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GENERALIZATION OF A THEOREM OF PETER J. CAMERON

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Peter J. Cameron [3] has shown that a primitive permutation group G has rank at most 4 if the stabilizer G_α of a point α is doubly transitive on all its nontrivial suborbits except one.

The purpose of this paper is to prove the following two theorems, one of which extends the Cameron's result.

Theorem 1. *Let G be a primitive permutation group on a finite set Ω, and all nontrivial G-obrits in Cartesian product $\Omega\times\Omega$ be $\Gamma_1, \cdots, \Gamma_s, \Delta_1, \cdots, \Delta_t$, where G_α is doubly transitive on $\Gamma_i(\alpha)=\{\beta\,|\,(\alpha, \beta)\ni\Gamma_i\}$, $1\leqq i\leqq s$ and not doubly transitive on $\Delta_i(\alpha)$, $1\leqq i\leqq t$. Suppose that G has no subdegree smaller than 4 and that $t>1$. Then, we have*

$$s\leqq 2t-r\,,$$

where $r=\#\{\Delta_i\,|\,\Delta_i=\Gamma_j^*\circ\Gamma_j\ 1\leqq j\leqq s\}$. Moreover if $r=1$, then we have

$$s\leqq 2t-2\,.$$

(For the notation $\Gamma_j^*\circ\Gamma_j$, see the section 1)

Theorem 2. *Under the hypothesis of Theorem 1, if $r=t$, then $s=t=2$, and G is isomorphic to the small Janko simple group and G_α is isomorphic to $PSL(2, 11)$.*

For the case of $t\geqq 3$, I don't know the example satisfying the equality $s=2t-r$, and when $r=1$, the exampel satisfying the equality $s=2t-2$. I know only three examples with $t=2$ and $s=2$.

The small Janko simple group J_1 of order 175560 has a primitive rank 5 represestation of degree 266 in which the stabilizer of a point is isomorphic to PSL (2, 11) and acts doubly transitively on suborbits of lengths 11 and 12; the other suborbit lengths are 110 and 132 (See Livingstone [7]). The Mathieu group M_{12} has a primitive rank 5 representation of degree 144 in which the stabilizer of a point is isomorphic to PSL (2, 11) and acts doubly transitively on two suborbits of length 11; the other suborbit lengths are 55 and 66 (See Cameron [4]).

The group $[Z_3\times Z_3\times Z_3]S_4$ has a primitive rank 5 representation of degree 27 in which the stabilizer of a point is S_4 and acts doubly transitively on two suborbits of length 4; the other suborbit lengths are 6 and 12. I conjecture that it may even be true that s is at most t.

1. Preliminaries

Let G be a transitive permutation group on a finite set Ω, and Δ be a subset of the Cartesian product $\Omega\times\Omega$ which is fixed by G (acting in the natural way on $\Omega\times\Omega$), then $\Delta(\alpha)\{\beta\in\Omega\,|\,(\alpha,\beta)\in\Delta\}$ is a subset of Ω fixed by G_α. This procedure sets up a one-to-one correspondence between G-orbits in $\Omega\times\Omega$ and G_α-orbits in Ω. The number of such orbits is called the rank of G. $\Delta^*=\{(\beta,\alpha)\,|\,(\alpha,\beta)\ni\Delta\}$ is the subset of $\Omega\times\Omega$ fixed by G paired with Δ; Δ is self-paired if $\Delta=\Delta^*$. Note that $|\Delta(\alpha)|=\Delta^*(\alpha)|=|\Delta|/|\Omega|$. If Γ and Δ are fixed sets of G in $\Omega\times\Omega$, let $\Gamma\circ\Delta$ denote the set $\{(\alpha,\beta)\}$ there exists $\gamma\in\Omega$ with $(\alpha,\gamma)\in\Gamma$, $(\gamma,\beta)\in\Delta$; $\alpha\neq\beta\}$; this is also a fixed set of G. The diagonal $\{(\alpha,\alpha)\,|\,\alpha\in\Omega\}$ is a trivial G-orbit. If Γ is a nontrivial G-orbits in $\Omega\times\Omega$, the Γ-graph is the reqular directed graph whose point set is Ω and whose edges are precisely the ordered pairs in Γ. A connected component of any such graph is a block of imprimitivity for G. G is primitive if and only if each such graph is connected.

For a G-orbit Γ in $\Omega\times\Omega$, the basis matrix $C=C(\Gamma)$ is the matrix whose rows and columns are indexed by Ω, with (α,β) entry 1 if $(\alpha,\beta)\in\Gamma$, 0 otherwise. All of the basis matrices form a basis of the centralizer algebra of the permutation matrices in G.

Let G be a group which acts as a permutation group on Ω, and π the permutation character of G i.e. the integer-valued function on G defined by $\pi(g)=$ number of fixed points of g. The formula

$$(\pi,1)_G=\frac{1}{|G|}\sum_{g\in G}\pi(g)=\text{number of orbits of } G\,,$$

is well-known. If G acts as a permutation group on Ω_1 and Ω_2, with permutation characters π_1 and π_2, the number m of G-orbits in $\Omega_1\times\Omega_2$ is

$$m=(\pi_1\pi_2,1)_G=(\pi_1,\pi_2)_G\,.$$

In particular, if G is a transitive permutation group on Ω with permutation character π, the rank r of G is given by

$$r=(\pi,\pi)_G=\text{sum of squares of multiplicities of irreducible consitituents of } \pi$$

If G acts doubly transitively on Ω_1 and Ω_2,

$$(\pi_1,\pi_2)_G=2 \text{ or } 1 \text{ according as } \pi_1=\pi_2 \text{ or } \pi_1\neq\pi_2\,.$$

Lastly, we note that if G is a primitive permutation group on Ω, then for α, β $(\neq)\in\Omega$, either $G_\alpha\neq G_\beta$ or G is a regular group of prime degree ([8], Prop. 8.6); primitive groups with a sudbegree 2 are Frobenius groups of prime degree ([8], Theorem 18.7); primitive groups with a subdegree 3 are classified by W.J. Wong [9].

2. Lemmata

Throughout this section, we suppose that G is a primitive but not doubly transitive group on a finite set Ω, and Γ_1, Γ_2, $\cdots$ are G-orbits in $\Omega\times\Omega$ such that G_α is doubly transitive on $\Gamma_i(\alpha)$, $i=1, 2, \cdots$; π_i and π_i^* are the permutation characters of G_α on $\Gamma_i(\alpha)$ and $\Gamma_i^*(\alpha)$, respectively, and let $C_i=C(\Gamma_i)$, $C_i^*=C(\Gamma_i^*)$.

Lemma 1. (P. J. Cameron [2]. Proposition 1.2)
G_α is doubly transitive on $\Gamma_i^*(\alpha)$.

Lemma 2. (P. J. Cameron [3]. Lemma 1)
$\Gamma_i^*\circ\Gamma_i$ is a G-orbit in $\Omega\times\Omega$, and if $|\Gamma_i(\alpha)|>2$, then G_α is not doubly transitive on $\Gamma_i^*\circ\Gamma_i(\alpha)$.

Lemma 3. (P. J. Cameron [2]. Theorem 2.2)
For $(\alpha, \beta)\in\Gamma_i\circ\Gamma_i^*$, we put $v_i=|\Gamma_i(\alpha)|$ and $k_i=|\Gamma_i(\alpha)\cap\Gamma_i(\beta)|$. Then $k_i<v_i$ and $|\Gamma_i\circ\Gamma_i^*(\alpha)|=\frac{v_i(v_i-1)}{k_i}$. If $v_i>2$, then $k_i\leqq\frac{v_i-1}{2}$; when particulary $k_i=\frac{v_i-1}{2}$, then $v_i=3$ or 5.

In the following, we set

$$|\Gamma_i(\alpha)|=v_i\,,\quad |\Gamma_i\circ\Gamma_i^*(\alpha)|=\frac{v_i(v_i-1)}{k_i}\,.$$

Lemma 4. (P. J. Cameron [2]. Lemma 2.1)

$$|\Gamma_i^*\circ\Gamma_i(\alpha)|=|\Gamma_i\circ\Gamma_i^*(\alpha)|\,.$$

Lemma 5. *$\Gamma_i^*\circ\Gamma_i\neq\Gamma_2^*\circ\Gamma_2$ if and only if $|(\Gamma_1\circ\Gamma_2^*(\alpha)|=|\Gamma_1(\alpha)|\cdot|\Gamma_2(\alpha)|$.*

Proof. If $|\Gamma_1\circ\Gamma_2^*(\alpha)|<|\Gamma_1(\alpha)|\cdot|\Gamma_2(\alpha)|$, we have $|\Gamma_1(\alpha)\cap\Gamma_2(\beta)|>1$ for some $(\alpha, \beta)\in\Gamma_1\circ\Gamma_2^*$. For γ_1, $\gamma_2(\neq)\in\Gamma_1(\alpha)\cap\Gamma_2(\beta)$, $(\gamma_1, \gamma_2)\in\Gamma_1^*\circ\Gamma_1$ and $(\gamma_1, \gamma_2)\in\Gamma_2^*\circ\Gamma_2$. So $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2$. Conversely, if $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2$ for $(\gamma_1, \gamma_2)\in\Gamma_2^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2$ we can choose α and β such that $\alpha\in\Gamma_1^*(\gamma_1)\cap\Gamma_1^*(\gamma_2)$, $\beta\in\Gamma_2^*(\gamma_1)\cap\Gamma_2^*(\gamma_2)$. Since $\Gamma_1(\alpha)\cap\Gamma_2(\beta)\in\gamma_1, \gamma_2$, $|\Gamma_1(\alpha)\cap\Gamma_2(\beta)|>1$. Therefore $|\Gamma_1\circ\Gamma_2^*(\alpha)|<|\Gamma_1(\alpha)|\cdot|\Gamma_2(\alpha)|$.

Lemma 6. *$\Gamma_1^*\circ\Gamma_2$ is the union of at most two G-orbits in $\Omega\times\Omega$, and*

$\pi_1=\pi_2$ if and only if $\Gamma_1^*\circ\Gamma_2$ is the union of two G-orbits in $\Omega\times\Omega$.

Proof. Since $(\pi_1\pi_2, 1)_G=(\pi_1, \pi_2)_G\leqq 2$, and $\pi_1\pi_2$ is the permutation character of G_α on $\Gamma_1(\alpha)\times\Gamma_2(\alpha)$, G has at most two orbits in $\{(\alpha, \gamma, \delta)\,|\,(\alpha, \gamma)\in\Gamma_1, (\alpha, \delta)\in\Gamma_2\}$, and hence, $\Gamma_1^*\circ\Gamma_2$ is the union of at most two G-orbits. If $\pi_1\neq\pi_2$, then G is transitive on $\{(\alpha, \gamma, \delta)\,|\,(\alpha, \gamma)\in\Gamma_1, (\alpha, \delta)\in\Gamma_2\}$, and hence, $\Gamma_1^*\circ\Gamma_2$ is a G-orbit in $\Omega\times\Omega$. Now, we shall assume that $\pi_1=\pi_2$ and $\Gamma_1^*\circ\Gamma_2$ is a G-orbit in $\Omega\times\Omega$. We put $v=v_2=v_1$, and $m=|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|$ for $(\alpha, \delta)\in\Gamma_1^*\circ\Gamma_2$. If $m=1$, then since $\Gamma_1^*\circ\Gamma_2$ is a G-orbit, G is transitive on $\{(\alpha, \gamma, \delta)\,|\,(\gamma, \alpha)\in\Gamma_1, (\gamma, \delta)\in\Gamma_2\}$. Therefore $(\pi_1, \pi_2)_G=1$, and hence, $\pi_1\neq\pi_2$, this is contrary to the assumption If $m>1$, then there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_2^* and Γ_1; and whose vertices are all distinct. Counting all of them in two ways, we have

$$|\Omega|\frac{vv}{m}m(m-1)=|\Omega|\frac{v(v-1)}{k_1}k_1k_2\,,$$

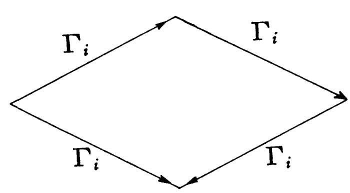
so

$$v(m-1)=(v-1)k_2\,.$$

Hence, $v=k_2$. This is impossible by Lemma 3.

Lemma 7. *If $\Gamma_i\circ\Gamma_i^*\neq\Gamma_i^*\circ\Gamma_i$, then $\Gamma_i\circ\Gamma_i\not\supset\Gamma_i\circ\Gamma_i^*, \Gamma_i^*\circ\Gamma_i$.*

Proof. Now assume $\Gamma_i\circ\Gamma_i\supset\Gamma_i\circ\Gamma_i^*$ or $\Gamma_i^*\circ\Gamma_i$, then we have the following figure,



and hence, $\Gamma_i\circ\Gamma_i\supset\Gamma_i\circ\Gamma_i^*\cup\Gamma_i^*\circ\Gamma_i$. Since $\Gamma_i\circ\Gamma_i$ is the union of at most two G-orbits in $\Omega\times\Omega$, we have $\Gamma_i\circ\Gamma_i=\Gamma_i\circ\Gamma_i^*\cup\Gamma_i^*\circ\Gamma_i$. By the assummption of this lemma, $|(\Gamma_i\circ\Gamma_i)(\alpha)|=|\Gamma_i(\alpha)|\cdot|\Gamma_i(\alpha)|=v_i^2$. So

$$v_i^2=|\Gamma_i\circ\Gamma_i(\alpha)|=|\Gamma_i\circ\Gamma_i^*(\alpha)|+|\Gamma_i^*\circ\Gamma_i(\alpha)|=\frac{2v_i(v-1)}{k_i}\,,$$

$$v_ik_i=2(v_i-1)\,.$$

Therefore, $v_i=2$. All of the suborbits of the primitive group with a subdegree 2 are self-paired. This is contrary to the assumption of this Lemma.

