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Author(s)	Nagao, Hirosi
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Osaka University

**On the topologies of homeomorphism groups of  
 topological spaces <sup>1)</sup>**

By Hiroshi NAGAO

1. Let  $R$  be a topological space. Then, by the usual definition of the multiplication, all the homeomorphic mappings of  $R$  onto itself form an abstract group  $A_R$ . The present note is devoted to the question of in respect to what topology  $A_R$  may become a topological group, provided that  $R$  is regular and locally bicomact. <sup>2)</sup>

In section 2, under the condition that  $R$  is regular and locally bicomact, we obtain the weakest one of topologies of  $A_R$  in respect to which  $A_R$  becomes a topological group and the mapping  $(f, a) \rightarrow f(a)$  from the topological product of  $A_R$  and  $R$  to  $R$  is continuous for both  $f$  and  $a$ .

Furthermore, in section 3, supposing that  $R$  is a uniform space which is locally bicomact, we shall show that the topology of  $A_R$  introduced in section 2 coincides with the topology introduced analogously to that of character groups.

2. Let  $R$  be a regular and locally bicomact topological space, and let  $A_R$  have the same significance as section 1. Let further  $\{U_\alpha\}$  be a basis of  $R$  (that is, a system of open sets such that every open set of  $R$  can be obtained as a sum of open sets belonging to it) such that the closure  $\bar{U}_\alpha$  of  $U_\alpha$  is bicomact. If we denote by  $V_{\alpha_1 \dots \alpha_r; \beta_1 \dots \beta_r}$  the set of homeomorphisms which transform  $\bar{U}_{\alpha_i}$  into  $U_{\beta_i}$  ( $i=1, \dots, r$ ), then we have the following theorem.

**Theorem 1.** Denote  $V_{\alpha_1 \dots \alpha_r; \beta_1 \dots \beta_r} \wedge V_{\alpha_1' \dots \alpha_s'; \beta_1' \dots \beta_s'}$  by  $W_{\alpha_1 \dots \alpha_r; \beta_1 \dots \beta_r}^{\alpha_1' \dots \alpha_s'; \beta_1' \dots \beta_s'}$  (where  $S^{-1}$  means the set of inverses of elements belonging to a subset  $S$  of  $A_R$ ), and let  $\Sigma$  be the system consisting of all  $W_{\alpha_1 \dots \alpha_r; \beta_1 \dots \beta_r}^{\alpha_1' \dots \alpha_s'; \beta_1' \dots \beta_s'}$ .

<sup>1)</sup> The writer is grateful to Prof. K. Shoda, who gave an impulse to the present paper.

<sup>2)</sup> After having written this paper, the writer became aware of the fact that J. Dieudonné, R. Arens, J. Braconnier and J. Cholmez have already investigated this problem, but in the present situation, the writer cannot see their papers except the paper of J. Dieudonné, which appeared in Amer. Journ. Vo 1. 70 No. 3 (1948).

(occasionally abbreviated  $W_{(\alpha);(\beta)}^{(\alpha');(\beta')}$ ) which are non-empty. Then, taking  $\Sigma$  as a basis of  $A_R$ , we obtain the weakest topology of  $A_R$  in respect to which  $A_R$  becomes a topological group and the mapping  $(f, a) \rightarrow f(a)$  from the topological product of  $A_R$  and  $R$  to  $R$  is continuous for both  $f$  and  $a$ .<sup>3)</sup>

*Proof.* In order to prove that  $A_R$  becomes a topological group regarding  $\Sigma$ , it will be sufficient to show that if we take the system  $\Sigma'$  of all  $V_{\alpha_1 \dots \alpha_r; \beta_1 \dots \beta_r}$  (occasionally abbreviated  $V_{(\alpha);(\beta)}$ ) which are non-empty, then  $A_R$  becomes a topological space and the topology is continuous respecting the product.

Let  $f$  and  $g$  be any two different elements of  $A_R$ . Then there exists  $a \in R$  such that  $f(a) \neq g(a)$ , and an open set  $U_\alpha$  belonging to  $\{U_\alpha\}$  such that  $f(a) \in U_\alpha$ ,  $g(a) \notin U_\alpha$ . If we take an open set  $U_\beta$  from  $\{U_\alpha\}$  such that  $a \in U_\beta$  and  $f(\overline{U_\beta}) \subset U_\alpha$ , then  $f \in V_{\beta; \alpha}$  and  $g \in V_{\beta; \alpha}$ . Furthermore, the intersection of any two sets which belong to  $\Sigma'$  and contain some element of  $A_R$  belongs also to  $\Sigma'$ . Hence  $A_R$  is a topological space regarding  $\Sigma'$ .

