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<th><strong>Title</strong></th>
<th>Generalization of a theorem of Peter J. Cameron</th>
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Peter J. Cameron [3] has shown that a primitive permutation group $G$ has rank at most 4 if the stabilizer $G_a$ of a point $\alpha$ is doubly transitive on all its nontrivial suborbits except one.

The purpose of this paper is to prove the following two theorems, one of which extends the Cameron's result.

**Theorem 1.** Let $G$ be a primitive permutation group on a finite set $\Omega$, and all nontrivial $G$-orbits in Cartesian product $\Omega \times \Omega$ be $\Gamma_1, \ldots, \Gamma_i, \Delta_1, \ldots, \Delta_t$, where $G_a$ is doubly transitive on $\Gamma_i(\alpha) = \{\beta | (\alpha, \beta) \in \Gamma_i\}, 1 \leq i \leq s$ and not doubly transitive on $\Delta_j(\alpha), 1 \leq j \leq t$. Suppose that $G$ has no subdegree smaller than 4 and that $t > 1$. Then, we have

$$s \leq 2t - r,$$

where $r = \#\{\Delta_j | \Delta_j = \Gamma_j^{\#} \Gamma_j, 1 \leq j \leq s\}$. Moreover if $r = 1$, then we have

$$s \leq 2t - 2.$$

(For the notation $\Gamma_j^{\#} \Gamma_j$, see the section 1)

**Theorem 2.** Under the hypothesis of Theorem 1, if $r = t$, then $s = t + 2$, and $G$ is isomorphic to the small Janko simple group and $G_a$ is isomorphic to $PSL(2, 11)$.

For the case of $t \geq 3$, I don't know the example satisfying the equality $s = 2t - r$, and when $r = 1$, the example satisfying the equality $s = 2t - 2$. I know only three examples with $t = 2$ and $s = 2$.

The small Janko simple group $J_1$ of order 175560 has a primitive rank 5 representation of degree 266 in which the stabilizer of a point is isomorphic to $PSL(2, 11)$ and acts doubly transitively on suborbits of lengths 11 and 12; the other suborbit lengths are 110 and 132 (See Livingstone [7]). The Mathieu group $M_{12}$ has a primitive rank 5 representation of degree 144 in which the stabilizer of a point is isomorphic to $PSL(2, 11)$ and acts doubly transitively on two suborbits of length 11; the other suborbit lengths are 55 and 66 (See Cameron [4]).
The group \([Z_3 \times Z_3 \times Z_3]S_4\) has a primitive rank 5 representation of degree 27 in which the stabilizer of a point is \(S_4\) and acts doubly transitively on two suborbits of length 4; the other suborbit lengths are 6 and 12. I conjecture that it may even be true that \(s\) is at most \(t\).

1. Preliminaries

Let \(G\) be a transitive permutation group on a finite set \(\Omega\), and \(\Delta\) be a subset of the Cartesian product \(\Omega \times \Omega\) which is fixed by \(G\) (acting in the natural way on \(\Omega \times \Omega\)), then \(\Delta(\alpha) = \{\beta \in \Omega | (\alpha, \beta) \in \Delta\}\) is a subset of \(\Omega\) fixed by \(G\). This procedure sets up a one-to-one correspondence between \(G\)-orbits in \(\Omega \times \Omega\) and \(G\)-orbits in \(\Omega\). The number of such orbits is called the rank of \(G\). \(\Delta^* = \{(\beta, \alpha) | (\alpha, \beta) \in \Delta\}\) is the subset of \(\Omega \times \Omega\) fixed by \(G\) paired with \(\Delta\); \(\Delta\) is self-paired if \(\Delta = \Delta^*\). Note that \(|\Delta(\alpha)| = |\Delta^*(\alpha)| = |\Delta| / |\Omega|\). If \(\Gamma\) and \(\Delta\) are fixed sets of \(G\) in \(\Omega \times \Omega\), let \(\Gamma \circ \Delta\) denote the set \(\{(\alpha, \beta)\}\) there exists \(\gamma \in \Omega\) with \((\alpha, \gamma) \in \Gamma\), \((\gamma, \beta) \in \Delta\); \(\alpha \neq \beta\); this is also a fixed set of \(G\). The diagonal \(\{((\alpha, \alpha)) | \alpha \in \Omega\}\) is a trivial \(G\)-orbit. If \(\Gamma\) is a nontrivial \(G\)-orbits in \(\Omega \times \Omega\), the \(\Gamma\)-graph is the regular directed graph whose point set is \(\Omega\) and whose edges are precisely the ordered pairs in \(\Gamma\). A connected component of any such graph is a block of imprimitivity for \(G\). \(G\) is primitive if and only if each such graph is connected.

For a \(G\)-orbit \(\Gamma\) in \(\Omega \times \Omega\), the basis matrix \(C = C(\Gamma)\) is the matrix whose rows and columns are indexed by \(\Omega\), with \((\alpha, \beta)\) entry 1 if \((\alpha, \beta) \in \Gamma\), 0 otherwise. All of the basis matrices form a basis of the centralizer algebra of the permutation matrices in \(G\).

Let \(G\) be a group which acts as a permutation group on \(\Omega\), and \(\pi\) the permutation character of \(G\) i.e. the integer-valued function on \(G\) defined by \(\pi(g) = \) number of fixed points of \(g\). The formula

\[
(\pi, 1)_G = \frac{1}{|G|} \sum_{g \in G} \pi(g) = \text{number of orbits of } G,
\]

is well-known. If \(G\) acts as a permutation group on \(\Omega_1\) and \(\Omega_2\), with permutation characters \(\pi_1\) and \(\pi_2\), the number \(m\) of \(G\)-orbits in \(\Omega_1 \times \Omega_2\) is

\[
m = (\pi_1, \pi_2, 1)_G = (\pi_1, \pi_2)_G.
\]

In particular, if \(G\) is a transitive permutation group on \(\Omega\) with permutation character \(\pi\), the rank \(r\) of \(G\) is given by

\[
r = (\pi, \pi)_G = \text{sum of squares of multiplicities of irreducible constituents of } \pi.
\]

If \(G\) acts doubly transitively on \(\Omega_1\) and \(\Omega_2\),

\[
(\pi_1, \pi_2)_G = 2 \text{ or } 1 \text{ according as } \pi_1 = \pi_2 \text{ or } \pi_1 \neq \pi_2.
\]
Lastly, we note that if $G$ is a primitive permutation group on $\Omega$, then for $\alpha, \beta \in \Omega$, either $G_{\alpha} \neq G_{\beta}$ or $G$ is a regular group of prime degree ([8], Prop. 8.6); primitive groups with a subdegree 2 are Frobenius groups of prime degree ([8], Theorem 18.7); primitive groups with a subdegree 3 are classified by W.J. Wong [9].

2. Lemmata

Throughout this section, we suppose that $G$ is a primitive but not doubly transitive group on a finite set $\Omega$, and $\Gamma_{1}, \Gamma_{2}, \ldots$ are $G$-orbits in $\Omega \times \Omega$ such that $G_{\alpha}$ is doubly transitive on $\Gamma_{i}(\alpha), i=1, 2, \ldots; \pi_{i}$ and $\pi_{i}^{*}$ are the permutation characters of $G_{\alpha}$ on $\Gamma_{i}(\alpha)$ and $\Gamma_{i}^{*}(\alpha)$, respectively, and let $C_{i} = C(\Gamma_{i}), C_{i}^{*} = C(\Gamma_{i}^{*})$.

**Lemma 1.** (P. J. Cameron [2]. Proposition 1.2) $G_{\alpha}$ is doubly transitive on $\Gamma_{i}^{*}(\alpha)$.

**Lemma 2.** (P. J. Cameron [3]. Lemma 1) $\Gamma_{i}^{*} \circ \Gamma_{i}$ is a $G$-orbit in $\Omega \times \Omega$, and if $|\Gamma_{i}(\alpha)| > 2$, then $G_{\alpha}$ is not doubly transitive on $\Gamma_{i}^{*} \circ \Gamma_{i}(\alpha)$.