Lemma 8. *Let $\Gamma_i^*\circ\Gamma_2$ be the union of two G-orbits $\sum_1$ and $\sum_2$. We set $v=v_1=v_2$, $S_i=C(\sum_i)$, $s_i=|\sum_i(\alpha)|$, $i=1, 2$, and $C_i^*C_2=a_1S_1+a_2S_2$. Then we have*

i) $s_1, s_2 \geqq v$. *If* $s_1 = v$, G_α *is double transitive on* $\Sigma_1(\alpha)$

ii) $v^2 = a_1 s_1 + a_2 s_2$

iii) $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ *if and only if* $a_1 = a_2 = 1$

iv) *if* $s_1 = v(v-1)$, *then* $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ *and* $\Gamma_1^* \circ \Gamma_2$ *containes some* Γ_i

Proof. i) Assume $s_1 \leqq v$. Then $(\pi_1^*, \pi(\Sigma_1)) = 1$ or 2 according as $\pi_1^* \neq \pi(\Sigma_1)$ or $\pi_1^* = \pi(\Sigma_1)$ where $\pi(\Sigma_1)$ is the permutation character of G_α on $\Sigma_1(\alpha)$. If $\pi_1^* \neq \pi(\Sigma_1)$, for $\delta \in \Sigma_1(\alpha)$, $G_{\alpha,\delta}$ is transitive on $\Gamma_1^*(\alpha)$. Thus $\Gamma_1^*(\alpha) = \Gamma_1^*(\delta)$. Therefore $G_\alpha = G_{\{\Gamma_1^*(\alpha)\}} = G_{\{\Gamma_1^*(\delta)\}} = G_\delta$. This is impossible. So we have $\pi_1^* = \pi(\Sigma_1)$, and hence, $s_1 = v$ and G_α is doubly transitive on $\Sigma_1(\alpha)$.

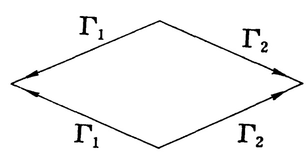
ii) For the matrix F such that any entry is 1, we have

$$F(C_1^* C_2) = v^2 F \quad \text{and} \quad F(a_1 S_1 + a_2 S_2) = (a_1 s_1 + a_2 s_2) F\,,$$

so

$$v^2 = a_1 s_1 + a_2 s_2\,.$$

iii) The existence of the following figure is equvalent to $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$.



It holds also that the figure exists if and only if $a_i \geqq 2$ for $i = 1$ or 2.

iv) By ii), $v^2 = a_1 v(v-1) + a_2 s_2$. Since $s_2 \geqq v$, $a_1 = a_2 = 1$ and $s_2 = v$. Therefore we conclude that $\Gamma_1^* \circ \Gamma_2$ containes some Γ_i by i), and $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ by iii).

Lemma 9. *If* $\pi_1 \neq \pi_2$, G_α *is not doubly transitive on* $\Gamma_1^* \circ \Gamma_2(\alpha)$.

Proof. Assume that G_α is doubly transitive on $\Gamma_2^* \circ \Gamma_2(\alpha)$. If $|\Gamma_1^* \circ \Gamma_2(\alpha)| \neq |\Gamma_1(\alpha)|$, then G_α has different permutation characters on $\Gamma_1^*(\alpha)$ and $\Gamma_1^* \circ \Gamma_2(\alpha)$. Hence, for $(\alpha, \gamma) \in \Gamma_1^*$, $G_{\alpha,\gamma}$ is transitive on $\Gamma_1^* \circ \Gamma_2(\alpha)$, so, $\Gamma_2(\gamma) = \Gamma_1^* \circ \Gamma_2(\alpha)$. Therefore $G_\gamma = G_{\{\Gamma_2(\gamma)\}} = G_{\{\Gamma_1^* \circ \Gamma_2(\alpha)\}} = G_\alpha$. This is impossible. Thus, we obtain $|\Gamma_2^* \circ \Gamma_1(\alpha)| = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)|$. On the other hand, for $(\delta, \gamma) \in \Gamma_2^*$, $\Gamma_1(\gamma) \subset \Gamma_2^* \circ \Gamma_1(\delta)$. So, $\Gamma_2^* \circ \Gamma_1(\delta) = \Gamma_1(\gamma)$. This is alos impossible.

Lemma 10. *Assume* $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$ *and* $\Gamma_1^* \circ \Gamma_2$ *be the union of two* G-*orbits* Σ_1 *and* Σ_2; *put* $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v$, $|\Gamma_i \circ \Gamma_i^*(\alpha) = \dfrac{v(v-1)}{k}$, $|\Sigma_i(\alpha)| = s_i$, $i = 1, 2$; *and* $|\Gamma_2(\gamma) \cap \Sigma_2(\alpha)| = t$ *for* $\gamma \in \Gamma_1^*(\alpha)$. *Then, we have the following quadratic equation for* t

$$\frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1) = 0\,.$$

Particulary, i) *when* $s_1 \geqq \dfrac{v(v-1)}{k}$, *the quadratic equation has at most one root for*

$0<t<v$; ii) *when* $t=1$, *then* $s_2=v$, $s_1=\dfrac{v(v-1)}{(k+1)}$ *and* G_α *is doubly transitive on* $\sum_2(\alpha)$.

Proof. For $\gamma_1, \gamma_2(\neq)\in\Gamma_1^*(\alpha)$, counting arguments show that

$$|\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)\cap\textstyle\sum_1(\alpha)| = \dfrac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1},$$

$$|\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)\cap\textstyle\sum_2(\alpha)| = \dfrac{t(vt-s_2)}{(v-1)s_2},$$

so

$$k = |\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)| = \dfrac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1}+\dfrac{t(vt-s_2)}{(v-1)s_2},$$

$$(v-1)k = \dfrac{v(v+t)^2}{s_1}-(v-t)+\dfrac{vt^2}{s_2}-t = \dfrac{v(v-t)^2}{s_1}+\dfrac{vt^2}{s_2}-v,$$

$$0 = \dfrac{v(v-t)^2}{s_1}+\dfrac{vt^2}{s_2}-v-k(v-1).$$

We shall prove the latter assertions. We put

$$f(t) = \dfrac{v(v-t)^2}{s_1}+\dfrac{vt^2}{s_2}-v-k(v-1).$$

When $s_1\geqq\dfrac{v(v-1)}{k}$, then $f(0)<0$. Since the coefficient of t^2 in $f(t)$ is positive, $f(t)$ has at most one root for $0<t<v$. When $t=1$, then $s_2\leqq v$. By Lemma 8, i) $s_2\geqq v$. So $s_2=v$, and hence, G_α is doubly transitive on $\sum_2(\alpha)$, and $s_1=\dfrac{v(v-1)}{k+1}$.

Lemma 11. *Let* $\Gamma_1^*\circ\Gamma_2$ *be the union of two G-orbits* $\sum_1$ *and* $\sum_2$, *and* G_α *doubly transitive on* $\sum_1(\alpha)$ *and* $\sum_2(\alpha)$, *then* $|\Gamma_1(\alpha)|=|\Gamma_2(\alpha)|\leqq 3$.

Proof. This lemma due to P. J. Cameron. ([3], Lemma 4.) We put $|\Gamma_1(\alpha)|=|\Gamma_2(\alpha)|=v$, and assume $|\sum_1(\alpha)|\neq v$. Then, G_α has the different permutation characters on $\Gamma_1^*(\alpha)$ and $\sum_1(\alpha)$, so, for $(\alpha,\delta)\in\sum_1$, $G_{\alpha,\delta}$ is transitive on $\Gamma_1^*(\alpha)$. Hence, $\Gamma_1^*(\alpha)=\Gamma_2^*(\delta)$. Therefore, $G_\alpha=G_{\{\Gamma_1^*(\alpha)\}}=G_{\{\Gamma_2^*(\delta)\}}=G_\delta$. This is impossible. Thus we conclude that $|\sum_1(\alpha)|=v$. In the same way, we have $|\sum_2(\alpha)|=v$.

Now, if $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$, then by Lemma 5 $|\Gamma_1^*\circ\Gamma_2(\alpha)|=|\Gamma_1^*(\alpha)|\ |\Gamma_2(\alpha)| =v^2$. Therefore, $v^2=|\Gamma_1^*\circ\Gamma_2(\alpha)|=|\sum_1(\alpha)|+|\sum_2(\alpha)|=2v$, so $v=2$. Thus, when $v>2$, we obtain that $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$. For $\gamma\in\Gamma_1^*(\alpha)$, we put $t=|\Gamma_2(\gamma)\cap\sum_1(\alpha)|$. Then for $(\gamma_1,\gamma_2)\in\Gamma_1\circ\Gamma_1^*$, by Lemma 10 we have the following equation

$$k_2 = |\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)| = \frac{1}{v-1}\{(v-t)^2+t^2-v\}$$
$$= v-\frac{2t(v-t)}{v-1}.$$

If $t=\frac{v}{2}$, $|\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)| = v+\frac{v^2}{2(v-1)}$ is not integer, so $t\leqq\frac{v-1}{2}$ or $t\geqq\frac{v+1}{2}$. Hence $k_2=v-\frac{2t(v-t)}{v-1}\geqq v-\frac{1}{2}(v+1)=\frac{1}{2}(v-1)$. But $k_2\leqq\frac{1}{2}(v-1)$ by Lemma 3, so equality holds, and thus $v=3$ or 5 by Lemma 3, and $t=\frac{1}{2}(v+1)$ or $\frac{1}{2}(v-1)$. Counting arguments show that $|\Gamma_2(\gamma_1)\cap\Gamma_2(\gamma_2)\cap\sum_1(\alpha)|=\frac{t(t-1)}{v-1}$ for γ_1, $\gamma_2(\neq)\in\Gamma_1^*(\alpha)$. Therefore $v-1$ divides $t(t-1)$; this excludes $v=5$, and so $v=3$.

Lemma 12. *For $\Gamma_1, \Gamma_2, \Gamma_3$, if $\sum$ is a G-orbit contained in $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3$, and $|\Gamma_1(\alpha)|>3$; then G_α is not doubly transitive on $\sum(\alpha)$.*

Proof. $\sum^*\circ\Gamma_1^*\supset\Gamma_2^*\cup\Gamma_3^*$. If G_α is doubly transitive on $\sum(\alpha)$, $\sum^*\circ\Gamma_1^*$ is the union of at most two G-orbits by Lemma 6, so $\sum^*\circ\Gamma_1^*=\Gamma_2^*\cup\Gamma_3^*$. This is contrary to Lemma 11.

Lemma 13. *If $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$ and $\pi_1\neq\pi_2$ then, $|v_1-v_2|\geqq 2$, and $|\Gamma_1\circ\Gamma_1^*(\alpha)|>|\Gamma_1^*\circ\Gamma_2(\alpha)|$.*

Proof. For $(\alpha, \delta)\in\Gamma_1^*\circ\Gamma_2$, we put

$$m = |\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|.$$

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ with $\gamma_1\neq\gamma_2$ whose edges are successively Γ_1^*, Γ_2, Γ_2^*, and Γ_1; then we have

$$|\Omega|\frac{v_2(v_2-1)}{k_2}k_2k_1 = |\Omega|\frac{v_1v_2}{m}m(m-1),$$

so

$$(v_2-1)k_1 = v_1(m-1). \tag{1}$$

If $v_1=v_2$, then $k_1=v_1$. This is impossible. If $v_1=v_2+1$, then $k_1\geqq\frac{v_1}{2}$, and hence, by Lemma 3 $v_1=2$, $v_2=1$. This is also impossible. Thus we can conclude that $|v_1-v_2|\geqq 2$.

Assume $|\Gamma_1\circ\Gamma_1^*(\alpha)|=\frac{v_1(v_1-1)}{k_1}=|\Gamma_1^*\circ\Gamma_2(\alpha)|=\frac{v_1v_2}{m}$. Then

$$k_1v_2\geqq m(v_1-1). \tag{2}$$

From $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$, we have also

$$k_2v_1\geqq m(v_2-1)\,. \tag{3}$$

Therefore, (1) and (2) yield

$$v_1\leqq k_1+m\,. \tag{4}$$

By Lemma 3 and (3), we have

$$2v_2\leqq\frac{v_2(v_2-1)}{k_2}\leqq\frac{v_1v_2}{m}\,,$$

so

$$2\leqq m\leqq\frac{v_1}{2}\,. \tag{5}$$

Thus (4) and (5) yield

$$k_1\geqq\frac{1}{2}v_1\,.$$

This is contrary to Lemma 3.

Lemma 14. (P. J. Cameron [3]) *If* $\Gamma_1\circ\Gamma_1^*=\Gamma_1\circ\Gamma_2^*$, *then* $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$.

