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Osaka University
GENERALIZATION OF A THEOREM OF 
PETER J. CAMERON

MINORU NUMATA

(Received October 29, 1976)

Peter J. Cameron [3] has shown that a primitive permutation group $G$ has rank at most 4 if the stabilizer $G_\alpha$ of a point $\alpha$ is doubly transitive on all its nontrivial suborbits except one.

The purpose of this paper is to prove the following two theorems, one of which extends the Cameron's result.

**Theorem 1.** Let $G$ be a primitive permutation group on a finite set $\Omega$, and all nontrivial $G$-orbits in Cartesian product $\Omega \times \Omega$ be $\Gamma_1, \ldots, \Gamma_r, \Delta_1, \ldots, \Delta_s$, where $G_\alpha$ is doubly transitive on $\Gamma_i(\alpha) = \{ \beta | (\alpha, \beta) \in \Gamma_i \}$, $1 \leq i \leq s$ and not doubly transitive on $\Delta_i(\alpha), 1 \leq i \leq t$. Suppose that $G$ has no subdegree smaller than 4 and that $t > 1$. Then, we have

$$s \leq 2t - r,$$

where $r = \# \{ \Delta_i | \Delta_i = \Gamma_j \circ \Gamma_j, 1 \leq j \leq s \}$. Moreover if $r = 1$, then we have

$$s \leq 2t - 2.$$

(For the notation $\Gamma_j \circ \Gamma_j$, see the section 1)

**Theorem 2.** Under the hypothesis of Theorem 1, if $r = t$, then $s = t = 2$, and $G$ is isomorphic to the small Janko simple group and $G_\alpha$ is isomorphic to $PSL(2, 11)$.

For the case of $t \geq 3$, I don't know the example satisfying the equality $s = 2t - r$, and when $r = 1$, the example satisfying the equality $s = 2t - 2$. I know only three examples with $t = 2$ and $s = 2$.

The small Janko simple group $J_1$ of order 175560 has a primitive rank 5 representation of degree 266 in which the stabilizer of a point is isomorphic to $PSL(2, 11)$ and acts doubly transitively on suborbits of lengths 11 and 12; the other suborbit lengths are 110 and 132 (See Livingstone [7]). The Mathieu group $M_{12}$ has a primitive rank 5 representation of degree 144 in which the stabilizer of a point is isomorphic to $PSL(2, 11)$ and acts doubly transitively on two suborbits of length 11; the other suborbit lengths are 55 and 66 (See Cameron [4]).
The group $[Z_3 \times Z_3 \times Z_3]S_4$ has a primitive rank 5 representation of degree 27 in which the stabilizer of a point is $S_4$ and acts doubly transitively on two suborbits of length 4; the other suborbit lengths are 6 and 12. I conjecture that it may even be true that $s$ is at most $t$.

1. Preliminaries

Let $G$ be a transitive permutation group on a finite set $\Omega$, and $\Delta$ be a subset of the Cartesian product $\Omega \times \Omega$ which is fixed by $G$ (acting in the natural way on $\Omega \times \Omega$), then $\Delta(\alpha) = \{\beta \in \Omega | (\alpha, \beta) \in \Delta\}$ is a subset of $\Omega$ fixed by $G$. This procedure sets up a one-to-one correspondence between $G$-orbits in $\Omega \times \Omega$ and $G$-orbits in $\Omega$. The number of such orbits is called the rank of $G$. $\Delta^* = \{(\beta, \alpha) | (\alpha, \beta) \in \Delta\}$ is the subset of $\Omega \times \Omega$ fixed by $G$ paired with $\Delta$. $\Delta$ is self-paired if $\Delta = \Delta^*$. Note that $|\Delta(\alpha)| = |\Delta^*(\alpha)| = |\Delta|/|\Omega|$. If $\Gamma$ and $\Delta$ are fixed sets of $G$ in $\Omega \times \Omega$, let $\Gamma \circ \Delta$ denote the set $\{(\alpha, \beta)\}$ there exists $\gamma \in \Omega$ with $(\alpha, \gamma) \in \Gamma$, $(\gamma, \beta) \in \Delta; \alpha \neq \beta$; this is also a fixed set of $G$. The diagonal $\{(\alpha, \alpha) | \alpha \in \Omega\}$ is a trivial $G$-orbit. If $\Gamma$ is a nontrivial $G$-orbits in $\Omega \times \Omega$, the $\Gamma$-graph is the regular directed graph whose point set is $\Omega$ and whose edges are precisely the ordered pairs in $\Gamma$. A connected component of any such graph is a block of imprimitivity for $G$. $G$ is primitive if and only if each such graph is connected.

For a $G$-orbit $\Gamma$ in $\Omega \times \Omega$, the basis matrix $C = C(\Gamma)$ is the matrix whose rows and columns are indexed by $\Omega$, with $(\alpha, \beta)$ entry 1 if $(\alpha, \beta) \in \Gamma$, 0 otherwise. All of the basis matrices form a basis of the centralizer algebra of the permutation matrices in $G$.

Let $G$ be a group which acts as a permutation group on $\Omega$, and $\pi$ the permutation character of $G$ i.e. the integer-valued function on $G$ defined by $\pi(g) =$ number of fixed points of $g$. The formula

$$(\pi, 1)_G = \frac{1}{|G|} \sum_{g \in G} \pi(g) = \text{number of orbits of } G,$$

is well-known. If $G$ acts as a permutation group on $\Omega_1$ and $\Omega_2$, with permutation characters $\pi_1$ and $\pi_2$, the number $m$ of $G$-orbits in $\Omega_1 \times \Omega_2$ is

$$m = (\pi_1 \pi_2, 1)_G = (\pi_1, \pi_2)_G.$$

In particular, if $G$ is a transitive permutation group on $\Omega$ with permutation character $\pi$, the rank $r$ of $G$ is given by

$$r = (\pi, \pi)_G = \text{sum of squares of multiplicities of irreducible constituents of } \pi.$$

If $G$ acts doubly transitively on $\Omega_1$ and $\Omega_2$,

$$(\pi_1, \pi_2)_G = 2 \text{ or } 1 \text{ according as } \pi_1 = \pi_2 \text{ or } \pi_1 \neq \pi_2.$$
Lastly, we note that if $G$ is a primitive permutation group on $\Omega$, then for $\alpha, \beta (\neq) \in \Omega$, either $G_\alpha \neq G_\beta$ or $G$ is a regular group of prime degree ([8], Prop. 8.6); primitive groups with a subdegree 2 are Frobenius groups of prime degree ([8], Theorem 18.7); primitive groups with a subdegree 3 are classified by W.J. Wong [9].

2. Lemmata

Throughout this section, we suppose that $G$ is a primitive but not doubly transitive group on a finite set $\Omega$, and $\Gamma \subseteq \Gamma_2, \ldots$ are $G$-orbits in $\Omega \times \Omega$ such that $G_\alpha$ is doubly transitive on $\Gamma_i(\alpha), i=1, 2, \ldots$; $\pi_i$ and $\pi_i^*$ are the permutation characters of $G_\alpha$ on $\Gamma_i(\alpha)$ and $\Gamma_i^*(\alpha)$, respectively, and let $C_i=C(\Gamma_i), C_i^*=C(\Gamma_i^*)$.

**Lemma 1.** (P. J. Cameron [2]. Proposition 1.2)

$G_\alpha$ is doubly transitive on $\Gamma_i^*(\alpha)$.

**Lemma 2.** (P. J. Cameron [3]. Lemma 1)

$\Gamma_i^* \circ \Gamma_i$ is a $G$-orbit in $\Omega \times \Omega$, and if $|\Gamma_i(\alpha)| > 2$, then $G_\alpha$ is not doubly transitive on $\Gamma_i(\alpha)$.

**Lemma 3.** (P. J. Cameron [2]. Theorem 2.2)

For $(\alpha, \beta) \in \Gamma_1 \circ \Gamma_i^*$, we put $v_i = |\Gamma_i(\alpha)|$ and $k_i = |\Gamma_i(\alpha) \cap \Gamma_i(\beta)|$. Then $k_i < v_i$ and $|\Gamma_i \circ \Gamma_i^*(\alpha)| = \frac{v_i(v_i-1)}{k_i}$. If $v_i > 2$, then $k_i \leq \frac{v_i-1}{2}$; when particularly $k_i = \frac{v_i-1}{2}$, then $v_i = 3$ or 5.

In the following, we set

$|\Gamma_i(\alpha)| = v_i, \quad |\Gamma_i \circ \Gamma_i^*(\alpha)| = \frac{v_i(v_i-1)}{k_i}$.

**Lemma 4.** (P. J. Cameron [2]. Lemma 2.1)

$|\Gamma_i^* \circ \Gamma_i(\alpha)| = |\Gamma_i \circ \Gamma_i^*(\alpha)|$.

**Lemma 5.** $\Gamma_i^* \circ \Gamma_i \neq \Gamma_i^* \circ \Gamma_2$ if and only if $|\Gamma_i \circ \Gamma_i^*(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$.

Proof. If $|\Gamma_1 \circ \Gamma_i^*(\alpha)| < |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$, we have $|\Gamma_1(\alpha) \cap \Gamma_2(\beta)| > 1$ for some $(\alpha, \beta) \in \Gamma_1 \circ \Gamma_i^*$. For $\gamma_1, \gamma_2 \in \Gamma_1 \cap \Gamma_2$, $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_2$. So $\Gamma_i^* \circ \Gamma_1 = \Gamma_i^* \circ \Gamma_2$. Conversely, if $\Gamma_i^* \circ \Gamma_1 = \Gamma_i^* \circ \Gamma_2$ for $(\gamma_1, \gamma_2) \in \Gamma_i^* \circ \Gamma_1 = \Gamma_i^* \circ \Gamma_2$ we can choose $\alpha$ and $\beta$ such that $\alpha \in \Gamma_i^*(\gamma_1) \cap \Gamma_i^*(\gamma_2)$, $\beta \in \Gamma_i^*(\gamma_1) \cap \Gamma_i^*(\gamma_2)$. Since $\Gamma_1(\alpha) \cap \Gamma_2(\beta) \subseteq \gamma_1, \gamma_2, |\Gamma_1(\alpha) \cap \Gamma_2(\beta)| > 1$. Therefore $|\Gamma_1 \circ \Gamma_i^*(\alpha)| < |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$.

**Lemma 6.** $\Gamma_i^* \circ \Gamma_2$ is the union of at most two $G$-orbits in $\Omega \times \Omega$, and
\( \pi_1 = \pi_2 \text{ if and only if } \Gamma^*_1 \circ \Gamma^*_2 \text{ is the union of two } G \text{-orbits in } \Omega \times \Omega. \)

Proof. Since \((\pi_1, \pi_2, 2) = (\pi_1, \pi_2, 2), \) and \( \pi_1 \pi_2 \) is the permutation character of \( G_\alpha \) on \( \Gamma_1(\alpha) \times \Gamma_2(\alpha), \) \( G \) has at most two orbits in \( \{(\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1, (\alpha, \delta) \in \Gamma_2, \} \), and hence, \( \Gamma^*_1 \circ \Gamma^*_2 \) is the union of at most two \( G \)-orbits. If \( \pi_1 \neq \pi_2, \) then \( G \) is transitive on \( \{(\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1, (\alpha, \delta) \in \Gamma_2, \} \), and hence, \( \Gamma^*_1 \circ \Gamma^*_2 \) is a \( G \)-orbit in \( \Omega \times \Omega. \) Now, we shall assume that \( \pi_1 = \pi_2 \) and \( \Gamma^*_1 \circ \Gamma^*_2 \) is a \( G \)-orbit in \( \Omega \times \Omega. \) We put \( v = v_1 = v_2, \) and \( m = |\Gamma^*_1(\alpha) \cap \Gamma^*_2(\delta)| \) for \( (\alpha, \delta) \in \Gamma^*_1 \circ \Gamma^*_2. \) If \( m = 1, \) then since \( \Gamma^*_1 \circ \Gamma^*_2 \) is a \( G \)-orbit, \( G \) is transitive on \( \{(\alpha, \gamma, \delta) | (\gamma, \alpha) \in \Gamma_1, (\gamma, \delta) \in \Gamma_2, \} \). Therefore \( (\pi_1, \pi_2, 2) = 1, \) and hence, \( \pi_1 = \pi_2, \) this is contrary to the assumption. If \( m > 1, \) then there exist quadrilaterals \( (\alpha, \gamma, \delta, \gamma_2) \) whose edges are successively \( \Gamma^*_1, \Gamma^*_2, \Gamma^*_1 \) and \( \Gamma_1; \) and whose vertices are all distinct. Counting all of them in two ways, we have

\[
|\Omega| \frac{v(v-1)}{m} (m-1) = |\Omega| \frac{v(v-1)}{k_1} k_2 ,
\]

so

\[
v(v-1) = (v-1)k_2 .
\]

Hence, \( v = k_2. \) This is impossible by Lemma 3.