**Lemma 3.** (P. J. Cameron [2]. Theorem 2.2) For $(\alpha, \beta) \in \Gamma_{i} \circ \Gamma_{i}^{*}$, we put $v_{i} = |\Gamma_{i}(\alpha)|$ and $k_{i} = |\Gamma_{i}(\alpha) \cap \Gamma_{i}(\beta)|$. Then $k_{i} < v_{i}$ and $|\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)| = \frac{v_{i}(v_{i}-1)}{k_{i}}$. If $v_{i} > 2$, then $k_{i} \leq \frac{v_{i}-1}{2}$; when particularity $k_{i} = \frac{v_{i}-1}{2}$, then $v_{i} = 3$ or 5.

In the following, we set $|\Gamma_{i}(\alpha)| = v_{i}, |\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)| = \frac{v_{i}(v_{i}-1)}{k_{i}}$.

**Lemma 4.** (P. J. Cameron [2]. Lemma 2.1) $|\Gamma_{i}^{*} \circ \Gamma_{i}(\alpha)| = |\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)|$.

**Lemma 5.** $\Gamma_{i}^{*} \circ \Gamma_{i} \neq \Gamma_{i}^{*} \circ \Gamma_{i}^{2}$ if and only if $|\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)| = |\Gamma_{i}(\alpha) \cdot |\Gamma_{2}(\alpha)|$.

Proof. If $|\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)| < |\Gamma_{i}(\alpha) \cdot |\Gamma_{2}(\alpha)|$, we have $|\Gamma_{i}(\alpha) \cap \Gamma_{2}(\beta)| > 1$ for some $(\alpha, \beta) \in \Gamma_{i} \circ \Gamma_{i}^{*}$. For $(\gamma_{1}, \gamma_{2}) \in \Gamma_{i}(\alpha) \cap \Gamma_{2}(\beta), (\gamma_{1}, \gamma_{2}) \in \Gamma_{2}^{*} \circ \Gamma_{1}$ and $(\gamma_{1}, \gamma_{2}) \in \Gamma_{i}^{*} \circ \Gamma_{2}$. So $\Gamma_{i}^{*} \circ \Gamma_{1} = \Gamma_{2}^{*} \circ \Gamma_{2}$. Conversely, if $\Gamma_{i}^{*} \circ \Gamma_{1} = \Gamma_{i}^{*} \circ \Gamma_{2}$ for $(\gamma_{1}, \gamma_{2}) \in \Gamma_{i}^{*} \circ \Gamma_{1} = \Gamma_{i}^{*} \circ \Gamma_{2}$ we can choose $\alpha$ and $\beta$ such that $\alpha \in \Gamma_{i}^{*}(\gamma_{1}) \cap \Gamma_{i}^{*}(\gamma_{2}), \beta \in \Gamma_{i}^{*}(\gamma_{1}) \cap \Gamma_{i}^{*}(\gamma_{2})$. Since $\Gamma_{i}(\alpha) \cap \Gamma_{2}(\beta) \subseteq \gamma_{1}, \gamma_{2}, |\Gamma_{i}(\alpha) \cap \Gamma_{2}(\beta)| > 1$. Therefore $|\Gamma_{i} \circ \Gamma_{i}^{*}(\alpha)| < |\Gamma_{i}(\alpha) \cdot |\Gamma_{2}(\alpha)|$.

**Lemma 6.** $\Gamma_{i}^{*} \circ \Gamma_{2}$ is the union of at most two $G$-orbits in $\Omega \times \Omega$, and
\( \pi_1 = \pi_2 \) if and only if \( \Gamma f \circ \Gamma_2 \) is the union of two \( G \)-orbits in \( \Omega \times \Omega \).

Proof. Since \( (\pi_1, \pi_2, 1) = (\pi_1, \pi_2, 0) \neq 2 \), and \( \pi_1 \pi_2 \) is the permutation character of \( G \) on \( \Gamma_1(\alpha) \times \Gamma_2(\alpha) \), \( G \) has at most two orbits in \( \{ (\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1, (\alpha, \delta) \in \Gamma_2 \} \), and hence, \( \Gamma f \circ \Gamma_2 \) is the union of at most two \( G \)-orbits. If \( \pi_1 \neq \pi_2 \), then \( G \) is transitive on \( \{ (\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1, (\alpha, \delta) \in \Gamma_2 \} \), and hence, \( \Gamma f \circ \Gamma_2 \) is a \( G \)-orbit in \( \Omega \times \Omega \). Now, we shall assume that \( \pi_1 = \pi_2 \) and \( \Gamma f \circ \Gamma_2 \) is a \( G \)-orbit in \( \Omega \times \Omega \). We put \( v = v_1 = v_2 \), and \( m = |\Gamma f(\alpha) \cap \Gamma_2(\delta)| \) for \( (\alpha, \delta) \in \Gamma f \circ \Gamma_2 \). If \( m = 1 \), then since \( \Gamma f \circ \Gamma_2 \) is a \( G \)-orbit, \( G \) is transitive on \( \{ (\alpha, \gamma, \delta) | (\gamma, \alpha) \in \Gamma_1, (\gamma, \delta) \in \Gamma_2 \} \). Therefore \( (\pi_1, \pi_2) = 1 \), and hence, \( \pi_1 = \pi_2 \), this is contrary to the assumption. If \( m > 1 \), then there exist quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma f \), \( \Gamma_2 \), \( \Gamma f \) and \( \Gamma_1 \); and whose vertices are all distinct. Counting all of them in two ways, we have

\[
\frac{v m(m - 1)}{2} = \frac{v(v - 1)k_1k_2}{k_i},
\]

so

\[
v(m - 1) = (v - 1)k_2.
\]

Hence, \( v = k_2 \). This is impossible by Lemma 3.

**Lemma 7.** If \( \Gamma f \circ \Gamma_1 \neq \Gamma f \circ \Gamma_2 \), then \( \Gamma f \circ \Gamma_1 \neq \Gamma f \circ \Gamma_2 \).

Proof. Now assume \( \Gamma f \circ \Gamma_1 \neq \Gamma f \circ \Gamma_2 \), then we have the following figure,

\[
\Gamma_1 \circ \Gamma_i \circ \Gamma_2 \circ \Gamma_1 \circ \Gamma_2,
\]

and hence, \( \Gamma f \circ \Gamma_1 \circ \Gamma_2 \circ \Gamma_1 \circ \Gamma_2 \). Since \( \Gamma f \circ \Gamma_1 \) is the union of at most two \( G \)-orbits in \( \Omega \times \Omega \), we have \( \Gamma f \circ \Gamma_1 = \Gamma f \circ \Gamma_2 \). By the assumption of this lemma, \( |(\Gamma f \circ \Gamma_i)(\alpha)| = |\Gamma f(\alpha)| \cdot |\Gamma_i(\alpha)| = v_2 \). So

\[
v_2 = |\Gamma f \circ \Gamma_i(\alpha)| = |\Gamma f(\alpha)| + |\Gamma f \circ \Gamma_i(\alpha)| = \frac{2v(v - 1)}{k_i},
\]

\[
v_i k_i = 2(v - 1).
\]

Therefore, \( v_i = 2 \). All of the suborbits of the primitive group with a subdegree 2 are self-paired. This is contrary to the assumption of this Lemma.