Proof. We shall prove this lemma in a different way from P. J. Cameron's. Assume $\Gamma_1\circ\Gamma_1^*=\Gamma_1\circ\Gamma_2^*=\Gamma_2\circ\Gamma_2^*$. We put

$$|\Gamma_1\circ\Gamma_1^*(\alpha)|=\frac{v_1(v_1-1)}{k_1}=|\Gamma_2\circ\Gamma_2^*(\alpha)|=\frac{v_2(v_2-1)}{k_2}=|\Gamma_1\circ\Gamma_2^*(\alpha)|=\frac{v_1v_2}{m}\,,$$

where $m=|\Gamma_1(\alpha)\cap\Gamma_2(\delta)|$ for $(\alpha,\,\delta\in\Gamma_1\circ\Gamma_2^*)$. Then it is trivial that $m>1$ from the above formula, and hence, $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2$. Thus, by Lemma 13, $|\Gamma_1\circ\Gamma_2^*(\alpha)|<|\Gamma_1^*\circ\Gamma_1(\alpha)|=|\Gamma_1\circ\Gamma_1^*(\alpha)|$. This is contrary to assumption.

Now we shall investigate from Lemma 15 to Lemma 22 the necessary condition that the intersection of $\Gamma_1^*\circ\Gamma_2$ and $\Gamma_1^*\circ\Gamma_3$ for Γ_1, Γ_2, Γ_3 $(\neq)$ is not empty.

Lemma 15. *If* $\pi_1=\pi_2\neq\pi_3$ *and* $\pi_2^*=\pi_3^*$, *or* $\pi_1=\pi_2=\pi_3$ *and* $\pi_2^*\neq\pi_3^*$, *then* $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3=\emptyset$.

Proof. Assume $\pi_1=\pi_2\neq\pi_3$ and $\pi_2^*=\pi_3^*$. Then we have $v_1=v_2=v_3$. We put $v=v_1=v_2=v_3$. By Lemma 13, $\Gamma_1\circ\Gamma_1^*\neq\Gamma_3\circ\Gamma_3^*$, and hence, $|\Gamma_1^*\circ\Gamma_3(\alpha)|=|\Gamma_1^*(\alpha)|\cdot|\Gamma_3(\alpha)|=v^2$ by Lemma 5. If $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3\neq\emptyset$, then since $\Gamma_1^*\circ\Gamma_3$ is a G-orbit and $\Gamma_1^*\circ\Gamma_2$ is a union of two G-orbits, we have $\Gamma_1^*\circ\Gamma_2\supsetneqq\Gamma_1^*\circ\Gamma_3$. Therefore $|\Gamma_1^*\circ\Gamma_2(\alpha)|>|\Gamma_1^*\circ\Gamma_3(\alpha)|=v^2$. This is impossible. Similarly, we can prove the lemma for the case of $\pi_1=\pi_2=\pi_3$ and $\pi_2^*\neq\pi_3^*$.

Lemma 16. *If* $\pi_1^* \neq \pi_2^*$, $\pi_1^* \neq \pi_3^*$ *and* $\pi_2 \neq \pi_3$, *then* $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* = \emptyset$.

Proof. By the assumption, $\Gamma_1 \circ \Gamma_2^*$, $\Gamma_1 \circ \Gamma_3^*$ and $\Gamma_2^* \circ \Gamma_3$ are G-orbits. Assume $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_2^*$, we put

$$|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = m_2 \quad \text{and} \quad |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| = m_3 .$$

For $\gamma_1, \gamma_2 (\neq) \in \Gamma_1(\alpha)$, we put

$$|\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2)| = x .$$

Then, since $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_3$, we have

$$\frac{v_1(v_1-1)}{k_1} = |\Gamma_1^* \circ \Gamma_1(\alpha)| = |\Gamma_2^* \circ \Gamma_3(\alpha)| = \frac{v_2 v_3}{x} ,$$

so

$$v_1(v_1-1)x = v_2 v_3 k_1 . \tag{1}$$

Count in two ways quadrilaterals $(\alpha, \gamma', \delta, \gamma)$ whose edges are successively Γ_1, Γ_2^*, Γ_3 and Γ_1^*, then we have

$$|\Omega| \frac{v_1({}_1v-1)}{k_1} k_1 x = |\Omega| \frac{v_1 v_3}{m_3} m_2 m_3 ,$$

so

$$(v_1-1)x = v_3 m_2 . \tag{2}$$

(1) and (2) yield

$$v_1 m_2 = k_1 v_2 . \tag{3}$$

If $m_2 > 1$, there exist quadrilaterals $(\alpha, \beta_1, \delta, \beta_2)$ whose edges are successively Γ_1, Γ_2^*, Γ_2 and Γ_1^*, whose vetices are all distinct; count all of them in two ways, we have

$$|\Omega| \frac{v_1(v_1-1)}{k_1} k_1 k_2 = |\Omega| \frac{v_1 v_2}{m_2} m_2(m_2-1) ,$$

so

$$(v_1-1)k_2 = v_2(m_2-1) .$$

On the other hand, from $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$,

$$v_2(v_2-1)k_1 = v_1(v_1-1)k_2 = v_1 v_2(m_2-1) ,$$

so

$$v_1(m_2-1) = (v_2-1)k_1 .$$

(3) and (4) yield

$$v_1 = k_1.$$

This is contrary to Lemma 3.

Thus, we have $m_2=m_3=1$ and $v_1=k_1v_2$. For $(\alpha, \gamma)\in\Gamma_1$, $G_{\alpha,\gamma}$ is transitive on $\Gamma_1(\alpha)\setminus\{\gamma\}$ and since $\pi_1^*\neq\pi_2^*$, it is also transitive on $\Gamma_2^*(\gamma)$. Count in two ways (γ', δ) such that $\gamma'\in\Gamma_1(\alpha)\setminus\{\gamma\}$, $\delta\in\Gamma_2^*(\gamma)$ and $(\gamma', \delta)\in\Gamma_3^*$, then we have

$$(v_1-1)x = v_2 = \frac{v_1}{k_1}\,.$$

This is impossible.

Lemma 17. *If* $\pi_1\neq\pi_2$, $\pi_1\neq\pi_3$ *and* $\Gamma_1\circ\Gamma_1^*=\Gamma_2\Gamma_2^*$, *then* $\Gamma_2^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3=\emptyset$.

Proof. Assume $\Gamma_1^*\circ\Gamma_2=\Gamma_1^*\circ\Gamma_3$. By Lemma 16, $\pi_2^*=\pi_3^*$. We put $v=v_1$, $w=v_2=v_3$, $m=|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|=|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|>1$ for $(\alpha, \delta)\in\Gamma_1^*\circ\Gamma_2$, and $x=|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)|$ for $\gamma_1, \gamma_2(\neq)\in\Gamma_1^*(\alpha)$.

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_3^* and Γ_1; then we have

$$|\Omega|\frac{v(v-1)}{k_1}k_1x = |\Omega|\frac{vw}{m}mm\,,$$

so

$$(v-1)x = wm\,. \tag{1}$$

Next, count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_2^*, Γ_1 and whose vertices are all distinct; then

$$|\Omega|\frac{v(v-1)}{k_1}k_1k_2 = |\Omega|\frac{vw}{m}m(m-1)\,,$$

$$(v-1)k_2 = w(m-1)\,. \tag{2}$$

(1) and (2) yield

$$(v-1)(x-k_2) = w, \text{ that is, } x>k_2\geqq 1\,. \tag{3}$$

Since $x\geqq 2$, there exist quadrilaterals $(\gamma, \delta_1, \gamma', \delta_2)$ whose edges successively Γ_3, Γ_2^*, Γ_2 and Γ_3^*, whose vertices are all distinct, and $(\gamma, \gamma')\in\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$; count all of them in two ways, then

$$|\Omega|w(w-1)\lambda = |\Omega|\frac{w(w-1)}{k_2}x(x-1)\,,$$

$$(\lambda = |\Gamma_2^*(\delta_1)\cap\Gamma_2^*(\delta_2)\cap\Gamma_1\circ\Gamma_1^*(\gamma)| \quad \text{for } \delta_1, \delta_2\ (\neq)\in\Gamma_3(\gamma))$$

so

$$\lambda = \frac{x(x-1)}{k_2}\,.$$

By the definition of λ, $\lambda \leqq k_2$. On the other hand, since $x>k_2$, $\lambda=\frac{x(x-1)}{k_2}>k_2$. This is a contradiction.

Lemma 18. *If* $\pi_1^* \neq \pi_2^*$, $\pi_1^* \neq \pi_3^*$ *and* $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$, *then* $C_1 C_2^* = C_1 C_3^*$.

Proof. By Lemma 6 $\sum = \Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$ is a G-orbit. Let $S=C(\sum)$, $C_1 C_2^* = m_2 S$, $C_1 C_3^* = m_3 S$ and $|\sum(\alpha)|=s$.
For the matrix F such that the value of any entry is 1, we have

$$v_1 v_2 F = F(C_1 C_2^*) = F(m_2 S) = m_2 s F\,,$$

so

$$v_1 v_2 = m_2 s\,.$$

Similarly

$$v_1 v_3 = m_3 s\,.$$

On the other hand, by Lemma 16, $\pi_2 = \pi_3$, and hence, $v_2 = v_3$. So, $m_2 = m_3$. Thus we can conclude that $C_1 C_2^* = C_1 C_3^*$.

Lemma 19. *If* $C_1 C_2^* = C_1 C_3^*$ *and* $|\Gamma_1(\alpha)| = v_1 > 3$, *then we have*

i) $\pi_2 = \pi_3$, $\pi_1^* \neq \pi_2^*$, π_3^*.

ii) $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$, $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_3^* \circ \Gamma_3$.

iii) $v_1 = v_2 + 1 = v_3 + 1$, $|\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2)| = 1$ *for* $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$.

iv) $|\Gamma_1^* \circ \Gamma_1(\alpha)| = \frac{v_1(v_1-1)}{2}$.

Proof. By the assumption $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For the matrix F such that the value of any entry is 1, we have

$$F(C_1 C_2^*) = (F C_1) C_2^* = (v_1 F) C_2^* = v_1 (F C_2^*) = v_1 v_2 F\,.$$

Similarly

$$F(C_1 C_3^*) = v_1 v_3 F\,.$$

So

$$v_2 = v_3\,.$$

We shall show that $v_1 \neq v_2 = v_3$. Assume $v = v_1 = v_2 = v_3$ ana put $D = C(\Gamma_1^* \circ \Gamma_1)$. If $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$, then $|\Gamma_1 \circ \Gamma_3^*(\alpha)| = |\Gamma_1 \circ \Gamma_2^*(\alpha)| \neq |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_3(\alpha)|$, therefore $\Gamma_1^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3$ by Lemma 5. We put $k = k_1 = k_2 = k_3$.

$$C_1^*(C_1 C_2^*) = (C_1^* C_1) C_2^* = (vE + kD) C_2^* = v C_2^* + k(v-1) C_2^* + \text{terms not involving } C_2^*.$$

Similarly

$$C_1^*(C_1C_3^*)=vC_3^*+k(v-1)C_3^*+\text{terms not involving } C_3^* .$$

So

$$(vE+kD)C_2^* = \{v+k(v-1)\}C_3^*+\text{terms not involving } C_3^* .$$

Since the coefficients of the basis matrices in DC_2^* are at most v, the above formula is impossible.

Next, if $\Gamma_1^*\circ\Gamma_1\neq\Gamma_2^*\circ\Gamma_2$, then $\Gamma_1^*\circ\Gamma_1\neq\Gamma_3^*\circ\Gamma_3$, and DC_3^* does not ivnolve C_3^*. Now

$$C_1^*(C_1C_2^*) = (C_1^*C_1)C_2^* = (vE+k_1D)C_2^* ,$$
$$C_1^*(C_1C_3^*) = (C_1^*C_1)C_3^* = (vE+kD)C_3^* = vC_3^*+\text{terms not invloving } C_3^* ,$$

and hence, $k_1DC_2^*=vC_3^*+$terms not involving C_3^*.

For $(\gamma_1, \gamma_2)\in\Gamma_1^*\circ\Gamma_1$ and $(\gamma_1, \delta)\in\Gamma_3^*$, we put

$$x = |\Gamma_3^*(\gamma_1)\cap\Gamma_2^*(\gamma_2)| \quad \text{and} \quad t = |\Gamma_1^*\circ\Gamma_1(\gamma_1)\cap\Gamma_2(\delta)| .$$

Then from the above formula we have

$$t = \frac{v}{k_1} . \tag{1}$$

Counting in two ways triplilaterals $(\gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_3^*, Γ_2 and $\Gamma_1^*\circ\Gamma_1$, we have

$$\frac{v(v-1)}{k_1}x = vt .$$

(1) and (2) yield

$$(v-1)x = v ,$$

which is a contradiction. Thus we can conclude that $v_1\neq v_2=v_3$, and hence, $\pi_2^*\neq\pi_1^*\neq\pi_3^*$. Therefore, we obtain $\pi_2=\pi_3$ by Lemma 16, $\Gamma_2^*\circ\Gamma_2\neq\Gamma_1^*\circ\Gamma_1\neq\Gamma_3^*\circ\Gamma_3$ by Lemma 17, and hence we have i) and ii) of Lemma.

