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Author(s)	Akiyama, Hiroshi
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NONSTANDARD REPRESENTATIONS OF GENERALIZED SECTIONS OF VECTOR BUNDLES

HIROSHI AKIYAMA

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Introduction

In terms of nonstandard analysis, Todorov ([8], [9]) showed that every Schwartz distribution on $\boldsymbol{R}^n$ can be represented by a $*$-integral with ${}^*C^\infty$ internal kernel function without the necessity of saying, "up to an infinitesimal" (for the case $n=1$, see also [5]). From the differential-geometric viewpoint, it would be desirable to obtain nonstandard representations of generalized sections of vector bundles in an intrinsic manner.

The main purpose of this note is to prove in a simple way that every generalized section (see §1 or [2]) T of a C^∞ vector bundle E over a σ-compact manifold M can be represented by a $*$-integral in the sense that there exists a ${}^*C^\infty$ internal section β_T of the nonstandard extension ${}^*E$ of E such that $T(u)=\int_{{}^*M}\beta_T\cdot{}^*u$ for every compactly supported C^∞ section u of $E^\dagger\otimes|\wedge_M|$, where $E^\dagger$ is the dual bundle of E and $|\wedge_M|$ stands for the density bundle over M.

After devoting §1 to some notational preliminaries, we obtain in §2 nonstandard representations of linear maps from the space of C^∞ sections of E with compact support and then the desired representations of generalized sections. In § 3 we get a result on nonstandard representations of linear maps either from the space of C^k sections of E or from its subspace consisting of compactly supported sections.

As for nonstandard analysis, see, e.g., [1], [3], or [4]; we work with a sufficiently saturated nonstandard model.

1. Notational preliminaries

Throughout the paper we let $\boldsymbol{K}$ be either $\boldsymbol{R}$ (the real numbers) or $\boldsymbol{C}$ (the complex numbers). Furthermore, let $\boldsymbol{N}$ be the (strictly) positive integers and ${}^*\boldsymbol{N}_\infty$ the infinite elements in ${}^*\boldsymbol{N}$.

By a vector bundle π_E: $E\rightarrow M$ we mean a C^∞ vector bundle with typical fiber $\boldsymbol{K}^p$ (for some $p\in\boldsymbol{N}$) over a σ-compact C^∞ manifold M with dim $M\in\boldsymbol{N}$. For each

$x\in M$, we write $E_x:=\pi_E^{-1}(x)$, the fiber of E over x. The dual bundle of E is denoted by $E^\dagger$. For $k\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$, let $\Gamma^k(E)$ be the space of all C^k sections of E and $\Gamma_0^k(E)$ the space of C^k sections of E with compact support.

The $\boldsymbol{K}$-line bundle of densities over M is denoted by $|\wedge_M|$. Given a C^∞ Riemannian metric g on M, we let $dv_g\in\Gamma^\infty(|\wedge_M|)$ be the Riemannian volume density associated with g (see [7]). A *generalized section* of a vector bundle $E\to M$ is defined as a continuous linear functional on the space $\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ (endowed with the canonical LF-topology); cf. [2].

For two vector bundles $E\to M$ and $F\to N$, we denote by $E\boxtimes F$ the vector bundle over $M\times N$ such that $(E\boxtimes F)_{(x,y)}=E_x\otimes F_y$ for every $(x, y)\in M\times N$. We will write $\otimes$ and $\boxtimes$ for $^*\otimes$ and $^*\boxtimes$, respectively.