Let  $f, g \in V_{\alpha; \beta}$ . Then for any  $a \in \overline{U_\alpha}$ , there exist  $U_{\sigma(a)}$  and  $U_{\rho(a)}$  such that  $a \in U_{\rho(a)}$ ,  $g(a) \in U_{\sigma(a)}$ ,  $g(\overline{U_{\rho(a)}}) \subset U_{\sigma(a)}$ , and  $f(\overline{U_{\sigma(a)}}) \subset U_\beta$ . Since  $\overline{U_\alpha}$  is bicomact, there exists a finite set  $\{a_1, a_2, \dots, a_n\}$  of elements belonging to  $\overline{U_\alpha}$  such that  $\overline{U_\alpha} \subset U_{\rho(a_1)} \cup U_{\rho(a_2)} \cup \dots \cup U_{\rho(a_n)}$ . For brevity, let us denote  $\rho(a_i)$  and  $\sigma(a_i)$  by  $\rho_i$  and  $\sigma_i$  respectively. Then  $g \in V_{\rho_1 \dots \rho_n; \sigma_1 \dots \sigma_n}$ ,  $f \in V_{\sigma_1 \dots \sigma_n; \beta \dots \beta}$  and  $V_{\sigma_1 \dots \sigma_n; \beta \dots \beta} V_{\rho_1 \dots \rho_n; \sigma_1 \dots \sigma_n} \subset V_{\alpha; \beta}$ .

From this fact, it can readily be seen that for any  $V_{(\alpha);(\beta)}$  containing  $f, g$  there exist  $V_{(\alpha');(\beta')}$  and  $V_{(\alpha'');(\beta'')}$  containing  $f$  and  $g$  respectively and satisfying  $V_{(\alpha');(\beta')} V_{(\alpha'');(\beta'')} \subset V_{(\alpha);(\beta)}$ . That is, the topology of  $A_R$  introduced by  $\Sigma'$  is continuous respecting the product.

Now we shall prove the remaining part of the theorem. The mapping  $(f, a) \rightarrow f(a)$  is clearly continuous for  $f$  and  $a$  in respect to the topology of  $A_R$  introduced by  $\Sigma$ . Let  $\Sigma^* = \{\overline{W^*}\}$  be a system of subsets of  $A_R$  such that, when we take it as a basis,  $A_R$  becomes a topological group

<sup>3)</sup> The author is grateful to Prof. T. Tannaka, who suggested that this topology is the weakest one of such topologies.

and the mapping  $(f, a) \rightarrow f(a)$  is continuous for  $f$  and  $a$ . Then, in order to prove our proposition, it is sufficient to show that for any  $V_{\alpha}; \beta$  containing  $f$  there exists  $W^* \in \Sigma^*$  such that  $f \in W^* \subset V_{\alpha}; \beta$ . For any element  $a$  of  $\bar{U}_{\alpha}$  there exist  $W a^* \in \Sigma^*$  and  $U_{\rho(a)} \in \{U_{\alpha}\}$  such that  $f \in W a^*$ ,  $a \in U_{\rho(a)}$ , and  $W a^* U_{\rho(a)} \subset U_{\beta}$ . Since  $\bar{U}_{\alpha}$  is bicomact, there exists a finite system  $\{a_1, a_2, \dots, a_n\}$  such that  $\bar{U}_{\alpha} \subset U_{\rho(a_1)} \cup U_{\rho(a_2)} \cup \dots \cup U_{\rho(a_n)}$ . Let  $W^*$  be a subset belonging to  $\Sigma^*$  and having the property  $W^* \subset \bigcap_{i=1}^n W a_i^*$ , then obviously  $f \in W^* \subset V_{\alpha}; \beta$ , q. e. d.

