Ikegami, T. and Nishio, M.
Osaka J. Math.
39 (2002), 931–944

Q-COMPACTIFICATION OF HARMONIC SPACES AND THE CHOQUET SIMPLEX

TERUO IKEGAMI and MASAHARU NISHIO

(Received January 30, 2001)

Introduction

In 1976, P.A. Loeb [10] constructed a compactification of a harmonic space X (see also [11]). This new compactification is deeply connected with Choquet theory. When X has the Green function we can construct the Martin compactification X^M, but Loeb's compactification is different from X^M. Afterwards, in the joint-works [1], [2], J. Bliedtner and P.A. Loeb deepened the simplicial consideration of Loeb's compactification, and extended their theory in more extensive framework including the notion of sturdy harmonic functions.

As for the compactification of harmonic spaces, extending the idea of Loeb, we construct for an arbitrary metrizable and resolutive compactification X^* a compactification $\widehat{X}$ that enables simplicial considerations. In the case where every bounded harmonic function is obtained by PWB method, $\widehat{X}$ is just Loeb's compactification. Examples in what follows will reveal the relation between $\widehat{X}$ and the Choquet simplex.

0. Notations and Assumptions

X : a $\mathcal{P}$-harmonic space of Constantinescu-Cornea with a countable base.
λ : a normalized reference measure, i.e., $\lambda \in M^+(X) = \{$the set of positive Borel measures on $X\}$, $\lambda(X) = 1$, the smallest absorbing set containing the support $S(\lambda)$ of λ is X.
X^* : a metrizable and resolutive compactification, i.e., every continuous function f on $\Delta^* = X^* \setminus X$ has the PWB solution $H_f^{X^*}$.
χ_x^*: the harmonic measure of X^* at $x \in X$, i.e., $H_f^{X^*}(x) = \int f \, d\chi_x^*$.
μ^*: the dilation of λ, i.e., $\mu^* = \int \chi_x^* \, d\lambda$.
Γ^*: the harmonic boundary of X^*, i.e., the closure of $[\bigcup_{x\in X}(\text{support of } \chi_x^*)]$.
$\mathcal{H}_1^\lambda$: $= \{u \in H^+(X); \int u \, d\lambda \leq 1\}$.

We note that $\mathcal{H}_1^\lambda$ is a Choquet simplex ([9]).

For $Q \subset \mathbf{C}(X)$, the Q-compactification X^Q of X is the compactification $\overline{X}$ of X where every function of Q is extended continuously to $\overline{X}$ and the set of extensions separates points of $\Delta := \overline{X} \setminus X$ ([4]).

We assume:

(1) $1 \in H(X)$,
(2) the Doob convergence axiom ([5]).

1. General Theory

Proposition 1.1. *χ_x^* is absolutely continuous with respect to μ^* for every $x \in X$. Therefore $d\chi_x^*/d\mu^*$ has a Borel representative $k(x, z) = k^x(z)$.*

Proof. Fix $A \in \mathcal{B}(\Delta^*) = \{$the σ-field of Borel subsets of $\Delta^*\}$ with $\mu^*(A) = 0$. By MCT ([3, Theorem 2.3]),

$$\mu^*(A) = \int \left[\int 1_A \, d\chi_x^* \right] d\lambda(x) = 0,$$

which implies

$$H_{1_A}^{X^*}(x) = \int 1_A \, d\chi_x^* = 0 \qquad \text{for } \lambda\text{-a.e. } x.$$

Thus we have

$$S(\lambda) \subset \{x \in X;\ H_{1_A}^{X^*}(x) = 0\},$$

and since the latter set is absorbing

$$H_{1_A}^{X^*}(x) = \int 1_A \, d\chi_x^* \equiv 0,$$

i.e., $\chi_x^*(A) = 0$ for every $x \in X$. □

Proposition 1.2. *$k^x \in L^\infty(\mu^*)$ for every $x \in X$, and thus $d\chi_x^*/d\mu^*$ has a bounded Borel representative.*

Proof. First we remark that for $f \in \mathcal{B}(\Delta^*)$, $f \in L^1(\mu^*)$ implies $f \in L^1(\chi_x^*)$ for every $x \in X$. In fact

$$\int |f| \, d\mu^* = \int \left[\int |f| \, d\chi_x^* \right] d\lambda(x) < \infty$$

implies $S(\lambda)$ is included in the closure of $\{x \in X;\ H_{|f|}^{X^*}(x) < +\infty\}$, which is absorbing ([5, Proposition 6.1.4]). Therefore $|f|$ is χ_x^*-integrable for every x in a dense subset of X, and we conclude that $|f|$ is resolutive, and $H_f^{X^*}(x)$ is a linear functional on $L^1(\mu^*)$ for every $x \in X$.

Next we shall show that this functional is bounded, i.e., $|H_f^{X^*}(x)| \le c \int |f| \, d\mu^*$

for every $f \in L^1(\mu^*)$. By a version of Harnack's inequality ([5, Proposition 6.1.5])

$$|H_f^{X^*}(x)| \leq H_{|f|}^{X^*}(x) \leq c \int H_{|f|}^{X^*}(x)\, d\lambda = c \int \left[\int |f|\, d\chi_x^* \right] d\lambda(x) = c \int |f|\, d\mu^*.$$

Thus, there exists $\varphi_x \in L^\infty(\mu^*) \cap \mathcal{B}(\Delta^*)$ such that $H_f^{X^*}(x) = \int f\, \varphi_x\, d\mu^*$. This implies $\varphi_x = k^x \quad \mu^*$-a.e. □

Now we define

$$q(x, y) = \int k^x\, k^y\, d\mu^*,$$
$$\widehat{X} = X^{\{q(x,\,\cdot\,);\, x \in X\}} \qquad .$$

We remark $q(x, y) = q(y, x) = H_{k^y}^{X^*}(x) \in HB(X)$ for every $y \in X$.