2. Nonstandard representations of generalized sections of vector bundles

Let $E\to M$ be a vector bundle. Choose a C^∞ Riemannian metric g on M. By the saturation principle, there exists a hyperfinite-dimensional vector subspace V of the internal vector space $^*(\Gamma_0^\infty(E))$ such that $^\sigma(\Gamma_0^\infty(E)):=\{^*s: s\in\Gamma_0^\infty(E))\}$ is an external subset of V. Take a C^∞ fiber metric h in E and pick $\psi_i\in V$ $(i=1, 2, \dots, \eta$ with $\eta={}^*\dim V\in{}^*\boldsymbol{N}_\infty)$ such that

$$(2.1)\qquad \int_{^*M}{}^*h(\psi_i, \psi_j)\,{}^*dv_g=\delta_{ij}\text{ (Kronecker delta)};\ i, j=1, 2, \dots, \eta.$$

Regard $\psi_j^h:={}^*h(\cdot, \psi_j)$ as an element of $^*(\Gamma_0^\infty(E^\dagger))$ in a natural manner. Define an internal section $\Psi\in{}^*(\Gamma_0^\infty(E^\dagger\boxtimes E))$ by

$$(2.2)\qquad \Psi(x, y):=\sum_{i=1}^{\eta}\psi_i^h(x)\otimes\psi_i(y)\ (x, y\in{}^*M).$$

Moreover, define $\widetilde{\Psi}\in{}^*(\Gamma_0^\infty((E^\dagger\otimes|\wedge_M|)\boxtimes E))$ by

$$(2.3)\qquad \widetilde{\Psi}(x, y):=\sum_{i=1}^{\eta}(\psi_i^h(x)\otimes{}^*dv_g(x))\otimes\psi_i(y)\ (x, y\in{}^*M).$$

Proposition 2.1. *Given a vector bundle $E\to M$ and a C^∞ Riemannian metric g on M, let Ψ and $\widetilde{\Psi}$ be as in* (2.2) *and* (2.3), *respectively.*

(1) *For every $s\in\Gamma_0^\infty(E)$ and every $y\in{}^*M$,*

$$(2.4)\qquad {}^*s(y)=\int_{x\in{}^*M}{}^*s(x)\cdot\widetilde{\Psi}(x, y).$$

[*The map $^*M\times{}^*M\ni(x, y)\mapsto{}^*s(x)\cdot\widetilde{\Psi}(x, y)\in{}^*(|\wedge_M|\boxtimes E)_{(x,y)}$ gives an element of $^*(\Gamma_0^\infty(|\wedge_M|\boxtimes E))$ obtained from $^*s(x)\otimes\widetilde{\Psi}(x, y)$ by the canonical pairing between $^*E_x:={}^*(\pi_E)^{-1}(x)$ and $^*E_x^\dagger$.*]

(2) *For every $x, z\in{}^*M$,*

$$(2.5)\qquad \int_{y\in{}^*M}\Psi(x, y)\cdot\Psi(y, z)\otimes{}^*dv_g(y)=\Psi(x, z).$$

Proof. Since every ${}^*s \in {}^{\sigma}(\Gamma_0^\infty(E))$ is expressed as

$$(2.6)\qquad {}^*s=\sum_{j=1}^{\eta} c_j({}^*s)\psi_j \text{ with } c_j({}^*s)=\int_{{}^*M} {}^*s\cdot\psi_j^h\, {}^*dv_g,$$

we have (2.4). Formula (2.5) follows immediately from (2.1). □

Proposition 2.2. *Let $E\to M$ and $F\to N$ be vector bundles. Let $\widetilde{\Psi}\in {}^*(\Gamma_0^\infty((E^\dagger\otimes|\wedge_M|)\boxtimes E))$ be as in* (2.3). *For a $\boldsymbol{K}$-linear map $L: \Gamma_0^\infty(E)\to\Gamma^\infty(F)$, define $\Psi_L\in{}^*(\Gamma^\infty((E^\dagger\otimes|\wedge_M|)\boxtimes F))$ by*

$$\Psi_L(y, z):=(I_{{}^*(E^\dagger\otimes|\wedge_M|)_y}\otimes{}^*L)(\widetilde{\Psi}(y,\cdot))(z)=\sum_{i=1}^{\eta}(\psi_i^h(y)\otimes{}^*dv_g(y))\otimes{}^*L(\psi_i)(z)$$

for $y\in{}^*M$ and $z\in{}^*N$, where $I_{{}^*(E^\dagger\otimes|\wedge_M|)_y}$ is the identity transformation of ${}^*(E^\dagger\otimes|\wedge_M|)_y$. Then L is represented as

$${}^*(L(s))(z)=\int_{y\in{}^*M}{}^*s(y)\cdot\Psi_L(y, z)\ (s\in\Gamma_0^\infty(E),\ z\in{}^*N).$$