**Lemma 8.** Let \( \Gamma f \circ \Gamma_2 \) be the union of two \( G \)-orbits \( \Sigma_1 \) and \( \Sigma_2 \). We set \( v = v_1 = v_2 \), \( S_i = C(\Sigma_i) \), \( s_i = |\Sigma_i(\alpha)| \), \( i = 1, 2 \), and \( C f C_2 = a_1S_1 + a_2S_2 \). Then we have
i) \( s_1, s_2 \geq v \). If \( s_1 = v \), \( G_a \) is double transitive on \( \Sigma_1(\alpha) \)

ii) \( v^2 = a_1s_1 + a_2s_2 \)

iii) \( \Gamma_1 \circ \Gamma^*_1 \neq \Gamma_2 \circ \Gamma^*_2 \) if and only if \( a_1 = a_2 = 1 \)

iv) If \( s_1 = v(v-1) \), then \( \Gamma_1 \circ \Gamma^*_1 \neq \Gamma_2 \circ \Gamma^*_2 \) and \( \Gamma^*_1 \circ \Gamma_2 \) contains some \( \Gamma_i \)

Proof. i) Assume \( s_1 \leq v \). Then \( (\pi_1, \pi(\Sigma_1)) = 1 \) or 2 according as \( \pi(\Sigma_1) \neq \pi(\Sigma_2) \) where \( \pi(\Sigma_i) \) is the permutation character of \( G_a \) on \( \Sigma_i(\alpha) \). If \( \pi(\Sigma_1) \neq \pi(\Sigma_2) \), for \( \delta \in \Sigma_1(\alpha) \), \( G_{a, \delta} \) is transitive on \( \Gamma^*_1(\alpha) \). Thus \( \Gamma^*_1(\alpha) = \Gamma^*_2(\delta) \). Therefore \( G_a = G_{(\Gamma^*_1(\alpha))} = G_{(\Gamma^*_2(\delta))} = G_\delta \). This is impossible. So we have \( \pi(\Sigma_1) = \pi(\Sigma_2) \), and hence, \( s_1 = v \) and \( G_a \) is doubly transitive on \( \Sigma_1(\alpha) \).

ii) For the matrix \( F \) such that any entry is 1, we have

\[
F(C_1^*C_2) = v^2F \quad \text{and} \quad F(a_1S_1 + a_2S_2) = (a_1s_1 + a_2s_2)F,
\]

so

\[
v^2 = a_1s_1 + a_2s_2.
\]

iii) The existence of the following figure is equivalent to \( \Gamma_1 \circ \Gamma^*_1 = \Gamma_2 \circ \Gamma^*_2 \).

\[
\begin{array}{c}
\Gamma_1 \\
\downarrow \\
\Gamma_{12} = \Gamma_2 \\
\uparrow \\
\Gamma_1 \\
\end{array}
\]

It holds also that the figure exists if and only if \( a_i \geq 2 \) for \( i = 1 \) or 2.

iv) By ii), \( v^2 = a_1s_1 + a_2s_2 \). Since \( s_2 \geq v \), \( a_1 = a_2 = 1 \) and \( s_2 = v \). Therefore we conclude that \( \Gamma_1 \circ \Gamma^*_1 \) contains some \( \Gamma_1 \) by i), and \( \Gamma_1 \circ \Gamma^*_1 \neq \Gamma_2 \circ \Gamma^*_2 \) by iii).

Lemma 9. If \( \pi_1 = \pi_2 \), \( G_a \) is not doubly transitive on \( \Gamma^*_1 \circ \Gamma_2(\alpha) \).

Proof. Assume that \( G_a \) is doubly transitive on \( \Gamma^*_1 \circ \Gamma_2(\alpha) \). If \( |\Gamma^*_1 \circ \Gamma_2(\alpha)| \neq |\Gamma_1(\alpha)| \), then \( G_a \) has different permutation characters on \( \Gamma^*_1(\alpha) \) and \( \Gamma^*_2 \circ \Gamma_2(\alpha) \).

Hence, for \( (\alpha, \gamma) \in \Gamma^*_1(\alpha) \), \( G_{a, \gamma} \) is transitive on \( \Gamma^*_2 \circ \Gamma_2(\alpha) \), so \( \Gamma_2(\gamma) = \Gamma^*_2 \circ \Gamma_2(\alpha) \).

Therefore \( G_a = G_{(\Gamma_2(\gamma))} = G_{(\Gamma^*_2 \circ \Gamma_2(\alpha))} = G_\delta \). This is impossible. Thus, we obtain \( |\Gamma^*_1 \circ \Gamma_1(\alpha)| = |\Gamma^*_1 \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \). On the other hand, for \( (\delta, \gamma) \in \Gamma^*_1 \), \( \Gamma_1(\gamma) \leq \Gamma^*_1 \circ \Gamma_2(\alpha) \). So, \( \Gamma^*_1 \circ \Gamma_2(\alpha) = \Gamma_1(\gamma) \). This is also impossible.

Lemma 10. Assume \( \Gamma_1 \circ \Gamma^*_1 = \Gamma_2 \circ \Gamma^*_2 \) and \( \Gamma^*_1 \circ \Gamma_2 \) be the union of two \( G \)-orbits \( \Sigma_a \) and \( \Sigma^a_1 \); put \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v \), \( |\Gamma^*_1 \circ \Gamma(\alpha)| = v(v-1) \), \( |\Sigma^a_1(\alpha)| = s_i \), \( i=1, 2 \); and \( |\Gamma_2(\gamma) \cap \Sigma^a_2(\alpha)| = t \) for \( \gamma \in \Gamma^*_1(\alpha) \). Then, we have the following quadratic equation for \( t \)

\[
\frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1) = 0.
\]

Particular, i) when \( s_1 \leq \frac{v(v-1)}{k} \), the quadratic equation has at most one root for
0 < t < v; ii) when t = 1, then \( s_2 = v, s_1 = \frac{v(v-1)}{(k+1)} \) and \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \).

Proof. For \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma(f(\alpha)) \), counting arguments show that

\[
|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_2(\alpha)| = \frac{(v-t)(v(t-v)-s_1)}{(v-1)s_1},
\]

\[
|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_2(\alpha)| = \frac{t(vt-s_2)}{(v-1)s_2},
\]

so

\[
k = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{(v-t)(v(t-v)-s_1)}{(v-1)s_1} + \frac{t(vt-s_2)}{(v-1)s_2},
\]

\[
(v-1)k = \frac{v(v+t)^2}{s_1} - (v-t) + \frac{vt^2}{s_2} - t = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v,
\]

\[
0 = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).
\]

We shall prove the latter assertions. We put

\[
f(t) = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).
\]

When \( s_1 \geq \frac{v(v-1)}{k} \), then \( f(0) < 0 \). Since the coefficient of \( t^2 \) in \( f(t) \) is positive, \( f(t) \) has at most one root for \( 0 < t < v \). When \( t = 1 \), then \( s_2 = v \). By Lemma 8,

i) \( s_2 \geq v \). So \( s_2 = v \), and hence, \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \), and \( s_1 = \frac{v(v-1)}{(k+1)} \).

**Lemma 11.** Let \( \Gamma_1 \circ \Gamma_2 \) be the union of two \( G \)-orbits \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \), and \( G_\alpha \) doubly transitive on \( \Sigma_2(\alpha) \) and \( \Sigma_2(\alpha) \), then \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| \leq 3 \).

Proof. This lemma due to P. J. Cameron. ([3], Lemma 4.) We put \( |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v \), and assume \( |\Sigma_1(\alpha)| = d \). Then, \( G_\alpha \) has the different permutation characters on \( \Gamma_1(\alpha) \) and \( \Sigma_1(\alpha) \), so, for \((\alpha, \delta) \in \Sigma_1 \), \( G_{\alpha, \delta} \) is transitive on \( \Gamma_1(\alpha) \). Hence, \( \Gamma_1(\alpha) = \Gamma_2(\delta) \). Therefore, \( G_{\alpha} = G_{\Gamma_1(\alpha)} = G_{\Gamma_2(\delta)} = G_\delta \). This is impossible. Thus we conclude that \( |\Sigma_1(\alpha)| = v \). In the same way, we have \( |\Sigma_2(\alpha)| = v \).