Proposition 1.3. *$\widehat{X}$ is a metrizable and resolutive compactification of X.*

Proof. The resolutivity of $\widehat{X}$ is derived as in [4, Satz 9.3]. To see the metrizability, we take a countable dense subset $\{x_n\}$ of X and construct the Q-compactification $X^{Q'} = X^{\{q(x_n, y)\}}$, which is metrizable. We assert $X^{Q'} \simeq \widehat{X}$. In fact, $\{q(x, y);\, y \in X\}$ is locally uniformly bounded. For a compact subset K of X, there exists a constant α_K satisfying

$$\sup\{q(x, y);\, x \in K\} \leq \alpha_K \int q(x, y)\, d\lambda(x) = \alpha_K \int k^y\, d\mu^* = \alpha_K \int d\chi_y^* = \alpha_K \forall y \in X.$$

Therefore $\{q(x, y);\, y \in X\}$ is equicontinuous ([5, Theorem 11.1.1]). Now for $x \in X$ and $z \in \Delta^{Q'} = X^{Q'} \setminus X$, let $\{x_n'\} \subset \{x_n\}$ with $x_n' \to x$ and $\{y_n\} \subset X$ with $y_n \to z$ in $X^{Q'}$. Then $\{q(x, y_n)\}$ is a Cauchy sequence. For in the inequality

$$\begin{aligned} &|q(x, y_m) - q(x, y_{m'})| \\ &\qquad \leq |q(x, y_m) - q(x_n', y_m)| + |q(x_n', y_m) - q(x_n', y_{m'})| + |q(x_n', y_{m'}) - q(x, y_{m'})|, \end{aligned}$$

the first and third terms of the right hand side become arbitrary small when n is sufficiently large, and the second term is small for fixed n when m and m' are suficiently large. Thus $\lim_{n\to\infty} q(x, y_n)$ exists for every $x \in X$, which means that $q(x, y)$ is extended to $X^{Q'}$ continuously for every $x \in X$ and the extensions separate points of $\Delta^{Q'}$. □

We denote by $\hat{q}(x, \hat{z}) = \hat{q}^x(\hat{z}) = \hat{q}_{\hat{z}}(x)$ the continuous extension of $q(x, y)$ to $\hat{z} \in \widehat{\Delta} = \widehat{X} \setminus X$. Obviously we have $H_{\hat{q}^y}^{\widehat{X}}(x) = q(x, y)$.

The following proposition is clear.

Proposition 1.4. *The mapping $T : T\hat{z} = \hat{q}_{\hat{z}}$ is a continuous injection of $\widehat{\Delta}$ to $\mathcal{H}_1^\lambda$.*

We define

$$\widehat{\Delta}_1 = \left\{\hat{z} \in \widehat{\Delta};\ \hat{q}_{\hat{z}} \text{ is minimal harmonic, } \int \hat{q}_{\hat{z}}\, d\lambda = 1\right\}.$$

Then, $T(\widehat{\Delta}_1) \subset \mathrm{ext}.\mathcal{H}_1^\lambda$ (the set of extreme points of $\mathcal{H}_1^\lambda$).

Now we have a result from the simplicial aspect of the theory.

Theorem 1.1. *If $\widehat{\Gamma} \subset \widehat{\Delta}_1$, then $\widehat{X}$ is a semi-regular compactification, i.e., $H_{\hat{f}}^{\widehat{X}}$ is extended to $\widehat{X}$ continuously for every $\hat{f} \in \mathbf{C}(\widehat{\Delta})$.*

Proof. $\overline{X} = X^{\{\widehat{F}|_X;\, \widehat{F}\in\mathbf{C}(\widehat{\Delta})\}\cup\{H_{\hat{f}}^{\widehat{X}};\, \hat{f}\in\mathbf{C}(\widehat{\Delta})\}}$ is semi-regular ([7, p. 890]). Let π be the canonical mapping of $\overline{X}$ to $\widehat{X}$, i.e., π is a surjection and $\pi(x) = x$ for every $x \in X$. We assert that π is the bijection. If this is not the case, then for some $\hat{z}_0 \in \widehat{\Delta}$ there exist $\overline{z}_1, \overline{z}_2 \in \overline{\Delta}$ $(= \overline{X} \setminus X)$ such that $\overline{z}_1 \neq \overline{z}_2$ and $\pi(\overline{z}_1) = \pi(\overline{z}_2) = \hat{z}_0$. Further we may find $\{y_j^{(1)}\}$, $\{y_j^{(2)}\} \subset X$ and $\hat{f} \in \mathbf{C}(\widehat{\Delta})$ satisfying $y_j^{(i)} \to \overline{z}_i$ in $\overline{X}$ and $y_j^{(i)} \to \hat{z}_i$ in $\widehat{X}$ $(i = 1, 2)$, and $\lim_{j\to\infty} H_{\hat{f}}^{\widehat{X}}(y_j^{(1)}) \neq \lim_{j\to\infty} H_{\hat{f}}^{\widehat{X}}(y_j^{(2)})$. If necessary, taking subsequences, we may assume $\widehat{\chi}_{y_j^{(i)}} \to \nu_i$ vaguely $(i = 1, 2)$. $\nu_i \in M^+(\widehat{\Delta})$, $\nu_i(\widehat{\Delta} \setminus \widehat{\Gamma}) = 0$ $(i = 1, 2)$ and $\nu_1 \neq \nu_2$.

$$\hat{q}_{\hat{z}_0}(x) = \lim_{j\to\infty} q(x, y_j^{(1)}) = \lim_{j\to\infty} H_{\hat{q}^x}^{\widehat{X}}(y_j^{(1)}) = \lim_{j\to\infty} \int \hat{q}^x\, d\widehat{\chi}_{y_j^{(1)}} = \int \hat{q}^x\, d\nu_1 = \int_{\widehat{\Delta}_1} \hat{q}^x\, d\nu_1.$$

Similarly $\hat{q}_{\hat{z}_0}(x) = \int_{\widehat{\Delta}_1} \hat{q}^x\, d\nu_2$, which implies that ν_1 and ν_2 are the canonical reprezentation measures of $\hat{q}_{\hat{z}_0} \in \mathcal{H}_1^\lambda$. Since $\mathcal{H}_1^\lambda$ is the Choquet simplex $\nu_1 = \nu_2$, thus we have the contradiction. □

Corollary 1. *If $\widehat{\Gamma} \subset \widehat{\Delta}_1$ then $\widehat{\Gamma} = \widehat{\Delta}_1 = \mathrm{reg}(\widehat{X})$, where $\mathrm{reg}(\widehat{X})$ denotes the set of all regular points of $\widehat{X}$.*