In theorem 1. starting from a definite basis of  $R$ , we have defined one topology of  $A_R$ , but now it is shown that this topology does not depend on the choice of the bases, namely

Theorem 2. *Let  $\{U_{\alpha}\}$  and  $\{U_{\rho^*}\}$  be any two bases. Then the two topologies of  $A_R$  introduced in the same way as above coincide with each other.*

*Proof.* Let  $V_{(\alpha)}; (\beta)$  and  $V_{(\rho^*); (\sigma)}$  have the similar meaning to the previous case according to  $\{U_{\alpha}\}$  and  $\{U_{\rho^*}\}$  respectively. In order to prove the proposition it is sufficient to show that for any  $f \in A_R$  and any  $V_{\alpha}; \beta$  containing  $f$  there exists  $V_{(\rho^*); (\sigma)}$  such that  $f \in V_{(\rho^*); (\sigma)} \cup V_{\alpha}; \beta$ , and conversely. If  $f \in V_{\alpha}; \beta$ , then for any  $a \in \bar{U}_{\alpha}$  there exists  $U_{\rho(a)}^*$  such that  $f(a) \in U_{\rho(a)}^* \subset U_{\beta}$ , and for such  $U_{\rho^*}$  there exists  $U_{\sigma(a)}^*$  such that  $a \in U_{\sigma(a)}^*$  and  $f(U_{\sigma(a)}^*) \subset U_{\rho(a)}^*$ . Since  $\bar{U}_{\alpha}$  is bicomact, we can select certain finite elements  $\{a_1, a_2, \dots, a_n\}$  from  $\bar{U}_{\alpha}$  such that  $\bar{U}_{\alpha} \subset U_{\sigma(a_1)}^* \cup \dots \cup U_{\sigma(a_n)}^*$ . Then obviously  $f \in V_{\sigma(a_1) \dots \sigma(a_n); \rho(a_1) \dots \rho(a_n)}$ . The converse will be proved similarly, q. e. d

The following proposition is almost evident.

Theorem 3. *If  $R$  satisfies the second axiom of countability, then the topological group  $A_R$  defined as above satisfies the same axiom.*

3. In this section, we shall assume that  $R$  is a uniform space which is locally bicomact. Let  $\{U_{\alpha}(a) | a \in \Gamma, a \in R\}$  be a uniform system of neighborhoods of  $R$  satisfying the following conditions:

- $n_1)$   $a \in U_{\alpha}(a)$
- $n_2)$   $\bigcap_{\alpha \in \Gamma} U_{\alpha}(a) = a$
- $n_3)$  For any  $\alpha, \beta \in \Gamma$ , there exists  $\gamma \in \Gamma$  such that  $U_{\gamma}(a) \subseteq U_{\alpha}(a)$

$\bigcap U_\beta(a)$  for any  $a \in R$ .

$n_4$ ) For any  $\alpha \in \Gamma$ , there exists  $\beta \in \Gamma$  such that  $a \in U_\beta(b)$  implies  $U_\beta(b) \subset U_\alpha(a)$ .

$n_5$ ) For any  $\alpha \in \Gamma$ , there exists  $\beta \in \Gamma$  such that  $b \in U_\beta(a)$  implies  $U_\beta(b) \subset U_\alpha(a)$ .

$n_6$ ) For any  $U_\gamma(a)$  and  $b \in U_\alpha(a)$ , there exists  $\beta \in \Gamma$  (which depends on  $a$  and  $b$ ) such that  $c \in U_\beta(a)$  implies  $U_\beta(b) \subset U_\alpha(c)$ .

For each bicomcompact subset  $F$  of  $R$ ,  $\alpha \in \Gamma$ , and  $f \in A_R$ , we denote by  $V_{F, \alpha}(f)$  the set of homeomorphisms  $g$  of  $R$  such that  $a \in F$  implies  $g(a) \in U_\alpha(f(a))$ . Set  $W_{F, \alpha}(f) = V_{F, \alpha}(f) \cap V_{F, \alpha}(f^{-1})^{-1}$ . Then we have

**Theorem 4.** *If we take  $\{W_{F, \alpha}(f)\}$  as a complete system of neighborhoods of  $f$ , then  $A_R$  is a topological group.*

*Proof.* In order to prove the proposition it is sufficient to show that if we take  $\{V_{F, \alpha}(f)\}$  as a complete system of neighborhoods of  $f$  then  $A_R$  becomes a topological space and the topology of  $A_R$  is continuous respecting the product.

$a$ ) Let  $f$  and  $g$  be any two different elements of  $A_R$ , then there exists  $a \in R$  such that  $f(a) \neq g(a)$ . Let  $g(a) \in U_\alpha(f(a))$ . Then, if we denote by  $F$  the set consisting of the single element  $a$  we have  $g \in V_{F, \alpha}(f)$ .