For $(\alpha, \gamma)\in\Gamma_1$, count in two ways the ordered pairs (γ', δ) such that $\gamma'\in\Gamma_1(\alpha)\backslash\{\gamma\}$, $\delta\in\Gamma_2^*(\gamma)$ and $(\gamma', \delta)\in\Gamma_3^*$; then since $\Gamma_1^*\circ\Gamma_1\neq\Gamma_3^*\circ\Gamma_3$ we have

$$(v_1-1)x = v_2 . \tag{3}$$

Now, we shall show that $x=1$. Assume $x>1$, then there exist quadrilaterals $(\gamma, \delta_1, \gamma', \delta_2)$ whose edges are successively Γ_2^*, Γ_3, Γ_3^* and Γ_2 whose edges are all distinct, and $(\gamma, \gamma')\in\Gamma_1^*\circ\Gamma_1$; count all of them in two ways, then we have

$$|\Omega|v_2(v_2-1)\lambda = |\Omega|\frac{v_1(v_1-1)}{k_1}x(x-1) ,$$

$(\lambda=|\Gamma_1^*\circ\Gamma_1(\gamma)\cap\Gamma_3(\delta_1)\cap\Gamma_3(\delta_2)|$ for $(\gamma, \delta_1), (\gamma, \delta_2)\ (\neq)\in\Gamma_2^*, (\delta_1, \delta_2)\in\Gamma_2\circ\Gamma_2^*)$,

so

$$(v_2-1)\lambda k_1 = v_1(x-1) = (v_1-1)x+x-v_1 = v_2+x-v_1\,.$$

Therefore, $x \geqq v_1-1$. If $x=v_1$ then $(v_2-1)\lambda k_1=v_2$, which is a contradiction. If $x>v_1$, then $v_2=(v_1-1)x>\frac{v_1(v_1-1)}{k_1}$. So $(\pi_2^*, \pi(\Gamma_1^* \circ \Gamma_1(\gamma)))_{G_\gamma}=1$, where $\pi(\Gamma_1^* \circ \Gamma_1(\gamma))$ is the permutation character of G_γ on $\Gamma_1^* \circ \Gamma_1(\gamma)$. Hence, for $(\gamma, \gamma') \in \Gamma_1^* \circ \Gamma_1$, $G_{\gamma,\gamma'}$ is transitive on $\Gamma_2^*(\gamma)$. So $\Gamma_2^*(\gamma)=\Gamma_3^*(\gamma')$. This is impossible.

Thus we have $x=v_1-1$, $k_1=\lambda=1$, $v_2=(v_1-1)^2$ and $|\Gamma_1^* \circ \Gamma_1(\gamma) \cap \Gamma_3(\delta)|=v_1$ for $(\gamma, \delta) \in \Gamma_2^*$.

Now, count in two ways quadrilaterals $(\alpha, \gamma_1, \gamma_2, \gamma_3)$ such that $(\alpha, \gamma_1) \in \Gamma_2$, $(\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3$, and $(\gamma_1, \gamma_2), (\gamma_1, \gamma_3) \in \Gamma_1^* \circ \Gamma_1$, $\gamma_2 \neq \gamma_3$; then we have

$$|\Omega|v_3(v_3-1)\lambda' = |\Omega|v_2v_1(v_1-1)\,,$$
$$(\lambda' = |\Gamma_1^* \circ \Gamma_1(\gamma_2) \cap \Gamma_1^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)| \text{ for } \gamma_2, \gamma_3\ (\neq) \in \Gamma_3(\alpha))$$

so

$$\lambda' = \frac{v_1(v_1-1)}{v_3-1} = \frac{v_1(v_1-1)}{(v_1-1^2)-1} = \frac{v_1-1}{v_1-2}\,.$$

Therefore, $v_1=3$. This is contrary to the hypossesis of Lemma. Thus we can conclude that $x=1$, and hence, by (3) we have $v_1=v_2+1=v_3+1$. This proves Lemma iii).

Lastly, we shall show that $k_1=2$. If $k_1=1$, then $|\Gamma_1^* \circ \Gamma_1(\alpha)|=v_1(v_1-1) \leqq |\Gamma_2^* \Gamma \circ_3(\alpha)| \leqq v_2v_3=(v_1-1)^2$ This is impossible. Now, we have

$$u = |\Gamma_1^* \circ \Gamma_1(\gamma) \cap \Gamma_3(\delta)| = \frac{v_1}{k_1} \qquad \text{for } (\gamma, \delta) \in \Gamma_2^* \text{ and } 2 \leqq k_1 < \frac{v_1}{2}\,.$$

Count again in two ways quadrilaterals $(\alpha, \gamma_1, \gamma_2, \gamma_3)$ such that $(\alpha, \gamma_1) \in \Gamma_2$, $(\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3$ and $(\gamma_1, \gamma_2), (\gamma_1, \gamma_3) \in \Gamma_1^* \circ \Gamma_1$, $\gamma_2 \neq \gamma_3$; then

$$|\Omega|(v_1-1)(v_1-2)\lambda'' = |\Omega|(v_1-1)\left(\frac{v_1}{k_1}-1\right)\frac{v_1}{k_1}\,,$$
$$(\lambda'' = |\Gamma_1^* \circ \Gamma_1(\gamma_2) \cap \Gamma_1^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)| \text{ for } \gamma_2, \gamma_3 (\neq) \in \Gamma_3(\alpha))$$

so

$$\lambda'' = \frac{v_1(v_1-k_1)}{(v_1-2)k_1^2} = \frac{u(u-1)k_1^2}{(k_1u-2)k_1^2} = \frac{u(u-1)}{k_1u-2}\,.$$

If u is odd, then k_1u-2 divides $u-1$. This is impossible. We put $u=2u_0$, then

$$\lambda'' = \frac{2u_0(2u_0-1)}{2k_1u_0-2} = \frac{u_0(2u_0-1)}{k_1u_0-1}\,.$$

Therefore, we conclude that $k_1=2$.

Lemma 20. *If* $\pi_1=\pi_2\neq\pi_3$ *and* $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3\neq\emptyset$, *then* $v_1=v_2=v_3+1$, $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$ *and* $\Gamma_1^*\circ\Gamma_2=\Gamma_1^*\circ\Gamma_3\cup\Gamma_i$ *for some* Γ_i.

Proof. By assumption, $\sum=\Gamma_1^*\circ\Gamma_3$ is a G-orbit contained in $\Gamma_1^*\circ\Gamma_2$. We put $v=v_1=v_2$, $w=v_3$, $|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)|=x$ for $(\gamma_1, \gamma_2)\in\Gamma_1\circ\Gamma_1^*$, $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|=y$ and $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|=m$ for $(\alpha, \delta)\in\sum$, $|\Gamma_2(\gamma)\cap\sum(\alpha)|=t$ for $(\alpha, \gamma)\in\Gamma_1^*$. By Lemma 15, $\pi_2^*\neq\pi_3^*$, and hence, $\Gamma_2\circ\Gamma_3^*$ is a G-orbit. We have

$$\frac{v(v-1)}{k_1} = |\Gamma_1\circ\Gamma_1^*(\gamma_1)| = |\Gamma_2\circ\Gamma_3^*(\gamma_1)| = \frac{vw}{x}\,,$$

so

$$(v-1)x = wk_1\,. \tag{1}$$

We have also $|\sum(\alpha)|=\dfrac{vw}{m}=\dfrac{vt}{y}$, and so

$$wy = tm\,. \tag{2}$$

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_3^* and Γ_1, then we have

$$|\Omega|\frac{v(v-1)}{k_1}k_1x = |\Omega|\frac{vw}{m}my\,,$$

so

$$(v-1)x = wy\,. \tag{3}$$

(1) and (3) yield

$$y = k_1\,. \tag{4}$$

From (2) and (3),

$$(v-1)x = tm\,. \tag{5}$$

We shall show that $m=1$. If $m>1$, then there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_3, Γ_3^* and Γ_1, whose vertices are all distinct; count all of them in two ways, then we have

$$|\Omega|\frac{w(w-1)}{k_3}k_3k_1 = |\Omega|\frac{wv}{m}m(m-1)\,,$$

so

$$(w-1)k_1 = v(m-1)\,.$$

On the other hand, from (3) and (4)

$$(w-1)k_1 = wk_1 - k_1 = (v-1)x - k_1\,,$$

therefore

$$v(m-1) = (v-1)x - k_1\,,$$

so

$$0 \lneqq v(x-m+1) = x+k_1 < 2v\,. \tag{6}$$

(6) yields

$$x = m\,,\quad v = m+k_1\,. \tag{7}$$

From (5) and (7),

$$t = v-1\,. \tag{8}$$

Thus $|\sum(\alpha)| = \frac{vt}{y} = \frac{v(v-1)}{k_1}$.

If $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$, then by Lemma 10, $|\sum(\alpha)| = \frac{v(v-1)}{k_1+1}$. This is a contradiction. So we have $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, and hence,

$$1 = y = k_1\,. \tag{9}$$

Therefore we have $m=v-1$ from (7) and (9), and $w=(v-1)^2$ from (2) and (8). So

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = |\Gamma_3 \circ \Gamma_3^*(\alpha)| = \frac{w(w-1)}{k_3} \geqq 2w = 2(v-1)^2 > v(v-1)\,.$$

This is impossible. Thus, we can conclude that $m=1$, and then by (5) $t=v-1$, $x=1$ and $|\sum(\alpha)| = \frac{v(v-1)}{k_1}$. By Lemma 10, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, and hence, $1=y=k_1$. Therefore, by (2) $w=v-1$, $|\sum(\alpha)| = v(v-1)$. By Lemma 8 iv), $\Gamma_1^* \circ \Gamma_2 = \sum \cup \Gamma_i$ for some Γ_i.

Lemma 21. *If $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 \neq \emptyset$, and $v_1, v_2, v_3 > 3$, then the following hold;*

i) *if $\pi_1 = \pi_2 = \pi_3$, then $\pi_2^* = \pi_3^*$*

ii) *if $\pi_1 = \pi_2 \neq \pi_3$, then $\pi_2^* \neq \pi_3^*$ and $v_1 = v_2 = v_3 + 1$.*

iii) *if $\pi_1 \neq \pi_2, \pi_3$, then $\pi_2^* = \pi_3^*$, $C_1^* C_2 = C_1^* C_3$ and $v_1 = v_2 + 1 = v_3 + 1$.*

Proof. We have this assertion by arranging from Lemma 15 to Lemma 20.

Lemma 22. *Suppose that $\Gamma_1^* \circ \Gamma_2$ and $\Gamma_1^* \circ \Gamma_3$ contain a G-orbit $\sum$ in $\Omega \times \Omega$, and $\pi_1 = \pi_2 = \pi_3$, $|\Gamma_1(\alpha)| > 3$. For $\gamma_1, \gamma_2 (\neq) \Gamma_1^*(\alpha)$ and $\delta \in \sum(\alpha)$, the following hold;*

i) *if $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$, then $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| > 1$, $|\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| > 1$ and $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha)| > 1$.*

ii) *if* $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$, *then* $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)| > |\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = |\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = 1$, $|\sum(\alpha)| = \frac{v(v-1)}{k_1+1}$, *and* $\Gamma_1^*\circ\Gamma_2$ *contains some* Γ_k.

iii) *if* $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$, $\Gamma_3\circ\Gamma_3^*$, *then* $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)| = |\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = |\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = 1$, $|\sum(\alpha)| = v(v-1)$, *and* $\Gamma_1^*\circ\Gamma_2$ *contains some* Γ_i *and* $\Gamma_1^*\circ\Gamma_3$ *contains another* Γ_j.

Proof. Put $|\sum(\alpha)\cap\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = \lambda$ for $\gamma_1, \gamma_2(\neq)\in\Gamma_1^*(\alpha)$. $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)| = x_2$, $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = x_3$ for $(\alpha, \delta)\in\sum$. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_3^* and Γ_1, and $(\alpha, \delta)\in\sum$, then we have

$$|\Omega|\frac{v(v-1)}{k_1}k_1\lambda = |\Omega|\,|\textstyle\sum(\alpha)|\,x_2x_3\,,$$

so

$$v(v-1)\lambda = |\textstyle\sum(\alpha)|\,x_2x_3\,. \tag{1}$$

Assume $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$, $\Gamma_3\circ\Gamma_3^*$. Then we have $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)| = |\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = 1$. By (1)

$$v(v-1)\lambda = |\textstyle\sum(\alpha)|\,.$$

Since $|\sum(\alpha)| \leqq v(v-1)$, we have $\lambda=1$ and $|\sum(\alpha)| = v(v-1)$. By Lemma 8 iv), $\Gamma_1^*\circ\Gamma_2 = \sum\cup\Gamma_i$ and $\Gamma_1^*\circ\Gamma_3 = \sum\cup\Gamma_j$ for some Γ_i, Γ_j. By Lemma 8, iii), we have $C_1^*C_2 = S + C_i$, $C_1^*C_3 = S + C_j$. ($S = C(\sum)$) If $C_i = C_j$, then $C_1^*C_2 = C_1^*C_3$, and hence, by Lemma 19 $\pi_1 \neq \pi_2$, π_3. This is contrary to the hypothesis of this lemma. Thus $C_i \neq C_j$, that is, $\Gamma_i \neq \Gamma_j$. So $\sum(\alpha)\cap\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2) = \Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)$. Therefore $|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = |\sum(\alpha)\cap\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = \lambda = 1$. Thus we have iii) of Lemma.

Next assume $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$. Then we have $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = 1$. By (1)

$$v(v-1)\lambda = |\textstyle\sum(\alpha)|\,x_2\,. \tag{2}$$

Count in two ways triplilaterals (α, δ, γ) whose edges are successibly $\sum$, Γ_2^*, and Γ_1 then we have

$$|\textstyle\sum(\alpha)|\,x_2 \leqq v(v-1)\,. \tag{3}$$

If $x_2=1$, then $|\sum(\alpha)| = v(v-1)$ by (2) and (3). By Lemma 8. iv), $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$. This is contrary to the assumption. Therefore we have $x_2>1$, $\lambda=1$ and $|\sum(\alpha)|x_2 = v(v-1)$. Since $|\sum(\alpha)|x_2 = v(v-1)$, $|\sum(\alpha)\cap\Gamma_2(\gamma)| = v-1$ for $(\alpha, \gamma)\in\Gamma_1^*$. By Lemma 10. ii), $|\sum(\alpha)| = \frac{v(v-1)}{k_1+1}$ and $\Gamma_1^*\circ\Gamma_2$ contains some Γ_i.