Proof. Fix $\hat{z}_0 \in \widehat{\Delta}_1$ and $\{y_j\} \subset X$ such that $y_j \to \hat{z}_0$ in $\widehat{X}$ and $\widehat{\chi}_{y_j} \to \nu$ vaguely. Then

$$\hat{q}_{\hat{z}_0}(x) = \int_{\widehat{\Delta}_1} \hat{q}_{\hat{z}}(x)\, d\nu(\hat{z}) = \int_{\widehat{\Delta}_1} \hat{q}_{\hat{z}}(x)\, d\varepsilon_{\hat{z}_0}(\hat{z}),$$

which means $\nu = \varepsilon_{\hat{z}_0}$, i.e., $\hat{z}_0 \in \mathrm{reg}(\widehat{X})$. Thus $\widehat{\Gamma} \subset \widehat{\Delta}_1 \subset \mathrm{reg}(\widehat{X}) = \widehat{\Gamma}$. □

We are going to investigate the relation between X^* and $\widehat{X}$. To this purpose, we define the Q-compactification $\tilde{X} = X^{\{F|_X;\, F\in\mathbf{C}(X^*)\}\cup\{q(x,y);\, x\in X\}}$ and denote by $\tilde{q}(x, \tilde{z})$ the continuous extention of $q(x, y)$ to $\tilde{z} \in \widetilde{\Delta} = \widetilde{X} \setminus X$.

$\widetilde{X}$ is a refinement of X^* and $\widehat{X}$, i.e., there exists the canonical mapping π^* (resp. $\hat{\pi}$ of $\widetilde{X}$ onto X^* (resp. $\widehat{X}$), which is a continuous surjection, and $\pi^*(x) = x$ (resp. $\hat{\pi}(x) = x$) for every $x \in X$.

Theorem 1.2. *The following assertions are equivalent*:
i) $\tilde{z} \mapsto \tilde{q}_{\tilde{z}}$ *is a* (*continuous*) *injection of* $\widetilde{\Delta}$ *to* $\mathcal{H}_1^\lambda$.
ii) $\widetilde{X}$ *is homeomorphic to* $\widehat{X}$.
iii) $\widehat{X}$ *is a refinement of* X^*.

Proof. i) $\Rightarrow$ ii): letting $\{y_j\} \subset X$ with $y_j \to \hat{z} \in \widehat{\Delta}$ is $\widehat{X}$, if there exist two subsequences $\{y'_j\} \subset \{y_j\}$, $\{y''_j\} \subset \{y_j\}$ and two points $\tilde{z}' \in \hat{\pi}^{-1}(\hat{z})$, $\tilde{z}'' \in \hat{\pi}^{-1}(\hat{z})$ such that $y'_j \to \tilde{z}'$ in $\widetilde{X}$, $y''_j \to \tilde{z}''$ in $\widetilde{X}$, then $\lim_{j\to\infty} q(x, y'_j) = \tilde{q}(x, \tilde{z}')$ and $\lim_{j\to\infty} q(x, y''_j) = \tilde{q}(x, \tilde{z}'')$. On the other hand $\lim_{j\to\infty} q(x, y'_j) = \lim_{j\to\infty} q(x, y''_j) = \hat{q}(x, \hat{z})$. Hence, $\tilde{q}_{\tilde{z}'} = \tilde{q}_{\tilde{z}''}$ and $\tilde{z}' = \tilde{z}''$ by the assumption, which means $\hat{\pi}$ is the bijection of $\widetilde{X}$ onto $\widehat{X}$.

ii) $\Rightarrow$ iii) is trivial and ii) $\Rightarrow$ i) is a consequence of Proposition 1.4.

To complete the proof, it remains to show iii) $\Rightarrow$ ii). Let $\{y_j\} \subset X$, $y_j \to \hat{z}$ in $\widehat{X}$ and let π be the canonical mapping of $\widehat{X}$ to X^*. Since $y_j \to \pi(\hat{z})$ in X^*, the functions in $\{F|_X;\ F \in \mathbf{C}(X^*)\} \cup \{q(x, y);\ x \in X\}$ are extended continuously to $\widehat{X}$ and separate points of $\widehat{\Delta}$, which implies $\widehat{X} \simeq \widetilde{X}$. $\square$

REMARKS. 1. If Γ^* is a singleton then $\widehat{X}$ is the one-point compactification.
2. Let $\{y_j\} \subset X$, $y_j \to z \in \Delta^*$ in X^*; then

$$\left[\exists \hat{z} \in \widehat{\Delta}; y_j \to \hat{z} \text{ in } \widehat{X}\right] \rightleftarrows \left[\exists \tilde{z} \in \widetilde{\Delta}; y_j \to \tilde{z} \text{ in } \widetilde{X}\right].$$

Here we have a question: "$\widehat{\chi}_x = \hat{q}(x, \hat{z})\,\hat{\mu},\ \forall x \in X,\ \forall \hat{z} \in \widehat{\Gamma}$?"

This is solved affirmatively in the following two cases:
1. $d\chi_x^*/d\mu^*$ has a continuous representative $k(x, z)$,
2. every bounded harmonic function is a Dirichlet solution in X^*.

The second case will be treated in the next section. For the first case, since $q(x, y) = H_{\tilde{q}^x}^{\widetilde{X}}(y) = H_{k^x}^{X^*}(y) = H_{k^x\circ\pi^*}^{\widetilde{X}}(y)$, we have $\tilde{q}^x = k^x \circ \pi^*$ $\tilde{\mu}$-a.e. on $\widetilde{\Gamma}$. From the continuity of k^x and $\tilde{q}^x$, $k^x \circ \pi^* = \tilde{q}^x$ on $\widetilde{\Gamma}$. In the same way, $\hat{q}^x \circ \hat{\pi} = \tilde{q}^x$ on $\widetilde{\Gamma}$. Further, $q(x, y) = H_{k^x}^{X^*}(y) \in \{H_f^{X^*};\ f \in \mathbf{C}(\Delta^*)\}$ implies $\widetilde{\Gamma} \simeq \Gamma^*$ ([7, Proposition 3.5]). For every $\hat{f} \in \mathbf{C}(\widehat{\Delta})$,

$$\begin{aligned}\int \hat{f}\, d\widehat{\chi}_x = H_{\hat{f}}^{\widehat{X}}(x) = H_{\hat{f}\circ\hat{\pi}}^{\widetilde{X}}(x) = \int [\hat{f}\circ\hat{\pi}]\, d\tilde{\chi}_x = \int [\hat{f}\circ\hat{\pi}]\, \frac{d\tilde{\chi}_x}{d\tilde{\mu}}\, d\tilde{\mu} \\ = \int [\hat{f}\circ\hat{\pi}]\, \tilde{q}^x\, d\tilde{\mu} = \int [\hat{f}\circ\hat{\pi}]\, [\hat{q}^x\circ\hat{\pi}]\, d\tilde{\mu} = \int \hat{f}\, \hat{q}^x\, d\hat{\mu},\end{aligned}$$