**Lemma 7.** If \( \Gamma^*_1 \circ \Gamma^*_1 = \Gamma^*_1 \circ \Gamma^*_1, \) then \( \Gamma^*_1 \circ \Gamma^*_1 = \Gamma^*_1 \circ \Gamma^*_1, \Gamma^*_1 \circ \Gamma^*_1, \Gamma^*_1 \circ \Gamma^*_1. \)

Proof. Now assume \( \Gamma^*_1 \circ \Gamma^*_1 = \Gamma^*_1 \circ \Gamma^*_1, \) then we have the following figure,

![Diagram](image)

and hence, \( \Gamma^*_1 \circ \Gamma^*_1 = \Gamma^*_1 \circ \Gamma^*_1 \cup \Gamma^*_1 \circ \Gamma^*_1 \). Since \( \Gamma^*_1 \circ \Gamma^*_1 \) is the union of at most two \( G \)-orbits in \( \Omega \times \Omega, \) we have \( \Gamma^*_1 \circ \Gamma^*_1 = \Gamma^*_1 \circ \Gamma^*_1 \cup \Gamma^*_1 \circ \Gamma^*_1. \) By the assumption of this lemma, \( |(\Gamma^*_1 \circ \Gamma^*_1)\Gamma_1(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_1(\alpha)| = v_1^n. \) So

\[
v_1^n = |\Gamma^*_1 \circ \Gamma_1(\alpha)| = |\Gamma^*_1 \circ \Gamma^*_1(\alpha)| + |\Gamma^*_1 \circ \Gamma^*_1(\alpha)| = \frac{2v(v-1)}{k_1} ,
\]

\[
v_1k_1 = 2v(v-1) .
\]

Therefore, \( v_1 = 2. \) All of the suborbits of the primitive group with a subdegree 2 are self-paired. This is contrary to the assumption of this Lemma.

**Lemma 8.** Let \( \Gamma^*_1 \circ \Gamma^*_2 \) be the union of two \( G \)-orbits \( \Sigma_1 \) and \( \Sigma_2. \) We set \( v = v_1 = v_2, S_1 = C(\Sigma_1), s_i = |\Sigma_i(\alpha)|, i = 1, 2, \) and \( C^* C_2 = a_1 S_1 + a_2 S_2. \) Then we have
i) If \( s_1 = v \), \( G_a \) is double transitive on \( \Sigma_1(\alpha) \)

ii) \( v^2 = a_1s_1 + a_2s_2 \)

iii) If \( \Gamma_1 \circ \Gamma_2 \neq \Gamma_2 \circ \Gamma_1^* \) if and only if \( a_1 = a_2 = 1 \)

iv) If \( s_1 = v(v - 1) \), then \( \Gamma_1 \circ \Gamma_2^* \neq \Gamma_2 \circ \Gamma_2^* \) and \( \Gamma_1^* \circ \Gamma_2 \) contains some \( \Gamma_i \)

Proof. i) Assume \( s_1 \leq v \). Then \( (\pi \Gamma, \pi(\Sigma_1)) = 1 \) or \( 2 \) according as \( \pi(\Sigma_1) = \pi(\Sigma_1) \) where \( \pi(\Sigma_1) \) is the permutation character of \( G_a \) on \( \Sigma_1(\alpha) \).

If \( \pi(\Sigma_1) \neq \pi(\Sigma_1) \), for \( \delta \in \Sigma_1(\alpha) \), \( G_{a, \delta} \) is transitive on \( \Gamma_1^*(\alpha) \). Thus \( \Gamma_1^*(\alpha) = \Gamma_2^*(\delta) \).

Therefore \( G_a = G_1 \circ \Gamma_2(\delta) = G_2 \circ \Gamma_1^*(\delta) \). This is impossible. So we have \( \pi(\Sigma_1) = \pi(\Sigma_1) \), and hence, \( s_1 = v \) and \( G_a \) is doubly transitive on \( \Sigma_1(\alpha) \).

ii) For the matrix \( F \) such that any entry is 1, we have

\[
F(C_1 C_2) = v^2 F \quad \text{and} \quad F(a_1S_1 + a_2S_2) = (a_1s_1 + a_2s_2)F,
\]

so \( v^2 = a_1s_1 + a_2s_2 \).

iii) The existence of the following figure is equivalent to \( \Gamma_1^* \circ \Gamma_2^* \).

\[
\begin{array}{c}
\Gamma_1 \\
\downarrow \\
\Gamma_2 \\
\downarrow \\
\Gamma_1 \\
\downarrow \\
\Gamma_2
\end{array}
\]

It holds also that the figure exists if and only if \( a_1 \geq 2 \) for \( i = 1 \) or 2.

iv) By ii), \( v^2 = a_1s_1 + a_2s_2 \). Since \( s_2 \geq v \), \( a_1 = a_2 = 1 \) and \( s_2 = v \). Therefore we conclude that \( \Gamma_1^* \circ \Gamma_2^* \) contains some \( \Gamma_i \), and hence, \( s_1 = v \) and \( G_a \) is doubly transitive on \( \Sigma_1(\alpha) \).

**Lemma 9.** If \( \pi_1 \neq \pi_2 \), \( G_a \) is not doubly transitive on \( \Gamma_1^* \circ \Gamma_2(\alpha) \).

Proof. Assume that \( G_a \) is doubly transitive on \( \Gamma_1^* \circ \Gamma_2(\alpha) \). If \( |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \) or \( |\Gamma_2(\alpha)| \), then \( G_a \) has different permutation characters on \( \Gamma_1^* \) and \( \Gamma_2^* \) and \( \Gamma_1^* \circ \Gamma_2(\alpha) \).

Hence, for \( (\alpha, \gamma) \in \Gamma_1^* \circ \Gamma_2(\alpha) \), \( G_{a, \gamma} \) is transitive on \( \Gamma_1^* \circ \Gamma_2(\alpha) \), so \( \Gamma_3(\gamma) = \Gamma_2^* \circ \Gamma_1(\alpha) \).

Therefore \( G_a = G_1 \circ \Gamma_2(\gamma) = G_2 \circ \Gamma_1^*(\gamma) \). This is impossible. Thus, we obtain \( |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \). On the other hand, for \( (\delta, \gamma) \in \Gamma_2^* \), \( \Gamma_2(\gamma) \subset \Gamma_2^* \circ \Gamma_1(\delta) \). So, \( \Gamma_1^* \circ \Gamma_1(\delta) = \Gamma_1(\gamma) \). This is also impossible.

**Lemma 10.** Assume \( \Gamma_1 \circ \Gamma_2^* \neq \Gamma_2 \circ \Gamma_2^* \) and \( \Gamma_1^* \circ \Gamma_2 \) be the union of two G-orbits \( \Sigma_1 \) and \( \Sigma_2 \); put \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v \), \( |\Gamma_1 \circ \Gamma_2^*(\alpha)| = v(v - 1) \), \( |\Sigma_1(\alpha)| = s_1 \), \( s_2 \), \( i = 1, 2 \); and \( |\Gamma_2(\alpha) \cap \Sigma_2(\alpha)| = t \) for \( \gamma \in \Gamma_1^*(\alpha) \). Then, we have the following quadratic equation for \( t \)

\[
\frac{v(v - 1)^2}{s_1} + \frac{vt^2}{s_2} - v^2 - k(v - 1) = 0.
\]

Particularly, i) when \( s_1 = \frac{v(v - 1)}{k} \), the quadratic equation has at most one root for
0 < t < v; ii) when \( t = 1 \), then \( s_2 = v \), \( s_1 = \frac{v(v-1)}{(k+1)} \) and \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \).

**Proof.** For \( \gamma_1, \gamma_2 \in \Gamma^f(\alpha) \), counting arguments show that

\[
|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_1(\alpha)| = \frac{(v-t)(v-t-s_1)}{(v-1)s_1},
\]

\[
|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_2(\alpha)| = \frac{t(vt-s_2)}{(v-1)s_2},
\]

so

\[
k = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{(v-t)(v(t-t)-s_1)}{(v-1)s_1} + \frac{t(vt-s_2)}{(v-1)s_2},
\]

\[
(v-1)k = \frac{v(v+t)^2}{s_1} - (v-t) + \frac{vt^2 - t}{s_2} = \frac{v(v-t)^2}{s_1} + \frac{vt^2 - v}{s_2} + k(v-1).
\]

We shall prove the latter assertions. We put

\[
f(t) = \frac{v(v-t)^2}{s_1} + \frac{vt^2 - v}{s_2} - k(v-1).
\]

When \( s_1 \geq \frac{v(v-1)}{k} \), then \( f(0) < 0 \). Since the coefficient of \( t^2 \) in \( f(t) \) is positive, \( f(t) \) has at most one root for \( 0 < t < v \). When \( t = 1 \), then \( s_2 \leq v \). By Lemma 8, i) \( s_2 = v \). So \( s_2 = v \), and hence, \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \), and \( s_1 = \frac{v(v-1)}{(k+1)} \).

**Lemma 11.** Let \( \Gamma^f_1 \circ \Gamma_2 \) be the union of two \( G \)-orbits \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \), and \( G_\alpha \) doubly transitive on \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \), then \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| \leq 3 \).

**Proof.** This lemma due to P. J. Cameron. ([3], Lemma 4.) We put \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v \), and assume \( |\Sigma_1(\alpha)| \neq v \). Then, \( G_\alpha \) has the different permutation characters on \( \Gamma^f_1(\alpha) \) and \( \Sigma_1(\alpha) \), so, for \( (\alpha, \delta) \in \Sigma_1 \), \( G_{\alpha,\delta} \) is transitive on \( \Gamma^f \). Hence, \( \Gamma^f(\alpha) = \Gamma^f(\delta) \). Therefore, \( G_{\alpha} = G_{\Gamma^f_1(\alpha)} = G_{\Gamma^f(\delta)} \). This is impossible. Thus we conclude that \( |\Sigma_1(\alpha)| = v \). In the same way, we have \( |\Sigma_2(\alpha)| = v \).

Now, if \( \Gamma_1 \circ \Gamma^f_1 \neq \Gamma_2 \circ \Gamma^f_2 \), then by Lemma 5 \( |\Gamma^f_1 \circ \Gamma_2(\alpha)| = |\Gamma^f_1(\alpha)| \cdot |\Gamma_2(\alpha)| = v^2 \). Therefore, \( v^2 = |\Gamma^f_1 \circ \Gamma_2(\alpha)| = |\Sigma_1(\alpha)| + |\Sigma_2(\alpha)| = 2v \), so \( v = 2 \). Thus, when \( v > 2 \), we obtain that \( \Gamma_1 \circ \Gamma^f_1 = \Gamma_2 \circ \Gamma^f_2 \). For \( \gamma \in \Gamma^f, \) we put \( t = |\Gamma_2(\gamma) \cap \Sigma_1(\alpha)| \). Then for \( (\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma^f, \) by Lemma 10 we have the following equation
\[ k_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{1}{v-1} ((v-t)^2 + t^2 - v) \]

\[ = v - \frac{2t(v-t)}{v-1}. \]

If \( t = \frac{v}{2} \), \( |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = v + \frac{v^2}{2(v-1)} \) is not integer, so \( t \leq \frac{v-1}{2} \) or \( t \geq \frac{v+1}{2} \).

Hence \( k_2 = v - \frac{2t(v-t)}{v-1} \leq v - \frac{1}{2}(v+1) = \frac{1}{2}(v-1) \). But \( k_2 \leq \frac{1}{2}(v-1) \) by Lemma 3, so equality holds, and thus \( v = 3 \) or 5 by Lemma 3, and \( t = \frac{1}{2}(v+1) \) or \( \frac{1}{2}(v-1) \). Counting arguments show that \( |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \sum_i(\alpha)| = \frac{t(t-1)}{v-1} \) for \( \gamma_1, \gamma_2(\neq) \in \Gamma^*(\alpha) \). Therefore \( v - 1 \) divides \( t(t-1) \); this excludes \( v = 5 \), and so \( v = 3 \).

**Lemma 12.** For \( \Gamma_1, \Gamma_2, \Gamma_3 \), if \( \sum \) is a \( G \)-orbit contained in \( \Gamma_1^* \circ \Gamma_2 \cap \Gamma_2^* \circ \Gamma_3 \), and \( |\Gamma_1(\alpha)| > 3 \); then \( G_\alpha \) is not doubly transitive on \( \sum(\alpha) \).

**Proof.** \( \sum \circ \Gamma_1^* \supseteq \Gamma_2^* \cup \Gamma_3^* \). If \( G_\alpha \) is doubly transitive on \( \sum(\alpha) \), \( \sum \circ \Gamma_1^* \) is the union of at most two \( G \)-orbits by Lemma 6, so \( \sum \circ \Gamma_1^* = \Gamma_2^* \cup \Gamma_3^* \). This is contrary to Lemma 11.

**Lemma 13.** If \( \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \circ \Gamma_2^* \) and \( \pi_1 \neq \pi_2 \) then, \( |\pi_1, \pi_2| \geq 2 \), and \( |\Gamma_1 \circ \Gamma_2^*(\alpha)| > |\Gamma_2^* \circ \Gamma_2(\alpha)| \).

**Proof.** For \( (\alpha, \beta) \in \Gamma_1 \circ \Gamma_2 \), we put

\[ m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\beta)|. \]

Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) with \( \gamma_1 \neq \gamma_2 \) whose edges are successively \( \Gamma_1^*, \Gamma_2, \Gamma_2^*, \) and \( \Gamma_1 \); then we have

\[ |\Omega| \frac{v_3(v_2-1)}{k_2} k_1 k_2 = |\Omega| \frac{v_1 v_2}{m} m(m-1), \]

so

\[ (v_2-1)k_1 = v_1(m-1). \quad (1) \]

If \( v_1 = v_2 \), then \( k_1 = v_1 \). This is impossible. If \( v_1 = v_2 + 1 \), then \( k_1 \geq \frac{v_1}{2} \), and hence, by Lemma 3 \( v_1 = 2, v_2 = 1 \). This is also impossible. Thus we can conclude that \( |v_1 - v_2| \geq 2 \).

Assume \( |\Gamma_1 \circ \Gamma_2^*(\alpha)| = \frac{v_1(v_1-1)}{k_1} = |\Gamma_2^* \circ \Gamma_2(\alpha)| = \frac{v_1 v_2}{m} \). Then

\[ k_1 v_2 \geq m(v_1 - 1). \quad (2) \]
From $\Gamma_1 \circ \Gamma^\# = \Gamma_2 \circ \Gamma_3^\#$, we have also

$$k_2 v_1 \geq m(v_2 - 1) .$$  \hspace{1cm} (3)

Therefore, (1) and (2) yield

$$v_1 \leq k_1 + m .$$  \hspace{1cm} (4)

By Lemma 3 and (3), we have

$$2v_2 = \frac{v_2(v_2 - 1)}{k_2} \leq \frac{v_1 v_2}{m} ,$$

so

$$2 \leq m \leq \frac{v_1}{2} .$$  \hspace{1cm} (5)

Thus (4) and (5) yield

$$k_1 \geq \frac{1}{2} v_1 .$$

This is contrary to Lemma 3.

