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On the topologies of homeomorphism groups of topological spaces [1]

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1. Let R be a topological space. Then, by the usual definition of the multiplication, all the homeomorphic mappings of R onto itself form an abstract group A_R. The present note is devoted to the question of in respect to what topology A_R may become a topological group, provided that R is regular and locally bicompact. [2]

In section 2, under the condition that R is regular and locally bicompact, we obtain the weakest one of toplogies of A_R in respect to which A_R becomes a topological group and the mapping $(f, a)\rightarrow f(a)$ from the topological product of A_R and R to R is continuous for both f and a.

Furthermore, in section 3, supposing that R is a uniform space which is locally bicompact, we shall show that the topology of A_R introduced in section 2 coincides with the topology introduced analogously to that of character groups.

2. Let R be a regular and locally bicompact topological space, and let A_R have the same significance as section 1. Let further $\{U_\alpha\}$ be a basis of R (that is, a system of open sets such that every open set of R can be obtained as a sum of open sets belonging to it) such that the closure $\overline{U}_\alpha$ of U_α is bicompact. If we denote by $V_{\alpha_1 \ldots \alpha_r : \beta_1 \ldots \beta_r}$ the set of homeomorphisms which transform $\overline{U}_{\alpha_i}$ into U_{β_i} $(i=1, \ldots, r)$, then we have the following theorem.

Theorem 1. *Denote* $V_{\alpha_1 \ldots \alpha_r : \beta_1 \ldots \beta_r} \wedge \overset{-1}{V}_{\alpha_1' \ldots \alpha_s' ; \beta_1' \ldots \beta_s'}$ *by* $W^{\alpha_1' \ldots \alpha_s' ; \beta_1' \ldots \beta_s'}_{\alpha_1 \ldots \alpha_r ; \beta_1 \ldots \beta_r}$ *(where* S^{-1} *means the set of inverses of elements belonging to a subset* S *of* A_R.*), and let* Σ *be the system consisting of all* $W^{\alpha_1' \ldots \alpha_s' ; \beta_1' \ldots \beta_s'}_{\alpha_1 \ldots \alpha_r : \beta_1 \ldots \beta_r}$

[1] The writer is grateful to Prof. K. Shoda, who gave an impulse to the present paper.

[2] After having written this paper, the writer became aware of the fact that J. Dieudonné, R. Arens, J. Braconnier and J. Cholmez have already investigated this problem, but in the present situation, the writer cannot see their papers except the paper of J. Dieudonné, which appeared in Amer. Journ. Vo 1. 70 No. 3 (1948).

(occasionally abbreviated $W^{(\alpha');(\beta')}_{(\alpha);(\beta)}$ *) which are non-empty. Then, taking* Σ *as a basis of* A_R*, we obtain the weakest topology of* A_R *in respect to which* A_R *becomes a topological group and the mapping* $(f, a) \rightarrow f(a)$ *from the topological product of* A_R *and* R *to* R *is continutous for both* f *and* a.[3])

Proof. In order to prove that A_R becomes a topological group regarding Σ, it will be sufficient to show that if we take the system Σ' of all $V_{\alpha_1 \ldots \alpha_r ; \beta_1 \ldots \beta_r}$ (occasionally abbrevited $V_{(\alpha);(\beta)}$) which are non-empty, then A_R becomes a topological space and the topology is continuous respecting the product.

Let f and g be any two different elements of A_R. Then there exists $a \varepsilon R$ such that $f(a) \neq g(a)$, and an open set U_α belonging to $\{U_\alpha\}$ such that $f(a) \varepsilon U_\alpha$, $g(a) \bar{\varepsilon} U_\alpha$. If we take an open set U_β from $\{U_\alpha\}$ such that $a \varepsilon U_\beta$ and $f(\overline{U_\beta}) \subset U_\alpha$, then $f \varepsilon V_{\beta;\alpha}$ and $g \varepsilon V_{\beta;\alpha}$. Furthermore, the intersection of any two sets which belong to Σ' and contain some element of A_R belongs also to Σ'. Hence A_R is a topologcal space regarding Σ'.