so we have $d\widehat{\chi}_x = \hat{q}^x\, d\hat{\mu}$ on $\widehat{\Gamma}$. Here we have used $d\tilde{\chi}_x/d\tilde{\mu} = k^x \circ \pi^* = \tilde{q}^x$, and this is easily derived from the following argument: for $\tilde{f} \in \mathbf{C}(\widetilde{\Delta})$ there exists an $f \in \mathcal{B}(\Delta^*)$ such that $f|_{\Gamma^*} \in \mathbf{C}(\Gamma^*)$, $\tilde{f} = f \circ \pi^*$ on $\widetilde{\Gamma}$. Then

$$\int \tilde{f}\, d\tilde{\chi}_x = H_{\tilde{f}}^{\widetilde{X}}(x) = H_{f\circ\pi^*}^{\widetilde{X}}(x) = H_f^{X^*}(x) = \int f\, d\chi_x^* = \int f\, k^x\, d\mu^*$$
$$= \int [f \circ \pi^*]\,[k^x \circ \pi^*]\, d\tilde{\mu} = \int \tilde{f}\,[k^x \circ \pi^*]\, d\tilde{\mu},$$

which implies $d\tilde{\chi}_x = [k^x \circ \pi^*]\, d\tilde{\mu} = \tilde{q}^x\, d\tilde{\mu}$ on $\widetilde{\Gamma}$.

If we set $\Psi(z) = \hat{\pi}[(\pi^*)^{-1}(z)]$ on Γ^*, Ψ is a continuous surjection of Γ^* to $\widehat{\Gamma}$. In this case,

$$k(x,z) = \tilde{q}\left(x, (\pi^*)^{-1}(z)\right) = \hat{q}\left(x, \hat{\pi}[(\pi^*)^{-1}(z)]\right) = \hat{q}\left(x, \Psi(z)\right) \quad \forall x \in X,\ \forall z \in \Gamma^*.$$

2. The case where $HB(X) \subset \{H_f^{X^*};\, f$ is resolutive$\}$

In what follows, we consider the case where every bounded harmonic function is a Dirichlet solution.

We recall the Loeb compactification X^L ([11]). Let X^W be the Wiener compactification, χ_x^W be the harmonic measure and $\mu^W = \int \chi_x^W\, d\lambda(x)$. Then $d\chi_x^W/d\mu^W$ has a continuous representative $\omega^x(\zeta)$ on $\Delta^W = X^W \setminus X$. We define

$$h(x,y) = \int \omega^x(\zeta)\, \omega^y(\zeta)\, d\mu^W(\zeta)$$

and the Loeb compactification $X^L = X^{\{h(x,y);\, x\in X\}}$. It is known that X^L is a metrizable and resolutive compactification and $\Delta_1^L \subset \mathrm{reg}(X^L)$. For every $u \in HB(X)$ there exists the canonical representation measure ν_u such that

$$u(x) = \int_{\Delta_1^L} h(x,\cdot)\, \nu_u(\cdot), \qquad \nu_u(\Delta^L \setminus \Delta_1^L) = 0.$$

Theorem 2.1. $\widehat{X} = X^L$.

Proof. Let π^W be the canonical mapping of X^W to X^*. By MCT

(1) $$\int f\, d\mu^* = \int f \circ \pi^W\, d\mu^W \qquad \text{for every} \quad f \in L^1(\mu^*).$$

For every $\varphi \in \mathbf{C}(\Delta^W)$ there exists $f_\varphi \in \mathcal{B}(\Delta^*)$, which is bounded, and $H_\varphi^{X^W} = H_{f_\varphi}^{X^*}$ by the hypothesis of this section. $H_\varphi^{X^W} = H_{f_\varphi}^{X^*} = H_{f_\varphi\circ\pi^W}^{X^W}$ implies $\varphi = f_\varphi \circ \pi^W$ $d\chi_x^W$-a.e. for every $x \in X$. $A = \{\zeta \in \Delta^W;\, \varphi(\zeta) \neq f_\varphi[\pi^W(\zeta)]\} \in \mathcal{B}(\Delta^W)$ and $\mu^W(A) = 0$. Thus,

(2) $$\varphi = f_\varphi \circ \pi^W \qquad d\mu^W\text{-a.e.}$$

Since $\int \varphi(\zeta)\,\omega^x(\zeta)\,d\mu^W(\zeta) = \int \varphi(\zeta)\,d\chi_x^W(\zeta) = H_\varphi^{X^W}(x) = H_{f_\varphi}^{X^*}(x) = \int f_\varphi\,d\chi_x^* = \int f_\varphi\,k^x\,d\mu^*$, by (1) and (2), $\int f_\varphi\,k^x\,d\mu^* = \int [f_\varphi \circ \pi^W][k^x \circ \pi^W]\,d\mu^W$ and

$$\int \varphi(\zeta)\,\omega^x(\zeta)\,d\mu^W(\zeta) = \int f_\varphi\,k^x\,d\mu^* = \int [f_\varphi \circ \pi^W][k^x \circ \pi^W]\,d\mu^W$$
$$= \int \varphi(\zeta)\,[k^x \circ \pi^W](\zeta)\,d\mu^W(\zeta),$$

which implies

$$\omega^x = k^x \circ \pi^W \qquad \mu^W\text{-a.e.}$$

and

$$h(x,y) = \int \omega^x(\zeta)\,\omega^y(\zeta)\,d\mu^W(\zeta) = \int [k^x \circ \pi^W][k^y \circ \pi^W]\,d\mu^W$$
$$= \int k^x\,k^y\,d\mu^* = q(x,y).$$

Therefore $X^L = X^{\{h(x,y);\,x\in X\}} = X^{\{q(x,y);\,x\in X\}} = \widehat{X}$. $\square$

The following corollary is clear.

Corollary 2.

$$\widehat{\Delta}_1 \subset \operatorname{reg}(\widehat{X}).$$

Now we can resolve the second question.