**Lemma 14.** (P.J. Cameron [3]) If $\Gamma_1 \circ \Gamma^\# = \Gamma_1 \circ \Gamma_2^\#$, then $\Gamma_1 \circ \Gamma^\# = \Gamma_2 \circ \Gamma^\#$.

**Proof.** We shall prove this lemma in a different way from P.J. Cameron's. Assume $\Gamma_1 \circ \Gamma^\# = \Gamma_1 \circ \Gamma_2^\# = \Gamma_2 \circ \Gamma_3^\#$. We put

$$|\Gamma_1 \circ \Gamma^\#(\alpha)| = \frac{v_1(v_1 - 1)}{k_1} = |\Gamma_2 \circ \Gamma_3^\#(\alpha)| = \frac{v_2(v_2 - 1)}{k_2} = |\Gamma_1 \circ \Gamma^\#(\alpha)| = \frac{v_1 v_2}{m} ,$$

where $m = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)|$ for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma^\#$. Then it is trivial that $m > 1$ from the above formula, and hence, $\Gamma_1 \circ \Gamma^\# = \Gamma_2 \circ \Gamma_3$. Thus, by Lemma 13, $|\Gamma_1 \circ \Gamma^\#(\alpha)| < |\Gamma_1 \circ \Gamma_3(\alpha)| = |\Gamma_1 \circ \Gamma^\#(\alpha)|$. This is contrary to assumption.

Now we shall investigate from Lemma 15 to Lemma 22 the necessary condition that the intersection of $\Gamma_1 \circ \Gamma_2$ and $\Gamma_1 \circ \Gamma_3$ for $\Gamma_1, \Gamma_2, \Gamma_3 (\neq)$ is not empty.

**Lemma 15.** If $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^\# = \pi_3^\#$, or $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^\# = \pi_3^\#$, then $\Gamma_1 \circ \Gamma_2 \cap \Gamma_1 \circ \Gamma_3 = \emptyset$.

**Proof.** Assume $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^\# = \pi_3^\#$. Then we have $v_1 = v_2 = v_3$. We put $v = v_1 = v_2 = v_3$. By Lemma 13, $\Gamma_1 \circ \Gamma_2^\# = \Gamma_3 \circ \Gamma_3^\#$, and hence, $|\Gamma_1 \circ \Gamma_3(\alpha)| = |\Gamma_1 \circ \Gamma_3(\alpha)| = |\Gamma_3(\alpha)| = v_3^2$ by Lemma 5. If $\Gamma_1 \circ \Gamma_2 \cap \Gamma_1 \circ \Gamma_3 = \emptyset$, then since $\Gamma_2 \circ \Gamma_3$ is a $G$-orbit and $\Gamma_1 \circ \Gamma_3$ is a union of two $G$-orbits, we have $\Gamma_1 \circ \Gamma_2 \cap \Gamma_1 \circ \Gamma_3$. Therefore $|\Gamma_1 \circ \Gamma_2(\alpha)| > |\Gamma_1 \circ \Gamma_3(\alpha)| = v_3^2$. This is impossible. Similarly, we can prove the lemma for the case of $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^\# = \pi_3^\#$. 

Lemma 16. If $\pi^\ast_1 \neq \pi^\ast_2$, $\pi^\ast_1 = \pi^\ast_3$ and $\pi_2 \neq \pi_3$, then $\Gamma_1 \circ \Gamma^*_2 \cap \Gamma_1 \circ \Gamma^*_3 = \emptyset$.

Proof. By the assumption, $\Gamma_1 \circ \Gamma^*_2$, $\Gamma_1 \circ \Gamma^*_3$ and $\Gamma^*_2 \circ \Gamma_3$ are $G$-orbits. Assume $\Gamma_1 \circ \Gamma^*_2 = \Gamma_1 \circ \Gamma^*_3$. For $(\alpha, \delta) \in \Gamma_1 \circ \Gamma^*_2$, we put

$$|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = m_2 \quad \text{and} \quad |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| = m_3.$$ 

For $\gamma_1, \gamma_2(\neq) \in \Gamma_1(\alpha)$, we put

$$|\Gamma^*_2(\gamma_1) \cap \Gamma^*_3(\gamma_2)| = x.$$ 

Then, since $\Gamma^*_2 \circ \Gamma_1 = \Gamma^*_3 \circ \Gamma_3$, we have

$$\frac{v_1(v_1-1)}{k_1} = |\Gamma^*_2 \circ \Gamma_1(\alpha)| = |\Gamma^*_3 \circ \Gamma_3(\alpha)| = \frac{v_2 v_3}{x},$$

so

$$v_1(v_1-1)x = v_2 v_3 k_1.$$  \hspace{1cm} \text{(1)}$$

Count in two ways quadrilaterals $(\alpha, \gamma, \delta, \gamma)$ whose edges are successively $\Gamma_1, \Gamma^*_2, \Gamma_3$ and $\Gamma^*_3$, then we have

$$|\Omega| \frac{v_1(v_1-1)}{k_1} k_1 x = |\Omega| \frac{v_1 v_3 m_2 m_3}{m_3},$$

so

$$(v_1-1)x = v_2 m_2.$$ \hspace{1cm} \text{(2)}$$

(1) and (2) yield

$$v_2 m_2 = k_1 v_2.$$ \hspace{1cm} \text{(3)}$$

If $m_2 > 1$, there exist quadrilaterals $(\alpha, \beta_1, \delta, \beta_2)$ whose edges are successively $\Gamma_1, \Gamma^*_2, \Gamma_2$ and $\Gamma^*_3$, whose vertices are all distinct; count all of them in two ways, we have

$$|\Omega| \frac{v_1(v_1-1)}{k_1} k_2 = |\Omega| \frac{v_1 v_2 m_3(m_2-1)}{m_2},$$

so

$$(v_1-1)k_2 = v_2(m_2-1).$$

On the other hand, from $\Gamma^*_1 \circ \Gamma_1 = \Gamma^*_2 \circ \Gamma_2$,

$$v_2(v_2-1)k_1 = v_1(v_1-1)k_2 = v_1 v_2(m_2-1),$$

so

$$v_1(m_2-1) = (v_2-1)k_1.$$$$\text{(3) and (4) yield}$$

$$v_1 = k_1.$$
This is contrary to Lemma 3. Thus, we have \( m_2 = m_3 = 1 \) and \( v_1 = k_1 v_2 \). For \((\alpha, \gamma) \in \Gamma_1\), \( G_{\alpha, \gamma} \) is transitive on \( \Gamma_1(\alpha) \setminus \{\gamma\} \) and since \( \pi_1^* = \pi_2^* \), it is also transitive on \( \Gamma_2^*(\gamma) \). Count in two ways \((\gamma', \delta)\) such that \( \gamma' \in \Gamma_1(\alpha) \setminus \{\gamma\}, \delta \in \Gamma_2^*(\gamma) \) and \((\gamma', \delta) \in \Gamma_3^*\), then we have

\[
(v_1 - 1)x = v_2 = \frac{v_1}{k_1}.
\]

This is impossible.

**Lemma 17.** If \( \pi_1 \neq \pi_2, \pi_1 \neq \pi_3 \) and \( \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \Gamma_3^* \), then \( \Gamma_3^* \circ \Gamma_2 \cap \Gamma_3^* \circ \Gamma_3 = \emptyset \).

Proof. Assume \( \Gamma_3^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3 \). By Lemma 16, \( \pi_2^* = \pi_3^* \). We put \( v = v_1, w = v_2 = v_3, m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| > 1 \) for \((\alpha, \delta) \in \Gamma_1^* \circ \Gamma_2\), and \( x = |\Gamma_2^*(\gamma_1) \cap \Gamma_2^*(\gamma_2)| \) for \( \gamma_1, \gamma_2 (\neq) \in \Gamma_2^*(\alpha) \).

Count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma_1^*, \Gamma_2, \Gamma_2^* \) and \( \Gamma_1 \); then we have

\[
|\Omega| \frac{v(v-1)}{k_1} k_2 = |\Omega| \frac{vw}{m} m(m-1),
\]

so

\[
(v-1)x = wm . \tag{1}
\]

Next, count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma_1^*, \Gamma_2, \Gamma_2^* \) and \( \Gamma_1 \) and whose vertices are all distinct; then

\[
|\Omega| \frac{v(v-1)}{k_1} k_2 = |\Omega| \frac{vw}{m} m(m-1), \quad (v-1)k_2 = w(m-1). \tag{2}
\]

(1) and (2) yield

\[
(v-1)(x-k_2) = w, \text{ that is, } x > k_2 \geq 1 . \tag{3}
\]

Since \( x \geq 2 \), there exist quadrilaterals \((\gamma, \delta_1, \gamma', \delta_2)\) whose edges successively \( \Gamma_3, \Gamma_2^*, \Gamma_2 \) and \( \Gamma_2^* \), whose vertices are all distinct, and \((\gamma, \gamma') \in \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \circ \Gamma_2^* \); count all of them in two ways, then

\[
|\Omega| w(w-1)\lambda = |\Omega| \frac{w(w-1)}{k_2} x(x-1),
\]

\[
(\lambda = |\Gamma_2^*(\delta_1) \cap \Gamma_2^*(\delta_2) \cap \Gamma_1 \circ \Gamma_2^*(\gamma)| \text{ for } \delta_1, \delta_2 (\neq) \in \Gamma_3(\gamma))
\]

so

\[
\lambda = \frac{x(x-1)}{k_2} .
\]
By the definition of $\lambda$, $\lambda \leq k_2$. On the other hand, since $x > k_2$, $\lambda = \frac{x(x-1)}{k_2} > k_2$.

This is a contradiction.

**Lemma 18.** If $\pi^*_1 = \pi^*_2$, $\pi^*_1 = \pi^*_3$ and $\Gamma_1 \circ \Gamma_3^* = \Gamma_1 \circ \Gamma_3^*$, then $C_1C^*_1 = C_1C^*_3$.

**Proof.** By Lemma 6 $\sum \Gamma_1 \circ \Gamma_3^* = \Gamma_1 \circ \Gamma_3^*$ is a $G$-orbit. Let $S = C(\sum)$, $C_1C^*_1 = m_2S$, $C_1C^*_3 = m_3S$ and $|\sum(\alpha)| = s$.

For the matrix $F$ such that the value of any entry is 1, we have

$$v_1v_2F = F(C_1C^*_3) = F(m_2S) = m_2F,$$

so

$$v_1v_2 = m_2.$$

Similarly

$$v_1v_3 = m_3.$$

On the other hand, by Lemma 16, $\pi_2 = \pi_3$, and hence, $v_2 = v_3$. So, $m_2 = m_3$.

Thus we can conclude that $C_1C^*_3 = C_1C^*_1$.

**Lemma 19.** If $C_1C^*_2 = C_1C^*_3$ and $|\Gamma_1(\alpha)| = v_1 > 3$, then we have

i) $\pi_2 = \pi_3$, $\pi^*_1 = \pi^*_2$, $\pi^*_3$.

ii) $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$, $\Gamma_3^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3$.

iii) $v_1 = v_2 + 1 = v_3 + 1$, $|\Gamma_1^*(\gamma_1) \cap \Gamma_2^*(\gamma_2)| = 1$ for $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$.

iv) $|\Gamma_1^* \circ \Gamma_1(\alpha)| = v_1(v_1 - 1)/2$.

**Proof.** By the assumption $\Gamma_1^* \circ \Gamma_1 = \Gamma_1 \circ \Gamma_3^*$. For the matrix $F$ such that the value of any entry is 1, we have

$$F(C_1C^*_3) = (FC_1)C^*_3 = (v_1F)C^*_3 = v_1(FC^*_3) = v_1v_2F.$$  

Similarly

$$F(C_1C^*_3) = v_1v_3F.$$  

So

$$v_2 = v_3.$$

We shall show that $v_1 = v_2 = v_3$. Assume $v = v_1 = v_2 = v_3$ and put $D = C(\Gamma_1^* \circ \Gamma_1)$. If $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$, then $|\Gamma_1^* \circ \Gamma_1(\alpha)| = |\Gamma_1 \circ \Gamma_3^*(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_3(\alpha)|$, therefore $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_3$ by Lemma 5. We put $k = k_1 = k_2 = k_3$.

$$C_1^*(C_1C^*_3) = (C_1^*C_1)C^*_3 = (vE + kD)C^*_3 = vC^*_3 + k(v-1)C^*_3 +$$ 

terms not involving $C^*_3$.

Similarly
\[ C_\ast(C_1C_\ast) = vC_\ast + k(v-1)C_\ast + \text{terms not involving } C_\ast. \]

So
\[ (vE + kD)C_\ast = \{v + k(v-1)\} C_\ast + \text{terms not involving } C_\ast. \]

Since the coefficients of the basis matrices in \( DC_\ast \) are at most \( v \), the above formula is impossible.