Proof. Let $s\in\Gamma_0^\infty(E)$ and $z\in{}^*N$. By the expression (2.6),

$$\begin{aligned}{}^*(L(s))(z)&={}^*L({}^*s)(z)={}^*L(\sum_{i=1}^{\eta}c_i({}^*s)\psi_i)(z)\\&=\sum_{i=1}^{\eta}c_i({}^*s){}^*L(\psi_i)(z)=\int_{y\in{}^*M}{}^*s(y)\cdot\Psi_L(y,z).\end{aligned}$$ □

We can now represent every generalized section of E by a $*$-integral.

Theorem 2.3. *Let $E\to M$ be a vector bundle, and let $\widetilde{\Psi}$ be as in* (2.3). *For each generalized section T of E, define $\beta_T\in{}^*(\Gamma_0^\infty(E))$ by*

$$\beta_T(y):=({}^*T\otimes I_{{}^*E_y})(\widetilde{\Psi}(\cdot, y))=\sum_{i=1}^{\eta}{}^*T(\psi_i^h\otimes{}^*dv_g)\psi_i(y)\ (y\in{}^*M).$$

Then

$$T(u)=\int_{{}^*M}\beta_T\cdot{}^*u,\ u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|).$$

Proof. Apply Proposition 2.2 to the $\boldsymbol{K}$-linear map $L_T: \Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)\ni u \mapsto L_T(u)\in\Gamma^\infty(M\times\boldsymbol{K})$ defined by $L_T(u)(z)=(z, T(u))$ for $z\in M$. □

REMARK. If there exists a section $s\in\Gamma_0^\infty(E)$ such that $T(u)=\int_M s\cdot u$ for all $u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$, then ${}^*T(\psi_i^h\otimes{}^*dv_g)=c_i({}^*s)$ (see (2.6)) and thus $\beta_T={}^*s$.

EXAMPLE. Given a σ-compact C^∞ Riemannian manifold (M, g), we obtain a "nonstandard delta function" with respect to ${}^*dv_g$ using the above results. In fact, let $C^\infty(M\,;\,\boldsymbol{K})$ denote the space of $\boldsymbol{K}$-valued C^∞ functions on M and let $C_0^\infty(M\,;\,\boldsymbol{K}):=\{f\in C^\infty(M\,;\,\boldsymbol{K}):\ \mathrm{supp}(f)$ compact$\}$. Let V_0 be a hyperfinite-dimensional vector space over ${}^*\boldsymbol{R}$ such that

$$^\sigma(C_0^\infty(M\,;\,\boldsymbol{R})):=\{{}^*f:\ f\in C_0^\infty(M\,;\,\boldsymbol{R})\}\subset V_0\subset{}^*(C_0^\infty(M\,;\,\boldsymbol{R})).$$

Pick $\varphi_i\in V_0$ ($i=1,\,2,\,\ldots,\,\nu$ with $\nu={}^*\dim V_0$) such that $\int_{{}^*M}\varphi_i\varphi_j\,{}^*dv_g=\delta_{ij}$ ($i,\,j=1,\,2,\,\ldots,\,\nu$) and define an internal function $\delta\in{}^*(C_0^\infty(M\times M\,;\,\boldsymbol{R}))$ by $\delta(x,\,y):=\sum_{i=1}^{\nu}\varphi_i(x)\varphi_i(y)$ $(x,\ y\in{}^*M)$. Noting that $C_0^\infty(M\,;\,\boldsymbol{C})=C_0^\infty(M\,;\,\boldsymbol{R})+\sqrt{-1}C_0^\infty(M\,;\,\boldsymbol{R})$, for $x,\,y,\,z\in{}^*M$ we have:

(1) $\delta(x,\,x)\geq 0$, $\delta(x,\,y)=\delta(y,\,x)$, and $(\delta(x,\,y))^2\leq\delta(x,\,x)\delta(y,\,y)$.