Corollary 3.

$$\widehat{\chi}_x = \hat{q}(x,z)\hat{\mu} \qquad \forall x \in X\,\forall z \in \widehat{\Gamma}.$$

Proof. Letting $\hat{\pi}^W$ be the canonical mapping of X^W to $\widehat{X}$,

$$H_{\omega^x}^{X^W}(y) = \int \omega^x\,d\chi_y^W = \int \omega^x\,\omega^y\,d\mu^W = h(x,y) = q(x,y) = H_{\hat{q}^x}^{\widehat{X}}(y) = H_{\hat{q}^x\circ\hat{\pi}^W}^{X^W}(y).$$

Thus, $\omega^x = \hat{q}^x \circ \hat{\pi}^W$ on Γ^W. Now for $\hat{f} \in \mathbf{C}(\widehat{\Delta})$

$$\int \hat{f}\,\hat{q}^x\,d\hat{\mu} = \int [\hat{f} \circ \hat{\pi}^W][\hat{q}^x \circ \hat{\pi}^W]\,d\mu^W = \int [\hat{f} \circ \hat{\pi}^W]\,\omega^x\,d\mu^W = \int \hat{f} \circ \hat{\pi}^W\,d\chi_x^W$$
$$= H_{\hat{f}\circ\hat{\pi}^W}^{X^W}(x) = H_{\hat{f}}^{\widehat{X}}(x) = \int \hat{f}\,d\widehat{\chi}_x,$$

which implies $d\widehat{\chi}_x(\hat{z}) = \hat{q}(x, \hat{z})\, d\hat{\mu}(\hat{z})$ for every $\hat{z} \in \widehat{\Gamma}$ and $x \in X$. □

Corollary 4. *$\hat{\mu}$ is the canonical representation measure of* 1. *Therefore*

$$\hat{\mu}(\widehat{\Delta}) = \hat{\mu}(\widehat{\Delta}_1) = 1.$$

Proof. The canonical representation measure σ of 1 is characterized by

$$\int \hat{f}\, d\sigma = \int [\hat{f} \circ \hat{\pi}^W]\, d\mu^W$$

([11]), which is equal to

$$\int \left(\int [\hat{f} \circ \hat{\pi}^W]\, d\chi_x^W \right) d\lambda(x) = \int H^{X^W}_{\hat{f}\circ\hat{\pi}^W}(x)\, d\lambda(x) = \int H^{\widehat{X}}_{\hat{f}}(x)\, d\lambda(x) = \int \hat{f}\, d\hat{\mu}. \quad □$$

REMARK 1. For $u \in HB(X)$ there is $\hat{f}_u \in \mathcal{B}(\widehat{\Delta})$, which is bounded, and $u(x) = H^{\widehat{X}}_{\hat{f}_u}(x) = \int \hat{q}^x\, d\nu_u$. Thus $\nu_u = \hat{f}_u\, \hat{\mu}$. On the other hand,

$$\nu_u(A) = \int_{(\hat{\pi}^W)^{-1}(A)} u\, d\mu^W \qquad \forall A \in \mathcal{B}(\widehat{\Delta}).$$

We say $T(\widehat{\Delta}_1)$ covers $HB(X)$.

3. The case where X^* is of Martin type

In this section we treat the case where X^* is of Martin type [8]. We recall that $(X^*, k(x, z), \Delta_1^*, \mu^*)$ is of Martin type if

1) X^* is a metrizable and resolutive compactification of X,
2) $k(x, z) \in \mathbf{C}(X \times \Delta^*)$ and k_z is positive harmonic for every $z \in \Delta^*$,
3) $\Delta_1^* \subset \{z \in \Delta^*; k_z$ is minimal harmonic, $\int k_z\, d\lambda = 1\}$, where λ is a normalized reference measure.

$$\mu^* = \int \chi_x^*\, d\lambda(x), \qquad \mu^*(\Delta^* \setminus \Delta_1^*) = 0$$

4) for every $u \in HB(X)$ there exists a resolutive $f_u \in \mathcal{B}(\Delta^*)$ such that

$$u(x) = H^{X^*}_{f_u}(x) = \int k(x, z)\, f_u(z)\, d\mu^*(z) \qquad \forall x \in X.$$

When X has the Green function $G(x, y)$ such that

i) $G(x, y) = G_y(x)$ is non-negative and lower semi-continuous on $X \times X$ and finite continuous if $x \neq y$,
ii) G_y is a potential and harmonic on $X \setminus \{y\}$,

iii) for every potential p on X there exists $\nu \in M^+(X)$ such that

$$p(x) = \int G_y(x)\, d\lambda(y) \qquad \forall x \in X,$$

iv) $G^*\lambda(y) = \int G(x, y)\, d\lambda(x)$ is positive and continuous,
we set

$$K(x, y) = \frac{G(x, y)}{G^*\lambda(y)}$$

and call $X^M = X^{\{K(x,y);\, x\in X\}}$ the Martin compactification.

We note that X^M and X^L are of Martin type.

Though the following theorem is known in a general theory (e.g. [8, Theorem 8.3]), we shall give a direct proof.

Theorem 3.1. *If for distinct points z, z' of Γ^*, $k_z \neq k_{z'}$, then there exists a homeomorphism Ψ of Γ^* to $\widehat{\Gamma}$ such that*

$$k(x, z) = \hat{q}(x, \Psi(z)) \qquad \forall x \in X\, \forall z \in \Gamma^*.$$

Proof. Recall the consideration at the end of the first section. Since $d\chi_x^*/d\mu^*$ has a continuous repesentative $k(x, z)$, there exists a continuous surjection Ψ of Γ^* to $\widehat{\Gamma}$ such that $\Psi(z) = \hat{\pi}[(\pi^*)^{-1}(z)]$ and $k(x, z) = \hat{q}(x, \Psi(z))$. From the assumption of the theorem, Ψ is injective and is a homeomorphism of Γ^* to $\widehat{\Gamma}$. □

Corollary 5. $\int k(x, z)\, d\lambda(x) \leq 1$ *for every $z \in \Gamma^*$. Therefore $Tz = k_z$ is a continuous injection of Γ^* to $\mathcal{H}_1^\lambda$.*

Proof. $\int q(x, y)\, d\lambda(x) = \int [\int k(x, z)\, d\chi_y^*]\, d\lambda(x) = \int k(x, z)\, d\mu^*(z) = \int d\chi_x^* = H_1^{X^*}(x) = 1$. $\int \hat{q}(x, \hat{z})\, d\lambda(x) \leq \liminf_j \int q(x, y_j)\, d\lambda(x) = 1$ for some $\{y_j\} \subset X$, $y_j \to \hat{z}$ in $\widehat{X}$. □

The next theorem is an immediate consequence of Theorem 1.1 and Theorem 3.1. However, as an application, it has some importance, so we state it directly.