Next, if \( \Gamma_1 \circ \Gamma_2 \neq \Gamma_3 \circ \Gamma_4 \), then \( \Gamma_1 \circ \Gamma_2 \neq \Gamma_3 \circ \Gamma_4 \), and \( DC_\ast \) does not involve \( C_\ast \). Now
\[ C_\ast(C_1C_\ast) = (C_\ast C_1)C_\ast = (vE + kD)C_\ast, \]
\[ C_\ast(C_1C_\ast) = (C_\ast C_1)C_\ast = (vE + kD)C_\ast = vC_\ast + \text{terms not involving } C_\ast, \]
and hence, \( k_1 DC_\ast = vC_\ast + \text{terms not involving } C_\ast. \)

For \( (\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_2 \) and \( (\gamma_1, \delta) \in \Gamma_3 \), we put
\[ x = |\Gamma_3(\gamma_1) \cap \Gamma_3(\gamma_2)| \quad \text{and} \quad t = |\Gamma_3 \circ \Gamma_1(\gamma_1) \cap \Gamma_3(\delta)|. \]

Then from the above formula we have
\[ t = \frac{v}{k_1}. \quad (1) \]

Counting in two ways triplilaterals \( (\gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \circ \Gamma_1 \), we have
\[ \frac{v(v-1)x}{k_1} = vt. \]

(1) and (2) yield
\[ (v-1)x = v, \]
which is a contradiction. Thus we can conclude that \( v_1 + v_2 = v_3 \), and hence, \( \pi_\ast \neq \pi_\ast \neq \pi_\ast \). Therefore, we obtain \( \pi_0 = \pi_3 \) by Lemma 16, \( \Gamma_1 \circ \Gamma_2 \neq \Gamma_3 \circ \Gamma_3 \) by Lemma 17, and hence we have i) and ii) of Lemma.

For \( (\alpha, \gamma) \in \Gamma_1 \), count in two ways the ordered pairs \( (\gamma', \delta) \) such that \( \gamma' \in \Gamma_1(\alpha) \setminus \{\gamma\} \), \( \delta \in \Gamma_3(\gamma) \) and \( (\gamma', \delta) \in \Gamma_3 \); then since \( \Gamma_3 \circ \Gamma_1 \neq \Gamma_3 \circ \Gamma_3 \) we have
\[ (v_1 - 1)x = v_2. \quad (3) \]

Now, we shall show that \( x = 1 \). Assume \( x > 1 \), then there exist quadrilaterals \( (\gamma, \delta, \gamma', \delta') \) whose edges are successively \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_3 \) whose edges are all distinct, and \( (\gamma, \gamma') \in \Gamma_1 \circ \Gamma_1 \); count all of them in two ways, then we have
\[ |\Omega|v_2(v_2 - 1) = |\Omega|v_1(v_1 - 1)x(x - 1), \]
\[ (\lambda = |\Gamma_1(\gamma) \cap \Gamma_3(\delta_1) \cap \Gamma_3(\delta_2)| \quad \text{for} \quad (\gamma, \delta_1), (\gamma, \delta_2) \in \Gamma_3 \circ \Gamma_3), \]
so
\[(v_2 - 1) \lambda k_1 = v_1 (x - 1) = (v_1 - 1) x + x - v_1 = v_2 + x - v_1.\]

Therefore, \(x \geq v_1 - 1.\) If \(x = v_1\) then \((v_2 - 1) \lambda k_1 = v_2\), which is a contradiction.

If \(x > v_1\), then \(v_2 = (v_1 - 1) x > \frac{v_1 (v_1 - 1)}{k_1}.\) So \(\pi_2^* (\pi (\Gamma^* \circ \Gamma_1 (\gamma))) \nu_1 = 1\), where \(\pi (\Gamma^* \circ \Gamma_1 (\gamma))\) is the permutation character of \(G_\gamma\) on \(\Gamma^* \circ \Gamma_1 (\gamma)\). Hence, for \((\gamma, \gamma^{'}) \in \Gamma^* \circ \Gamma_1, G_\gamma, \gamma^{'}\) is transitive on \(\Gamma^* (\gamma)\). So \(\Gamma^* (\gamma) = \Gamma^* (\gamma^{'}).\) This is impossible.

Thus we have \(x = v_1 - 1, k_1 = \lambda = 1, v_2 = (v_1 - 1)^2\) and \(|\Gamma^* \circ \Gamma_1 (\gamma) \cap \Gamma_3 (\delta)| = v_1\) for \((\gamma, \delta) \in \Gamma^*\).

Now, count in two ways quadrilaterals \((\alpha, \gamma_1, \gamma_2, \gamma_3)\) such that \((\alpha, \gamma_1) \in \Gamma_2, (\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3, (\gamma_1, \gamma_2), (\gamma_1, \gamma_3), (\gamma_2, \gamma_3) \in \Gamma^* \circ \Gamma_1, \gamma_2 \neq \gamma_3;\) then we have
\[|\Omega| v_1 (v_3 - 1) \lambda' = |\Omega| v_2 v_1 (v_1 - 1),\]
\[\lambda' = \frac{|\Gamma^* \circ \Gamma_1 (\gamma_2) \cap \Gamma_3 (\gamma_3) \cap \Gamma_3 (\alpha)|}{v_2 (v_1 - 1)^2} = \frac{v_1 - 1}{v_1 - 2} + \frac{w}{v_1 - 1}.\]

Therefore, \(v_1 = 3.\) This is contrary to the hypothesis of Lemma. Thus we can conclude that \(x = 1,\) and hence, by (3) we have \(v_1 = v_2 - 1 = v_3 + 1.\) This proves Lemma iii).

Lastly, we shall show that \(k_1 = 2.\) If \(k_1 = 1,\) then \(|\Gamma^* \circ \Gamma_1 (\alpha)| = v_1 (v_1 - 1) \leq \frac{|\Gamma^* \circ \Gamma_3 (\alpha)|}{\nu_2 (v_3 - 1)^2} = (v_1 - 1)^2\) This is impossible. Now, we have
\[u = \frac{|\Gamma^* \circ \Gamma_1 (\gamma) \cap \Gamma_3 (\delta)|}{k_1} = \frac{v_1}{k_1} \quad \text{for} \quad (\gamma, \delta) \in \Gamma^* \quad \text{and} \quad 2 \leq k_1 < \frac{v_1}{2}.\]

Count again in two ways quadrilaterals \((\alpha, \gamma_1, \gamma_2, \gamma_3)\) such that \((\alpha, \gamma_1) \in \Gamma_2, (\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3, (\gamma_1, \gamma_2), (\gamma_1, \gamma_3), (\gamma_2, \gamma_3) \in \Gamma^* \circ \Gamma_1, \gamma_2 \neq \gamma_3;\) then
\[|\Omega| (v_1 - 1)(v_1 - 2) \lambda'' = |\Omega| (v_1 - 1)\left(\frac{v_1}{k_1} - 1\right)\frac{v_1}{k_1},\]
\[\lambda'' = \frac{|\Gamma^* \circ \Gamma_1 (\gamma_2) \cap \Gamma_3 (\gamma_3) \cap \Gamma_3 (\alpha)|}{v_2 (v_1 - 2)^2} = \frac{u (u - 1) k_1}{k_1 u - 2} .\]

If \(u\) is odd, then \(k_1 u - 2\) divides \(u - 1.\) This is impossible. We put \(u = 2u_0,\) then
\[ \chi'' = \frac{2u_0(2u_0 - 1)}{2k_1u_0 - 2} = \frac{u_0(2u_0 - 1)}{k_1u_0 - 1}. \]

Therefore, we conclude that \( k_1 = 2 \).

**Lemma 20.** If \( \pi_1 \neq \pi_2 \neq \pi_3 \) and \( \Gamma_1^* \cap \Gamma_2^* \cap \Gamma_3^* \neq \emptyset \), then \( v_1 = v_2 = v_3 + 1 \), \( \Gamma_1^* \neq \Gamma_2^* \neq \Gamma_3^* \) and \( \Gamma_1^* \cap \Gamma_2^* \cap \Gamma_3^* = \emptyset \) for some \( \Gamma_i \).

**Proof.** By assumption, \( \sum \Gamma^* \Gamma_2 \) is a \( G \)-orbit contained in \( \Gamma^* \Gamma_2 \). We put \( v = v_1 = v_2, w = v_3, |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = x \) for \( (\gamma_1, \gamma_2) \in \Gamma_1 \Gamma^*, |\Gamma^*(\alpha) \cap \Gamma^*(\delta)| = y \) and \( |\sum(\alpha) \cap \sum(\delta)| = m \) for \( (\alpha, \delta) \in \sum \), \( |\Gamma_2(\gamma) \cap \sum(\gamma)| = t \) for \( (\alpha, \gamma) \in \Gamma^* \).

By Lemma 15, \( \pi^* \neq \pi^*, \) and hence, \( \Gamma_2 \Gamma^* \) is a \( G \)-orbit. We have

\[ \frac{v(v-1)}{k_1} = |\Gamma_1 \Gamma^*(\gamma_1)| = |\Gamma_2 \Gamma^*(\gamma_1)| = \frac{v+1}{x}. \]

so

\[ (v-1)x = wk_1. \]  

(1)

We have also \( |\sum(\alpha)| = \frac{vw}{m} = \frac{vt}{y} \), and so

\[ wy = tm. \]  

(2)

Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1^*, \Gamma_2, \Gamma^* \) and \( \Gamma_1 \), then we have

\[ |\Omega| \frac{v(v-1)}{k_1} k_1 x = |\Omega| \frac{vw}{m} my, \]

so

\[ (v-1)x = wy. \]  

(3)

(1) and (3) yield

\[ y = k_1. \]  

(4)

From (2) and (3),

\[ (v-1)x = tm. \]  

(5)

We shall show that \( m = 1 \). If \( m > 1 \), then there exist quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1^*, \Gamma_2, \Gamma^* \) and \( \Gamma_1 \), whose vertices are all distinct; count all of them in two ways, then we have

\[ |\Omega| \frac{w(w-1)}{k_3} k_1 k_1 = |\Omega| \frac{vw}{m} m(m-1), \]

so

\[ (w-1)k_1 = v(m-1). \]

On the other hand, from (3) and (4)
(w - 1)k_1 = wk_1 - k_1 = (v - 1)x - k_1,

therefore

\[ v(m - 1) = (v - 1)x - k_1, \]

so

\[ 0 \leq v(x - m + 1) = x + k_1 < 2v. \]  \(6\)

(6) yields

\[ x = m, \quad v = m + k_1. \]  \(7\)

From (5) and (7),

\[ t = v - 1. \]  \(8\)

Thus \[ \left| \sum(\alpha) \right| = \frac{vt}{y}\frac{v(v - 1)}{k_1}. \]

If \( \Gamma_1 \circ \Gamma^*_2 = \Gamma_2 \circ \Gamma^*_2 \), then by Lemma 10, \( \left| \sum(\alpha) \right| = \frac{v(v - 1)}{k_1 + 1} \). This is a contradiction. So we have \( \Gamma_1 \circ \Gamma^*_2 \neq \Gamma_2 \circ \Gamma^*_2 \), and hence,

\[ 1 = y = k_1. \]  \(9\)

Therefore we have \( m = v - 1 \) from (7) and (9), and \( w = (v - 1)^2 \) from (2) and (8). So

\[ \left| \Gamma_1 \circ \Gamma^*_2(\alpha) \right| = \left| \Gamma_2 \circ \Gamma^*_2(\alpha) \right| = \frac{w(w - 1)}{k_3} \leq 2w = 2(v - 1)^2 > v(v - 1). \]

This is impossible. Thus, we can conclude that \( m = 1 \), and then by (5) \( t = v - 1 \), \( x = 1 \) and \( \left| \sum(\alpha) \right| = \frac{v(v - 1)}{k_1} \). By Lemma 10, \( \Gamma_1 \circ \Gamma^*_2 \neq \Gamma_2 \circ \Gamma^*_2 \), and hence, \( 1 = y = k_1 \). Therefore, by (2) \( w = v - 1 \), \( \left| \sum(\alpha) \right| = v(v - 1) \). By Lemma 8 iv), \( \Gamma^*_1 \circ \Gamma_2 = \sum \cup \Gamma_i \) for some \( \Gamma_i \).

**Lemma 21.** If \( \Gamma^*_1 \circ \Gamma_2 \cap \Gamma^* \circ \Gamma_3 \neq \emptyset \), and \( v_1, v_2, v_3 > 3 \), then the following hold:

i) if \( \pi_1 = \pi_2 = \pi_3 \), then \( \pi_1^* = \pi_2^* = \pi_3^* \)

ii) if \( \pi_1 = \pi_2 \neq \pi_3 \), then \( \pi_1^* \neq \pi_2^* = \pi_3^* \) and \( v_1 = v_2 = v_3 + 1 \).

iii) if \( \pi_1 \neq \pi_2, \pi_3 \), then \( \pi_1^* = \pi_2^* = \pi_3^* \), \( C_1^* C_2 = C_1^* C_3 \) and \( v_1 = v_2 + 1 = v_3 + 1 \).

**Proof.** We have this assertion by arranging from Lemma 15 to Lemma 20.

**Lemma 22.** Suppose that \( \Gamma^*_1 \circ \Gamma_2 \) and \( \Gamma^*_1 \circ \Gamma_3 \) contain a $G$-orbit \( \sum \) in \( \Omega \times \Omega \), and \( \pi_1 = \pi_2 = \pi_3 \), \( \left| \Gamma_i(\alpha) \right| > 3 \). For \( \gamma_1, \gamma_2 (\neq) \Gamma^*_i(\alpha) \) and \( \delta \in \sum(\alpha) \), the following hold:

i) if \( \Gamma_1 \circ \Gamma^*_2 = \Gamma_2 \circ \Gamma^*_2 = \Gamma_3 \circ \Gamma^*_2 \), then \( \left| \Gamma^*_1(\alpha) \cap \Gamma^*_2(\delta) \right| > 1 \), \( \left| \Gamma^*_1(\alpha) \cap \Gamma^*_3(\delta) \right| > 1 \) and \( \left| \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha) \right| > 1 \).
ii) if $\Gamma_1 \circ \Gamma^*_f = \Gamma_2 \circ \Gamma^*_g = \Gamma_3 \circ \Gamma^*_h$, then $|\Gamma^*_f(\alpha) \cap \Gamma^*_g(\delta)| > |\Gamma^*_f(\alpha) \cap \Gamma^*_h(\delta)| = |\Gamma_3(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$, and $\Gamma^*_f \circ \Gamma_2$ contains some $\Gamma_i$.

iii) if $\Gamma_1 \circ \Gamma^*_f = \Gamma_2 \circ \Gamma^*_i$, $\Gamma_3 \circ \Gamma^*_h$, then $|\Gamma^*_f(\alpha) \cap \Gamma^*_i(\delta)| = |\Gamma^*_f(\alpha) \cap \Gamma^*_h(\delta)| = |\Gamma_3(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = v(v-1)$, and $\Gamma^*_f \circ \Gamma_2$ contains some $\Gamma_i$ and $\Gamma^*_i \circ \Gamma_3$ contains another $\Gamma_j$.