$b$ ) For any two neighborhoods  $V_{F, \alpha}(f)$ ,  $V_{F', \beta}(f)$  of  $f$ , if we take  $\gamma$  satisfying the condition  $n_3$ ) for  $\alpha$  and  $\beta$ , then  $V_{F \cup F', \gamma}(f) \subset V_{F, \alpha}(f) \cap V_{F', \beta}(f)$ .

$c$ ) Let  $g \in V_{F, \alpha}(f)$ . Then  $a \in F$  implies  $g(a) \in U_\alpha(f(a))$ . From the condition  $n_6$ ), there exists  $\beta(a) \in \Gamma$  such that  $c \in U_{\beta(a)}(f(a))$  implies  $U_{\beta(a)}(g(a)) \subset U_\alpha(c)$ . If we take  $\gamma(a) \in \Gamma$  satisfying  $n_5$ ) for  $\beta(a)$ , then from  $c \in U_{\gamma(a)}(f(a))$  and  $d \in U_{\gamma(a)}(g(a))$  it follows that  $U_{\gamma(a)}(d) \subset U_{\beta(a)}(g(a)) \subset U_\alpha(c)$ .

Let  $U(a)$  be an open set containing  $a$  such that its closure is bicomcompact and  $f(U(a)) \subset U_{\gamma(a)}(f(a))$ ,  $g(U(a)) \subset U_{\gamma(a)}(f(a))$ . Since  $F$  is bicomcompact,  $F$  may be covered by a certain finite system of  $U(a)$ :  $F \subset U(a_1) \cup U(a_2) \cup \dots \cup U(a_n)$ . Let  $\gamma$  be an element from  $\Gamma$  such that for any  $x \in R$   $U_\gamma(x) \subset U_{\gamma(a_i)}(x)$  ( $i = 1, 2, \dots, n$ ), and set  $F' = \overline{U(a_1)} \cup \dots \cup \overline{U(a_n)}$ . Then if  $h$  and  $a$  belong to  $V_{F', \gamma}(g)$  and  $F'$

respectively, there exists  $U(a_i)$  such that  $a \in U(a_i)$ , and hence  $g(a) \in U_{\gamma(a_i)}(g(a_i))$  and  $f(a) \in U_{\gamma(a_i)}(f(a_i))$ . Accordingly  $U_{\gamma(a_i)}(g(a_i)) \subset U_{\gamma}(f(a))$ . Since  $h(a) \in U_{\gamma}(g(a)) \subset U_{\gamma(a_i)}(g(a))$  we have  $h(a) \in U_{\alpha}(f(a))$ , that is,  $V_{F', \gamma}(g) \subset V_{F, \alpha}(f)$ .

From a), b), c), we can conclude that  $A_R$  is a topological space regarding  $\{V_{F, \alpha}(f)\}$ .

Now we shall show that for any two elements from  $A_R$  and any neighborhood  $V_{F, \alpha}(fg)$  of  $fg$  there exist  $V_{F_1, \alpha_1}(f)$  and  $V_{F_2, \alpha_2}(g)$  such that  $V_{F_1, \alpha_1}(f)V_{F_2, \alpha_2}(g) \subset V_{F, \alpha}(fg)$ .

Generally, it is easily verified that if  $F$  is a bicomact subset of  $R$  and  $f$  is an element of  $A_R$  then for any  $\alpha$  there exists  $\beta$  such that  $f(U_{\beta}(a)) \subset U_{\alpha}(f(a))$  for any  $a \in F$ .

Let  $V_{F, \alpha}(fg)$  be an arbitrary neighborhood of  $f$ , and let  $\beta$  satisfy the condition  $n_5$  for  $\alpha$ . Since  $g(F)$  is bicomact, there exists  $\gamma$  such that  $f(U_{\gamma}(g(a))) \subset U_{\beta}(fg(a))$  for any  $a \in F$ , and moreover  $U_{\gamma}(a)$  is bicomact.  $g(F)$  may be covered by a certain finite system of  $U_{\gamma}(a)$ :  $g(F) \subset U_{\gamma}(a_1) \cup \dots \cup U_{\gamma}(a_n)$ . Denote  $U_{\gamma}(a_1) \cup \dots \cup U_{\gamma}(a_n)$  by  $O$ . Then, since  $F' = \overline{O}$  is bicomact, there exists  $\gamma'$  such that for any  $x \in R$   $U_{\gamma'}(x) \subset U_{\gamma}(x)$ , and also  $U_{\gamma'}(g(a)) \subset O$  holds for any  $a \in F$ . Let  $f' \in V_{F', \beta}(f)$  and  $g' \in V_{F, \gamma'}(g)$ . Then, since for any  $a \in F$   $g'(a) \in U_{\gamma'}(g(a))$ ,  $f g'(a) \in U_{\beta}(fg(a))$  and  $f' g'(a) \in U_{\beta}(f g'(a))$ , hence  $f' g'(a) \in U_{\alpha}(fg(a))$ . That is  $V_{F', \beta}(f)V_{F, \gamma'}(g) \subset U_{F, \alpha}(fg)$ , q. e. d.