Now we shall show that $\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2) = \Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)\cap\sum(\alpha)$, for

$\gamma_1, \gamma_2 \in \Gamma_1^*(\alpha)$. If $\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \supsetneqq \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha)$, then $\Gamma_1^* \circ \Gamma_2 = \Gamma_1^* \circ \Gamma_3$. But $|\Gamma_1^* \circ \Gamma_2(\alpha)| = |\sum(\alpha)| + |\Gamma_i(\alpha)| = \frac{v(v-1)}{k_1+1} + v < v^2$ and $|\Gamma_1^* \circ \Gamma_3(\alpha)| = v^2$. This is impossible. Therefore, $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha)| = \lambda = 1$. Thus we have ii) of Lemma.

Last assume $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$. We shall show that $x_2 = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| > 1$ and $x_3 = |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| > 1$. We note that $k_1 = k_2 = k_3$, therefore we put $k = k_1 = k_2 = k_3$. If $x_2 = x_3 = 1$, by (1) we have $|\sum(\alpha)| = v(v-1)$. By Lemma 8. iv) $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, $\Gamma_3 \circ \Gamma_3^*$. This is contrary to the assumption. If $x_2 > x_3 = 1$, we have $|\sum(\alpha)| = \frac{v(v-1)}{k+1}$ as before, and $x_2 = k+1$. We put $\Gamma_1^* \circ \Gamma_3 = \sum \cup \sum'$,

$$x = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta')| \qquad \text{for } (\alpha, \delta') \in \textstyle\sum', \text{ and}$$

$$t = |\Gamma_3(\gamma_1) \cap \textstyle\sum(\alpha)| = \dfrac{v-1}{k+1} \qquad \text{for } (\alpha, \gamma_1) \in \Gamma_1^* .$$

Since $\Gamma_1 \circ \Gamma_1^* = \Gamma_3 \Gamma_3^*$ and $x_3 = 1$, there exist quadrilaterals $(\alpha, \gamma_1, \delta', \gamma_2)$, with $\gamma_1 \neq \gamma_2$ and $(\alpha, \delta') \in \sum'$, whose edges are successively Γ_1^*, Γ_3, Γ_3^* and Γ_1. Count all of them in two ways then we have

$$|\Omega| \frac{v(v-1)}{k} kk = |\Omega| \frac{v\left(v - \dfrac{v-1}{k+1}\right)}{x} x(x-1),$$

so

$$x-1 = \frac{(v-1)k}{v - \dfrac{v-1}{k+1}} = \frac{t(k+1)k}{t(k+1)+1-t} = \frac{tk(k+1)}{tk+1}.$$

Therefore $t=1$, and hence, $v = k+2$. This is impossible by Lemma 3. Thus we have $x_2 > 1$ and $x_3 > 1$.

Now we shall show that $\lambda > 1$. If $\lambda = 1$, by (1) we have

$$v(v-1) = |\textstyle\sum(\alpha)| x_2 x_3$$

Since $x_2 > 1$, there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$, with $\gamma_1 \neq \gamma_2$ and $(\alpha, \delta) \in \sum$, whose edges are successively Γ_1^*, Γ_2, Γ_2^* and Γ_1. Count all of them in two ways then we have

$$|\Omega| \frac{v(v-1)}{k} k\lambda_2 = |\Omega| \, |\textstyle\sum(\alpha)| x_2(x_2-1),$$

$$(\lambda_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \textstyle\sum(\alpha)| \qquad \text{for } \gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha))$$

so

$$\lambda_2=\frac{|\sum(\alpha)|x_2(x_2-1)}{v(v-1)},$$

and by (1),

$$\lambda_2=\frac{x_2-1}{x_3}.$$

Thus $\frac{x_2-1}{x_3}$ is a positive integer. Since $x_3>1$, in the same way, we have that $\frac{x_3-1}{x_2}$ is a positive integer. This is impossible. Thus we have i) of Lemma.

Lemma 23. *If* $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$ *and* $\pi_1\neq\pi_2$, *then for any* Γ_i, $\Gamma_j(\neq)$, $\Gamma_i\circ\Gamma_j^*\not\supset\Gamma_1\circ\Gamma_1^*$.

Proof. Assume $\Gamma_i\circ\Gamma_j^*\supset\Gamma_1\circ\Gamma_1^*$. Note that $|v_1-v_2|\geqq 2$ by Lemma 13, and hence, $\pi_1^*\neq\pi_2^*$. If $\{\Gamma_i,\Gamma_j\}=\{\Gamma_1,\Gamma_2\}$, then since $\Gamma_i\circ\Gamma_j^*$ is a G-orbit, $\Gamma_i\circ\Gamma_j^*=\Gamma_1\circ\Gamma_1^*=\Gamma_2\Gamma_2^*$. This is a contrary to Lemma 14. Therefore we can assume that $\Gamma_i\neq\Gamma_1$, Γ_2. If $\Gamma_j=\Gamma_1$ then, $\Gamma_2^*\circ\Gamma_1\cap\Gamma_2^*\circ\Gamma_i\neq\emptyset$. By Lemma 21 have $v_2=v_1-1$. This is a contradiction. Thus we have $\{\Gamma_i,\Gamma_j\}\cap\{\Gamma_1,\Gamma_2\}=\emptyset$.

From $v_1\neq v_2$, we may assume $v_i\neq v_1$. Since $\Gamma_1^*\circ\Gamma_i\cap\Gamma_1^*\circ\Gamma_j\neq\emptyset$, $v_i=v_1-1$ by Lemma 21. On the other hand, from $|v_1-v_2|\geqq 2$, $v_i\neq v_2$. Since $\Gamma_2^*\circ\Gamma_i\cap\Gamma_j^*\circ\Gamma_j\neq\emptyset$, in the same way, we have $v_i=v_2-1$. This is a contradiction.

Lemma 24. *If* $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*=\Delta$, $\pi_1=\pi_2=\pi_3$ *and* $|\Gamma_1(\alpha)|>3$, *then* $\Gamma_1\circ\Gamma_2^*\not\supset\Delta$ *or* $\Gamma_1\circ\Gamma_3^*\not\supset\Delta$.

Proof. Assume $\Gamma_1\circ\Gamma_2^*\supset\Delta$ and $\Gamma_1\circ\Gamma_3^*\supset\Delta$. We put $v=v_1=v_2=v_3$ and $k=k_1=k_2=k_3$. Since $\pi_1=\pi_2=\pi_3$, we have $\pi_1^*=\pi_2^*=\pi_3^*$ by Lemma 21. We shall show that $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2=\Gamma_3^*\circ\Gamma_3$. If $\Gamma_1^*\circ\Gamma_1\neq\Gamma_2^*\circ\Gamma_2$, $\Gamma_3^*\circ\Gamma_3$, $|\Delta(\alpha)|=v(v-1)$ by Lemma 22. iii). Since $|\Gamma_1^*\circ\Gamma_1(\alpha)|=|\Gamma_1\circ\Gamma_1^*(\alpha)|=|\Delta(\alpha)|=v(v-1)$, we have $\Gamma_2\circ\Gamma_2^*\neq\Gamma_3\circ\Gamma_3^*$ by Lemma 8. iv). If $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2\neq\Gamma_3^*\circ\Gamma_3$, $|\Delta(\alpha)|=\frac{v(v-1)}{k+1}$ by Lemma 22. ii). This is impossible. Thus we can conclude that $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2=\Gamma_3^*\circ\Gamma_3$. If $\Gamma_1\circ\Gamma_2^*=\Gamma_1\circ\Gamma_3^*$, then $C_1C_2^*=C_1C_3^*$ by Lemma 10, i); and hence, $v_1=v_2+1$ by Lemma 19, iii). This is contrary to the hypothesis of this lemma. We shall show that $k>1$. If $k=1$, $|\Gamma_1^*\circ\Gamma_1(\alpha)|=\frac{v(v-1)}{k}=v(v-1)$. Since $\Gamma_2^*\circ\Gamma_3\supset\Gamma_1^*\circ\Gamma_1$, $\Gamma_2\Gamma_2^*\neq\Gamma_3\circ\Gamma_3^*$ by Lemma 8, iv). This is contrary to the assumption. Count in two ways quadrilaterals $(\alpha,\gamma_1,\delta,\gamma_2)$ whose edges are successively Γ_1, Γ_2^*, Γ_3 and Γ_1^*; then we have

$$|\Omega|\frac{v(v-1)}{k}kx=|\Omega|\frac{v(v-1)}{k}x_2x_3,$$

so

$$kx = x_2 x_3 . \tag{1}$$

Here we put $x_2 = |\Gamma_1(\alpha) \cap \Gamma_2(\delta)|$, $x_3 = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)|$ for $(\alpha, \delta) \in \Delta$ and $x = |\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2) \cap \Delta(\alpha)|$ for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1(\alpha)$.

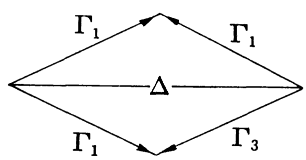
We shall show that x, x_2 and x_3 are smaller than k. If $x_2 \geqq k$, then for $(\alpha, \gamma) \in \Gamma_1$, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| \geqq v-1$. Of course, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| \leqq v-1$, and hence, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| = v-1$. By Lemma 10, ii), we have $|\Delta(\alpha)| = \dfrac{v(v-1)}{k+1}$, which is a contradiction. We can prove in the same way that $x_3 < k$. Then, (1) yields

$$x < x_2 , \quad x_3 < k . \tag{2}$$

Now

$$\begin{aligned}
&C_1(C_1^* C_3) = C_1(xD' + yS') , \\
&(C_1 C_2^*)C_3 = (x_2 D + y_2 S) C_3 = x_2(v-1)C_3 + \text{terms not involving } C_3 . \\
&(\Delta' = \Gamma_1^* \circ \Gamma_1,\ \Gamma_1 \circ \Gamma_2^* = \Delta \cup \textstyle\sum,\ \Gamma_2^* \circ \Gamma_3 = \Delta' \cup \textstyle\sum' , \\
&D = C(\Delta),\ D' = C(\Delta'),\ S = C(\textstyle\sum) \text{ and } S' = C(\textstyle\sum'))
\end{aligned}$$

Since $x_2 > x$ and the coefficient of C_3 contained in $C_1 D'$ is at most $v-1$, C_3 is contained in $C_1 S'$, that is, $\Gamma_1^* \circ \Gamma_3 \supset \sum'$. On the other hand, since $\Gamma_1 \circ \Gamma_3^* \supset \Delta$, there exists the following figure.



Therefore $\Gamma_1^* \circ \Gamma_3 \supset \Delta'$. Thus $\Gamma_1^* \circ \Gamma_3 = \Delta' \cap \sum' = \Gamma_2^* \circ \Gamma_3$. By Lemma 10, i) we have $C_1^* C_3 = C_2^* C_3$. So, $\pi_1 \neq \pi_3$ by Lemma 19, i). This is contrary to the hypothesis of this lemma.

Lemma 25. *If* v_1, v_2, v_3 *and* $v_4 > 3$, *then the following figures don't exist.*

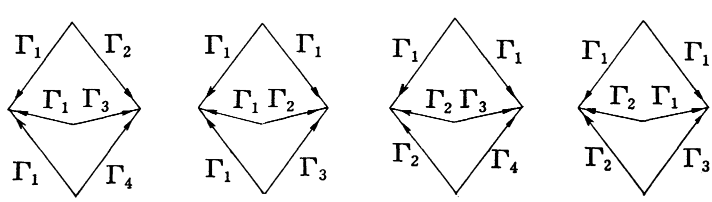


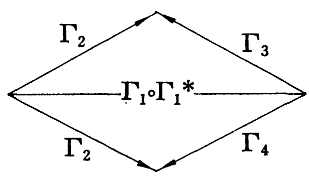
Fig. 1 Fig. 2 Fig. 3 Fig. 4

Proof. For each figure above, we assume its existence and show that it implies a contradiction.

Non-existence of Fig. 1.

Case I. $\pi_1 \neq \pi_2, \pi_3, \pi_4$.

By Lemma 18 and Lemma 19, $v_1 = v_2 + 1 = v_3 + 1 = v_4 + 1$, $|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2}$, $|\Gamma_2(\alpha) \cap \Gamma_3(\delta)| = |\Gamma_2(\alpha) \cap \Gamma_4(\delta)| = 1$ for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_1^*$ and $\pi_2^* = \pi_3^* = \pi_4^*$. Now let us consider the following figure.



Then by Lemma 22, i) and iii), we have

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v_2(v_2 - 1) = (v_1 - 1)(v_1 - 2) .$$

Thus,

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2} = (v_1 - 1)(v_1 - 2) ,$$

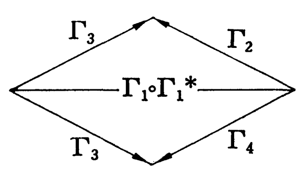
so

$$v_1 = 4 , \quad v_2 = v_3 = v_4 = 3 .$$

This is contrary to the hypothesis of this lemma.