Let $f g \varepsilon V_{\alpha;\beta}$. Then for any $a \varepsilon \overline{U_\alpha}$, there exist $U_{\sigma(a)}$ and $U_{\rho(a)}$ such that $a \varepsilon U_{\rho(a)}$, $g(a) \varepsilon U_{\sigma(a)}$, $g(\overline{U_{\rho(\alpha)}}) \subset U_{\sigma(a)}$, and $f(\overline{U_{\sigma(a)}}) \subset U_\beta$. Since $\overline{U_\alpha}$ is bicompact, there exists a finite set $\{a_1, a_2, \ldots, a_n\}$ of elements belonging to $\overline{U_\alpha}$ such that $\overline{U_\alpha} \subset U_{\rho(a_1)} \cup U_{\rho(a_2)} \cup \cdots \cup U_{\rho(a_n)}$. For brevity, let us denote $\rho(a_i)$ and $\sigma(a_i)$ by ρ_i and σ_i respectively. Then $g \varepsilon V_{\rho_1 \ldots \rho_n ; \sigma_1 \ldots \sigma_n}$, $f \varepsilon V_{\sigma_1 \ldots \sigma_n ; \beta \ldots \beta}$ and $V_{\sigma_1 \ldots \sigma_n ; \beta \ldots \beta} V_{\rho_1 \ldots \rho_n ; \sigma_1 \ldots \sigma_n} \subset V_{\alpha;\beta}$.

From this fact, it can readily be seen that for any $V_{(\alpha);(\beta)}$ containing $f g$ there exist $V_{(\alpha');(\beta')}$ and $V_{(\alpha'');(\beta'')}$ containing f and g respctively and satisfying $V_{(\alpha');(\beta')} V_{(\alpha'');(\beta'')} \subset V_{(\alpha);(\beta)}$. That is, the topology of A_R introduced by Σ' is continuous respecting the product.

Now we shall prove the remaining part of the theorem. The mapping $(f, a) \rightarrow f(a)$ is clearly continuous for f and a in respect to the topology of A_R introduced by Σ. Let $\Sigma^* = \{W^*\}$ be a system of subsets of A_R such that, when we take it as a basis, A_R becomes a topological group

3) The author is grateful to Prof. T. Tannaka, who suggested that this topology is the weakest one of such topologies.

and the mapping $(f, a) \rightarrow f(a)$ is continuous for f and a. Then, in order to prove our proposition, it is sufficient to show that for any $V_{\alpha;\beta}$ containiny f there exists $W^* \varepsilon \Sigma^*$ such that $f \varepsilon W^* \subset V_{\alpha;\beta}$. For any element a of $\bar{U}_\alpha$ there exist $W_a^* \varepsilon \Sigma^*$ and $U_{\rho(a)} \varepsilon \{U_\alpha\}$ such that $f \varepsilon W_a^*$, $a \varepsilon U_{\rho(a)}$, and $W_a^* U_{\rho(a)} \subset U_\beta$. Since $\bar{U}_\alpha$ is bicompact, there exists a finite system $\{a_1, a_2, \ldots, a_n\}$ such that $\bar{U}_\alpha \subset U_{\rho(a_1)} \cup U_{\rho(a_2)} \cup \ldots \cup U_{\rho(a_n)}$. Let W^* be a subset belonging to Σ^* and having the property $W^* \subset \bigcap_{i=1}^{n} W_{ai}^*$, then obviously $f \varepsilon W^* \subset V_{\alpha;\beta}$, q. e. d.

