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NONSTANDARD REPRESENTATIONS OF GENERALIZED SECTIONS OF VECTOR BUNDLES

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Introduction

In terms of nonstandard analysis, Todorov ([8], [9]) showed that every Schwartz distribution on R^n can be represented by a $*$-integral with $*C^\infty$ internal kernel function without the necessity of saying, "up to an infinitesimal" (for the case $n=1$, see also [5]). From the differential-geometric viewpoint, it would be desirable to obtain nonstandard representations of generalized sections of vector bundles in an intrinsic manner.

The main purpose of this note is to prove in a simple way that every generalized section (see §1 or [2]) T of a C^∞ vector bundle E over a σ-compact manifold M can be represented by a $*$-integral in the sense that there exists a $*C^\infty$ internal section β_T of the nonstandard extension $*E$ of E such that $T(u)=\int_{*M} \beta_T \cdot *u$ for every compactly supported C^∞ section u of $E^\dagger \otimes |\wedge_M|$, where $E^\dagger$ is the dual bundle of E and $|\wedge_M|$ stands for the density bundle over M.

After devoting §1 to some notational preliminaries, we obtain in §2 nonstandard representations of linear maps from the space of C^∞ sections of E with compact support and then the desired representations of generalized sections. In § 3 we get a result on nonstandard representations of linear maps either from the space of C^k sections of E or from its subspace consisting of compactly supported sections.

As for nonstandard analysis, see, e.g., [1], [3], or [4] ; we work with a sufficiently saturated nonstandard model.

1. Notational preliminaries

Throughout the paper we let K be either R (the real numbers) or C (the complex numbers). Furthermore, let N be the (strictly) positive integers and $*N_\infty$ the infinite elements in $*N$.

By a vector bundle $\pi_E : E \to M$ we mean a C^∞ vector bundle with typical fiber K^p (for some $p \in N$) over a σ-compact C^∞ manifold M with dim $M \in N$. For each

$x \in M$, we write $E_x := \pi_E^{-1}(x)$, the fiber of E over x. The dual bundle of E is denoted by $E^{\dagger}$. For $k \in \{0\} \cup N \cup \{\infty\}$, let $\Gamma^k(E)$ be the space of all C^k sections of E and $\Gamma_0^k(E)$ the space of C^k sections of E with compact support.

The K-line bundle of densities over M is denoted by $|\wedge_M|$. Given a C^{∞} Riemannian metric g on M, we let $dv_g \in \Gamma^{\infty}(|\wedge_M|)$ be the Riemannian volume density associated with g (see [7]). A *generalized section* of a vector bundle $E \to M$ is defined as a continuous linear functional on the space $\Gamma_0^{\infty}(E^{\dagger} \otimes |\wedge_M|)$ (endowed with the canonical LF-topology); cf. [2].

For two vector bundles $E \to M$ and $F \to N$, we denote by $E \boxtimes F$ the vector bundle over $M \times N$ such that $(E \boxtimes F)_{(x,y)} = E_x \otimes F_y$ for every $(x, y) \in M \times N$. We will write $\otimes$ and $\boxtimes$ for $^*\otimes$ and $^*\boxtimes$, respectively.

2. Nonstandard representations of generalized sections of vector bundles

Let $E \to M$ be a vector bundle. Choose a C^{∞} Riemannian metric g on M. By the saturation principle, there exists a hyperfinite-dimensional vector subspace V of the internal vector space $^*(\Gamma_0^{\infty}(E))$ such that $^{\sigma}(\Gamma_0^{\infty}(E)) := \{^*s : s \in \Gamma_0^{\infty}(E)\}$ is an external subset of V. Take a C^{∞} fiber metric h in E and pick $\psi_i \in V$ ($i = 1, 2, \ldots,$ η with $\eta = ^*\dim V \in ^*N_{\infty}$) such that

$$(2.1) \qquad \int_{^*M} {}^*h(\psi_i, \psi_j) \, {}^*dv_g = \delta_{ij} \text{ (Kronecker delta)}; \quad i, j = 1, 2, \ldots, \eta.$$

Regard $\psi_j^h := {}^*h(\cdot, \psi_j)$ as an element of $^*(\Gamma_0^{\infty}(E^{\dagger}))$ in a natural manner. Define an internal section $\Psi \in {}^*(\Gamma_0^{\infty}(E^{\dagger} \boxtimes E))$ by

$$(2.2) \qquad \Psi(x, y) := \sum_{i=1}^{\eta} \psi_i^h(x) \otimes \psi_i(y) \quad (x, y \in {}^*M).$$