Case II. $\pi_1 = \pi_2 \neq \pi_3, \pi_4$.

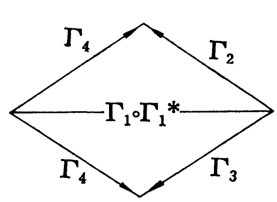
By Lemma 21, $v_1 = v_2 = v_3 + 1 = v_4 + 1$ and $\pi_3^* = \pi_4^* \neq \pi_2^*$. But considering the following figure,



we have $v_3 = v_2 + 1$ by Lemma 20. This is impossible.

Case III. $\pi_1 = \pi_2 = \pi_3 \neq \pi_4$.

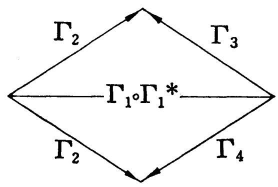
By Lemma 20, $v_1 = v_2 = v_3 = v_4 + 1$. But since there exists the following figure,



we have $v_4=v_3+1=v_2+1$ by Lemma 21, which is a contradiction.

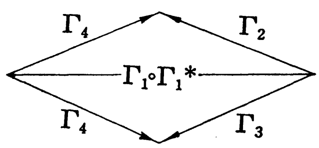
Case IV. $\pi_1=\pi_2=\pi_3=\pi_4$, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*=\Gamma_4\circ\Gamma_4^*$.

Existence of the following figure is contrary to Lemma 24.



Case V. $\pi_1=\pi_2=\pi_3=\pi_4$, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*\neq\Gamma_4\circ\Gamma_4^*$.

Since $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*$, we have by Lemma 22, i) $|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)|>1$ for $(\gamma_1, \gamma_2)\in\Gamma_1\circ\Gamma_1^*$, and hence, $\Gamma_2^*\circ\Gamma_2=\Gamma_3^*\circ\Gamma_3$. So, we have $|\Gamma_1\circ\Gamma_1^*(\alpha)|<v_1(v_1-1)$ by Lemma 8, iv). On the other hand, since $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*\neq\Gamma_4\circ\Gamma_4^*$ we have by Lemma 22, ii) $|\Gamma_4(\gamma_1)\cap\Gamma_2(\gamma_2)|=|\Gamma_4(\gamma_1)\cap\Gamma_3(\gamma_2)|=1$ for $(\gamma_1, \gamma_2)\in\Gamma_1\circ\Gamma_1^*$. Then from the existence of the following figure,



we have $|\Gamma_1\circ\Gamma_1^*(\alpha)|=v_1(v_1-1)$ by Lemma 22, which is a contradiction.

Case VI. $\pi_1=\pi_2=\pi_3=\pi_4$, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*\neq\Gamma_3\circ\Gamma_3^*$, $\Gamma_4\circ\Gamma_4^*$.

There exist the following figures, where $\sum$ is a G-orbit.

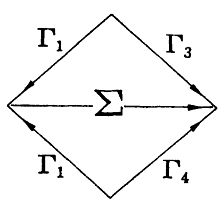


Fig. a

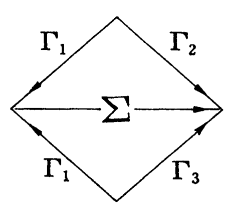


Fig. b

From Fig. a, we have $|\sum(\alpha)|=v_1(v_1-1)$ by Lemma 22, iii). On the other hand, from Fig. b, we have $|\sum(\alpha)|=\dfrac{v_1(v_1-1)}{k_1+1}$ by Lemma 22, ii), which is a contradiction.

Case VII. $\pi_1=\pi_2=\pi_3=\pi_4$, $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$, $\Gamma_3\circ\Gamma_3^*$, $\Gamma_4\circ\Gamma_4^*$.

From $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$, $\Gamma_3\circ\Gamma_3^*$, we have $|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)|=1$ for $\gamma_1, \gamma_2(\neq)\in\Gamma_1^*(\alpha)$, by Lemma 22, iii). Similarly from $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$, $\Gamma_4\circ\Gamma_4^*$, we have

$|\Gamma_2(\gamma_1)\cap\Gamma_4(\gamma_2)|=1$ for $\gamma_1, \gamma_2(\neq)\in\Gamma_1^*(\alpha)$. From $\Gamma_2\circ\Gamma_3^*\cap\Gamma_2\circ\Gamma_4^*\supset\Gamma_1\circ\Gamma_1^*$, we have by Lemma 22

$$|\Gamma_1\circ\Gamma_1^*(\alpha)| = v_1(v_1-1)\,. \tag{1}$$

By Lemma 21, $\pi_2^*=\pi_3^*=\pi_4^*$. Therefore we have by Lemma 8, iv)

$$\begin{gathered}\Gamma_2^*\circ\Gamma_2\neq\Gamma_3^*\circ\Gamma_3,\ \Gamma_3^*\circ\Gamma_3\neq\Gamma_4^*\circ\Gamma_4 \text{ and } \Gamma_4^*\circ\Gamma_4\neq\Gamma_2^*\circ\Gamma_2 \\ \text{and } \Gamma_i\circ\Gamma_j^*\ (2\leqq i, j(\neq)\leqq 4) \text{ contains some } \Gamma_k\,.\end{gathered} \tag{2}$$

We put

$$\begin{aligned}&v=v_1=v_2=v_3=v_4,\ \Gamma_1\circ\Gamma_1^*=\Delta_1,\ \Gamma_2^*\circ\Gamma_2=\Delta_2\,,\\ &\Gamma_2\circ\Gamma_3^*=\Delta_1\cup\Gamma_i,\ \Gamma_3^*\circ\Gamma_4=\Delta_2\cup\Sigma',\text{ and } D_1=C(\Delta_1)\,,\\ &D_2=C(\Delta_2),\ S'=C(\Sigma') \text{ and } s'=|\Sigma'(\alpha)|\,.\end{aligned}$$

Now,

$$(C_2C_3^*)C_4=(D_1+C_i)C_4=(v-1)C_3+\cdots\,.$$

The coefficient of C_3 of the above equation is $v-1$ or v by (2). Next,

$$C_2(C_3^*C_4)=C_2(D_2+xS')\,,$$

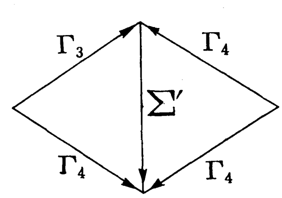
so

$$v^2=\frac{v(v-1)}{k_2}+xs'\,.$$

By Lemma 8, i), $s'\geqq v$, so

$$x\leqq v-\frac{v-1}{k_2}\leqq v-2\,. \tag{3}$$

We shall show that $\Gamma_4^*\circ\Gamma_4\neq\Sigma'$. If $\Gamma_4^*\circ\Gamma_4=\Sigma'$, there exists the following figure.



Since $\Gamma_3\circ\Gamma_4^*=\Delta_1\cup\Gamma_j$, we have $\Gamma_4\circ\Gamma_4^*=\Delta_1=\Gamma_1\circ\Gamma_1^*$. This is contrary to the assumption of this case. From $\Gamma_2\circ\Gamma_4^*\cap\Gamma_3\circ\Gamma_4^*\supset\Delta_1$ and (2), for $\gamma_1, \gamma_2(\neq)\in\Gamma_4(\alpha)$ we have by Lemma 22, iii)

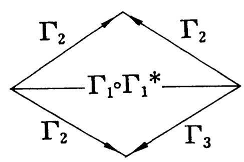
$$|\Gamma_2^*(\gamma_1)\cap\Gamma_3^*(\gamma_2)|=1\,. \tag{4}$$

If $\Gamma_2\circ\Sigma'$ contains Γ_3, then we have $\Gamma_2^*\circ\Gamma_3=\Gamma_4^*\circ\Gamma_4\cap\Sigma'$, and by (4)

$$C_2S'=\left(v-\frac{v-1}{k_4}\right)C_3+\text{terms not involving } C_3\,.$$

When $k_4=1$, $v-\dfrac{v-1}{k_4}=1$. So $\Gamma_2\circ\Delta_2$ contains Γ_3, by (3). When $k_4>1$, $v-1>v-\dfrac{v-1}{k_4}>\dfrac{v}{2}$. So, $x=1$, and hence $\Gamma_2\circ\Delta_2$ contains Γ_3.

In all cases, we can conclude that $\Gamma_2\circ\Delta_2$ contains Γ_3, and hence, $\Gamma_2^*\circ\Gamma_3\supset\Delta_2$. Thus, we have the following figure.



So, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*$. This is contrary to the assumption.

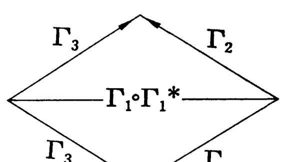
Non-existence of Fig. 2.

Case I. $\pi_1\neq\pi_2, \pi_3$.

From $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3\neq\emptyset$ and $\pi_1\neq\pi_2, \pi_3$, we have $|\Gamma_1\circ\Gamma_1^*(\alpha)|=\dfrac{v_1(v_1-1)}{2}$, $v_1=v_2+1$ and $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$ by Lemma 21 and Lemma 19. On the other hand, $|\Gamma_1\circ\Gamma_1^*(\alpha)|=|\Gamma_1^*\circ\Gamma_1(\alpha)|=|\Gamma_1^*\circ\Gamma_2(\alpha)|=|\Gamma_1^*(\alpha)|\cdot|\Gamma_2(\alpha)|=v_1(v_1-1)$. This is impossible.

Case II. $\pi_1=\pi_2\neq\pi_3$.

By Lemma 20, $v_1=v_2=v_3+1$. On the other hand from the existence of following figure,



we have $v_3=v_2+1=v_1+1$ by Lemma 21, iii). This is impossible.

Case III. $\pi_1=\pi_2=\pi_3$, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*=\Gamma_3\circ\Gamma_3^*$.

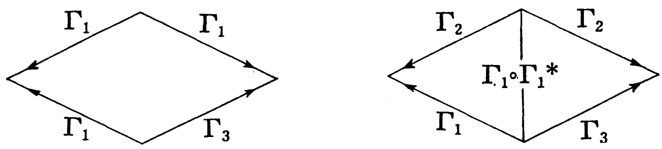
By Lemma 22, for $(\alpha,\delta)\in\Gamma_1^*\circ\Gamma_1$, $1<|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|$ and $1<|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|$. The counting auguments show that $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|=|\Gamma_1(\gamma_1)\cap\Gamma_2(\gamma_2)|$ and $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|=|\Gamma_1(\gamma_1)\cap\Gamma_3(\gamma_2)|$ for $(\gamma_1, \gamma_2)\in\Gamma_1\circ\Gamma_1^*$. Therefore, $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2=\Gamma_3^*\circ\Gamma_3$. Now $\Gamma_1^*\circ\Gamma_2\supset\Gamma_1^*\circ\Gamma_1$ and $\Gamma_1^*\circ\Gamma_3\supset\Gamma_1^*\circ\Gamma_1$. Since we can show that $\pi_1^*=\pi_2^*=\pi_3^*$ by Lemma 21, we have a contradiction by Lemma 24.

Case IV. $\pi_1=\pi_2=\pi_3$, $\Gamma_1\circ\Gamma_1^*=\Gamma_2\circ\Gamma_2^*\neq\Gamma_3\circ\Gamma_3^*$.

From $\Gamma_1^*\circ\Gamma_2\cap\Gamma_1^*\circ\Gamma_3\supset\Gamma_1^*\circ\Gamma_1$, we have $|\Gamma_1^*\circ\Gamma_1(\alpha)|=\dfrac{v(v-1)}{k_1+1}$ by Lemma 22. This is impossible.

Case V. $\pi_1=\pi_2=\pi_3$, $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*$, $\Gamma_3\circ\Gamma_2^*$.

By Lemma 21, we have $\pi_1^*=\pi_2^*=\pi_3^*$. By Lemma 22, iii), $|\Gamma_1\circ\Gamma_1^*(\alpha)|=v(v-1)$, and by Lemma 8, iv), $\Gamma_1^*\circ\Gamma_1\neq\Gamma_2^*\circ\Gamma_2$.

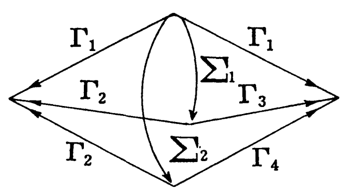


From the existence of the above figures, we have $\Gamma_1^*\circ\Gamma_3=\Gamma_1^*\circ\Gamma_1\cup\Gamma_2^*\circ\Gamma_2$. Therefore,

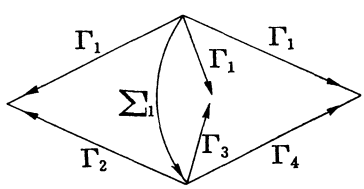
$$\begin{aligned} v^2 &= |\Gamma_1(\alpha)|\cdot|\Gamma_3(\alpha)|=|\Gamma_1^*\circ\Gamma_3(\alpha)| \\ &= |\Gamma_1^*\circ\Gamma_1(\alpha)|+|\Gamma_2^*\circ\Gamma_2(\alpha)| = v(v-1)+\frac{v(v-1)}{k_2}\,. \end{aligned}$$

This is impossible.

Non-existence of Fig. 3.



For the above figure, if $\Sigma_1=\Sigma_2$ then there exists the following figure.