Proof. Put $|\Sigma(\alpha) \cap \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = \lambda$ for $\gamma_1, \gamma_2 \in \Gamma^*_f(\alpha)$. $|\Gamma^*_f(\alpha) \cap \Gamma^*_i(\delta)| = |\Gamma^*_f(\alpha) \cap \Gamma^*_h(\delta)| = |\Gamma_3(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma^*_f, \Gamma_2, \Gamma^*_i$ and $\Gamma_3$, and $(\alpha, \delta) \in \Sigma$. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma^*_f, \Gamma_2, \Gamma^*_i$ and $\Gamma_3$, and $(\alpha, \delta) \in \Sigma$, then we have
\[ |\Omega| \frac{v(v-1)}{k_1} \lambda = |\Sigma(\alpha)| x_2 x_3, \]
so
\[ v(v-1) \lambda = |\Sigma(\alpha)|. \]

Assume $\Gamma_1 \circ \Gamma^*_f = \Gamma_2 \circ \Gamma^*_i$, $\Gamma_3 \circ \Gamma^*_h$. Then we have $|\Gamma^*_f(\alpha) \cap \Gamma^*_i(\delta)| = |\Gamma^*_f(\alpha) \cap \Gamma^*_h(\delta)| = 1$. By (1)
\[ v(v-1) \lambda = |\Sigma(\alpha)|. \]

Since $|\Sigma(\alpha)| \leq v(v-1)$, we have $\lambda = 1$ and $|\Sigma(\alpha)| = v(v-1)$. By Lemma 8 iv), $\Gamma^*_f \circ \Gamma_2 = \Gamma^*_i \circ \Gamma_3 = \Sigma \cup \Gamma_i$ and $\Gamma^*_f \circ \Gamma_3 = \Sigma \cup \Gamma_j$ for some $\Gamma_i, \Gamma_j$. By Lemma 8, iii), we have $C^*_i C_3 = S + C_1$, $C^*_i C_2 = S + C_2$. (S = C(\Sigma)) If $C_i = C_j$, then $C^*_i C_2 = C^*_i C_3$, and hence, by Lemma 19, $\pi_1 = \pi_2, \pi_3$. This is contrary to the hypothesis of this lemma. Thus $C_i \neq C_j$, that is, $\Gamma_i \neq \Gamma_j$. So $\Sigma(\alpha) \cap \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) = \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)$. Therefore $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = |\Sigma(\alpha) \cap \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = \lambda = 1$. Thus we have iii) of Lemma.

Next assume $\Gamma_1 \circ \Gamma^*_f = \Gamma_2 \circ \Gamma^*_i + \Gamma_3 \circ \Gamma^*_h$. Then we have $|\Gamma^*_f(\alpha) \cap \Gamma^*_i(\delta)| = 1$. By (1)
\[ v(v-1) \lambda = |\Sigma(\alpha)| x_2. \]

Count in two ways triplilaterals $(\alpha, \delta, \gamma)$ whose edges are successively $\Sigma, \Gamma^*_f$, and $\Gamma_1$ then we have
\[ |\Sigma(\alpha)| x_2 \leq v(v-1). \]

If $x_2 = 1$, then $|\Sigma(\alpha)| = v(v-1)$ by (2) and (3). By Lemma 8. iv), $\Gamma_1 \circ \Gamma^*_f = \Gamma_2 \circ \Gamma^*_i$. This is contrary to the assumption. Therefore we have $x_2 > 1$, $\lambda = 1$ and $|\Sigma(\alpha)| x_2 = v(v-1)$. Since $|\Sigma(\alpha)| x_2 = v(v-1)$, $|\Sigma(\alpha) \cap \Gamma_2(\gamma)| = v-1$ for $(\alpha, \gamma) \in \Gamma^*_i$. By Lemma 10. ii), $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$ and $\Gamma^*_f \circ \Gamma_2$ contains some $\Gamma_i$.

Now we shall show that $\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) = \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)$, for
\( \gamma_1, \gamma_2 \in \Gamma^\#(\alpha) \). If \( \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) = \gamma_1 \cap \Gamma_3(\gamma_2) = \sum(\alpha) \), then \( \Gamma^\#_2 \circ \Gamma_2 = \Gamma^\# \circ \Gamma_3 \).

But \( |\Gamma^\#_2 \circ \Gamma_2(\alpha)| = |\sum(\alpha)| + |\Gamma_1(\alpha)| = \frac{\nu(v-1)}{k+1} + \nu < \nu^2 \) and \( |\Gamma^\#_2 \circ \Gamma_3(\alpha)| = \nu^2 \).

This is impossible. Therefore, \( |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha)| = \lambda = 1 \). Thus we have ii) of Lemma.

Last assume \( \Gamma_1 \circ \Gamma^\#_2 = \Gamma_2 \circ \Gamma^\#_2 = \Gamma_3 \circ \Gamma^\#_2 \). We shall show that \( x_2 = |\Gamma^\#_2(\alpha) \cap \Gamma^\#_3(\delta)| > 1 \) and \( x_3 = |\Gamma^\#_3(\alpha) \cap \Gamma^\#_3(\delta)| > 1 \). We note that \( k_1 = k_2 = k_3 \), therefore we put \( k = k_1 = k_2 = k_3 \). If \( x_2 = x_3 = 1 \), by (1) we have \( |\sum(\alpha)| = \nu(v-1) \). By Lemma 8. iv) \( \Gamma_1 \circ \Gamma^\#_2 = \Gamma_2 \circ \Gamma^\#_2, \Gamma_3 \circ \Gamma^\#_2 \). This is contrary to the assumption. If \( x_2 > x_3 = 1 \), we have \( |\sum(\alpha)| = \frac{\nu(v-1)}{k+1} \) as before, and \( x_2 = k+1 \). We put \( \Gamma^\#_2 \circ \Gamma_3 = \sum' \cup \sum' \),

\[
x = |\Gamma^\#_2(\alpha) \cap \Gamma^\#_3(\delta')| \quad \text{for } (\alpha, \delta') \in \sum', \text{and}
\]

\[
t = |\Gamma_3(\gamma_1) \cap \sum(\alpha)| = \frac{\nu-1}{k+1} \quad \text{for } (\alpha, \gamma_1) \in \Gamma^\#_2.
\]

Since \( \Gamma_1 \circ \Gamma^\#_2 = \Gamma_2 \circ \Gamma^\#_2 \) and \( x_3 = 1 \), there exist quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \), with \( \gamma_1 \neq \gamma_2 \) and \( (\alpha, \delta') \in \sum' \), whose edges are successively \( \Gamma^\#_2, \Gamma_2, \Gamma^\#_2 \) and \( \Gamma_1 \). Count all of them in two ways then we have

\[
|\Omega| \frac{\nu(v-1)}{k} kk = |\Omega| \frac{\nu(v-1)}{x} x(x-1),
\]

so

\[
x-1 = \frac{(v-1)k}{v-1} = \frac{t(k+1)k}{t(k+1)+1-t} = \frac{tk(k+1)}{tk+1}.
\]

Therefore \( t = 1 \), and hence, \( v = k+2 \). This is impossible by Lemma 3. Thus we have \( x_2 > 1 \) and \( x_3 > 1 \).

Now we shall show that \( \lambda > 1 \). If \( \lambda = 1 \), by (1) we have

\[
\nu(v-1) = |\sum(\alpha)| x_2 x_3
\]

Since \( x_2 > 1 \), there exist quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \), with \( \gamma_1 \neq \gamma_2 \) and \( (\alpha, \delta) \in \sum \), whose edges are successively \( \Gamma^\#_2, \Gamma_2, \Gamma^\#_2 \) and \( \Gamma_1 \). Count all of them in two ways then we have

\[
|\Omega| \frac{\nu(v-1)}{k} k\lambda = |\Omega| |\sum(\alpha)| x_2 (x_2 - 1),
\]

\[
(\lambda = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \sum(\alpha)| \quad \text{for } \gamma_1, \gamma_2 \neq \in \Gamma^\#_2(\alpha))
\]

so
\[ \lambda_2 = \frac{\sum (\alpha) |x_2(x_2-1)|}{v(v-1)}, \]

and by (1),

\[ \lambda_2 = \frac{x_2-1}{x_3}. \]

Thus \( \frac{x_2-1}{x_3} \) is a positive integer. Since \( x_3 > 1 \), in the same way, we have that \( \frac{x_2-1}{x_3} \) is a positive integer. This is impossible. Thus we have i) of Lemma.

**Lemma 23.** If \( \Gamma_1 \circ \Gamma_2 = \Gamma_3 \circ \Gamma_4 \) and \( \pi_1 = \pi_2 \), then for any \( \Gamma_0, \Gamma_0(\pm) \), \( \Gamma_1 \circ \Gamma_2 \supset \Gamma_0 \circ \Gamma_4 \).

**Proof.** Assume \( \Gamma_1 \circ \Gamma_2 \supset \Gamma_0 \circ \Gamma_4 \). Note that \( |v_1 - v_2| \geq 2 \) by Lemma 13, and hence, \( \pi_1 = \pi_2 \). If \( \{ \Gamma_0, \Gamma_4 \} = \{ \Gamma_1, \Gamma_2 \} \), then since \( \Gamma_1 \circ \Gamma_4 \) is a \( G \)-orbit, \( \Gamma_1 \circ \Gamma_4 = \Gamma_1 \circ \Gamma_4 = \Gamma_2 \circ \Gamma_4 \). This is a contrary to Lemma 14. Therefore, we can assume that \( \Gamma_1 = \Gamma_2, \Gamma_0 \). If \( \Gamma_j = \Gamma_1 \) then, \( \Gamma_1 \circ \Gamma_1 \cap \Gamma_2 \circ \Gamma_4 = \emptyset \). By Lemma 21, we have \( v_2 = v_1 - 1 \). This is a contradiction. Thus we have \( \{ \Gamma_1, \Gamma_4 \} \cap \{ \Gamma_1, \Gamma_2 \} = \emptyset \).

From \( v_1 = v_2 \), we may assume \( v_1 = v_2 \). Since \( \Gamma_1 \circ \Gamma_1 \cap \Gamma_1 \circ \Gamma_4 = \emptyset \), \( v_1 = v_2 - 1 \) by Lemma 21. On the other hand, from \( |v_1 - v_2| \geq 2 \), \( v_1 \neq v_2 \). Since \( \Gamma_2 \circ \Gamma_1 \cap \Gamma_3 \circ \Gamma_4 \neq \emptyset \), in the same way, we have \( v_1 = v_2 - 1 \). This is a contradiction.

**Lemma 24.** If \( \Gamma_1 \circ \Gamma_2 = \Gamma_3 \circ \Gamma_4 \) and \( \pi_1 = \pi_2 = \pi_3 \) and \( |\Gamma_1(\alpha)| > 3 \), then \( \Gamma_1 \circ \Gamma_3 \supset \Delta \) or \( \Gamma_3 \circ \Gamma_1 \supset \Delta \).

**Proof.** Assume \( \Gamma_1 \circ \Gamma_3 \supset \Delta \) and \( \Gamma_3 \circ \Gamma_1 \supset \Delta \). We put \( v = v_1 = v_2 = v_3 \) and \( k = k_1 = k_2 = k_3 \). Since \( \pi_1 = \pi_2 = \pi_3 \), we have \( \pi_1 = \pi_2 = \pi_3 \) by Lemma 21. We shall show that \( \Gamma_1 \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2 = \Gamma_3 \circ \Gamma_3 \). If \( \Gamma_1 \circ \Gamma_2 = \Gamma_3 \circ \Gamma_4 \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 22, i). If \( \Gamma_1 \circ \Gamma_2 = \Gamma_1 \circ \Gamma_4 \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 8, iv). If \( \Gamma_2 \circ \Gamma_4 = \Gamma_1 \circ \Gamma_1 \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 22, ii). This is impossible. Thus we can conclude that \( k > 1 \). If \( k = 1 \), \( |\Gamma_1 \circ \Gamma_1(\alpha)| = \frac{v(v-1)}{k} = v(v-1) \). Since \( \Gamma_2 \circ \Gamma_3 \supset \Gamma_1 \circ \Gamma_1 \), \( \Gamma_3 \circ \Gamma_2 \supset \Gamma_3 \circ \Gamma_4 \) by Lemma 8, iv). This is contrary to the assumption. Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_4 \); then we have

\[ |\Omega| \frac{v(v-1)k}{k} = |\Omega| \frac{v(v-1)}{k} x_2 x_3, \]
Here we put \( x_2 = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| \), \( x_3 = |\Gamma_3(\delta) \cap \Delta(\alpha)| \) for \((\alpha, \delta) \in \Delta\) and \( x = |\Gamma_\delta(\gamma_1) \cap \Gamma_\delta(\gamma_2) \cap \Delta(\alpha)| \) for \( \gamma_1, \gamma_2 (\neq) \in \Gamma_1(\alpha)\).