Moreover, define $\widetilde{\Psi} \in {}^*(\Gamma_0^{\infty}((E^{\dagger} \otimes |\wedge_M|) \boxtimes E))$ by

$$(2.3) \qquad \widetilde{\Psi}(x, y) := \sum_{i=1}^{\eta} (\psi_i^h(x) \otimes {}^*dv_g(x)) \otimes \psi_i(y) \quad (x, y \in {}^*M).$$

Proposition 2.1. *Given a vector bundle $E \to M$ and a C^{∞} Riemannian metric g on M, let Ψ and $\widetilde{\Psi}$ be as in (2.2) and (2.3), respectively.*
  (1) *For every $s \in \Gamma_0^{\infty}(E)$ and every $y \in {}^*M$,*

$$(2.4) \qquad {}^*s(y) = \int_{x \in {}^*M} {}^*s(x) \cdot \widetilde{\Psi}(x, y).$$

[The map $^*M \times {}^*M \ni (x, y) \mapsto {}^*s(x) \cdot \widetilde{\Psi}(x, y) \in {}^*(|\wedge_M| \boxtimes E)_{(x,y)}$ gives an element of $^*(\Gamma_0^{\infty}(|\wedge_M| \boxtimes E))$ obtained from $^*s(x) \otimes \widetilde{\Psi}(x, y)$ by the canonical pairing between $^*E_x := {}^*(\pi_E)^{-1}(x)$ and $^*E_x^{\dagger}$.]
  (2) *For every $x, z \in {}^*M$,*

$$(2.5) \qquad \int_{y \in {}^*M} \Psi(x, y) \cdot \Psi(y, z) \otimes {}^*dv_g(y) = \Psi(x, z).$$

Proof. Since every $*s\in{}^\sigma(\Gamma_0^\infty(E))$ is expressed as

$$(2.6) \qquad *s=\sum_{j=1}^\eta c_j(*s)\psi_j \text{ with } c_j(*s)=\int_{*M}*s\cdot\psi_j^h \,*dv_g,$$

we have (2.4). Formula (2.5) follows immediately from (2.1). $\square$

Proposition 2.2. *Let $E\to M$ and $F\to N$ be vector bundles. Let $\widetilde{\Psi}\in *(\Gamma_0^\infty((E^\dagger\otimes|\wedge_M|)\boxtimes E))$ be as in (2.3). For a K-linear map $L:\Gamma_0^\infty(E)\to\Gamma^\infty(F)$, define $\Psi_L\in*(\Gamma^\infty((E^\dagger\otimes|\wedge_M|)\boxtimes F))$ by*

$$\Psi_L(y,z):=(I_{*(E^\dagger\otimes|\wedge_M|)_y}\otimes *L)(\widetilde{\Psi}(y,\cdot))(z)=\sum_{i=1}^\eta(\psi_i^h(y)\otimes *dv_g(y))\otimes *L(\psi_i)(z)$$

for $y\in*M$ and $z\in*N$, where $I_{*(E^\dagger\otimes|\wedge_M|)_y}$ is the identity transformation of $*(E^\dagger\otimes|\wedge_M|)_y$. Then L is represented as

$$*(L(s))(z)=\int_{y\in*M}*s(y)\cdot\Psi_L(y,z)\quad(s\in\Gamma_0^\infty(E),\ z\in*N).$$

Proof. Let $s\in\Gamma_0^\infty(E)$ and $z\in*N$. By the expression (2.6),

$$*(L(s))(z)=*L(*s)(z)=*L(\sum_{i=1}^\eta c_i(*s)\psi_i)(z)$$
$$=\sum_{i=1}^\eta c_i(*s)*L(\psi_i)(z)=\int_{y\in*M}*s(y)\cdot\Psi_L(y,z). \qquad \square$$

We can now represent every generalized section of E by a $*$-integral.

Theorem 2.3. *Let $E\to M$ be a vector bundle, and let $\widetilde{\Psi}$ be as in (2.3). For each generalized section T of E, define $\beta_T\in*(\Gamma_0^\infty(E))$ by*

$$\beta_T(y):=(*T\otimes I_{*E_y})(\widetilde{\Psi}(\cdot,y))=\sum_{i=1}^\eta *T(\psi_i^h\otimes *dv_g)\psi_i(y)\quad(y\in*M).$$

Then

$$T(u)=\int_{*M}\beta_T\cdot*u,\ u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|).$$

Proof. Apply Proposition 2.2 to the K-linear map $L_T:\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)\ni u\mapsto L_T(u)\in\Gamma^\infty(M\times K)$ defined by $L_T(u)(z)=(z,T(u))$ for $z\in M$. $\square$

REMARK. If there exists a section $s\in\Gamma_0^\infty(E)$ such that $T(u)=\int_M s\cdot u$ for all $u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$, then $*T(\psi_i^h\otimes *dv_g)=c_i(*s)$ (see (2.6)) and thus $\beta_T=*s$.