This is contrary to non-existence of Fig. 1. Thus we have $\Sigma_1\neq\Sigma_2$, $\pi_1^*=\pi_2^*$, $\Gamma_1\circ\Gamma_2^*=\Sigma_1\cup\Sigma_2$ and G_α is not doubly transitive on $\Sigma_1(\alpha)$ and $\Sigma_2(\alpha)$ by Lemma 12. So, by Lemma 20 we have $\pi_1^*=\pi_2^*=\pi_3^*=\pi_4^*$. Also $\Gamma_1^*\circ\Gamma_1=\Gamma_2^*\circ\Gamma_2=\Gamma_3^*\circ\Gamma_3=\Gamma_4^*\circ\Gamma_4$ by Lemma 22. From $\Gamma_2^*\circ\Gamma_3\cap\Gamma_2^*\circ\Gamma_4\supset\Gamma_1^*\circ\Gamma_1$, this is contrary to Lemma 24.

Non-existence of Fig. 4.

There exist the following figures.

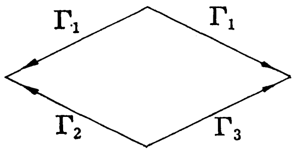


Fig. a

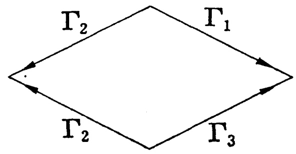


Fig. b

Case I. $\pi_1^* \neq \pi_2^*$.

By Lemma 21, we have $v_1 = v_2 + 1$ from Fig. a, and $v_2 = v_1 + 1$ from Fig. b. This is impossible.

Case II. $\pi_1^* = \pi_2^* \neq \pi_3^*$.

By Lemma 20, we have $v_1 = v_2 = v_3 + 1$ and $\Gamma_2 \circ \Gamma_2^* \neq \Gamma_1 \circ \Gamma_1^*$ from Fig. b. On the other hand, $\Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_1^* \subset \Gamma_2 \circ \Gamma_1^* \supset \Gamma_1 \circ \Gamma_1^*$, and $\Gamma_2 \circ \Gamma_1^*$ has some Γ_i by Lemma 20, and hence, $\Gamma_2 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_1^*$. This is impossible.

Case III. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3$.

By assumption, $\Gamma_2^* \circ \Gamma_1 \cap \Gamma_2^* \circ \Gamma_3 \supset \Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3$, which is contrary to Lemma 24.

Case IV. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 \neq \Gamma_3^* \circ \Gamma_3$.

From Fig. a, $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_1^* \cup \Gamma_i$ for some Γ_i by Lemma 22. So, $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* = \Gamma_1 \circ \Gamma_1^*$ and $\Gamma_1 \circ \Gamma_1^*(\alpha) = \dfrac{v(v-1)}{k_1+1}$. This is impossible.

Case V. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3 \neq \Gamma_2^* \circ \Gamma_2$.

We put $\sum = \Gamma_1 \circ \Gamma_3^* \cap \Gamma_1 \circ \Gamma_2^*$.

By Lemma 22, $|\sum(\alpha)| = \dfrac{v(v-1)}{k_1+1}$.

From that $\Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_1^*$, we have $\Gamma_1 \circ \Gamma_2^* = \sum \cup \Gamma_1 \circ \Gamma_1^*$. So $v^2 = \dfrac{v(v-1)}{k_1+1} + \dfrac{v(v-1)}{k_1}$. Therefore $k_1 = 1$ and $v - 1 = k_1 + 1 = 2$. This is contrary to the hypothesis of this lemma.

Case VI. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$, $\Gamma^*{}_3 \circ \Gamma_3$.

We put $\sum_1 = \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^*$. By Lemma 22, we have $\Gamma_1 \circ \Gamma_2^* = \sum \cup \Gamma_i$, $\Gamma_1 \circ \Gamma_3^* = \sum \cup \Gamma_j$ from some Γ_i, Γ_j and $|\sum(\alpha)| = v(v-1)$. Since $\Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_1^*$ and $\Gamma_1 \circ \Gamma_3^* \supset \Gamma_2 \circ \Gamma_2^*$, we have $\Gamma_1 \circ \Gamma_1^* = \sum = \Gamma_2 \circ \Gamma_2^*$. On the other hand, since $\Gamma_1^* \circ \Gamma_2 \supset \Gamma_1^* \circ \Gamma_1$ and $|\Gamma_1^* \circ \Gamma_1(\alpha)| = v(v-1)$, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ by Lemma 8, iv). This is impossible.

Lemma 26. *For Γ_1, Γ_2 and Γ_3, suppose that $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^*$ contains a G-orbit $\sum$ in $\Omega \times \Omega$, and v_1, v_2, $v_3 > 3$. Then, there does not exist Γ_i such that $\Gamma_i \circ \Gamma_i^* = \sum$.*

Proof. From non-existences of Fig. 2, Fig. 3, Fig. 4 of Lemma 24, we have this assertion.

Lemma 27. (P. J. Cameron [3], Prop.)
If $\Gamma_i^* \neq \Gamma_i$ *and* $\Gamma_i \circ \Gamma_i \subseteqq \Gamma_i \cup \Gamma_i^* \cup (\Gamma_i \cup \Gamma_i^*) \cup (\Gamma_i^* \circ \Gamma_i)$, *then* G *has rank* 4.

3. Proof of Theorem 1

We put

$$x_i = \#\{\Gamma_j \mid \Delta_i = \Gamma_j \circ \Gamma_j^*\}\ ,$$
$$y_i = \#\{(\Gamma_k, \Gamma_l) \mid \Gamma_k \circ \Gamma_l^* \supset \Delta_i\}$$

and assume that $x_1 \geqq \cdots \geqq x_r > x_{r+1} = \cdots = x_t = 0$. Counting in two ways triplilaterals $(\Gamma_k, \Gamma_l, \Delta_i)$ such that $\Gamma_k \circ \Gamma_l^* \supset \Delta_i$, we have by Lemma 9 and 11

$$s^2 \leqq \sum_{i=1}^{t} y_i\ .$$

The equality means that, for any Γ_i and Γ_j, we cannot have $\Gamma_i \circ \Gamma_j^* = \Delta_k \cup \Delta_l$, $\Delta_k \neq \Delta_l$.

When $x_i > 0$, by Lemma 26 $y_i \leqq x_i + s$. When $x_i = 0$, by non-existence of Fig. 1 of Lemma 25 $y_i \leqq 2s$. Therefore

$$s^2 \leqq \sum_{i=1}^{t} y_i \leqq \sum_{i=1}^{r} (x_i + s) + 2(t-r)s\ ,$$

so

$$s^2 \leqq (r+1)s + (t-r)s\ ,$$
$$s \leqq 2t - r + 1\ . \tag{1}$$

Now, let $\Delta_1 = \Gamma_{i_0} \circ \Gamma_{i_0}^*$ and we put

$$A = \{\{\Gamma_i, \Gamma_j\} : \text{unordered pair} \mid \Gamma_i \circ \Gamma_j^* \supset \Delta_1,\ \Gamma_i \neq \Gamma_j\}\ ,$$
$$B = \{\Gamma_i \mid \{\Gamma_i, \Gamma_j\} \in A\}\ .$$

For $\{\Gamma_i, \Gamma_j\}$, $\{\Gamma_k, \Gamma_l\} (\neq) \in A$, $\{\Gamma_i, \Gamma_j\} \cap \{\Gamma_k, \Gamma_l\} = \emptyset$ by Lemma 26. Therefore $|B| = 2|A|$. Furthermore, for $\{\Gamma_i, \Gamma_j\}$, $\{\Gamma_k, \Gamma_l\} (\neq) \in A$, and for Γ_m, Γ_n $(\neq) \notin B$, $\Gamma_{i_0}^* \circ \Gamma_i \cap \Gamma_{i_0}^* \circ \Gamma_j$, $\Gamma_{i_0}^* \circ \Gamma_k \cap \Gamma_{i_0}^* \circ \Gamma_l$, $\Gamma_{i_0}^* \circ \Gamma_m$, $\Gamma_{i_0}^* \circ \Gamma_n$ are disjoint to each other by non-existence of Fig. 1 of Lemma 25. Thus we have

$$|A| + (s - |B|) = s - |A| \leqq t\ , \tag{2}$$

and by Lemma 26

$$|A| - 1 \leqq t - r\ . \tag{3}$$

Assume $s = 2t - r + 1$. Since the equality of (1) hold $y_1 = x_1 + s$, and hence

$|A|=\frac{s}{2}$ and $\frac{s}{2}-1\leqq t-r$ by (3), and hence, $2t-r+1=s\leqq 2t-2r+2$. So $r=1$. Therefore, if $r>1$, we conclude that $s\leqq 2t-r$.

We shall show that when $r=1$, $s\leqq 2t-2$. Assume $r=1$ and $2t\geqq s\geqq 2t-1$, and put $\Delta=\Gamma_i\circ\Gamma_i^*$, $1\leqq i\leqq s$. If $\pi_i\neq\pi_j$ for some Γ_i and Γ_j, then by Lemma 23, $\Delta\not\subset\Gamma_k\circ\Gamma_l^*$ for any Γ_k, $\Gamma_l(\neq)$, and hence, $\Gamma_i^*\circ\Gamma_k\cap\Gamma_i^*\circ\Gamma_l=\emptyset$. So $s\leqq t$. This is contrary to the assumption that $t\geqq 2$. Thus, it holds that $\pi_1=\pi_2=\cdots=\pi_s$.

Now, Suppose $\Gamma_i\circ\Gamma_j=\Delta\cup\Gamma_k^*$ for some Γ_i, Γ_j and Γ_k, and put $D=C(\Delta)$, $\Gamma_j\circ\Gamma_k=\Delta'\cup\Gamma_i^*$, $D'=C(\Delta')$, $t=|\Gamma_i(\alpha)\cap\Gamma_j^*(\beta)|$ for $(\alpha,\beta)\in\Gamma_k^*$, $x=|\Gamma_i(\alpha)\cap\Gamma_j^*(\delta)|$ for $(\alpha,\delta)\in\Delta$, $v=v_1=v_2=\cdots$, $k=k_1=k_2=\cdots$. Then we have

$$
\begin{aligned}
&(C_iC_j)C_k=(tC_k^*+xD)C_k=tvI+tkD+xDC_k\,,\\
&C_i(C_jC_k)=C_i(t'C_i^*+x'D')=t'vI+t'kD+x'C_iD'\,.\\
&(t'=|\Gamma_j(\alpha)\cap\Gamma_k^*(\beta)| \text{ for } (\alpha,\beta)\in\Gamma_i^*,\ x'=|\Gamma_j(\alpha)\cap\Gamma_k^*(\delta)|\\
&\text{for } (\alpha,\delta)\in\Delta'.)
\end{aligned}
$$

We have $t=t'$ by counting in two ways triplilaterals (β,α,γ) whose edges are successively Γ_i, Γ_j and Γ_k, and have $|\Delta(\alpha)|=|\Delta'(\alpha)|$ and $x=x'$ by Lemma 10. So,

$$C_iD'=DC_k=(v-1)C_k+\cdots\,.$$

If $C_i\neq C_k$ $|\Delta'(\alpha)|=\dfrac{v(v-1)}{k+1}$ by Lemma 10. This is impossible. Thus $C_i=C_k$. Similarly, $C_j=C_k$.

When $S=2t$, then the equality of (1) holds. Therefore, for any Γ_i, there exists Γ_j such that $\Gamma_i\circ\Gamma_j=\Delta\cup\Gamma_k^*$ for some Γ_k^*. So, as is shown above, $\Gamma_i=\Gamma_j=\Gamma_k$. Therefore we have any Γ_i.

$$\Gamma_i\neq\Gamma_i^*,\ \Gamma_i\circ\Gamma_i=\Delta\cup\Gamma_i^* \text{ and } \Gamma_i\circ\Gamma_m^*\cap\Gamma_i\circ\Gamma_n^*=\emptyset \qquad \text{for } \Gamma_m\neq\Gamma_n,\ \Gamma_n^*\,.$$

When $s=2t-1$, then $|A|\leqq t-1$, and from (2) $s-|A|\leqq t$. So $|A|=t-1$. Therefore, there is a unique Γ_u such that for any $\Gamma_i\,(\neq\Gamma_u)$, $\Gamma_i\circ\Gamma_u^*\not\subset\Delta$. We shall show that for any Γ_i, Γ_j $(\neq)$, $\Gamma_i\circ\Gamma_j^*$ contains some Γ_k. Assume $\Gamma_i\circ\Gamma_j^*=\Delta_k\cup\Delta_l$ for some Γ_i, Γ_j $(\neq)$. Count in two ways the paired (Γ_m,Δ_n) such that $\Gamma_i\circ\Gamma_m^*$ contains Δ_n, then by Lemma 25, we have

$$2t=s+1\leqq\#\{(\Gamma_m,\Delta_n)\,|\,\Gamma_i\circ\Gamma_m^*\supset\Delta_n\}\leqq 2t\,.$$

So, equality holds. Thus for any Δ_k, there exist Γ_p and Γ_q $(\neq)$ such that $\Gamma_i\circ\Gamma_p^*$ and $\Gamma_i\circ\Gamma_q^*$ contains Δ_k. Therefore we may choose Γ_a such that $\Gamma_i\circ\Gamma_u^*\cap\Gamma_i\circ\Gamma_a^*\neq\emptyset$ and $\Gamma_a\neq\Gamma_u$. Then $\Gamma_a\circ\Gamma_u^*\supset\Gamma_i\circ\Gamma_i^*=\Delta$. This is impossible. Thus, again as is shown above, we can conclude that for any Γ_i $(\neq\Gamma_u)$,

$$\Gamma_i \neq \Gamma_i^*,\ \Gamma_i\circ\Gamma_i = \Delta\cup\Gamma_i^* \text{ and}$$
$$\Gamma_i\circ\Gamma_m^* \cap \Gamma_i\circ\Gamma_n^* = \emptyset \text{ for } \Gamma_m\neq\Gamma_n.\ \Gamma_n^*$$

Thus if $s\geqq 2t-1$, there exists Γ_i such that

$$\Gamma_i\neq\Gamma_i^* \text{ and } \Gamma_i\circ\Gamma_i = \Gamma_i\circ\Gamma_i^*\cup\Gamma_i^* .$$

By Lemma 27, this show that G has rank 4. This is impposible for $s\geqq 2t-1$ and $t\geqq 2$.