In theorem 1. starting from a definite basis of R, we have defined one topology of A_R, but now it is shown that this topology does not depend on the choice of the bases, namely

Theorem 2. *Let $\{U_\alpha\}$ and $\{U_\rho^*\}$ be any two bases. Then the two topologies of A_R introduced in the same way as above coincide with each other.*

Proof. Let $V_{(\alpha);(\beta)}$ and $V^*_{(\rho);(\sigma)}$ have the similar meaning to the previous case according to $\{U_\alpha\}$ and $\{U_\rho^*\}$ respectively. In order to prove the proposition it is sufficient to show that for any $f \varepsilon A_R$ and any $V_{\alpha:\beta}$ containing f there exists $V^*_{(\rho);(\sigma)}$ such that $f \varepsilon V^*_{(\rho).(\sigma)} \cup V_{\alpha;\beta}$, and conversely. If $f \varepsilon V_{\alpha;\beta}$, then for any $a \varepsilon \bar{U}_\alpha$ there exists $U^*_{\rho(a)}$ such that $f(a) \varepsilon U^*_{\rho(a)} \subset U_\beta$, and for such U_ρ^* there exists $U^*_{\sigma(a)}$ such that $a \varepsilon U^*_{\sigma(a)}$ and $f(\bar{U}^*_{\sigma(a)}) \subset U^*_{\rho(\sigma)}$. Since $\bar{U}_\alpha$ is bicompact, we can select certain finite elements $\{a_1, a_2, \ldots, a_n\}$ from $\bar{U}_\sigma$ such that $\bar{U}_\alpha \subset U^*_{\sigma(a_1)} \cup \ldots \cup U^*_{\sigma(a_n)}$. Then obviously $f \varepsilon V^*_{\sigma(a_1)\ldots\sigma(a_n);\rho(a_1)\ldots\rho(a_n)}$. The converse will be proved similarly, q. e. d

The following proposition is almost evident.

Theorem 3. *If R satisfies the second axiom of countability, then the topological group A_R defined as above satisfies the same axiom.*

3. In this section, we shall assume that R is a uniform space which is locally bicompact. Let $\{U_\alpha(a) \mid \alpha \varepsilon \Gamma, a \varepsilon R\}$ be a uniform system of neighborhoods of R satifysing the following conditions:

$n_1)$ $a \varepsilon U_\alpha(a)$

$n_2)$ $\bigcap_{\alpha \varepsilon \Gamma} U_\alpha(a) = a$

$n_3)$ For any $\alpha, \beta \varepsilon \Gamma$, there exists $\gamma \varepsilon \Gamma$ such that $U_\gamma(a) \subseteq U_\alpha(a)$

$\frown U_\beta(a)$ for any $a \varepsilon R$.

n_4) For any $\alpha \varepsilon \Gamma$, there exists $\beta \varepsilon \Gamma$ such that $a \varepsilon U_\beta(b)$ implies $U_\beta(b) \subset U_\alpha(a)$.

n_5) For any $\alpha \varepsilon \Gamma$, there exists $\beta \varepsilon \Gamma$ such that $b \varepsilon U_\beta(a)$ implies $U_\beta(b) \subset U_\alpha(a)$.

n_6) For any $U_\gamma(a)$ and $b \varepsilon U_\alpha(a)$, there exists $\beta \varepsilon \Gamma$ (which depends on a and b) such that $c \varepsilon U_\beta(a)$ implies $U_\beta(b) \subset U_\alpha(c)$.

For each bicompact subset F of R, $\alpha \varepsilon \Gamma$, and $f \varepsilon A_R$, we denote by $V_{F,\alpha}(f)$ the set of homeomorphisms g of R such that $a \varepsilon F$ implies $g(a) \varepsilon U_\alpha(f(a))$. Set $W_{F,\alpha}(f) = V_{F,\alpha}(f) \frown V_{F,\alpha}(f^{-1})^{-1}$. Then we have

Theorem 4. *If we take* $\{W_{F,\alpha}(f)\}$ *as a complete system of neghborhoods of* f, *then* A_R *is a topological group.*

Proof. In order to prove the proposition it is sufficient to show that if we take $\{V_{F,\alpha}(f)\}$ as a complete system of neighborhoods of f then A_R becomes a topological space and the topology of A_R is continuous respecting the product.

a) Let f and g be any two different elements of A_R, then there exists $a \varepsilon R$ such that $f(a) \neq g(a)$. Let $g(a) \bar{\varepsilon} U_\alpha(f(a))$. Then, if we denote by F the set consisting of the single element a we have $g \bar{\varepsilon} V_{F,\alpha}(f)$.