Now, if \( \Gamma_1 \circ \Gamma_2 \neq \Gamma_2 \circ \Gamma_2 \), then by Lemma 5 \( |\Gamma_2(\alpha)| = |\Gamma_1(\alpha)| = |\Gamma_1(\alpha)| = v^2 \). Therefore, \( v^2 = |\Gamma_1(\alpha)| = |\Sigma_1(\alpha)| + |\Sigma_2(\alpha)| = 2v \), so \( v = 2 \). Thus, when \( v > 2 \), we obtain that \( \Gamma_1 \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2 \). For \( \gamma \in \Gamma_1(\alpha) \), we put \( t = |\Gamma_1(\gamma) \cap \Sigma_2(\alpha)| \). Then for \( (\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_2 \), by Lemma 10 we have the following equation
\[ k_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{1}{v-1} \{(v-t)^2 + t^2 - v\} \]
\[ = v - \frac{2t(v-t)}{v-1}. \]
If \( t = \frac{v}{2} \), \( |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = v + \frac{v^2}{2(v-1)} \) is not integer, so \( t \leq \frac{v-1}{2} \) or \( t \geq \frac{v+1}{2} \).

Hence \( k_2 = v - \frac{2t(v-t)}{v-1} \leq v - \frac{1}{2} (v+1) = \frac{1}{2} (v-1) \). But \( k_2 \leq \frac{1}{2} (v-1) \) by Lemma 3, so equality holds, and thus \( v = 3 \) or 5 by Lemma 3, and \( t = \frac{1}{2} (v+1) \) or \( \frac{1}{2} (v-1) \). Counting arguments show that \( |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \sum_1(\alpha)| = \frac{t(t-1)}{v-1} \) for \( \gamma_1, \gamma_2(\pm) \in \Gamma^*(\alpha) \). Therefore \( v-1 \) divides \( t(t-1) \); this excludes \( v = 5 \), and so \( v = 3 \).

**Lemma 12.** For \( \Gamma_1, \Gamma_2, \Gamma_3 \), if \( \sum \) is a \( G \)-orbit contained in \( \Gamma_1^* \circ \Gamma_2 \cap \Gamma_3^* \circ \Gamma_3 \), and \( |\Gamma_1(\alpha)| > 3 \); then \( G_* \) is not doubly transitive on \( \sum(\alpha) \).

Proof. \( \sum^* \circ \Gamma^* \supset \Gamma^*_2 \cup \Gamma^*_3 \). If \( G_* \) is doubly transitive on \( \sum(\alpha) \), \( \sum^* \circ \Gamma^*_1 \)

is the union of at most two \( G \)-orbits by Lemma 6, so \( \sum^* \circ \Gamma^*_1 = \Gamma^*_2 \cup \Gamma^*_3 \). This is contrary to Lemma 11.

**Lemma 13.** If \( \Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^*_2 \) and \( \pi_1 \neq \pi_2 \) then, \( |v_1 - v_2| \geq 2 \), and \( |\Gamma_1 \circ \Gamma^*(\alpha)| > |\Gamma^*_2 \circ \Gamma^*_d(\alpha)| \).

Proof. For \( (\alpha, \delta) \in \Gamma^*_2 \circ \Gamma_2 \), we put
\[ m = |\Gamma^*_d(\alpha) \cap \Gamma_d^*(\delta)|. \]
Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) with \( \gamma_1 \neq \gamma_2 \) whose edges are successively \( \Gamma^*_1, \Gamma_2, \Gamma^*_2 \), and \( \Gamma_1 \); then we have
\[ |\Omega| \frac{v_2(v_2-1)}{k_2} k_2 k_1 = |\Omega| \frac{\gamma_1 \gamma_2}{m} m(m-1), \]
so
\[ (v_2-1)k_1 = v_1(m-1). \] (1)

If \( v_1 = v_2 \), then \( k_1 = v_1 \). This is impossible. If \( v_1 = v_2 + 1 \), then \( k_1 \geq \frac{v_1}{2} \), and hence, by Lemma 3 \( v_1 = 2 \), \( v_2 = 1 \). This is also impossible. Thus we can conclude that \( |v_1 - v_2| \geq 2 \).

Assume \( |\Gamma_1 \circ \Gamma^*(\alpha)| = \frac{v_1(v_1-1)}{k_1} = |\Gamma^*_2 \circ \Gamma^*_d(\alpha)| = \frac{v_2 v_2}{m} \). Then
\[ k_1 v_2 \geq m(v_1 - 1). \] (2)
From \( \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* \), we have also
\[
k_2v_1 \geq m(v_2 - 1) \quad (3)
\]
Therefore, (1) and (2) yield
\[
v_1 \leq k_1 + m \quad (4)
\]
By Lemma 3 and (3), we have
\[
2v_2 \geq \frac{v_2(v_2 - 1)}{k_2} \geq \frac{v_1v_2}{m},
\]
so
\[
2 \leq m \leq \frac{v_1}{2} \quad (5)
\]
Thus (4) and (5) yield
\[
k_1 \geq \frac{1}{2} v_1.
\]
This is contrary to Lemma 3.

**Lemma 14.** (P. J. Cameron [3]) If \( \Gamma_1 \circ \Gamma_1^* = \Gamma_1 \circ \Gamma_2^* \), then \( \Gamma_1 \circ \Gamma_2^* \neq \Gamma_2 \circ \Gamma_2^* \).

**Proof.** We shall prove this lemma in a different way from P. J. Cameron's. Assume \( \Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \circ \Gamma_2^* \). We put
\[
|\Gamma_1 \circ \Gamma_2^*(\alpha)| = \frac{v_1(v_1 - 1)}{k_1} = |\Gamma_2 \circ \Gamma_2^*(\alpha)| = \frac{v_2(v_2 - 1)}{k_2} = |\Gamma_1 \circ \Gamma_2^*(\alpha)| = \frac{v_1v_2}{m},
\]
where \( m = |\Gamma_1(\alpha) \cap \Gamma_2(\delta)| \) for \( (\alpha, \delta) \in \Gamma_1 \circ \Gamma_2^* \). Then it is trivial that \( m > 1 \) from the above formula, and hence, \( \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \circ \Gamma_2^* \). Thus, by Lemma 13, \( |\Gamma_1 \circ \Gamma_2^*(\alpha)| < |\Gamma_2 \circ \Gamma_3^*(\alpha)| = |\Gamma_1 \circ \Gamma_2^*(\alpha)| \). This is contrary to assumption.

Now we shall investigate from Lemma 15 to Lemma 22 the necessary condition that the intersection of \( \Gamma_1 \circ \Gamma_2 \) and \( \Gamma_2 \circ \Gamma_3 \) for \( \Gamma_1, \Gamma_2, \Gamma_3 (\neq 0) \) is not empty.

**Lemma 15.** If \( \pi_1 = \pi_2 = \pi_3 \) and \( \pi_2^* = \pi_3^* \), or \( \pi_1 = \pi_2 = \pi_3 \) and \( \pi_2^* = \pi_3^* \), then \( \Gamma_1 \circ \Gamma_2 \cap \Gamma_2 \circ \Gamma_3 \neq 0 \).

**Proof.** Assume \( \pi_1 = \pi_2 = \pi_3 \) and \( \pi_2^* = \pi_3^* \). Then we have \( v_1 = v_2 = v_3 \). We put \( v = v_1 = v_2 = v_3 \). By Lemma 13, \( \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* \), and hence, \( |\Gamma_1 \circ \Gamma_3^*(\alpha)| = |\Gamma_2 \circ \Gamma_3^*(\alpha)| = v^2 \) by Lemma 5. If \( \Gamma_1 \circ \Gamma_2 \cap \Gamma_2 \circ \Gamma_3 = 0 \), then since \( \Gamma_2 \circ \Gamma_3 \) is a \( G \)-orbit and \( \Gamma_2 \circ \Gamma_3 \) is a union of two \( G \)-orbits, we have \( \Gamma_1 \circ \Gamma_2 \supseteq \Gamma_2 \circ \Gamma_3 \). Therefore \( |\Gamma_1 \circ \Gamma_2(\alpha)| > |\Gamma_2 \circ \Gamma_3(\alpha)| = v^2 \). This is impossible. Similarly, we can prove the lemma for the case of \( \pi_1 = \pi_2 = \pi_3 \) and \( \pi_2^* = \pi_3^* \).
Lemma 16. If $\pi_1^* = \pi_2^*$, $\pi_1^* = \pi_3^*$ and $\pi_2^* = \pi_3^*$, then $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_2^* = \emptyset$.