4. Proof of Theorem 2

When $r=t$, we have $s\leqq t$ by Theorem 1. On the other hand, from $s\geqq r=t$, we conclude that $s=t=r$.

We put $\Gamma_i\circ\Gamma_i^*=\Delta_i$, $A_i=\{\{\Gamma_k;\ \Gamma_l\}$ unordered pair $|\,\Gamma_k\circ\Gamma_l^*\supset\Delta_i,\ \Gamma_k\neq\Gamma_l\}$. Then $|A_i|-1\leqq t-r=0$, so $|A_i|\leqq 1$.

Count in two ways triplilaterals $(\Gamma_i,\ \Gamma_j,\ \Delta_k)$ such that $\Gamma_i\circ\Gamma_j^*\supset\Delta_k$, we have

$$s^2\leqq 3s\,,$$

so

$$s\leqq 3\,. \tag{1}$$

Case $t=2$. If $|\Gamma_1(\alpha)|\neq|\Gamma_2(\alpha)|$, by T. Ito [6], G is isomorphic to the small Janko simple group and G_α is isomorphic to PSL(2.11). We shall prove that the case of $|\Gamma_1(\alpha)|=|\Gamma_2(\alpha)|$ does not occure. We put $|\Gamma_1(\alpha)|=|\Gamma_2(\alpha)|=v$. It is easy to prove that $\pi_1=\pi_2$. We shall show that Γ_1 and Γ_2 are self paired. If not, then $\Gamma_1^*=\Gamma_2$. Since $\Gamma_1\circ\Gamma_1^*\neq\Gamma_2\circ\Gamma_2^*=\Gamma_1^*\circ\Gamma_1$, we have that $\Gamma_1\circ\Gamma_1\not\supset\Gamma_1\circ\Gamma_1^*,\ \Gamma_1^*\circ\Gamma_1(=\Gamma_2\circ\Gamma_2^*)$ by Lemma 7. By Lemma 11, there exists a G-orbit $\sum$ in $\Gamma_1\circ\Gamma_1$ such that G_α is not 2-transitive on $\sum(\alpha)$, and $\sum\neq\Delta_1,\ \Delta_2$. This is impossible for $t=2$. Thus, we have $\Gamma_1\circ\Gamma_2=\Delta_1\cup\Delta_2$. So, $v^2=|\Gamma_1\circ\Gamma_2(\alpha)|=|\Gamma_1\circ\Gamma_1(\alpha)|+|\Gamma_2\circ\Gamma_2(\alpha)|=\dfrac{v(v-1)}{k_1}+\dfrac{v(v-1)}{k_2}$. This is impossible.

Case $t=3$. For this case, the equality of (1) holds. So we have $|A_i|=1$ for $1\leqq i\leqq 3$. We shall show that if $\Gamma_i\circ\Gamma_j^*=\Delta_1$ then $\Gamma_i=\Gamma_1$ or $\Gamma_j=\Gamma_1$. If $\Gamma_i,\ \Gamma_j\neq\Gamma_1$, then since $\Gamma_1^*\circ\Gamma_i\cap\Gamma_1^*\circ\Gamma_j\neq\emptyset$, there exists a G-orbit $\sum$ in $\Gamma_1^*\circ\Gamma_i\cap\Gamma_1^*\circ\Gamma_j$ such that G is not 2-transitive on $\sum(\alpha)$ by Lemma 12, and for any Γ_i, $\Gamma_i\circ\Gamma_i^*\neq\sum$ by Lemma 25. From $r=t$, this is impossible. Thus we may assume that there exist the following figures.

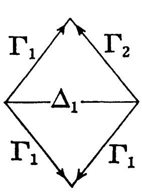


Fig. a

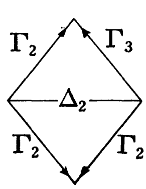


Fig. b

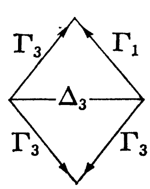


Fig. c

If $\pi_1 \neq \pi_2, \pi_3$, then $v_1v_2 = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1^* \circ \Gamma_1(\alpha)| = \frac{v_1(v_1-1)}{k_1}$ from Fig. a, so $v_1 > v_3$. Similarly, $v_3 > v_1$ from Fig. c. Therefore $v_3 > v_2$. On the other hand, $v_2v_3 = \frac{v_2(v_2-1)}{k_2}$ from Fig. b, so $v_2 > v_3$. This is impossible. Thus we have $\pi_1 = \pi_2 = \pi_3$. By Lemma 7, Γ_1, Γ_2 and Γ_3 are self-paired.

Thus $\Gamma_1 \circ \Gamma_2 = \Gamma_3 \cup \Delta_1$, $\Gamma_2 \circ \Gamma_3 = \Gamma_1 \cup \Delta_2$, $\Gamma_3 \circ \Gamma_1 = \Gamma_2 \cup \Delta_3$. Put $|\Gamma_1(\alpha)| = v$, then by Lemma 8, iii) we have

$$|\Delta_1(\alpha)| = |\Delta_2(\alpha)| = |\Delta_3(\alpha)| = v(v-1).$$

We put

$$D_i = C(\Delta_i) \text{ and } C_i = C(\Gamma_i),\ 1 \leqq i \leqq 3;$$
$$D_1C_3 = x_1D_1 + x_2D_2 + x_3D_3.$$

Then

$$x_1 + x_2 + x_3 = v$$
$$\begin{aligned} D_2C_3 &= x_2D_1 + \text{terms not involving } D_1, \\ D_3C_3 &= x_3D_1 + \text{terms not involving } D_1. \end{aligned} \tag{2}$$

Now

$$\begin{aligned} (C_1C_2)C_3 &= (D_1 + C_3)C_3 = vI + D_3 + D_1C_3, \\ C_1(C_2C_3) &= C_1(D_2 + C_1) = vI + D_1 + D_2C_1. \end{aligned}$$

So

$$D_2C_1 = D_1C_3 + D_3 - D_1 = (x_1-1)D_1 + x_2D_2 + (x_3+1)D_3.$$

Similarly

$$D_3C_2 = D_2C_1 + D_1 - D_2 = x_1D_1 + (x_2-1)D_2 + (x_3+1)D_3.$$

Next

$$\begin{aligned} (C_1C_1)C_3 &= (vI + D_1)C_3 = vC_3 + D_1C_3, \\ C_1(C_1C_3) &= C_1(D_3 + C_2) = C_3 + D_1 + D_3C_1. \end{aligned}$$

So

$$\begin{aligned} D_3C_1 &= D_1C_3 + (v-1)C_3 - D_1 \\ &= (x_1-1)D_1 + x_2D_2 + x_3D_3 + (v-1)C_3. \end{aligned}$$

Similarly

$$\begin{aligned} D_1C_2 &= D_2C_1 + (v-1)C_1 - D_2 \\ &= (x_1-1)D_1 + (x_2-1)D_2 + (x_3+1)D_3 + (v-1)C_1, \\ D_2C_3 &= D_3C_2 + (v-1)C_2 - D_3 \\ &= x_1D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_2. \end{aligned} \tag{3}$$

Furthermore

$$(C_1C_1)C_2 = (vI+D_1)C_2 = vC_2+D_1C_2\,,$$
$$C_1(C_1C_2) = C_1(C_3+D_1) = C_2+D_3+D_1C_1$$

So

$$\begin{aligned} D_1C_1 &= D_1C_2+(v-1)C_2-D_3 \\ &= (x_1-1)D_1+(x_2-1)D_2+x_3D_3+(v-1)C_1+(v-1)C_2\,. \end{aligned}$$

Similarly

$$\begin{aligned} D_2C_2 &= D_2C_3+(v-1)C_3-D_1 \\ &= (x_1-1)D_1+(x_2-1)D_2+x_3D_3+(v-1)C_2+(v-1)C_3\,, \\ D_3C_3 &= D_3C_1+(v-1)C_1-D_2 \\ &= (x_1-1)D_1+(x_2-1)D_2+x_3D_3+(v-1)C_3+(v-1)C_1\,. \end{aligned} \tag{4}$$

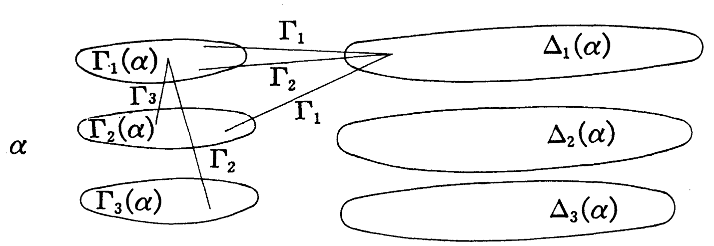
Thus (2), (3) and (4) yield

$$x_1 = x_2,\ x_1-1 = x_3\,.$$

We put $x_3=x$, then

$$v = x_1+x_2+x_3 = (x+1)+(x+1)+x = 3x+2\,. \tag{5}$$

It is easy to show that the graph $(\Omega,\ \Gamma_1\cup\Gamma_2\cup\Gamma_3)$ is a strongly regular graph with parameters $3v$, 2, 3.



From the conditions of the existence of the strongly regular graph, (see [1] p. 97) it holds that

$$(3-2)^2+4(3v-3)=12v-11=d^2\,, \tag{6}$$

(d is a positive integer)

$$m = \frac{3v}{2\cdot 3\cdot d}\,\{(3v-1+3-2)(d+3-2)-2\cdot 3\} = \frac{3}{2}v^2+\frac{3v(v-2)}{2d}\,. \tag{7}$$

(m is a positive integer)

From (7), $\dfrac{3v(v-2)}{d}$ is integer, and hence

$$12v-11 = d^2 \text{ is a divisor of } v_2(v-2)^2\,.$$

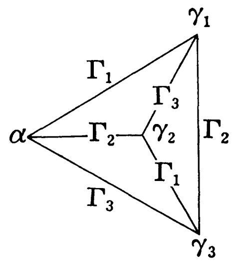
So

$$12v-11 \text{ is a divisor of } 11^2\cdot 13^2\,.$$

From $v=3x+2$, we conclude

$$v=11\,.$$

Lastly, we shall prove that the primitive group satisfying these conditions does not exist. It is easy to prove that G_α acts faithfully on $\Gamma_1(\alpha)$. We shall show that for γ_1, γ_1' $(\neq)\in\Gamma_1(\alpha)$, $G_{\alpha,\gamma_1,\gamma_1'}$ has the fixed points in $\Gamma_1(\alpha)\setminus\{\gamma_1, \gamma_1'\}$.



For $(\alpha, \gamma_1)\in\Gamma_1$, put $\{\gamma_2\}=\Gamma_2(\alpha)\cap\Gamma_3(\gamma_1)$ and $\{\gamma_3\}=\Gamma_3(\alpha)\cap\Gamma_2(\gamma_1)$. Then, G_{α,γ_1} fix γ_2 and γ_3. So we must have that $(\gamma_2, \gamma_3)\in\Gamma_1$. Now for γ_1, γ_1' $(\neq)\in$ $\Gamma_1(\alpha)$, put $\{\delta_1\}=\Gamma_1(\gamma_1)\cap\Gamma_2(\gamma_1')$, $\{\delta_2\}=\Gamma_2(\gamma_1)\cap\Gamma_1(\gamma_1')$. Then $G_{\alpha,\gamma_1,\gamma_1'}$ fiix δ_1 and δ_2. Since $(\gamma_1, \gamma_1')\not\ni\Gamma_3$, we have $(\delta_1, \delta_2)\not\ni\Gamma_3$. Therefore $\Gamma_1(\gamma_1)\cap\Gamma_3(\delta_2)=\{\delta\}\neq\{\delta_1\}$.

So, $G_{\alpha,\gamma\ \gamma_1'}$ fix δ_1 and δ. Since $\Gamma_1(\gamma_1)\ni\alpha$, δ_1, δ $(\neq)$, in the same way, we obtain that $G_{\alpha,\gamma_1\gamma,_1'}$, has the fix points in $\Gamma_1(\alpha)\setminus\{\gamma_1\cup\gamma_1'\}$. The order of G_α is at most one million. If G_α is non-solvable, then the minimal normal subgroup of G_α is non-solvable simple. From [5], it is isomorphic to the Mathieu group M_{11} or the transitive extension of the alternating group A_5 act on ten points. These groups have not the representation such that it is doubly-transitive on eleven points and it's stabilizer of two points has the additional fixed point. Thus, we can conclude that G_α is solvable and the order of G_α is 110. So $|G|=|\Omega|\cdot 11\cdot 10=364\cdot 11\cdot 10=2^3\cdot 5\cdot 7\cdot 11\cdot 13$. G is non-solvable group and $(|G|, 3)=1$. But there does not exist such group by M. Hall [5].

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