EXAMPLE. Given a σ-compact C^∞ Riemannian manifold (M,g), we obtain a "nonstandard delta function" with respect to $*dv_g$ using the above results. In fact,

let $C^\infty(M ; K)$ denote the space of K-valued C^∞ functions on M and let $C_0^\infty(M ; K) := \{f \in C^\infty(M ; K) : \mathrm{supp}(f) \text{ compact}\}$. Let V_0 be a hyperfinite-dimensional vector space over $*R$ such that

$$\sigma(C_0^\infty(M ; R)) := \{*f : f \in C_0^\infty(M ; R)\} \subset V_0 \subset *(C_0^\infty(M ; R)).$$

Pick $\varphi_i \in V_0$ $(i=1, 2, \ldots, \nu$ with $\nu = *\dim V_0)$ such that $\int_{*M} \varphi_i \varphi_j *dv_g = \delta_{ij}$ $(i, j = 1, 2, \ldots, \nu)$ and define an internal function $\delta \in *(C_0^\infty(M \times M ; R))$ by $\delta(x, y) := \sum_{i=1}^\nu \varphi_i(x) \varphi_i(y)$ $(x, y \in *M)$. Noting that $C_0^\infty(M ; C) = C_0^\infty(M ; R) + \sqrt{-1} C_0^\infty(M ; R)$, for $x, y, z \in *M$ we have:

(1) $\delta(x, x) \geq 0$, $\delta(x, y) = \delta(y, x)$, and $(\delta(x, y))^2 \leq \delta(x, x) \delta(y, y)$.

(2) $*f(x) = \int_{y \in *M} \delta(x, y) *f(y) *dv_g(y)$ for $f \in C_0^\infty(M ; C)$.

(3) $\int_{y \in *M} \delta(x, y) \delta(y, z) *dv_g(y) = \delta(x, z)$.

(4) If $T : \mathcal{D} = C_0^\infty(M ; C) \to C$ is a Schwartz distribution on M, then
$T(f) = \int_{y \in *M} *f(y) \gamma_T(y) *dv_g(y)$ $(f \in \mathcal{D})$, where $\gamma_T \in *(C_0^\infty(M ; C))$ is defined by $\gamma_T(y) := *T(\delta(\cdot, y)) = \sum_{i=1}^\nu *T(\varphi_i) \varphi_i(y)$ $(y \in *M)$.

3. Nonstandard representations of linear maps from $\Gamma^k(E)$ or $\Gamma_0^k(E)$ to $\Gamma^r(F)$

Let $E \to M$ and $F \to N$ be vector bundles. For $s \in \Gamma^0(E)$, define $T_s : \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|) \to K$ by

$$T_s(u) = \int_M s \cdot u, \quad u \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|).$$

Furthermore, for $k \in \{0\} \cup N \cup \{\infty\}$, let U_E^k be either $\Gamma^k(E)$ or $\Gamma_0^k(E)$.

We first note that if $s_1, s_2, \ldots, s_m$ $(m \in N)$ are linearly independent in U_E^k, then there exist m elements $\sigma_i \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ such that $T_{s_i}(\sigma_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$. Indeed, $T_{s_1}, \ldots, T_{s_m}$ are linearly independent in the vector space $\{T_s : s \in U_E^k\}$ over K. Therefore there exist m elements $\tau_i \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ $(i=1, 2, \ldots, m)$ such that the $m \times m$ matrix $A = (T_{s_i}(\tau_j))_{1 \leq i,j \leq m}$ is nonsingular; for a simple nonstandard proof (in a more general setting), see [6, Lemma 1.1]. Then we have only to put $\sigma_j = \sum_{i=1}^m b_{ij} \tau_i$ where (b_{ij}) is the inverse of the matrix A.