Proof. By the assumption, $\Gamma_1 \circ \Gamma_2^*$, $\Gamma_1 \circ \Gamma_2^*$ and $\Gamma_3^* \circ \Gamma_3^*$ are $G$-orbits. Assume $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_3^*$, we put

$$|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = m_2 \quad \text{and} \quad |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| = m_3.$$ 

For $\gamma_1, \gamma_2 (\neq) \in \Gamma_1(\alpha)$, we put

$$|\Gamma_3^*(\gamma_1) \cap \Gamma_3^*(\gamma_2)| = x.$$ 

Then, since $\Gamma_3^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3$, we have

$$v_1(v_1 - 1) = |\Gamma_3^* \circ \Gamma_1(\alpha)| = |\Gamma_3^* \circ \Gamma_3(\alpha)| = \frac{v_2v_3}{x},$$

so

$$v_1(v_1 - 1) = v_2v_3k_1. \quad (1)$$

Count in two ways quadrilaterals $(\alpha, \gamma', \delta, \gamma)$ whose edges are successively $\Gamma_1$, $\Gamma_2^*$, $\Gamma_3$ and $\Gamma_2^*$, then we have

$$|\Omega| \frac{v_1(v_1 - 1)}{k_1}k_1x = |\Omega| \frac{v_1v_3}{m_2}m_3,$$

so

$$(v_1 - 1)x = v_2m_2. \quad (2)$$

(1) and (2) yield

$$v_1m_2 = k_1v_2. \quad (3)$$

If $m_2 > 1$, there exist quadrilaterals $(\alpha, \beta_1, \delta, \beta_2)$ whose edges are successively $\Gamma_1$, $\Gamma_2^*$, $\Gamma_2$ and $\Gamma_2^*$, whose vertices are all distinct; count all of them in two ways, we have

$$|\Omega| \frac{v_1(v_1 - 1)}{k_1}k_1k_2 = |\Omega| \frac{v_1v_2}{m_2}m_2(m_2 - 1),$$

so

$$(v_1 - 1)k_2 = v_2(m_2 - 1).$$

On the other hand, from $\Gamma_2^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3^*$,

$$v_2(v_2 - 1)k_1 = v_1(v_1 - 1)k_2 = v_1v_2(m_2 - 1),$$

so

$$v_1(m_2 - 1) = (v_2 - 1)k_1.$$

(3) and (4) yield

$$v_1 = k_1.$$
This is contrary to Lemma 3. Thus, we have $m_2=m_3=1$ and $v_1=k_1v_2$. For $(\alpha, \gamma) \in \Gamma_1$, $G_{\alpha, \gamma}$ is transitive on $\Gamma_1(\alpha) \setminus \{\gamma\}$ and since $\pi_1^\ast = \pi_2^\ast$, it is also transitive on $\Gamma_\gamma^\ast(\gamma)$. Count in two ways $(\gamma', \delta)$ such that $\gamma' \in \Gamma_1(\alpha) \setminus \{\gamma\}$, $\delta \in \Gamma_\gamma^\ast(\gamma)$ and $(\gamma', \delta) \in \Gamma_\gamma^\ast$, then we have

$$(v_1-1)x = v_2 = \frac{v_1}{k_1}.$$  

This is impossible.

**Lemma 17.** If $\pi_1 \neq \pi_2$, $\pi_1 \neq \pi_3$ and $\Gamma_1 \circ \Gamma_\pi^\ast = \Gamma_2 \circ \Gamma_\pi^\ast$, then $\Gamma_\pi^\ast \circ \Gamma_1 \cap \Gamma_\pi^\ast \circ \Gamma_2 = \emptyset$.

**Proof.** Assume $\Gamma_\pi^\ast \circ \Gamma_1 = \Gamma_\pi^\ast \circ \Gamma_2$. By Lemma 16, $\pi_1^\ast = \pi_2^\ast$. We put $v=v_1$, $w=v_2=v_3$, $m=|\Gamma_\pi^\ast(\alpha) \cap \Gamma_\gamma^\ast(\delta)| = |\Gamma_\pi^\ast(\alpha) \cap \Gamma_\gamma^\ast(\delta)| > 1$ for $(\alpha, \delta) \in \Gamma_\gamma^\ast \circ \Gamma_2$, and $x=|\Gamma_1(\gamma_1) \cap \Gamma_1(\gamma_2)|$ for $\gamma_1, \gamma_2(\neq) \in \Gamma_\gamma^\ast(\alpha)$.

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma_1^\ast, \Gamma_2, \Gamma_\gamma^\ast$ and $\Gamma_1$; then we have

$$|\Omega| \frac{\nu(v-1)}{k_1}x = |\Omega| \frac{vw}{m} mm ,$$

so

$$(v-1)x = wv . \quad (1)$$

Next, count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma_1^\ast, \Gamma_2, \Gamma_\gamma^\ast, \Gamma_1$ and whose vertices are all distinct; then

$$|\Omega| \frac{\nu(v-1)}{k_1}k_2 = |\Omega| \frac{vw}{m} m(m-1) ,$$

$$(v-1)k_2 = w(m-1) . \quad (2)$$

(1) and (2) yield

$$(v-1)(x-k_2) = w , \text{ that is, } x \geq k_2 \geq 1 . \quad (3)$$

Since $x \geq 2$, there exist quadrilaterals $(\gamma, \delta_1, \gamma', \delta_2)$ whose edges successively $\Gamma_3, \Gamma_\gamma^\ast, \Gamma_2$ and $\Gamma_\delta^\ast$, whose vertices are all distinct, and $(\gamma, \gamma') \in \Gamma_1 \circ \Gamma_\gamma^\ast = \Gamma_2 \circ \Gamma_\gamma^\ast$; count all of them in two ways, then

$$|\Omega| \frac{\nu(w-1)}{k_2} = |\Omega| \frac{\nu(w-1)}{k_2} x(x-1) ,$$

$$(\lambda = |\Gamma_\gamma^\ast(\delta_1) \cap \Gamma_\gamma^\ast(\delta_2) \cap \Gamma_1 \circ \Gamma_\gamma^\ast(\gamma)| \text{ for } \delta_1, \delta_2 (\neq) \in \Gamma_3(\gamma))$$

so

$$\lambda = \frac{x(x-1)}{k_2} .$$
By the definition of $\lambda$, $\lambda \leq k_2$. On the other hand, since $x > k_2$, $\lambda = \frac{x(x-1)}{k_2} > k_2$. This is a contradiction.

**Lemma 18.** If $\pi_1^* = \pi_2^*$, $\pi_3^* = \pi_4^*$ and $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$, then $C_1 C_2^* = C_1 C_3^*$.

Proof. By Lemma 6, $\sum = \Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$ is a $G$-orbit. Let $S = C(\sum)$, $C_1 C_2^* = m_2 S$, $C_1 C_3^* = m_3 S$ and $|\sum(\alpha)| = s$.

For the matrix $F$ such that the value of any entry is 1, we have

$$v_1 v_2 F = F(C_1 C_2^*) = F(m_2 S) = m_2 F,$$

so

$$v_1 v_2 = m_2 .$$

Similarly

$$v_1 v_2 = m_3 .$$

On the other hand, by Lemma 16, $\pi_2 = \pi_3$, and hence, $v_2 = v_3$. So, $m_2 = m_3$.

Thus we can conclude that $C_1 C_2^* = C_1 C_3^*$.