We shall show that \( x, x_2 \) and \( x_3 \) are smaller than \( k \). If \( x_2 \geq k \), then for \((\alpha, \gamma) \in \Gamma_1\), \(|\Delta(\alpha) \cap \Gamma_\delta(\gamma)| \geq v - 1\). Of course, \(|\Delta(\alpha) \cap \Gamma_\delta(\gamma)| \leq v - 1\), and hence, \(|\Delta(\alpha) \cap \Gamma_\delta(\gamma)| = v - 1\). By Lemma 10, ii), we have \(|\Delta(\alpha)| = \frac{v(v-1)}{k+1}\), which is a contradiction. We can prove in the same way that \( x_3 < k \). Then, (1) yields

\[
x < x_2, \quad x_3 < k.
\]  

Now

\[
C_1(C_\delta C_3) = C_1(xD' + yS'),
\]

\[
(C_1C_\delta)C_3 = (x_2D + y_2S)C_3 = x_2(v-1)C_3 + \text{terms not involving } C_3.
\]

\[
(\Delta' = \Gamma_\delta \circ \Gamma_1, \Gamma_1 \circ \Gamma_\delta = \Delta \cup \Sigma, \Gamma_\delta \circ \Gamma_3 = \Delta' \cup \Sigma'),
\]

\[
D = C(\Delta), \quad D' = C(\Delta'), \quad S = C(\Sigma) \quad \text{and} \quad S' = C(\Sigma')
\]

Since \( x_2 > x \) and the coefficient of \( C_3 \) contained in \( C_1D' \) is at most \( v-1 \), \( C_3 \) is contained in \( C_1S' \), that is, \( \Gamma_\delta \circ \Gamma_3 \supset \Sigma' \). On the other hand, since \( \Gamma_1 \circ \Gamma_\delta \supset \Delta \), there exists the following figure.

\[
\begin{array}{c}
\Gamma_1 \\
\Delta \\
\Gamma_1 \\
\Gamma_3
\end{array}
\]

Therefore \( \Gamma_\delta \circ \Gamma_3 \supset \Delta' \). Thus \( \Gamma_\delta \circ \Gamma_3 = \Delta' \cap \Sigma' = \Gamma_\delta \circ \Gamma_3 \). By Lemma 10, i) we have \( C_\delta C_3 = C_\delta C_3 \). So, \( \pi_1 = \pi_3 \) by Lemma 19, i). This is contrary to the hypothesis of this lemma.

**Lemma 25.** If \( v_1, v_2, v_3 \) and \( v_4 > 3 \), then the following figures don't exist.

\[
\begin{array}{cccc}
\text{Fig. 1} & \text{Fig. 2} & \text{Fig. 3} & \text{Fig. 4}
\end{array}
\]

Proof. For each figure above, we assume its existence and show that it implies a contradiction.

Non-existence of Fig. 1.
Case I. \( \pi_1 \equiv \pi_2, \pi_3, \pi_4 \).

By Lemma 18 and Lemma 19, \( v_1 = v_2 + 1 = v_3 + 1 = v_4 + 1 \), \( |\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2} \), \( |\Gamma_2(\alpha) \cap \Gamma_3(\delta)| = |\Gamma_2(\alpha) \cap \Gamma_4(\delta)| = 1 \) for \( (\alpha, \delta) \in \Gamma_1 \circ \Gamma_4^* \) and \( \pi_2^* = \pi_4^* = \pi^*_1 \). Now let us consider the following figure.

\[
\begin{array}{c}
\Gamma_2 \\
\Gamma_1 \circ \Gamma_1^* \\
\Gamma_3 \\
\Gamma_2
\end{array}
\]

Then by Lemma 22, i) and iii), we have
\[
|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v_2(v_2 - 1) = (v_1 - 1)(v_1 - 2).
\]
Thus,
\[
|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1 - 1)}{2} = (v_1 - 1)(v_1 - 2),
\]
so
\[
v_1 = 4, \quad v_2 = v_3 = v_4 = 3.
\]
This is contrary to the hypothesis of this lemma.

Case II. \( \pi_1 = \pi_2 = \pi_3, \pi_4 \).

By Lemma 21, \( v_1 = v_2 = v_3 + 1 = v_4 + 1 \) and \( \pi_2^* = \pi_3^* = \pi_4^* \). But considering the following figure,

\[
\begin{array}{c}
\Gamma_3 \\
\Gamma_1 \circ \Gamma_1^* \\
\Gamma_2 \\
\Gamma_3
\end{array}
\]
we have \( v_3 = v_2 + 1 \) by Lemma 20. This is impossible.

Case III. \( \pi_1 = \pi_2 = \pi_4 \neq \pi_3, \pi_4 \).

By Lemma 20, \( v_1 = v_2 = v_3 = v_4 + 1 \). But since there exists the following figure,

\[
\begin{array}{c}
\Gamma_4 \\
\Gamma_1 \circ \Gamma_1^* \\
\Gamma_2 \\
\Gamma_4
\end{array}
\]
we have \( v_4 = v_3 + 1 = v_2 + 1 \) by Lemma 21, which is a contradiction.

Case IV. \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^* \).
Existence of the following figure is contrary to Lemma 24.

\[
\begin{array}{c}
\Gamma_2 \\
\Gamma_1^* \\
\Gamma_3 \\
\Gamma_4
\end{array}
\]

Case V. \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^* \).

Since \( \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \), we have by Lemma 22, i) \( |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| > 1 \) for \((\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^* \), and hence, \( \Gamma_1 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_3 \). So, we have \( |\Gamma_1 \circ \Gamma_1^*(a)| < v_i(v_i - 1) \) by Lemma 8, iv). On the other hand, since \( \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \) we have by Lemma 22, ii) \( |\Gamma_4(\gamma_1) \cap \Gamma_4(\gamma_2)| = 1 \) for \((\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^* \). Then from the existence of the following figure,

\[
\begin{array}{c}
\Gamma_4 \\
\Gamma_1^* \\
\Gamma_2 \\
\Gamma_3
\end{array}
\]

we have \( |\Gamma_1 \circ \Gamma_1^*(a)| = v_i(v_i - 1) \) by Lemma 22, which is a contradiction.

Case VI. \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^* \).

There exist the following figures, where \( \Sigma \) is a \( G \)-orbit.

\[
\begin{array}{c}
\Gamma_1 \\
\Sigma \\
\Gamma_4 \\
\Gamma_3
\end{array}
\quad
\begin{array}{c}
\Gamma_1 \\
\Sigma \\
\Gamma_2 \\
\Gamma_3
\end{array}
\]

Fig. a

Fig. b

From Fig. a, we have \( |\Sigma(a)| = v_i(v_i - 1) \) by Lemma 22, iii). On the other hand, from Fig. b, we have \( |\Sigma(a)| = \frac{v_i(v_i - 1)}{k_i + 1} \) by Lemma 22, ii), which is a contradiction.

Case VII. \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^*, \Gamma_4 \circ \Gamma_4^* \).

From \( \Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^*, \Gamma_4 \circ \Gamma_4^* \), we have \( |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1 \) for \( \gamma_1, \gamma_2 \in \Gamma_1 \circ \Gamma_1^*(a) \), by Lemma 22, iii). Similarly from \( \Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*, \Gamma_4 \circ \Gamma_4^* \), we have
\[ |\Gamma_1(\gamma_1) \cap \Gamma_4(\gamma_2)| = 1 \text{ for } \gamma_1, \gamma_2(\pm) \in \Delta (\alpha). \] From \( \Gamma_3 \circ \Gamma_4 \cap \Gamma_3 \circ \Gamma_3 \supseteq \Gamma_3 \circ \Gamma_3 \), we have by Lemma 22

\[ |\Gamma_1 \circ \Gamma_3(\alpha)| = v_1(v_1 - 1). \tag{1} \]

By Lemma 21, \( \pi_2^* = \pi_3^* = \pi_4^* \). Therefore we have by Lemma 8, iv)

\[ \Gamma_3 \circ \Gamma_2 = \Gamma_4 \circ \Gamma_3 \supseteq \Gamma_4 \circ \Gamma_4 \text{ and } \Gamma_3 \circ \Gamma_4 = \Gamma_4 \circ \Gamma_3 \]
and \( \Gamma_3 \circ \Gamma_4 (2 \leq i, j(\pm) \leq 4) \) contains some \( \Gamma_k \).

We put

\[ v = v_1 = v_2 = v_3 = v_4, \Gamma_1 \circ \Gamma_3 = \Delta_1, \Gamma_3 \circ \Gamma_2 = \Delta_2, \]
\[ \Gamma_1 \circ \Gamma_3 = \Delta_1 \cup \Gamma_1, \Gamma_1 \circ \Gamma_4 = \Delta_2 \cup \Sigma', \text{ and } D_i = C(\Delta_i), \]
\[ D_2 = C(\Delta_2), S' = C(\Sigma') \text{ and } s' = |\Sigma(\alpha)|. \]

Now,

\[ (C_2 C_3^*) C_4 = (D_1 + C_2) C_4 = (v - 1) C_3 + \cdots. \]

The coefficient of \( C_3 \) of the above equation is \( v - 1 \) or \( v \) by (2). Next,

\[ C_2 (C_3^* C_4) = C_2 (D_2 + x S'), \]

so

\[ v^2 = \frac{v(v - 1) + xs'}{k_2}. \]

By Lemma 8, i), \( s' \geq v \), so

\[ x \leq v - \frac{v - 1}{k_2} \leq v - 2. \tag{3} \]

We shall show that \( \Gamma_3 \circ \Gamma_4 \supseteq \Sigma' \). If \( \Gamma_3 \circ \Gamma_4 \supseteq \Sigma' \), there exists the following figure.

Since \( \Gamma_3 \circ \Gamma_3 = \Delta_1 \cup \Gamma_1 \), we have \( \Gamma_4 \circ \Gamma_3 = \Delta_1 = \Gamma_4 \circ \Gamma_3 \). This is contrary to the assumption of this case. From \( \Gamma_1 \circ \Gamma_3 \cap \Gamma_3 \circ \Gamma_4 \supseteq \Delta_1 \) and (2), for \( \gamma_1, \gamma_2(\pm) \in \Gamma_4(\alpha) \) we have by Lemma 22, iii)

\[ |\Gamma_3(\gamma_1) \cap \Gamma_4(\gamma_2)| = 1. \tag{4} \]

If \( \Gamma_2 \circ \Sigma' \) contains \( \Gamma_3 \), then we have \( \Gamma_3 \circ \Gamma_3 = \Gamma_4 \circ \Gamma_4 \cap \Sigma' \), and by (4)

\[ C_2 S' = \left( v - \frac{v - 1}{k_4} \right) C_3 + \text{terms not involving } C_3. \]
When $k_4=1$, $v - \frac{v-1}{k_4} = 1$. So $\Gamma_2 \circ \Delta_2$ contains $\Gamma_3$ by (3). When $k_4 > 1$, $v - \frac{v-1}{k_4} > \frac{v}{2}$. So, $x=1$, and hence $\Gamma_2 \circ \Delta_2$ contains $\Gamma_3$.

In all cases, we can conclude that $\Gamma_2 \circ \Delta_2$ contains $\Gamma_3$, and hence, $\Gamma_2^* \circ \Gamma_3 \supset \Delta_2$.

Thus, we have the following figure.

...}

So, $\Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^*$. This is contrary to the assumption.

Non-existence of Fig. 2.

Case I. $\pi_1 = \pi_2$, $\pi_3$.

From $\Gamma_1 \circ \Gamma_2 \cap \Gamma_1 \circ \Gamma_3 \neq \emptyset$ and $\pi_1 = \pi_2$, $\pi_3$, we have $|\Gamma_1 \circ \Gamma^*(\alpha)| = \frac{v_1(v_1-1)}{2}$, $v_1 = v_2 + 1$ and $\Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^*_2$ by Lemma 21 and Lemma 19. On the other hand, $|\Gamma_1 \circ \Gamma^*(\alpha)| = |\Gamma_1 \circ \Gamma^*_1(\alpha)| = |\Gamma_1 \circ \Gamma^*_2(\alpha)| = |\Gamma_1 \circ \Gamma^*_3(\alpha)| = v_1(v_1-1)$.

This is impossible.

Case II. $\pi_1 = \pi_2 \neq \pi_3$.

By Lemma 20, $v_1 = v_2 = v_3 + 1$. On the other hand from the existence of the following figure,

we have $v_3 = v_2 + 1 = v_1 + 1$ by Lemma 21, iii). This is impossible.

Case III. $\pi_1 = \pi_2 = \pi_3$, $\Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^*_2 = \Gamma_3 \circ \Gamma^*_3$.

By Lemma 22, for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_1$, $1 < |\Gamma_1 \circ \Gamma^*(\alpha) \cap \Gamma_2 \circ \Gamma^*_2| = |\Gamma_1(\gamma_1) \cap \Gamma_3(\gamma_2)|$ and $1 < |\Gamma_1(\alpha) \cap \Gamma_2(\delta)|$. The counting arguments show that $|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = |\Gamma_1(\gamma_1) \cap \Gamma_2(\gamma_2)|$ and $|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = |\Gamma_1(\gamma_1) \cap \Gamma_2(\gamma_2)|$ for $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma^*_2$. Therefore, $\Gamma_1 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_3$. Now $\Gamma_1 \circ \Gamma_2 \supset \Gamma_3 \circ \Gamma_1$ and $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_1$. Since we can show that $\pi_1^* = \pi_3^* = \pi_2^*$ by Lemma 21, we have a contradiction by Lemma 24.

Case IV. $\pi_1 = \pi_2 = \pi_3$, $\Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^*_2 = \Gamma_3 \circ \Gamma^*_3$.

From $\Gamma_1 \circ \Gamma_2 \cap \Gamma_1 \circ \Gamma_3 \supset \Gamma_1 \circ \Gamma_1$, we have $|\Gamma_1 \circ \Gamma_1(\alpha)| = \frac{v(v-1)}{k_1+1}$ by Lemma 22.