Theorem 3.2. *If $\Gamma^* \subset \{z \in \Delta^*;\, k_z$ is minimal harmonic, $\int k_z(x)\, d\lambda(x) \leq 1\}$ then $\widehat{X}$ is a semi-regular compactification and $\widehat{\Gamma} = \widehat{\Delta}_1 = \mathrm{reg}(\widehat{X})$.*

REMARK 2. In the definition of the compactification of Martin type, if we replace Δ_1^* by $\Delta_1^* \cap \Gamma^*$ then $T(\Delta_1^*)$ covers $HB(X)$.

4. Examples

4.1. In what follows, we denote by $\mathcal{H}$ the sheaf of continuous solutions of the Laplace equation.

EXAMPLE 1 (standard example). Let $X = \{x \in \mathbf{C}(\text{the complex plane}); 0<|x|<1\}$, a harmonic sheaf be $\mathcal{H}$ and $\lambda = \varepsilon_{x_0}$ with $0 < |x_0| < 1$. We take X^* as the topological closure of X. Then $\Delta^* = C \cup \{0\}$, where $C = \{x \in \mathbf{C}; |x| = 1\}$ and $k(x, z) = P_z(x)/P_z(x_0)$ where $P_z(x)$ is the Poisson kernel, i.e.,

$$P_z(x) = \frac{1}{2\pi}\frac{1 - |x|^2}{|x - z|^2}$$

and $\Gamma^* = C$.

In the compactification $\widehat{X}$, it is clear that $\widehat{\Delta} = \Delta^*$, $\widehat{\Gamma} = \Gamma^*$, $\widehat{\Delta}_1 = \widehat{\Gamma} \neq \widehat{\Delta}$ and

$$\hat{q}(x, \hat{z}) = \begin{cases} k(x, z) & \text{for } \hat{z} = z \in C \\ u(x) & \text{for } \hat{z} = 0, \end{cases}$$

where $u \in HB(X)$. In this case $T(\widehat{\Delta}_1)$ covers $HB(X)$.

EXAMPLE 2. Let $X = \{x \in \mathbf{C}; 0 < b < |x| < a\}$; let the harmonic sheaf be $\mathcal{H}$ and $\lambda = \varepsilon_{x_0}$ with $x_0 \in X$. We identify the circle $|z| = a$ as one point z_1 and $|z| = b$ as z_2. Then $\Gamma^* = \Delta^* = \{z_1, z_2\}$ and

$$k(x, z) = \begin{cases} h_1(x)/h_1(x_0) & \text{for } z = z_1 \\ h_2(x)/h_2(x_0) & \text{for } z = z_2, \end{cases}$$

where $h_1(x) = (\log|x| - \log b)/(\log a - \log b)$ and $h_2(x) = (\log a - \log|x|)/(\log a - \log b)$.

In the compactification $\widehat{X}$, we see that $\widehat{\Delta} = \widehat{\Gamma} = \mathrm{reg}(\widehat{X}) \simeq \Delta^*$ but $\widehat{\Delta}_1 = \emptyset$ and $\hat{q}(x, \hat{z}) = k(x, z)$. In this case $T(\widehat{\Delta}_1)$ covers the cone $\{c_1\,\hat{q}_{\hat{z}_1} + c_2\,\hat{q}_{\hat{z}_2}; c_1 > 0, c_2 > 0\} \subsetneq HB(X)$.

EXAMPLE 3. Let $X = \{x \in \mathbf{C}; |x| < 1\}$; let the harmonic sheaf be $\mathcal{H}/h$, $\lambda = \varepsilon_0$ and X^* be the topological closure of X, which is homeomorphic to the Martin compactification X^M. We consider three cases of h.

[i] $h = P_{z_0}$. Then $\Delta^* = C$, $\Gamma^* = \{z_0\}$ and $k(x, z_0) \equiv 1$. Hence $\widehat{X}$ is the one-point compactification and therefore $\widehat{\Delta} = \widehat{\Gamma} = \{\hat{z}_0\}$, where $\hat{z}_0$ corresponds to z_0 and $\widehat{\Delta} = \widehat{\Delta}_1$. $\hat{q}(x, \hat{z}_0) \equiv 1$.

[ii] $h = (1/2)(P_{z_1} + P_{z_2})$, where $z_j = e^{i\theta_j}$ ($j = 1, 2$) and $0 \leq \theta_1 < \theta_2 < 2\pi$. Then

$\Delta^* = C,\ \Gamma^* = \{z_1, z_2\}$ and

$$k(x,z) = \begin{cases} P_{z_1}(x)/h(x) & \text{for } z = z_1 \\ P_{z_2}(x)/h(x) & \text{for } z = z_2. \end{cases}$$

$\widehat{X}$ has the boundary $\widehat{\Delta} = \{\hat{z}_1, \hat{z}_2\} \cup \{\hat{z}_\theta;\ \theta_1 < \theta < \theta_2\}$, where $\hat{z}_j$ corresponds to z_j $(j = 1, 2)$ and $\hat{z}_\theta$ to $e^{i\,\theta}$ and some $e^{i\,\theta'}$ with $\theta' \in C \setminus [\theta_1 < \theta < \theta_2]$. $\widehat{\Gamma} = \widehat{\Delta}_1 = \{\hat{z}_1, \hat{z}_2\}$ and

$$\hat{q}(x,\hat{z}) = \begin{cases} k(x, z_j) & \text{for } \hat{z} = \hat{z}_j\ \ (j = 1,\ 2) \\ t_1(\hat{z})\, k(x, z_1) + t_2(\hat{z})\, k(x, z_2) & \text{for } \hat{z} = \hat{z}_\theta, \end{cases}$$

where $t_j(\hat{z}) = [|z - z_j|^2\, (|z - z_1|^{-2} + |z - z_2|^{-2})]^{-1}$ with $z = e^{i\,\theta}$.
[iii] $h = (1/n)\sum_{j=1}^{n} P_{z_j}$ $(n \geq 3)$, where z_j are distinct. Then $\Delta^* = C$, $\Gamma^* = \{z_1, \cdots, z_n\}$, $k(x,z) = P_{z_j}(x)/h(x)$ for $z = z_j$ $(1 \leq j \leq n)$, and $\widehat{\Delta} = C$, $\widehat{\Gamma} = \widehat{\Delta}_1 \simeq \Gamma^*$,

$$\hat{q}(x,\hat{z}) = \begin{cases} k(x, z_j) & \text{for } \hat{z} = \hat{z}_j\ \ (1 \leq j \leq n) \\ \sum_{j=1}^{n} t_j(\hat{z})\, k(x, z_j) & \text{for } \hat{z} \neq \hat{z}_j, \end{cases}$$

where $t_j(\hat{z}) = [|z - z_j|^2 \sum_{j=1}^{n} |z - z_j|^{-2}]^{-1}$ and z corresponds to $\hat{z}$.