**Lemma 19.** If $C_1 C_2^* = C_1 C_3^*$ and $|\Gamma_1(\alpha)| = v_1 > 3$, then we have

i) $\pi_2 = \pi_3$, $\pi_1^* \neq \pi_2^*$, $\pi_3^*$.

ii) $\Gamma_1^* \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_3 \circ \Gamma_3$.

iii) $v_1 = v_2 = v_3 = v_4 = 1$, $|\Gamma_1^*(\gamma_1) \cap \Gamma_2^*(\gamma_2)| = 1$ for $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$.

iv) $|\Gamma_1^* \circ \Gamma_1(\alpha)| = \frac{v_1(v_1-1)}{2}$.

Proof. By the assumption $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For the matrix $F$ such that the value of any entry is 1, we have

$$F(C_1 C_2^*) = (FC_1) C_2^* = (v_1 F) C_2^* = v_1 (FC_2) = v_1 v_2 F .$$

Similarly

$$F(C_1 C_3^*) = v_1 v_3 F .$$

So

$$v_2 = v_3 .$$

We shall show that $v_1 = v_2 = v_3$. Assume $v = v_1 = v_2 = v_3$ and put $D = C(\Gamma_1 \circ \Gamma_1)$. If $\Gamma_1^* \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2$, then $|\Gamma_1 \circ \Gamma_2(\alpha)| = |\Gamma_1 \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_3(\alpha)|$, therefore $\Gamma_1^* \circ \Gamma_1 = \Gamma_2 \circ \Gamma_3$ by Lemma 5. We put $k = k_1 = k_2 = k_3$.

$$C_1^* (C_1 C_2^*) = (C_1^* C_1) C_2^* = (v E + kD) C_2^* = v C_2^* + k(v-1) C_2^* + \text{terms not involving } C_3^* .$$

Similarly
C^+(C_1C_2^+) = vC_1^+ + k(v - 1)C_2^+ + \text{terms not involving } C_2^+.

So

\( (vE + kD)C_2^+ = \{v + k(v - 1)\} C_2^+ + \text{terms not involving } C_2^+ \).

Since the coefficients of the basis matrices in \( DC_2^+ \) are at most \( v \), the above formula is impossible.

Next, if \( \Gamma^+_2 \circ \Gamma_1 \neq \Gamma^+_3 \circ \Gamma_3 \), then \( \Gamma^+_2 \circ \Gamma_1 \neq \Gamma^+_3 \circ \Gamma_3 \), and \( DC_2^+ \) does not involve \( C_2^+ \).

Now

\[ C^+(C_1C_2^+) = (C_1C_2^+)C_2^+ = (vE + kD)C_2^+, \]

\[ C^+(C_1C_2^+) = (C_1C_2^+)C_2^+ = (vE + kD)C_2^+ = vC_2^+ + \text{terms not involving } C_2^+, \]

and hence, \( k_1DC_2^+ = vC_2^+ + \text{terms not involving } C_2^+. \)

For \( (\gamma_1, \gamma_2) \in \Gamma^+_2 \circ \Gamma_1 \) and \( (\gamma_1, \delta) \in \Gamma^+_3 \), we put

\[ x = |\Gamma^+_2(\gamma_1) \cap \Gamma^+_3(\gamma_2)| \quad \text{and} \quad t = |\Gamma^+_2 \circ \Gamma_1(\gamma_1) \cap \Gamma_3(\delta)|. \]

Then from the above formula we have

\[ t = \frac{v}{k_1}. \quad (1) \]

Counting in two ways triplilaterals \( (\gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma^+_2, \Gamma_2 \) and \( \Gamma^+_2 \circ \Gamma_1 \), we have

\[ \frac{v(v - 1)}{k_1} x = vt. \]

(1) and (2) yield

\[ (v - 1)x = v, \]

which is a contradiction. Thus we can conclude that \( v_1 + v_2 = v_3 \), and hence, \( \pi^+_2 + \pi^+_3 = \pi^+_3 \). Therefore, we obtain \( \pi_2 = \pi_3 \) by Lemma 16, \( \Gamma^+_2 \circ \Gamma_2 \neq \Gamma^+_2 \circ \Gamma_1 \neq \Gamma^+_3 \circ \Gamma_3 \) by Lemma 17, and hence we have i) and ii) of Lemma.

For \( (\alpha, \gamma) \in \Gamma_1 \), count in two ways the ordered pairs \( (\gamma', \delta) \) such that \( \gamma' \in \Gamma_1(\alpha) \setminus \{\gamma\} \), \( \delta \in \Gamma^+_2(\gamma) \) and \( (\gamma', \delta) \in \Gamma^+_2 \); then since \( \Gamma^+_2 \circ \Gamma_1 \neq \Gamma^+_3 \circ \Gamma_3 \) we have

\[ (v_1 - 1)x = v_2. \quad (3) \]

Now, we shall show that \( x = 1 \). Assume \( x > 1 \), then there exist quadrilaterals \( (\gamma, \delta_1, \gamma', \delta_2) \) whose edges are successively \( \Gamma^+_2, \Gamma_2, \Gamma^+_3 \) and \( \Gamma_2 \) whose edges are all distinct, and \( (\gamma, \gamma') \in \Gamma^+_2 \circ \Gamma_1 \); count all of them in two ways, then we have

\[ |\Omega|v_2(v_2 - 1) = |\Omega|v_1(v_1 - 1)\frac{x(x - 1)}{k_1}, \]

\[ (\lambda = |\Gamma^+_2 \circ \Gamma_1(\gamma) \cap \Gamma_2(\delta_1) \cap \Gamma_3(\delta_2)| \text{ for } (\gamma', \delta_1, \gamma, \delta_2)(\pm) \in \Gamma^+_2, (\delta_1, \delta_2) \in \Gamma_2 \circ \Gamma^+_2). \]
so

\[(v_2 - 1)\lambda k_1 = v_1(x - 1) = (v_1 - 1) x + x - v_1 = v_2 + x - v_1.\]

Therefore, \(x \geq v_1 - 1\). If \(x = v_1\), then \((v_2 - 1)\lambda k_1 = v_2\), which is a contradiction. If \(x > v_1\), then \(v_2 = (v_1 - 1) x > v_1(v_1 - 1)\). So \((\pi_1^\tau, \pi(\Gamma^* \circ \Gamma_1(\gamma)))_{G_1} = 1\), where \(\pi(\Gamma^* \circ \Gamma_1(\gamma))\) is the permutation character of \(G_1\) on \(\Gamma^* \circ \Gamma_1(\gamma)\). Hence, for \((\gamma, \gamma') \in \Gamma^* \circ \Gamma_1(\gamma), G_{\gamma, \gamma'}\) is transitive on \(\Gamma^* (\gamma)\). So \(\Gamma^* (\gamma) = \Gamma^* (\gamma')\). This is impossible.

Thus we have \(x = v_1 - 1, k_1 = \lambda = 1, v_2 = (v_1 - 1)^2\) and \(|\Gamma^* \circ \Gamma_1(\gamma) \cap \Gamma_2(\delta)| = v_1\) for \((\gamma, \delta) \in \Gamma^*\).

Now, count in two ways quadrilaterals \((\alpha, \gamma_1, \gamma_2, \gamma_3)\) such that \((\alpha, \gamma_1) \in \Gamma_2, (\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3\), and \((\gamma_1, \gamma_2), (\gamma_1, \gamma_3), (\gamma_2, \gamma_3) \in \Gamma^* \circ \Gamma_1, \gamma_2 \neq \gamma_3\); then we have

\[|\Omega| v_1(v_1 - 1)\nu' = |\Omega| v_2v_1(v_1 - 1),\]

\[(\lambda' = |\Gamma^* \circ \Gamma_1(\gamma_2) \cap \Gamma^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)| \text{ for } \gamma_2, \gamma_3 (\neq) \in \Gamma_3(\alpha))\]

so

\[\lambda' = \frac{v_1(v_1 - 1)}{v_1(v_1 - 1)^2 - 1} = \frac{v_1 - 1}{v_1 - 2}.\]

Therefore, \(v_1 = 3\). This is contrary to the hypothesis of Lemma. Thus we can conclude that \(x = 1\), and hence, by (3) we have \(v_1 = v_2 + 1 = v_3 + 1\). This proves Lemma iii).