b) For any two neighborhoods $V_{F,\alpha}(f)$, $V_{F',\beta}(f)$ of f, if we take γ satisfying the condition n_3) for α and β, then $V_{F \smile F',\gamma}(f) \subset V_{F,\alpha}(f) \frown V_{F',\beta}(f)$.

c) Let $g \varepsilon V_{F,\alpha}(f)$. Then $a \varepsilon F$ implies $g(a) \varepsilon U_\alpha(f(a))$. From the condition n_6), there exists $\beta(a) \varepsilon \Gamma$ such that $c \varepsilon U_{\beta(a)}(f(a))$ implies $U_{\beta(a)}(g(a)) \subset U_\alpha(c)$. If we take $\gamma(a) \varepsilon \Gamma$ satisfying n_5) for $\beta(a)$, then from $c \varepsilon U_{\gamma(a)}(f(a))$ and $d \varepsilon U_{\gamma(a)}(g(a))$ it follows that $U_{\gamma(a)}(d) \subset U_{\beta(a)}(g(a)) \subset U_\alpha(c)$.

Let $U(a)$ be an open set containing a such that its closure is bicompact and $f(U(a)) \subset U_{\gamma(a)}(f(a)$, $g(U(a)) \subset U_{\gamma(a)}(f(a))$. Since F is bicompact, F may be covered by a certain finite system of $U(a)$: $F \subset U(a_1) \smile U(a_2) \smile \cdots \smile U(a_n)$. Let γ be an element from Γ such that for any $x \varepsilon R$ $U_\gamma(x) \subset U_{\gamma(a_i)}(x)$ $(i = 1, 2, \ldots, n)$, and set $F' = \overline{U(a_1)} \smile \cdots \smile \overline{U(a_n)}$. Then if h and a belong to $V_{F',\gamma}(g)$ and F

respectively, there exists $U(a_i)$ such that $a \varepsilon U(a_i)$, and hence $g(a) \varepsilon U_{\gamma(a_i)}(g(a_i))$ and $f(a) \varepsilon U_{\gamma(a_i)}(f(a_i))$. Accordingly $U_{\gamma(a_i)}(g(a_i)) \subset U_\gamma(f(a))$. Since $h(a) \varepsilon U_\gamma((g(a)) \subset U_{\gamma(a_i)}(g(a))$ we have $h(a) \varepsilon U_\alpha(f(a))$, that is, $V_{F'}, \gamma(g) \subset V_F, \alpha(f)$.

From a), b), c), we can conclude that A_R is a topological space regarding $\{V_F, \alpha(f)\}$.

Now we shall show that for any two elements from A_R and any neighborhood $V_F, \alpha(fg)$ of fg there exist $V_{F_1}, \alpha_1(f)$ and $V_{F_2}, \alpha_2(g)$ such that $V_{F_1}, \alpha_1(f) V_{F_2}, \alpha_2(g) \subset V_F, \alpha(fg)$.

Generally, it is easily verified that if F is a bicompact subset of R and f is an element of A_R then for any α there exists β such that $f(U_\beta(a)) \subset U_\alpha(f(a))$ for any $a \varepsilon F$.

Let $V_F, \alpha(fg)$ be an arbitrary neighborhood of f, and let β satisfy the condition n_5) for α. Since $g(F)$ is bicompact, there exists γ such that $f(U_\gamma(g(a))) \subset U_\beta(fg(a))$ for any $a \varepsilon F$, and moreover $U_\gamma(a)$ is bicompact. $g(F)$ may be covered by a certain finite system of $U_\gamma(a)$: $g(F) \subset U_\gamma(a_1) \cup \cdots \cup U_\gamma(a_n)$. Denote $U_\gamma(a_1) \cup \cdots \cup U_\gamma(a^u)$ by O. Then, since $F' = \bar{O}$ is bicompact, there exists γ' such that for any $x \varepsilon R$ $U_{\gamma'}(x) \subset U_\gamma(x)$, and also $U_{\gamma'}(g(a)) \subset O$ holds for any $a \varepsilon F$. Let $f' \varepsilon V_{F'}, \beta(f)$ and $g' \varepsilon V_F, \gamma'(g)$. Then, since for any $a \varepsilon F$ $g'(a) \varepsilon U_{\gamma'}(g(a))$, $fg'(a)) \varepsilon U_\beta(fg(a))$ and $f'g'(a) \varepsilon U_\beta(fg'(a))$, hence $f'g'(a) \varepsilon U_\alpha(fg(a))$. That is $V_{F'}, \beta(f) V_F, \alpha(g) \subset U_F, \alpha(fg)$, q. e. d.