In these cases $T(\widehat{\Delta}_1)$ covers $HB(X)$.

EXAMPLE 4 (Cornea-Loeb [6]). Let $X = \{x \in \mathbf{C};\ |x| < 1\}$, $\{z_n\}$ be a countable dense subset of $C \setminus \{1\}$ and $\lambda = \varepsilon_0$. We put

$$\gamma_n = \sup\left\{ \frac{P_{z_n}(x)}{p(x)};\ 0 \leq x < 1 \right\},$$

where $p(x) = \min\{\log(1/|x|), 1\}$. It is easily checked that $0 < \gamma_n < +\infty$, therefore we may find $\alpha_n > 0$ such that $\sum_{n=1}^{\infty} \alpha_n\, \gamma_n < +\infty$ and $\sum_{n=1}^{\infty} \alpha_n = 1$. We set $h(x) = \sum_{n=1}^{\infty} \alpha_n\, P_{z_n}(x)$ and consider the harmonic sheaf $\mathcal{H}/h$. Letting X^* be the Martin compactification of X with the harmonic sheaf $\mathcal{H}/h$, we have $\Delta^* = \Delta_1^* = \Gamma^* = C \simeq \Delta^M$ (the usual Martin boundary). Further it is known that $1 \notin \mathrm{reg}(X^*)$ so $\mathrm{reg}(X^*) \subsetneq \Gamma^*$. By Theorem 3.2, we have $\widehat{X}$ is semi-regular and $\widehat{\Gamma} = \widehat{\Delta}_1 = \mathrm{reg}(\widehat{X})$. We have $\widehat{\Gamma} = \Gamma^*$ and that $\widehat{X}$ is a refinement of X^*.

EXAMPLE 5. Let X be a bounded domain in $\mathbf{R}^3$ with Lebesgue's spine at 0 and a harmonic sheaf $\mathcal{H}$ and $\lambda = \varepsilon_{x_0}$ with $x_0 \in X$. We suppose that every boundary point is regular except 0. We take the Martin compactification X^M as X^*. Then $\Delta^* = \Delta_1^* = \Gamma^*$ but $\mathrm{reg}(X^*) \subsetneq \Gamma^*$. In the same way as the above example $\widehat{X}$ is semi-regular and $\widehat{\Gamma} = \widehat{\Delta}_1 = \mathrm{reg}(\widehat{X}) \simeq \Gamma^*$. However $\widehat{\Gamma} \subsetneq \widehat{\Delta}$ and $\widehat{X}$ is a refinement of X^*.

4.2. In this section we conseder the disc $X = \{x \in \mathbf{C};\ |x| < 1\}$ with the harmonic sheaf $\mathcal{H}$ and $\lambda = \varepsilon_0$. Letting $k(x, z) = 2\pi P_z(x) = (1 - |x|^2)/(|x - z|^2)$ we form

$$q(x, y) = \int k(x, z)\, k(y, z)\, d\mu^*(z) \qquad \left(d\mu^*(z) = d\mu^*\left(e^{i\theta}\right) = \frac{1}{2\pi}\, d\theta \right).$$

Let us consider the following three cases:
[1°] X^{Q_0}; $Q_0 = \{q(0, y)\}$. $(q(0, y) \equiv 1)$
X^{Q_0} is the one point compactification of X.
[2°] X^{Q_1}; $Q_1 = \{q(x_1, y);\ 0 < x_1 < 1\}$.
$\Delta^{Q_1} = \{\check{z}_\theta;\ 0 \leq \theta \leq \pi\}$, where $e^{i\theta}$ and $e^{-i\theta}$ are identified as $\check{z}_\theta$. $H_f^{X^{Q_1}}(x)$ is equal to the usual Dirichlet solution of f such that $f(e^{i\theta}) = f(e^{-i\theta}) = \check{f}(\check{z}_\theta)$. $d\check{\chi}_x(\check{z}_\theta) = [P(x, e^{i\theta}) + P(x, e^{-i\theta})]\, d\theta$, $d\check{\mu}(\check{z}_\theta) = (1/\pi)\, d\theta$ and $\check{k}(x, \check{z}_\theta) = \pi\, [P(x, e^{i\theta}) + P(x, e^{-i\theta})]$ thus

$$\check{q}(x, y) = \int \check{k}(x, \check{z}_\theta)\, \check{k}(y, \check{z}_\theta)\, d\check{\mu}(\check{z}_\theta) = \frac{1}{2}\, [q(x, y) + q(x, \overline{y})],$$

where $\overline{y}$ is the conjugate complex number of y. Hence $(X^{Q_1})^\wedge = X^{Q_1}$ and the Dirichlet solutions in X^{Q_1} are the usual Dirichlet solutions such that $u(x) = u(\overline{x})$.
[3°] X^{Q_2}; $Q_2 = \{q(x_1, y), q(x_2, y)\}$, $\arg x_1 \not\equiv \arg x_2 \pmod{\pi}$.
$X^{Q_2} \simeq X^L \simeq X^M$.

For $Q \subset \{q(x, y);\ x \in X\}$, X^Q is one of the above type [1°], [2°] and [3°].

4.3. The heat equation

EXAMPLE 6. We consider the heat equation of one space dimension in the Euclidean space:

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \tau},$$

where we denote by $x = (\xi, \tau)$ a point in $\mathbf{R} \times \mathbf{R}$.