Lastly, we shall show that \(k_1 = 2\). If \(k_1 = 1\), then \(|\Gamma^* \circ \Gamma_1(\alpha)| = v_1(v_1 - 1) \leq |\Gamma^* \circ \Gamma_1(\delta)| \leq v_2v_1 = (v_1 - 1)^2\) This is impossible. Now, we have

\[u = |\Gamma^* \circ \Gamma_1(\gamma) \cap \Gamma_2(\delta)| = \frac{v_1}{k_1} \text{ for } (\gamma, \delta) \in \Gamma^*\text{ and } 2 \leq k_1 < \frac{v_1}{2}.\]

Count again in two ways quadrilaterals \((\alpha, \gamma_1, \gamma_2, \gamma_3)\) such that \((\alpha, \gamma_1) \in \Gamma_2, (\alpha, \gamma_2), (\alpha, \gamma_3) \in \Gamma_3\), and \((\gamma_1, \gamma_2), (\gamma_1, \gamma_3), (\gamma_2, \gamma_3) \in \Gamma^* \circ \Gamma_1, \gamma_2 \neq \gamma_3\); then

\[|\Omega|(v_1 - 1)(v_1 - 2)\nu'' = |\Omega|(v_1 - 1)\left(\frac{v_1 - 1}{k_1}\right)\frac{v_1}{k_1},\]

\[(\lambda'' = |\Gamma^* \circ \Gamma_1(\gamma_2) \cap \Gamma^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)| \text{ for } \gamma_2, \gamma_3 (\neq) \in \Gamma_3(\alpha))\]

so

\[\lambda'' = \frac{v_1(v_1 - k_1)}{(v_1 - 2)k_1^2} = \frac{u(u - 1)k_1^2}{(k_1 - u - 2)k_1^2} = \frac{u(u - 1)}{k_1u - 2}.\]

If \(u\) is odd, then \(k_1u - 2\) divides \(u - 1\). This is impossible. We put \(u = 2u_0\), then
\[ \chi'' = \frac{2u_0(2u_0-1)}{2k_1u_0-2} = \frac{u_0(2u_0-1)}{k_1u_0-1} . \]

Therefore, we conclude that \( k_1 = 2 \).

**Lemma 20.** If \( \pi_1 = \pi_2 = \pi_3 \) and \( \Gamma^*_1 \circ \Gamma_2 \cap \Gamma^*_2 \circ \Gamma_3 \neq \emptyset \), then \( v_1 = v_2 = v_3 + 1 \), \( \Gamma_1 \circ \Gamma^*_2 \neq \Gamma_2 \circ \Gamma^*_1 \) and \( \Gamma^*_1 \circ \Gamma_2 = \Gamma^*_2 \circ \Gamma_3 \cup \Gamma_i \) for some \( \Gamma_i \).

**Proof.** By assumption, \( \sum = \Gamma^*_1 \circ \Gamma_3 \) is a \( G \)-orbit contained in \( \Gamma^*_1 \circ \Gamma_2 \). We put \( v = v_1 = v_2, w = v_3, |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = x \) for \( (\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma^*_1 \), \( |\Gamma^*_2(\alpha) \cap \Gamma^*_3(\delta)| = y \) and \( |\Gamma^*_2(\alpha) \cap \Gamma^*_3(\delta)| = m \) for \( (\alpha, \delta) \in \sum \), \( |\Gamma_2(\gamma) \cap \sum(\alpha)| = t \) for \( (\alpha, \gamma) \in \Gamma^*_2 \).

By Lemma 15, \( \pi^*_1 \neq \pi^*_2 \), and hence, \( \Gamma_2 \circ \Gamma^*_1 \) is a \( G \)-orbit. We have

\[ \frac{v(v-1)}{k_1} = |\Gamma_1 \circ \Gamma^*_1(\gamma_1)| = |\Gamma_2 \circ \Gamma^*_2(\gamma_1)| = \frac{vw}{x} , \]

so

\[ (v-1)x = wk_1 . \tag{1} \]

We have also \( |\sum(\alpha)| = \frac{vw}{m} = \frac{vt}{y} \), and so

\[ wy = tm . \tag{2} \]

Count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma^*_1 \), \( \Gamma_2 \), \( \Gamma^*_2 \) and \( \Gamma_1 \), then we have

\[ |\Omega| \frac{v(v-1)}{k_1} k_1 x = |\Omega| \frac{vw}{m} my , \]

so

\[ (v-1)x = wy . \tag{3} \]

(1) and (3) yield

\[ y = k_1 . \tag{4} \]

From (2) and (3),

\[ (v-1)x = tm . \tag{5} \]

We shall show that \( m = 1 \). If \( m > 1 \), then there exist quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma^*_1 \), \( \Gamma_3 \), \( \Gamma^*_2 \) and \( \Gamma_1 \), whose vertices are all distinct; count all of them in two ways, then we have

\[ |\Omega| \frac{w(w-1)}{k_3} k_3 k_1 = |\Omega| \frac{vw}{m} m(m-1) , \]

so

\[ (w-1)k_1 = v(m-1) . \]

On the other hand, from (3) and (4)
(w−1)k_1 = wk_1−k_1 = (v−1)x−k_1,

therefore
\[ v(m−1) = (v−1)x−k_1, \]

so
\[ 0 \leq v(x−m+1) = x+k_1 < 2v. \]  \hspace{1cm} (6)

(6) yields
\[ x = m, \quad v = m+k_1. \]  \hspace{1cm} (7)

From (5) and (7),
\[ t = v−1. \]  \hspace{1cm} (8)

Thus \[ |\Sigma(\alpha)| = \frac{v−1}{k_1}. \]

If \( \Gamma_1 \circ \Gamma_2^\# = \Gamma_2 \circ \Gamma_2^\# \), then by Lemma 10, \[ |\Sigma(\alpha)| = \frac{v(v−1)}{k_1+1}. \] This is a contradiction. So we have \( \Gamma_1 \circ \Gamma_2^\# \neq \Gamma_2 \circ \Gamma_2^\# \), and hence,
\[ 1 = y = k_1. \]  \hspace{1cm} (9)

Therefore we have \( m=v−1 \) from (7) and (9), and \( w=(v−1)^2 \) from (2) and (8). So
\[ |\Gamma_1 \circ \Gamma_2^\#(\alpha)| = |\Gamma_2 \circ \Gamma_2^\#(\alpha)| = \frac{w(w−1)}{k_1} \geq 2w = 2(v−1)^2 > v(v−1). \]

This is impossible. Thus, we can conclude that \( m=1 \), and then by (5) \( t = v−1 \), \( x=1 \) and \[ |\Sigma(\alpha)| = \frac{v(v−1)}{k_1}. \] By Lemma 10, \( \Gamma_1 \circ \Gamma_2^\# \neq \Gamma_2 \circ \Gamma_2^\# \), and hence, \( 1 = y = k_1 \). Therefore, by (2) \( w=v−1 \), \[ |\Sigma(\alpha)| = v(v−1). \] By Lemma 8 iv), \( \Gamma_1 \circ \Gamma_2 = \bigcup \Gamma_i \) for some \( \Gamma_i \).

**Lemma 21.** If \( \Gamma_1 \circ \Gamma_2 \cap \Gamma_2 \circ \Gamma_3 = \emptyset \), and \( v_1, v_2, v_3 > 3 \), then the following hold;

i) if \( \pi_1 = \pi_2 = \pi_3 \), then \( \pi_2^\# = \pi_3^\# \)

ii) if \( \pi_1 = \pi_2 \neq \pi_3 \), then \( \pi_2^\# = \pi_3^\# \) and \( v_1 = v_2 = v_3 + 1 \).

iii) if \( \pi_1 \neq \pi_2, \pi_3 \), then \( \pi_2^\# = \pi_3^\# \), \( C_1^*C_2 = C_2^*C_3 \) and \( v_1 = v_2 + 1 = v_3 + 1 \).