(2) ${}^*f(x)=\int_{y\in{}^*M}\delta(x,\,y){}^*f(y)\,{}^*dv_g(y)$ for $f\in C_0^\infty(M\,;\,\boldsymbol{C})$.

(3) $\int_{y\in{}^*M}\delta(x,\,y)\delta(y,\,z)\,{}^*dv_g(y)=\delta(x,\,z)$.

(4) If $T:\ \mathcal{D}=C_0^\infty(M\,;\,\boldsymbol{C})\to\boldsymbol{C}$ is a Schwartz distribution on M, then $T(f)=\int_{y\in{}^*M}{}^*f(y)\gamma_T(y)\,{}^*dv_g(y)$ $(f\in\mathcal{D})$, where $\gamma_T\in{}^*(C_0^\infty(M\,;\,\boldsymbol{C}))$ is defined by $\gamma_T(y):={}^*T(\delta(\cdot.\ y))=\sum_{i=1}^{\nu}{}^*T(\varphi_i)\varphi_i(y)$ $(y\in{}^*M)$.

3. Nonstandard representations of linear maps from $\Gamma^k(E)$ or $\Gamma_0^k(E)$ to $\Gamma^r(F)$

Let $E\to M$ and $F\to N$ be vector bundles. For $s\in\Gamma^0(E)$, define $T_s:\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)\to\boldsymbol{K}$ by

$$T_s(u)=\int_M s\cdot u,\quad u\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|).$$

Furthermore, for $k\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$, let U_E^k be either $\Gamma^k(E)$ or $\Gamma_0^k(E)$.

We first note that if $s_1,\,s_2,\,\ldots,\,s_m$ $(m\in\boldsymbol{N})$ are linearly independent in U_E^k, then there exist m elements $\sigma_i\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ such that $T_{s_i}(\sigma_j)=\delta_{ij}$ $(i,\,j=1,\,2,\,\ldots,\,m)$. Indeed, $T_{s_1},\,\ldots,\,T_{s_m}$ are linearly independent in the vector space $\{T_s:\ s\in U_E^k\}$ over $\boldsymbol{K}$. Therefore there exist m elements $\tau_i\in\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|)$ $(i=1,\,2,\,\ldots,\,m)$ such that the $m\times m$ matrix $A=(T_{s_i}(\tau_j))_{1\leq i,j\leq m}$ is nonsingular; for a simple nonstandard proof (in a more general setting), see [6, Lemma 1.1]. Then we have only to put $\sigma_j=\sum_{i=1}^{m}b_{ij}\tau_i$ where (b_{ij}) is the inverse of the matrix A.

Now, for $r\in\{0\}\cup\boldsymbol{N}\cup\{\infty\}$, let $\mathcal{G}^r$ [resp. $\mathcal{G}_0^r$] be the internal set of all $\zeta\in{}^*(\Gamma^r((E^\dagger\otimes|\wedge_M|)\boxtimes F))$ of the form

$$\zeta(x,\,y)=\sum_{i=1}^{\nu}u_i(x)\otimes v_i(y)\quad(x\in{}^*M,\ y\in{}^*N)\tag{3.1}$$

for some $\nu\in{}^*\boldsymbol{N}$, $u_i\in{}^*(\Gamma_0^\infty(E^\dagger\otimes|\wedge_M|))$, $v_i\in{}^*(\Gamma^r(F))$ [resp. $v_i\in{}^*(\Gamma_0^r(F))$] $(i=1,\,2,\,\ldots,\,\nu)$.