For a uniform space  $R$  which is locally bicomact, two topologies may be introduced in  $A_R$  from theorem I and theorem 4. But, these two topologies coincide with each other.

To prove this, let  $\{U_{\alpha}(a)\}$  be a uniform system of neighborhoods of  $R$  and  $\{U_{\rho}^*\}$  a basis of  $R$ . Let further  $V_{F, \alpha}(f)$ ,  $W_{F, \alpha}(f)$ ,  $V_{(\rho)(\sigma)}$ ,  $W_{(\rho); (\sigma)}$  have the same significance as above.

Let  $f \in V_{\rho, \sigma}$ . Then, since  $f(\overline{U_{\rho}^*}) \subset U_{\sigma}^*$  and  $f(\overline{U_{\rho}^*})$  is bicomact, there exists  $\alpha$  such that  $a \in \overline{U_{\rho}^*}$  implies  $U_{\alpha}(f(a)) \subset U_{\rho}^*$ . Hence, if we put  $F' = \overline{U_{\beta}^*}$ , then  $V_{F, \alpha}(f) \subset V_{\rho; \sigma}$ . From this fact, now it is easily seen that for any  $W_{(\rho); (\sigma)}$  containing  $f$  there exists  $W_{F, \alpha}(f)$  which is contained in  $W_{(\rho); (\sigma)}$ .

Conversely, let us suppose that  $V_{F, \alpha}(f)$  is given arbitrarily. For  $\alpha$ , there exists  $\beta$  satisfying the condition  $n_i$ , and for any  $a \in F$  there exists  $U_{\sigma(a)}^*$  such that  $f(a) \in U_{\sigma(a)}^*$  and  $U_{\sigma(a)}^* \subset U_{\beta}(f(a))$ , and further for such  $U_{\sigma(a)}^*$  there exists  $U_{\rho(a)}^*$  such that  $a \in U_{\rho(a)}^*$  and  $f(U_{\rho(a)}^*) \subset U_{\sigma(a)}^*$ . Since  $F$  is bicomact,  $F$  is covered by a finite set of  $U_{\rho(a)}^* : F \subset U_{\rho(a_1)}^* \cup \dots \cup U_{\rho(a_n)}^*$ . Let  $g \in V_{\rho(a_1) \dots \rho(a_n); \sigma(a_1) \dots \sigma(a_n)}$  and  $a \in F$ . Then  $a$  is in some  $U_{\rho(a_i)}^*$ . Since  $g(a) \in U_{\sigma(a_i)}^* \subset U_{\beta}(f(a_i))$  and  $f(a) \in U_{\beta}(f(a_i))$ , we have  $g(a) \in U_{\alpha}(f(a))$ , that is  $g \in V_{F, \alpha}(f)$ . From this fact, it can be seen that for any  $W_{F, \alpha}(f)$  there exists  $W_{(\rho); (\sigma)}$  such that  $f \in W_{(\rho); (\sigma)}$  and  $W_{(\rho); (\sigma)} \subset W_{F, \alpha}(f)$ .

Therefore, our proposition is proved, and now we have the following

**Theorem 5.** *Let  $R$  be a uniform space which is locally bicomact. Then the topology of  $A_R$  defined in theorem 4 of course depends on the topology, but it is independent of the uniform structure of  $R$ . Furthermore, this topology coincides with the topology defined in theorem 1.*

Finally, we shall note that, in theorem 4, if  $R$  is bicomact then by taking  $\{V_{F, \alpha}(f)\}$  as a complete system of neighborhoods of  $f \in A_R$  becomes a topological group and this topology coincides with that defined by  $\{W_{F, \alpha}(f)\}$ .

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