This is impossible.
Case V. \( \pi_1=\pi_2=\pi_3, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^*. \)

By Lemma 21, we have \( \pi_1^* = \pi_2^* = \pi_3^* \). By Lemma 22, iii), \( |\Gamma_1 \circ \Gamma_1^*(\alpha)| = v(v-1) \), and by Lemma 8, iv), \( \Gamma_3 \circ \Gamma_1 \neq \Gamma_2 \circ \Gamma_2^* \).

\[
\begin{align*}
\Gamma_1 & \quad \Gamma_2 \\
\Gamma_1 & \quad \Gamma_1^* \circ \Gamma_1^* \\
\Gamma_3 & \quad \Gamma_3 \\
\Gamma_3 & \quad \Gamma_2
\end{align*}
\]

From the existence of the above figures, we have \( \Gamma_1^* \circ \Gamma_3 = \Gamma_1^* \circ \Gamma_1 \cup \Gamma_2 \circ \Gamma_2^* \).

Therefore,

\[
v^2 = |\Gamma_1(\alpha) \cdot \Gamma_3(\alpha)| = |\Gamma_1^* \circ \Gamma_3(\alpha)| = |\Gamma_1^* \circ \Gamma_1(\alpha)| + |\Gamma_2 \circ \Gamma_2^* (\alpha)| = v(v-1) + \frac{v(v-1)}{k_2}.
\]

This is impossible.

Non-existence of Fig. 3.

\[
\begin{align*}
\Gamma_1 & \quad \Gamma_1 \\
\Sigma_1 & \quad \Sigma_1 \\
\Sigma_2 & \quad \Sigma_2 \\
\Gamma_3 & \quad \Gamma_4 \\
\Gamma_2 & \quad \Gamma_2
\end{align*}
\]

For the above figure, if \( \Sigma_1 = \Sigma_2 \) then there exists the following figure.

\[
\begin{align*}
\Sigma_1 & \quad \Sigma_1 \\
\Sigma_1 & \quad \Sigma_1 \\
\Sigma_2 & \quad \Sigma_2 \\
\Sigma_2 & \quad \Sigma_2 \\
\Gamma_1 & \quad \Gamma_1 \\
\Gamma_2 & \quad \Gamma_2 \\
\Gamma_3 & \quad \Gamma_3 \\
\Gamma_4 & \quad \Gamma_4
\end{align*}
\]

This is contrary to non-existence of Fig. 1. Thus we have \( \Sigma_1 = \Sigma_2, \pi_1^* = \pi_2^* \), \( \Gamma_1 \circ \Gamma_1^* = \Sigma_1 \cup \Sigma_2 \), and \( G_0 \) is not doubly transitive on \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \) by Lemma 12. So, by Lemma 20 we have \( \pi_1^* = \pi_2^* = \pi_3^* \). Also \( \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4 \) by Lemma 22. From \( \Gamma_2 \circ \Gamma_3 \cap \Gamma_2 \circ \Gamma_4 \cap \Gamma_4 \circ \Gamma_1 \), this is contrary to Lemma 24.

Non-existence of Fig. 4.

There exist the following figures.
Case I. \( \pi^* = \pi^*_2 \).
By Lemma 21, we have \( v_1 = v_2 + 1 \) from Fig. a, and \( v_2 = v_3 + 1 \) from Fig. b. This is impossible.

Case II. \( \pi_1^* = \pi_2^* = \pi_3^* \).
By Lemma 20, we have \( v_1 = v_2 = v_3 + 1 \) and \( \Gamma_2 \circ \Gamma_1^* \neq \Gamma_1 \circ \Gamma_2^* \) from Fig. b. On the other hand, \( \Gamma_2 \circ \Gamma_1^* = \Gamma_1 \circ \Gamma_2^* \supseteq \Gamma_2 \circ \Gamma_1^* \supseteq \Gamma_1 \circ \Gamma_2^* \), and \( \Gamma_2 \circ \Gamma_1^* \) has some \( \Gamma \), by Lemma 20, and hence, \( \Gamma_2 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_1^* \). This is impossible.

Case III. \( \pi_1^* = \pi_2^* = \pi_3^* \).
By assumption, \( \Gamma_2 \circ \Gamma_1 \cap \Gamma_2 \circ \Gamma_3 \supseteq \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \), which is contrary to Lemma 24.

Case IV. \( \pi_1^* = \pi_2^* = \pi_3^* \).
From Fig. a, \( \Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^* \cap \Gamma_1 \circ \Gamma_1^* \) for some \( \Gamma_1 \) by Lemma 22. So, \( \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* \) and \( \Gamma_1 \circ \Gamma_1^*(\alpha) = \frac{\sigma(v-1)}{k_1+1} \). This is impossible.

Case V. \( \pi_1^* = \pi_2^* = \pi_3^* \).
We put \( \sum = \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* \).
By Lemma 22, \( |\sum(\alpha)| = \frac{\sigma(v-1)}{k_1+1} \).

From that \( \Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_3^* \), we have \( \Gamma_1 \circ \Gamma_2^* = \sum \cup \Gamma_1 \circ \Gamma_3^* \). So \( v^2 = \frac{\sigma(v-1)}{k_1+1} + \frac{\sigma(v-1)}{k_1} \). Therefore \( k_1 = 1 \) and \( v-1 = k_1+1 = 2 \). This is contrary to the hypothesis of this lemma.

Case VI. \( \pi_1^* = \pi_2^* = \pi_3^* \).
We put \( \sum = \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* \). By Lemma 22, we have \( \Gamma_1 \circ \Gamma_2^* = \sum \cup \Gamma_1 \circ \Gamma_3^* \). On the other hand, since \( \Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_1^* \), we have \( \Gamma_1 \circ \Gamma_2^* = \sum = \Gamma_2 \circ \Gamma_1^* \). This is impossible.

**Lemma 26.** For \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \), suppose that \( \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* \) contains a \( G \)-orbit \( \sum \) in \( \Omega \times \Omega \) and \( v_1, v_2, v_3 > 3 \). Then, there does not exist \( \Gamma \), such that \( \Gamma_1 \circ \Gamma_2^* = \sum \).
Proof. From non-existences of Fig. 2, Fig. 3, Fig. 4 of Lemma 24, we have this assertion.

**Lemma 27.** (P. J. Cameron [3], Prop.)

If \( \Gamma^*_i = \Gamma_i \) and \( \Gamma_i \circ \Gamma_i \subseteq \Gamma_i \cup \Gamma_i^* \cup (\Gamma_i \cup \Gamma_i^*) \cup (\Gamma_i^* \circ \Gamma_i) \), then \( G \) has rank 4.

3. **Proof of Theorem 1**

We put

\[
x_i = \# \{ \Gamma_i | \Delta_i = \Gamma_i \circ \Gamma_i^* \} ,
\]

\[
y_i = \# \{ (\Gamma_i, \Gamma_i) | \Gamma_i \circ \Gamma_i^* \supset \Delta_i \}
\]

and assume that \( x_1 \geq \cdots \geq x_r \geq x_{r+1} = \cdots = x_t = 0 \). Counting in two ways triplilaterals \( \Gamma_i, \Gamma_i, \Delta_i \) such that \( \Gamma_i \circ \Gamma_i^* \supset \Delta_i \), we have by Lemma 9 and 11

\[ s^2 \leq \sum_{i=1}^{t} y_i . \]

The equality means that, for any \( \Gamma_i \) and \( \Gamma_j \), we cannot have \( \Gamma_i \circ \Gamma_j^* = \Delta_k \cup \Delta_i, \Delta_k \neq \Delta_i \).

When \( x_i > 0 \), by Lemma 26 \( y_i \leq x_i + s \). When \( x_i = 0 \), by non-existence of Fig. 1 of Lemma 25 \( y_i \leq 2s \). Therefore

\[ s^2 \leq \sum_{i=1}^{t} y_i \leq \sum_{i=1}^{t} (x_i + s) + 2(t-r)s , \]

so

\[ s^2 \leq (r+1)s + (t-r)s , \]

\[ s \leq 2t - r + 1 . \]  \( (1) \)

Now, let \( \Delta_i = \Gamma_i \circ \Gamma_i^* \) and we put

\[ A = \{ \{ \Gamma_i, \Gamma_j \} : \text{unordered pair } \Gamma_i \circ \Gamma_j^* \supset \Delta_i, \Gamma_i \neq \Gamma_j \} , \]

\[ B = \{ \Gamma_i | \{ \Gamma_i, \Gamma_j \} \in A \} . \]

For \( \{ \Gamma_i, \Gamma_j \}, \{ \Gamma_k, \Gamma_l \} \in A \), \( \{ \Gamma_k, \Gamma_l \} \cap \{ \Gamma_i, \Gamma_j \} = \emptyset \) by Lemma 26. Therefore \( |B| = 2|A| \). Furthermore, for \( \{ \Gamma_i, \Gamma_j \}, \{ \Gamma_k, \Gamma_l \} \in A \), and for \( \Gamma_m, \Gamma_n \) \( \in B \), \( \Gamma_m \circ \Gamma_n \cap \Gamma_i \circ \Gamma_j, \Gamma_k \circ \Gamma_l \cap \Gamma_i \circ \Gamma_j, \Gamma_i \circ \Gamma_j \circ \Gamma_m, \Gamma_i \circ \Gamma_j \circ \Gamma_n \) are disjoint to each other by non-existence of Fig. 1 of Lemma 25. Thus we have

\[ |A| + |B| - |B| = s - |A| \leq t , \]  \( (2) \)

and by Lemma 26

\[ |A| - 1 \leq t - r . \]  \( (3) \)

Assume \( s = 2t - r + 1 \). Since the equality of (1) holds \( y_i = x_i + s \), and hence
\[ |A| = \frac{s}{2} \text{ and } \frac{s}{2} - 1 \leq t - r \text{ by (3)}, \text{ and hence, } 2t - r + 1 = s \leq 2t - 2r + 2. \] So \( r = 1 \). Therefore, if \( r > 1 \), we conclude that \( s \leq 2t - r \).

We shall show that when \( r = 1 \), \( s \leq 2t - 2 \). Assume \( r = 1 \) and \( 2t \geq s \geq 2t - 1 \), and put \( \Delta = \Gamma_t \circ \Gamma_e^* \), \( 1 \leq i \leq s \). If \( \pi_i \neq \pi_j \) for some \( \Gamma_t \) and \( \Gamma_e \), then by Lemma 23, \( \Delta \subset \Gamma_e \circ \Gamma_e^* \) for any \( \Gamma_e \), \( \Gamma_e (\pm) \), and hence, \( \Gamma_t \circ \Gamma_e \cap \Gamma_e \circ \Gamma_e = \emptyset \). So \( s \leq t \). This is contrary to the assumption that \( t \geq 2 \). Thus, it holds that \( \pi_1 = \pi_2 = \cdots = \pi_s \).

Now, Suppose \( \Gamma_t \circ \Gamma_e = \Delta \cup \Gamma_e^* \) for some \( \Gamma_t \), \( \Gamma_e \) and \( \Gamma_e^* \), and put \( D = C(\Delta) \), \( \Gamma_t \circ \Gamma_e = \Delta' \cup \Gamma_e^* \), \( D' = C(\Delta') \), \( t = |\Gamma_t(\alpha) \cap \Gamma_e^*(\beta)| \) for \( (\alpha, \beta) \in \Gamma_e^* \), \( x = |\Gamma_t(\alpha) \cap \Gamma_e^*(\delta)| \) for \( (\alpha, \delta) \in \Delta \), \( v = v_1 = v_2 = \cdots, k = k_1 = k_2 = \cdots \). Then we have

\[
(C, C_e)_{\psi} = (t C^* + x D) C_e = tvI + tk D + x DC_e, \\
C_e (C, C_e) = C_e (t' C^* + x' D') = tv'I + tk D + x'C_e D'. \\
(t' = |\Gamma_t'(\alpha) \cap \Gamma_e^*(\beta)| \text{ for } (\alpha, \beta) \in \Gamma_e^*, x' = |\Gamma_t'(\alpha) \cap \Gamma_e^*(\delta)| \text{ for } (\alpha, \delta) \in \Delta').
\]

We have \( t = t' \) by counting in two ways triplilaterals \((\beta, \alpha, \gamma)\) whose edges are successively \( \Gamma_t, \Gamma_e \) and \( \Gamma_e^* \), and have \( |\Delta(\alpha)| = |\Delta'(\alpha)| \) and \( x = x' \) by Lemma 10. So,

\[ C_e D' = D C_e = (v - 1) C_e + \cdots. \]

If \( C_i \neq C_k \), \( |\Delta'(\alpha)| = \frac{v(v - 1)}{k + 1} \) by Lemma 10. This is impossible. Thus \( C_i = C_k \).

Similarly, \( C_j = C_e \).

When \( s = 2t \), then the equality of (1) holds. Therefore, for any \( \Gamma_t \), there exists \( \Gamma_e \) such that \( \Gamma_t \circ \Gamma_e = \Delta \cup \Gamma_e^* \) for some \( \Gamma_e^* \). So, as is shown above, \( \Gamma_t = \Gamma_t' = \Gamma_e \). Therefore we have any \( \Gamma_t \).

\[ \Gamma_t \neq \Gamma_e, \Gamma_t \circ \Gamma_e = \Delta \cup \Gamma_e^* \text{ and } \Gamma_t \circ \Gamma_e \cap \Gamma_e \circ \Gamma_e = \emptyset \text{ for } \Gamma_m \neq \Gamma_e, \Gamma_e^* . \]

When \( s = 2t - 1 \), then \( |A| \leq t - 1 \), and from (2) \( s - |A| \leq t \). So \( |A| = t - 1 \).