Theorem 3.1. *Let k, $r \in \{0\} \cup \boldsymbol{N} \cup \{\infty\}$ be fixed. Let U_E^k be as above. Suppose that $L: U_E^k \to \Gamma^r(F)$ is a $\boldsymbol{K}$-linear map. Then there exists an element $\Phi_L \in \mathcal{G}^r$ such that*

$$*(L(s))(y) = \int_{x \in *M} *s(x) \cdot \Phi_L(x, y) \ (s \in U_E^k,\ y \in *N).$$

Moreover, if $L(s) \in \Gamma_0^r(F)$ for all $s \in U_E^k$, then Φ_L can be chosen from $\mathcal{G}_0^r$.

Proof. For $s \in U_E^k$, define an internal map $G_s: \mathcal{G}^r \to *(\Gamma^r(F))$ by

$$G_s(\zeta)(y) := \int_{x \in *M} *s(x) \cdot \zeta(x, y) = \sum_{i=1}^{\nu} (*(T_s)(u_i)) v_i(y) \ (y \in *N),$$

where $\zeta \in \mathcal{G}^r$ is as in (3.1). Let $\mathcal{B}_s$ be the internal set

$$\mathcal{B}_s := \{\zeta \in \mathcal{G}^r : G_s(\zeta) = *(L(s))\}.$$

We shall show that the family $\{\mathcal{B}_s : s \in U_E^k\}$ has the finite intersection property. To do this, let $P(m)$ $(m \in \boldsymbol{N})$ be the following proposition:

For $s_i \in U_E^k$, $i = 1, 2, \ldots, m$, the system of equations

$$G_{s_i}(\zeta) = *(L(s_i)) \ (i = 1, 2, \ldots, m) \tag{3.2}$$

has a solution ζ in $\mathcal{G}^r$.

Consider first the case $m=1$. If $s_1 = 0$, any $\zeta \in \mathcal{G}_0^r$ satisfies (3.2) for $m=1$. If $s_i \neq 0$, then there exists an element $\tau \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ with $T_{s_1}(\tau) \neq 0$ and thus we can choose $\sigma \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ such that $T_{s_1}(\sigma) = 1$; therefore, if we let

$$\zeta(x, y) = *\sigma(x) \otimes *(L(s_1))(y) \ (x \in *M,\ y \in *N),$$

then $\zeta \in \mathcal{G}^r$ and moreover this ζ satisfies (3.2) for $m=1$, since $*(T_{s_1})(*\sigma) = T_{s_1}(\sigma) = 1$. Hence $P(1)$ is true.

Next, assume that $m > 1$ and that $P(m-1)$ is true. If $s_1, \ldots, s_m$ are linearly dependent in U_E^k, then we may assume that $s_m = \sum_{i=1}^{m-1} a_i s_i$ $(a_i \in \boldsymbol{K})$ without loss of generality, so that the system (3.2) is equivalent to the system of equations for $i = 1, 2, \ldots, m-1$. If $s_1, \ldots, s_m$ are linearly independent, then, as noticed earlier, we can choose $\sigma_i \in \Gamma_0^\infty(E^\dagger \otimes |\wedge_M|)$ such that $T_{s_i}(\sigma_j) = \delta_{ij}$ $(i, j = 1, \ldots, m)$; so, if we let

$$\zeta(x, y) = \sum_{j=1}^{m} *\sigma_j(x) \otimes *(L(s_j))(y) \ (x \in *M,\ y \in *N),$$

then ζ belongs to $\mathcal{G}^r$ and satisfies (3.2). Thus $P(m-1)$ implies $P(m)$.

Hence $P(m)$ is true for all $m \in \boldsymbol{N}$. Then, by the saturation principle, the intersection $\mathcal{B} = \bigcap_{s \in U_E^k} \mathcal{B}_s$ is nonempty; accordingly, there exists an element $\Phi_L \in \mathcal{B}$.

If $L(s) \in \Gamma_0^r(F)$ for all $s \in U_E^k$, then by replacing $\mathcal{G}^r$ with $\mathcal{G}_0^r$ in the above discussion, we see that Φ_L can be chosen from $\mathscr{G}_0^r$. □

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Department of Applied Mathematics
Faculty of Engineering
Shizuoka University
Hamamatsu 432, Japan