Now, for $r \in \{0\} \cup N \cup \{\infty\}$, let $\mathcal{G}^r$ [resp. $\mathcal{G}_0^r$] be the internal set of all $\zeta \in *(\Gamma^r((E^\dagger \otimes |\wedge_M|) \boxtimes F))$ of the form

(3.1) $\zeta(x, y) = \sum_{i=1}^\nu u_i(x) \otimes v_i(y)$ $(x \in *M, y \in *N)$

for some $\nu \in *N$, $u_i \in *(\Gamma_0^\infty(E^\dagger \otimes |\wedge_M|))$, $v_i \in *(\Gamma^r(F))$ [resp. $v_i \in *(\Gamma_0^r(F))$] $(i = 1, 2, \ldots, \nu)$.

Theorem 3.1. *Let* k, $r \in \{0\} \cup N \cup \{\infty\}$ *be fixed. Let* U_E^k *be as above. Suppose that* $L : U_E^k \to \Gamma^r(F)$ *is a* K-*linear map. Then there exists an element* $\Phi_L \in \mathscr{G}^r$ *such that*

$$*(L(s))(y) = \int_{x \in {}^*M} *s(x) \cdot \Phi_L(x, y) \quad (s \in U_E^k, \ y \in {}^*N).$$

Moreover, if $L(s) \in \Gamma_0^r(F)$ *for all* $s \in U_E^k$, *then* Φ_L *can be chosen from* $\mathscr{G}_0^r$.

Proof. For $s \in U_E^k$, define an internal map $G_s : \mathscr{G}^r \to {}^*(\Gamma^r(F))$ by

$$G_s(\zeta)(y) := \int_{x \in {}^*M} *s(x) \cdot \zeta(x, y) = \sum_{i=1}^{\nu} (*(T_s)(u_i)) v_i(y) \quad (y \in {}^*N),$$

where $\zeta \in \mathscr{G}^r$ is as in (3.1). Let $\mathscr{B}_s$ be the internal set

$$\mathscr{B}_s := \{\zeta \in \mathscr{G}^r : G_s(\zeta) = *(L(s))\}.$$

We shall show that the family $\{\mathscr{B}_s : s \in U_E^k\}$ has the finite intersection property. To do this, let $P(m)$ $(m \in N)$ be the following proposition:

For $s_i \in U_E^k$, $i = 1, 2, \ldots, m$, the system of equations

$$(3.2) \qquad\qquad G_{s_i}(\zeta) = *(L(s_i)) \quad (i = 1, 2, \ldots, m)$$

has a solution ζ in $\mathscr{G}^r$.

Consider first the case $m = 1$. If $s_1 = 0$, any $\zeta \in \mathscr{G}_0^r$ satisfies (3.2) for $m = 1$. If $s_i \neq 0$, then there exists an element $\tau \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ with $T_{s_1}(\tau) \neq 0$ and thus we can choose $\sigma \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ such that $T_{s_1}(\sigma) = 1$; therefore, if we let

$$\zeta(x, y) = *\sigma(x) \otimes *(L(s_1))(y) \quad (x \in {}^*M, \ y \in {}^*N),$$

then $\zeta \in \mathscr{G}^r$ and moreover this ζ satisfies (3.2) for $m = 1$, since $*(T_{s_1})(*\sigma) = T_{s_1}(\sigma) = 1$. Hence $P(1)$ is true.

Next, assume that $m > 1$ and that $P(m-1)$ is true. If $s_1, \ldots, s_m$ are linearly dependent in U_E^k, then we may assume that $s_m = \sum_{i=1}^{m-1} a_i s_i$ $(a_i \in K)$ without loss of generality, so that the system (3.2) is equivalent to the system of equations for $i = 1, 2, \ldots, m-1$. If $s_1, \ldots, s_m$ are linearly independent, then, as noticed earlier, we can choose $\sigma_i \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ such that $T_{s_i}(\sigma_j) = \delta_{ij}$ $(i, j = 1, \ldots, m)$; so, if we let

$$\zeta(x, y) = \sum_{j=1}^{m} *\sigma_j(x) \otimes *(L(s_j))(y) \quad (x \in {}^*M, \ y \in {}^*N),$$

then ζ belongs to $\mathscr{G}^r$ and satisfies (3.2). Thus $P(m-1)$ implies $P(m)$.

Hence $P(m)$ is true for all $m \in N$. Then, by the saturation principle, the intersection $\mathscr{B} = \bigcap_{s \in U_E^k} \mathscr{B}_s$ is nonempty; accordingly, there exists an element $\Phi_L \in \mathscr{B}$.

If $L(s) \in \Gamma_0^r(F)$ for all $s \in U_E^k$, then by replacing $\mathscr{G}^r$ with $\mathscr{G}_0^r$ in the above

discussion, we see that Φ_L can be chosen from $\mathscr{G}_0^{\tau}$. $\square$

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