Therefore, there is a unique \( \Gamma_t \) such that for any \( \Gamma_t (\pm \Gamma_e) \), \( \Gamma_t \circ \Gamma_e \subset \Delta \). We shall show that for any \( \Gamma_t, \Gamma_e (\pm) \), \( \Gamma_t \circ \Gamma_e^* \) contains some \( \Gamma_e \). Assume \( \Gamma_t \circ \Gamma_e^* = \Delta_k \cup \Delta_l \) for some \( \Gamma_l, \Gamma_e (\pm) \). Count in two ways the paired \((\Gamma_m, \Delta_n)\) such that \( \Gamma_t \circ \Gamma_e^* \) contains \( \Delta_n \) then by Lemma 25, we have

\[ 2t = s + 1 \leq \# \{ (\Gamma_m, \Delta_n) \mid \Gamma_t \circ \Gamma_e^* \supset \Delta_n \} \leq 2t . \]

So, equality holds. Thus for any \( \Delta_n \), there exist \( \Gamma_k \) and \( \Gamma_q (\pm) \) such that \( \Gamma_t \circ \Gamma_e^* \) and \( \Gamma_t \circ \Gamma_{e'}^* \) contains \( \Delta_k \). Therefore we may choose \( \Gamma_k \) such that \( \Gamma_t \circ \Gamma_e^* \cap \Gamma_t \circ \Gamma_{e'}^* = \emptyset \) and \( \Gamma_k \neq \Gamma_q \). Then \( \Gamma_k \circ \Gamma_e^* \supset \Gamma_t \circ \Gamma_e^* = \Delta_l \). This is impossible. Thus, again as is shown above, we can conclude that for any \( \Gamma_t (\pm \Gamma_e) \),
Thus if \( s \geq 2t - 1 \), there exists \( \Gamma \) such that

\[ \Gamma \neq \Gamma^* \text{ and } \Gamma \circ \Gamma^* = \emptyset \text{ for } \Gamma_m \neq \Gamma_n. \]

By Lemma 27, this shows that \( G \) does not have rank 4. This is impossible for \( s \geq 2t - 1 \) and \( t \geq 2 \).

4. Proof of Theorem 2

When \( r = t \), we have \( s \leq t \) by Theorem 1. On the other hand, from \( s \geq r = t \), we conclude that \( s = t \).

We put \( \Gamma \circ \Gamma^* = \Delta_i \), \( A_i = \{ \{ \Gamma_i; \Gamma \} \} \) unordered pair \( | \Gamma \circ \Gamma^* \supset \Delta_i, \Gamma \neq \Gamma_i | \). Then \( | A_i | - 1 \leq t - r = 0 \), so \( | A_i | \leq 1 \).

Count in two ways triplilaterals \( (\Gamma_i, \Gamma_j, \Delta_i) \) such that \( \Gamma_i \circ \Gamma^* \supset \Delta_i \), we have

\[ s^2 \leq 3s, \]

so

\[ s \leq 3. \] (1)

Case \( t = 2. \) If \( | \Gamma_i(\alpha) | \neq | \Gamma_j(\alpha) | \), by T. Ito [6], \( G \) is isomorphic to the small Janko simple group and \( G_{\alpha}^* \) is isomorphic to \( \text{PSL}(2,11) \). We shall prove that the case of \( | \Gamma_i(\alpha) | = | \Gamma_j(\alpha) | \) does not occur. We put \( | \Gamma_i(\alpha) | = | \Gamma_j(\alpha) | = v. \) It is easy to prove that \( \pi_1 = \pi_2 \). We shall show that \( \Gamma_j \) and \( \Gamma_j \) are self paired. If not, then \( \Gamma \circ \Gamma^* = \Gamma_j \). Since \( \Gamma_j \circ \Gamma_j = \Gamma_j \circ \Gamma_j^* = \Gamma_j \circ \Gamma_j \), we have that \( \Gamma_j \circ \Gamma_j \supset \Gamma_j \circ \Gamma_j^* \), \( \Gamma_j \circ \Gamma_j^* (= \Gamma_j \circ \Gamma_j^*) \) by Lemma 7. By Lemma 11, there exists a \( G \)-orbit \( \Sigma \) in \( \Gamma_j \circ \Gamma_j \), such that \( \Sigma \) is not 2-transitive on \( \{ \alpha \} \) and \( \Sigma \neq \Delta_i \). This is impossible for \( t = 2 \). Thus, we have \( \Gamma_j \circ \Gamma_j = \Delta_i \cup \Delta_j \). So, \( \psi = | \Gamma_j \circ \Gamma_j(\alpha) | = | \Gamma_j \circ \Gamma_j(\alpha) | = \frac{v(v - 1)}{k_1} + \frac{v(v - 1)}{k_2} \). This is impossible.

Case \( t = 3. \) For this case, the equality of (1) holds. So we have \( | A_i | = 1 \) for \( 1 \leq i \leq 3 \). We shall show that if \( \Gamma_i \circ \Gamma_j \supset \Delta_i \) then \( \Gamma_i = \Gamma_j \) or \( \Gamma_j = \Gamma_i \). If \( \Gamma_i \neq \Gamma_j \), then since \( \Gamma_i \circ \Gamma_j \cap \Gamma_j \supset \emptyset \), there exists a \( G \)-orbit \( \Sigma \) in \( \Gamma_i \circ \Gamma_j \cap \Gamma_j \circ \Gamma_i \) such that \( G \) is not 2-transitive on \( \{ \alpha \} \) by Lemma 12, and for any \( \Gamma_i, \Gamma_j \circ \Gamma_i \supset \Sigma \) by Lemma 25. From \( r = t \), this is impossible. Thus we may assume that there exist the following figures.

![Fig. a](image1)

![Fig. b](image2)

![Fig. c](image3)
If \( \pi_1 = \pi_2 = \pi_3 \), then \( v_1v_2 = |\Gamma^\dagger \circ \Gamma_2(\alpha)| = |\Gamma^\dagger \circ \Gamma_3(\alpha)| = \frac{v_3(v_2-1)}{k_1} \) from Fig. a, so \( v_1 > v_3 \). Similarly, \( v_3 > v_1 \) from Fig. c. Therefore \( v_3 > v_2 \). On the other hand, \( v_2v_3 = \frac{v_3(v_2-1)}{k_2} \) from Fig. b, so \( v_2 > v_3 \). This is impossible. Thus we have \( \pi_1 = \pi_2 = \pi_3 \). By Lemma 7, \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are self-paired.

Thus \( \Gamma_1 \circ \Gamma_2 = \Gamma_3 = \Delta_1, \Gamma_2 \circ \Gamma_3 = \Gamma_1 = \Delta_2, \Gamma_3 \circ \Gamma_1 = \Gamma_2 = \Delta_3 \). Put \( |\Gamma_i(\alpha)| = v \), then by Lemma 8, iii) we have

\[
|\Delta_i(\alpha)| = |\Delta_2(\alpha)| = |\Delta_3(\alpha)| = v(v-1).
\]

We put

\[
D_i = C(\Delta_i) \text{ and } C_i = C(\Gamma_i), \ 1 \leq i \leq 3;
\]

\[
D_1C_3 = x_1D_1 + x_2D_2 + x_3D_3.
\]

Then

\[
x_1 + x_2 + x_3 = v
\]

\[
D_2C_3 = x_2D_1 + \text{terms not involving } D_1,
\]

\[
D_3C_3 = x_3D_1 + \text{terms not involving } D_1.
\]

Now

\[
(C_1C_2)C_3 = (D_1 + C_3)C_3 = vI + D_3 + D_1C_3,
\]

\[
C_1(C_2C_3) = C_1(D_2 + C_1) = vI + D_1 + D_2C_1.
\]

So

\[
D_2C_1 = D_1C_3 + D_3 - D_1 = (x_1 - 1)D_1 + x_2D_2 + (x_3 + 1)D_3.
\]

Similarly

\[
D_3C_2 = D_2C_3 + D_3 - D_2 = x_1D_1 + (x_2 - 1)D_2 + (x_3 + 1)D_3.
\]

Next

\[
(C_1C_2)C_3 = (vI + D_1)C_3 = vC_3 + D_1C_3,
\]

\[
C_1(C_2C_3) = C_1(D_3 + C_2) = C_3 + D_1 + D_3C_1.
\]

So

\[
D_2C_1 = D_1C_3 + (v-1)C_3 - D_1
\]

\[
= (x_1 - 1)D_1 + x_2D_2 + x_3D_3 + (v - 1)C_3.
\]

Similarly

\[
D_1C_2 = D_2C_1 + (v-1)C_1 - D_2
\]

\[
= (x_1 - 1)D_1 + (x_2 - 1)D_2 + (x_3 + 1)D_3 + (v - 1)C_1,
\]

\[
D_2C_3 = D_3C_2 + (v-1)C_2 - D_3
\]

\[
= x_1D_1 + (x_2 - 1)D_2 + x_3D_3 + (v - 1)C_2.
\]

Furthermore
\[(C_1C_3)C_2 = (vI + D_1)C_3 = vC_2 + D_1C_2 \]
\[C_1(C_3 + D_1) = C_2 + D_3 + D_1C_1 \]

So
\[D_1C_1 = D_1C_2 + (v - 1)C_2 - D_3 \]
\[= (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3D_3 + (v - 1)C_1 + (v - 1)C_2 . \]

Similarly
\[D_2C_2 = D_2C_3 + (v - 1)C_3 - D_1 \]
\[= (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3D_3 + (v - 1)C_2 + (v - 1)C_3, \]
\[D_3C_3 = D_3C_1 + (v - 1)C_1 - D_2 \]
\[= (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3D_3 + (v - 1)C_2 + (v - 1)C_1 . \]

Thus (2), (3) and (4) yield
\[x_1 = x_2, \quad x_1 - 1 = x_3. \]

We put \(x_3 = x\), then
\[v = x_1 + x_2 + x_3 = (x + 1) + (x + 1) + x = 3x + 2 . \]

It is easy to show that the graph \((\Omega, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)\) is a strongly regular graph with parameters \(3v, 2, 3\).

From the conditions of the existence of the strongly regular graph, (see [1] p. 97) it holds that
\[(3 - 2)^2 + 4(3v - 3) = 12v - 11 = d^2 \quad \text{ (6)} \]
\[(d \text{ is a positive integer)} \]
\[m = \frac{3v}{2 \cdot 3 \cdot d} \{(3v - 1 + 3 - 2)(d + 3 - 2) - 2 \cdot 3\} = \frac{3}{2} v^2 + \frac{3v(v - 2)}{2d}. \quad \text{ (7)} \]
\[(m \text{ is a positive integer)} \]

From (7), \(\frac{3v(v - 2)}{d}\) is integer, and hence
\[12v - 11 = d^2 \text{ is a divisor of } v_0(v - 2)^2. \]
So

$$12v - 11$$ is a divisor of $$11^3 \cdot 13^2$$.

From $$v = 3x + 2$$, we conclude

$$v = 11$$.

Lastly, we shall prove that the primitive group satisfying these conditions does not exist. It is easy to prove that $$G_a$$ acts faithfully on $$\Gamma_1(\alpha)$$. We shall show that for $$\gamma_1, \gamma'_1 (\neq) \in \Gamma_1(\alpha), G_{a, \gamma_1, \gamma'_1}$$ has the fixed points in $$\Gamma_1(\alpha) \setminus \{\gamma_1, \gamma'_1\}$$.

For $$(\alpha, \gamma_1) \in \Gamma_1$$, put $$\{\gamma_2\} = \Gamma_2(\alpha) \cap \Gamma_3(\gamma_1)$$ and $$\{\gamma_3\} = \Gamma_3(\alpha) \cap \Gamma_2(\gamma_1)$$. Then, $$G_{a, \gamma_1}$$ fixes $$\gamma_2$$ and $$\gamma_3$$. So we must have that $$(\gamma_2, \gamma_3) \in \Gamma_1$$. Now for $$\gamma_1, \gamma'_1 (\neq) \in \Gamma_1(\alpha)$$, put $$\{\delta_1\} = \Gamma_1(\gamma_1) \cap \Gamma_2(\gamma'_1), \{\delta_1\} = \Gamma_1(\gamma_1) \cap \Gamma_1(\gamma'_1)$$. Then $$G_{a, \gamma_1, \gamma'_1}$$ fixes $$\delta_1$$ and $$\delta_2$$. Since $$(\gamma_1, \gamma'_1) \nsubseteq \Gamma_3$$, we have $$(\delta_1, \delta_2) \nsubseteq \Gamma_3$$. Therefore $$\Gamma_1(\gamma_1) \cap \Gamma_3(\delta_2) = \{\delta\}$$.

So, $$G_{a, \gamma_1, \gamma'_1}$$ fixes $$\delta_1$$ and $$\delta$$. Since $$\Gamma_1(\gamma_1) \nsubseteq \alpha, \delta_1, \delta (\neq)$$, in the same way, we obtain that $$G_{a, \gamma_1, \gamma'_1, \delta}$$ has the fixed points in $$\Gamma_1(\alpha) \setminus \{\gamma_1, \gamma'_1\}$$. The order of $$G_a$$ is at most one million. If $$G_a$$ is non-solvable, then the minimal normal subgroup of $$G_a$$ is non-solvable simple. From [5], it is isomorphic to the Mathieu group $$M_{11}$$ or the transitive extension of the alternating group $$A_5$$ act on ten points. These groups have not the representation such that it is doubly-transitive on eleven points and it’s stabilizer of two points has the additional fixed point. Thus, we can conclude that $$G_a$$ is solvable and the order of $$G_a$$ is 110. So $$|G| = |\Omega| \cdot 11 \cdot 10 = 364 \cdot 11 \cdot 10 = 2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$$. $$G$$ is non-solvable group and $$(|G|, 3) = 1$$. But there does not exist such group by M. Hall [5].

References