In the first place, we shall treat a very simple case where $X = \{(\xi, \tau);\ 0 < \xi < 1, 0 < \tau < 1\}$, a reference measure λ is the restriction of the two dimensional Lebesgue measure to X, and X^* is the topological closure of X in $\mathbf{R}^2$. We put $O = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$ and $C = (1, 1)$. In this case, the Martin compactification X^M is obtained by identifying the closed segment $[BC]$ of X^* with one point J. The corresponding Martin kernel $K(x, z)$ is continuous and $K(x, J) = 0$, where $x \in X$, $z \in \Delta^M := X^M \setminus X$. It follows from the Green's formula that the harmonic measure χ_x^* at $x = (\xi, \tau) \in X$ is carried by $\Delta_\tau^* := \Delta^* \cap (\mathbf{R} \times (-\infty, \tau))$ and is absolutely continuous with respect to the length element of Δ^*; on each edge, the density function κ^x is continuous. Since κ_z is proportional to the Martin kernel K_z, we can write $\chi_x^* = K(x, z)\, \mu^*$, where $\mu^* = \int \chi_x^*\, d\lambda(x)$. A kernel $q(x, y) = \int K(x, z) K(y, z)\, d\mu^*(z) =$

$\int K^x \, d\chi_y^*$ on $X \times X$ has a continuous extension $q^*(x, z)$ to $X \times X^*$, because every boundary point in $\Delta^* \backslash (BC)$ is regular, K^x is continious there and $\lim_{z \to B, C} K^x(z) = 0$. Here (BC) is the open edge from B to C. If $z \in \Delta^* \setminus (BC)$, then $q^*(x, z) = K(x, z)$ and $\lim_{x \to z'} q^*(x, z) = 0$ for any $z' \in \Delta^* \backslash ((BC) \cup \{z\})$. On the other hand, if $z \in (BC)$, then $\lim_{x \to z'} q^*(x, z) = \lim_{y \to z} K(y, z') > 0$ for $z' \in \Delta^* \setminus [BC]$, which shows that $q^*(x, \cdot)$ separate a point in $\Delta^* \setminus (BC)$ from that in (BC). Finally, we show $q^*(x, \cdot)$ separate points in (BC). We take two points $z_i = (\xi_i, 1) \in (BC)$ arbitrary $(i = 1, 2)$. If $q^*(y, z_1) = q^*(y, z_2)$ for every $y \in X$, then $K(z_1, z') = K(z_2, z')$ for any $z' \in \Delta^* \backslash [BC]$, where $K(z_i, z') = \lim_{y \to z_i} K(y, z')$. For $0 < s < 1$, we put

$$u_s(x) = \int_{\Delta_s^*} \eta \, d\chi_x^*(\eta, t) = \int_{\Delta_s^*} \eta \, K\big(x, (\eta, t)\big) \, d\mu^*(\eta, t),$$

which can be considered as a function on $(0, 1) \times \mathbf{R}$. Since

$$u_s(\xi, 1) = \lim_{\tau \to 1} u_s(\xi, \tau) = \int_{\Delta_s^*} \eta \, K\big((\xi, 1), (\eta, t)\big) \, d\mu^*(\eta, t),$$

and since $\lim_{s \to 1} u_s(\xi, 1) = \xi$,we have $u_s(\xi_1, 1) = u_s(\xi_2, 1)$ and $\xi_1 = \xi_2$. Therefore we find

$$\widehat{\Delta} = [OA] \cup (OB) \cup (AC) \cup (BC) \cup \{B = C\}.$$

Next we insert a slit in X. Putting $E = (0, 1/2)$, $F = (1/4, 1/2)$ and $G = (1/2, 1/2)$, we consider two spaces $X_1 = X \setminus [EF]$ and $X_2 = X \setminus [FG]$. As a matter of convenience, we distinguish the upper side Λ_j^+ from the lower side Λ_j^- of the slits $(j = 1, 2)$. Using a similar argument to that above, we can find X_j^M and $\widehat{X}_j$:

$$X_1^M = [OA] \cup (AC) \cup (OE^-) \cup \{F\} \cup \Lambda_1^+ \cup (EB) \cup \{J\} \cup X_1,$$

where Λ_1^- shrinks to one point corresponding to F,

$$\widehat{X}_1 = [OA] \cup (AC) \cup (OE^-] \cup \Lambda_1^- \cup \widehat{F} \cup \Lambda_1^+ \cup [E^+B) \cup (BC) \cup \{B = C\} \cup X_1,$$

where E^-, E^+ denote the lower and upper side of E, respectively and $\widehat{F}$ corresponds to the irregularity of F.

$$X_2^M = [OA] \cup (AC) \cup (OB) \cup \{J\} \cup \Lambda_2^+ \cup \overline{FG} \cup X_2,$$

where $\overline{FG}$ corresponds to the convex combination of the Martin kernel at F and that at G, and

$$\widehat{X}_2 = [OA] \cup (AC) \cup (OB) \cup (BC) \cup \{B = C\} \cup \Lambda_2^- \cup \Lambda_2^+ \cup \widehat{F} \cup \widehat{G} \cup X_2.$$

References

[1] J. Bliedtner and P.A. Loeb: *An alternative to Martin boundary theory*, Semesterbericht Funktionalanalysis, Summer Semester 1988, Tübingen University, 55–60.
[2] J. Bliedtner and P.A. Loeb: *Sturdy harmonic functions and their integral representations*, preprint.
[3] R.M. Blumenthal and R.K. Getoor: Markov Processes and Potential Theory, Academic Press, 1968.
[4] C. Constantinescu and A. Cornea: Ideale Ränder Riemannscher Flächen, Ergebnisse der Math. Bd. 32, Springer, 1963.
[5] C. Constantinescu and A. Cornea: Potential Theory on Harmonic Spaces, Grundlehren der Math. Bd. 158, Springer, 1972.
[6] A. Cornea and P.A. Loeb: *On semiregular points of the Martin boundary*, Proc. Amer. Math. Soc. **103** (1988), 117–124.
[7] T. Ikegami: *On the simplicial cone of superharmonic functions in a resolutive compactification of a harmonic space*, Osaka J. Math. **20** (1983), 881–898.
[8] T. Ikegami: *Compactification of Martin type of harmonic spaces*, Osaka J. Math. **23** (1986), 653–680.
[9] K. Janßen: *Martin boundary and $\mathcal{H}^p$-theory of harmonic spaces*, Seminar on Potential Theory II, Lecture Notes in Math. 226, Springer, 1971.
[10] P.A. Loeb: *Applications of nonstandard analysis to ideal boundaries in potential theory*, Israel J. Math. **25** (1976), 154–187.
[11] P.A. Loeb: *A regular metrizable boundary for solutions of elliptic and parabolic differential equations*, Math. Ann. **251** (1980), 43–50.

Teruo Ikegami
Kitanagao, 1-2-8
Sakai 591-8043, Japan

Masaharu Nishio
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi
Osaka 558-8585, Japan