For a uniform space R which is locally bicompact, two topologies may be introduced in A_R from theorem I and theorem 4. But, these two topologies coicnide with each other.

To prove this, let $\{U_\alpha(a)\}$ be a uniform system of neighborhoods of R and $\{U_\rho{}^*\}$ a basis of R. Let further $V_F, \alpha(f)$, $W_F, \alpha(f)$, $V_{(\rho)(\sigma)}$, $W_{(\rho);(\rho)}$ have the same significance as above.

Let $f \varepsilon V_{\rho,\sigma}$. Then, since $f(\overline{U_\rho^*}) \subset U_\sigma^*$ and $f(\overline{U_\rho^*})$ is bicompact, there exists α such that $a \varepsilon \overline{U_\rho^*}$ implies $U_\alpha(f(a)) \subset U_\rho^*$. Hence, if we put $F = \overline{U_\beta^*}$, then $V_F, \alpha(f) \subset V_{\rho;\sigma}$. From this fact, now it is easily seen that for any $W_{(\rho);(\sigma)}$ containing f there exists $W_F, \alpha(f)$ which is contained in $W_{(\rho);(\sigma)}$.

Conversely, let us suppose that $V_{F},_{\alpha}(f)$ is given arbitrarily. For α, there exists β satisfying the conditin n_4), and for any $a \varepsilon F$ there exists $U^*_{\sigma(a)}$ such that $f(a) \varepsilon U^*_{\sigma(a)}$ and $U^*_{\sigma(a)} \subset U_\beta(f(a))$, and further for such $U^*_{\sigma(a)}$ there exists $U^*_{\rho(a)}$ such that $a \varepsilon U^*_{\rho(a)}$ and $f(U^*_{\rho(a)}) \subset U^*_{\sigma(a)}$. Since F is bicompact, F is covered by a finite set of $U^*_{\rho(a)}: F \subset U^*_{\rho(a_1)} \cup \dots \cup U^*_{\rho(a_n)}$. Let $g \varepsilon V_{\rho(a_1)\dots\rho(a_n);\sigma(a_1)\dots\sigma(a_n)}$ and $a \ni F$. Then a is in some $U^*_{\rho(a_i)}$. Since $g(a) \ni U^*_{\sigma(a_i)} \subset U_\beta(f(a_i))$ and $f(a) \varepsilon U_\beta(f(a_i))$, we have $g(a) \varepsilon U_\alpha(f(a))$, that is $g \varepsilon V_{F},_{\alpha}(f)$. From this fact, it can be seen that for any $W_{F},_{\alpha}(f)$ there exists $W_{(\rho);(\sigma)}$ such that $f \varepsilon W_{(\rho);(\sigma)}$ and $W_{(\rho);(\sigma)} \subset W_{F},_{\alpha}(f)$.

Therefore, our proposition is proved, and now we have the following

Theorem 5. *Let R be a uniform space which is locally bicompact. Then the topology of A_R defined in theorem 4 of course depends on the topology, but it is independent of the uniform structure of R. Furthermore, this topology coincides with the topology defined in theorem 1.*

Finally, we shall note that, in theorem 4, if R is bicompact then by taking $\{V_{F},_{\alpha}(f)\}$ as a complete system of neighborhoods of $f \varepsilon A_R$ becomes a topological group and this topology coincides with that defined by $\{W_{F},_{\alpha}(f)\}$.

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