu. \end{aligned}$$

But, since $\phi_l v_{(n)}$ belongs to D_p^2 for any l , we see that the right hand side of (3.14) is equal to $(v_{(n)}(\mathcal{L}v_{(n)} + \frac{2}{\rho} \langle D\rho, Dv_{(n)} \rangle_H) + \langle Dv_{(n)}, Dv_{(n)} \rangle_H, \phi_l) \rho^2 \mu$. On the other hand, $\phi_l v_{(n)} \in \mathcal{D}[\bar{S}^{(p)}]$, $1 < p < 2$, by Lemma 3. Hence

$$\begin{aligned}
\lim_{\beta \rightarrow \infty} \beta(v_{(n)} - \beta G_{\beta}^{(2)} v_{(n)}, \phi_l v_{(n)}) \rho^2 \mu &= \lim_{\beta \rightarrow \infty} \beta(v_{(n)}, \phi_l v_{(n)} - \beta G_{\beta}^{(p)}(\phi_l v_{(n)})) \rho^2 \mu \\
&= \lim_{\beta \rightarrow \infty} \beta(v_{(n)}, \phi_l v_{(n)} - \beta G_{\beta}^{(p)}(\phi_l v_{(n)})) \rho^2 \mu \\
&= (v_{(n)}, -\bar{S}^{(p)}(\phi_l v_{(n)})) \rho^2 \mu \\
&= \left(-\frac{\mathcal{L}}{2} v_{(n)} - \frac{1}{\rho} \langle D\rho, Dv_{(n)} \rangle_H, \phi_l v_{(n)}\right) \rho^2 \mu.
\end{aligned}$$

By noting $1 \in \mathcal{D}[\sqrt{-A}]$, we see that $\beta G_{\beta}^{(2)} 1 = 1$. Hence,

$$\begin{aligned}
\mathcal{E}_A(v, v) &\geq \mathcal{E}_A(v_{(n)}, v_{(n)}) \\
&\geq \lim_{\beta \rightarrow \infty} \frac{1}{2} (f_{\beta}, \phi_l) \rho^2 \mu \\
&= \frac{1}{2} \int \langle Dv_{(n)}, Dv_{(n)} \rangle_H \phi_l \rho^2 d\mu \\
&= \frac{1}{2} \int (b'_{(n)}(v))^2 \langle Dv, Dv \rangle_H \phi_l \rho^2 d\mu.
\end{aligned}$$

therefore, we can conclude that the function v belongs to \mathcal{H} by letting $l, n \rightarrow \infty$.
q.e.d.

REMARK. If ρ is a tame function represented as $\rho(x) = \tilde{\rho}(\langle e_1, x \rangle, \dots, \langle e_n, x \rangle)$, $\tilde{\rho} > 0$ $C^2(R^n)$, and $\int \rho^2 d\mu < \infty$, we can show that S is an essentially self-adjoint operator by using Wielens' idea. In fact, let $\psi_l(t)$ be a C_c^∞ -function satisfying that i) $0 \leq \psi_l(t) \leq 1$ ii) $\psi_l = \begin{cases} 1 & \text{on } t \leq l \\ 0 & \text{on } t \geq l+1 \end{cases}$ iii) $|\psi'_l(t)|, |\psi''_l(t)| < M$, and $\tilde{\psi}_l(r) = \psi_l(|r|)$, $r \in R^n$. Put $\phi_l(x) = \tilde{\psi}_l(\langle e_1, x \rangle, \dots, \langle e_n, x \rangle)$ and $\mathcal{M}_l = \{x \in B; (\langle e_1, x \rangle, \dots, \langle e_n, x \rangle) \in B_l (= \{r \in R^n; |r| < l\})\}$. Then, it holds that $\phi_l^2 v \in \mathcal{D}[\bar{S}]$, $v \in \mathcal{N}_a$, and that

$$\begin{aligned}
(v, (\alpha - \bar{S})(\phi_l^2 v)) \rho^2 \mu &= \alpha(\phi_l v, \phi_l v) \rho^2 \mu + \int \phi_l v \langle D\phi_l, Dv \rangle_H \rho^2 d\mu \\
&+ \frac{1}{2} \int \phi_l^2 \langle Dv, Dv \rangle_H \rho^2 d\mu = 0.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
(\phi_l v, (\alpha - \bar{S})(\phi_l v)) \rho^2 \mu &= \alpha(\phi_l v, \phi_l v) \rho^2 \mu + \int \phi_l v \langle D\phi_l, Dv \rangle_H \rho^2 d\mu \\
&+ \frac{1}{2} \int v^2 \langle D\phi_l, D\phi_l \rangle_H \rho^2 d\mu + \frac{1}{2} \int \phi_l^2 \langle Dv, Dv \rangle_H \rho^2 d\mu \\
&= \frac{1}{2} \int v^2 \langle D\phi_l, D\phi_l \rangle_H \rho^2 d\mu,
\end{aligned}$$

we have

$$\frac{1}{2} \int v^2 \langle D\phi_l, D\phi_l \rangle_H \rho^2 d\mu \geq \alpha \int \phi_l^2 v^2 \rho^2 d\mu.$$

therefore, $\frac{1}{2} M^2 \cdot n \int_{\mathcal{M}_l^c} \rho^2 d\mu \geq \alpha \int_{\mathcal{M}_{l+1}} v^2 \rho^2 d\mu$ and by letting $l \rightarrow \infty$, we obtain $v=0$.

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