Proof. We have this assertion by arranging from Lemma 15 to Lemma 20.

**Lemma 22.** Suppose that \( \Gamma_1 \circ \Gamma_2 \) and \( \Gamma_1 \circ \Gamma_3 \) contain a \( G \)-orbit \( \Sigma \) in \( \Omega \times \Omega \), and \( \pi_1 = \pi_2 = \pi_3 \), \[ |\Gamma_i(\alpha)| > 3. \] For \( \gamma_1, \gamma_2 \neq \gamma_3 \), \( \Gamma_i(\alpha) \) and \( \delta \in \Sigma(\alpha) \), the following hold;

i) if \( \Gamma_1 \circ \Gamma_2^\# = \Gamma_2 \circ \Gamma_3^\# = \Gamma_3 \circ \Gamma_2^\# \), then \( |\Gamma_1^\#(\alpha) \cap \Gamma_2^\#(\delta)| > 1 \), \( |\Gamma_1^\#(\alpha) \cap \Gamma_3^\#(\delta)| > 1 \) and \( |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)| > 1 \).
ii) if $\Gamma_1 \circ \Gamma^* = \Gamma_2 \circ \Gamma^* \neq \Gamma_3 \circ \Gamma^*$, then $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\beta)| > |\Gamma^*_i(\alpha) \cap \Gamma^*_j(\beta)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$, and $\Gamma^*_i \circ \Gamma_2$ contains some $\Gamma_i$.

iii) if $\Gamma_1 \circ \Gamma^*_i \neq \Gamma_2 \circ \Gamma^*_j \neq \Gamma_3 \circ \Gamma^*$, then $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\beta)| = |\Gamma^*_i(\alpha) \cap \Gamma^*_j(\beta)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$, and $\Gamma^*_i \circ \Gamma_2$ contains some $\Gamma_i$ and $\Gamma^*_i \circ \Gamma_3$ contains another $\Gamma_j$.

Proof. Put $|\Sigma(\alpha) \cap \Gamma_i(\gamma_1) \cap \Gamma_j(\gamma_2)| = \lambda$ for $\gamma_1, \gamma_2 \in \Gamma^*_i(\alpha)$. $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = |\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = x_2$, $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = x_3$ for $(\alpha, \delta) \in \Sigma$. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma^*_i, \Gamma_2, \Gamma^*_j$ and $\Gamma_1$, and $(\alpha, \delta) \in \Sigma$, then we have

$$| \Omega | \frac{v(v-1)}{k_1+1} x_2 x_3, \quad (1)$$

so

$$v(v-1) \lambda = | \Sigma(\alpha) | x_2 x_3.$$  

Assume $\Gamma_1 \circ \Gamma^*_i \neq \Gamma_2 \circ \Gamma^*_j \neq \Gamma_3 \circ \Gamma^*$. Then we have $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = |\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = 1$. By (1)

$$v(v-1) \lambda = | \Sigma(\alpha) |.$$  

Since $| \Sigma(\alpha) | \leq v(v-1)$, we have $\lambda = 1$ and $| \Sigma(\alpha) | = v(v-1)$. By Lemma 8 iv), $\Gamma^*_i \circ \Gamma_2 = \Sigma \cup \Gamma_i$ and $\Gamma^*_i \circ \Gamma_3 = \Sigma \cup \Gamma_j$ for some $\Gamma_i, \Gamma_j$. By Lemma 8, iii), we have $C_i C_2 = S + C_i, C_i C_3 = S + C_j$. (S = S($\Sigma$)) If $C_i = C_j$, then $C_i C_2 = C_i C_3$, and hence, by Lemma 19 $\pi_1 = \pi_2, \pi_3$. This is contrary to the hypothesis of this lemma. Thus $C_i \neq C_j$, that is, $\Gamma_i \neq \Gamma_j$. So $\Sigma(\alpha) \cap \Gamma_i(\gamma_1) \cap \Gamma_2(\gamma_2) = \Sigma(\alpha) \cap \Gamma_i(\gamma_1) \cap \Gamma_3(\gamma_2) = \lambda = 1$. Thus we have iii) of Lemma.

Next assume $\Gamma_1 \circ \Gamma^*_i = \Gamma_2 \circ \Gamma^*_j \neq \Gamma_3 \circ \Gamma^*$. Then we have $|\Gamma^*_i(\alpha) \cap \Gamma^*_j(\delta)| = 1$. By (1)

$$v(v-1) \lambda = | \Sigma(\alpha) | x_2. \quad (2)$$

Count in two ways tripililaterals $(\alpha, \beta, \gamma)$ whose edges are successively $\Sigma, \Gamma^*_i$, and $\Gamma_1$ then we have

$$| \Sigma(\alpha) | x_2 \leq v(v-1). \quad (3)$$

If $x_2 = 1$, then $| \Sigma(\alpha) | = v(v-1)$ by (2) and (3). By Lemma 8. iv), $\Gamma_1 \circ \Gamma^*_i \neq \Gamma_1 \circ \Gamma^*_j$. This is contrary to the assumption. Therefore we have $x_2 > 1, \lambda = 1$ and $| \Sigma(\alpha) | x_2 = v(v-1)$. Since $| \Sigma(\alpha) | x_2 = v(v-1)$, $| \Sigma(\alpha) \cap \Gamma_2(\gamma)| = v-1$ for $(\alpha, \gamma) \in \Gamma^*_i$. By Lemma 10. ii), $| \Sigma(\alpha) | = \frac{v(v-1)}{k_1+1}$ and $\Gamma^*_i \circ \Gamma_2$ contains some $\Gamma_i$.

Now we shall show that $\Gamma_1(\gamma_1) \cap \Gamma_3(\gamma_2) = \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)$, for
\( \gamma_1, \gamma_2 \in \Gamma_f^*(\alpha) \). If \( \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \neq \Gamma_3(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha) \), then \( \Gamma_f^* \circ \Gamma_2 = \Gamma_f^* \circ \Gamma_3 \).

But \( |\Gamma_f^* \circ \Gamma_2(\alpha)| = |\sum(\alpha)| + |\Gamma_1(\alpha)| = \frac{v(v-1)}{k_1+1} + v < v^2 \) and \( |\Gamma_f^* \circ \Gamma_3(\alpha)| = v^2 \).

This is impossible. Therefore, \( |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \sum(\alpha)| = 1 \). Thus we have ii) of Lemma.

Last assume \( \Gamma_1 \circ \Gamma_f^* = \Gamma_3 \circ \Gamma_f^* = \Gamma_f^* \circ \Gamma_f^* \). We shall show that \( x_2 = |\Gamma_f^* \circ \Gamma_f^*(\alpha) \cap \Gamma_f^*(\delta)| > 1 \) and \( x_3 = |\Gamma_f^* \circ \Gamma_f^*(\alpha) \cap \Gamma_f^*(\delta)| > 1 \). We note that \( k_1 = k_2 = k_3 \), therefore we put \( k = k_1 = k_2 = k_3 \). If \( x_2 = x_3 = 1 \), by (1) we have \( |\sum(\alpha)| = v(v-1) \). By Lemma 8. iv) \( \Gamma_1 \circ \Gamma_f^* \neq \Gamma_3 \circ \Gamma_f^* \), \( \Gamma_f^* \circ \Gamma_f^* \). This is contrary to the assumption. If \( x_2 > x_3 = 1 \), we have \( |\sum(\alpha)| = \frac{v(v-1)}{k+1} \) as before, and \( x_2 = k+1 \). We put \( \Gamma_f^* \circ \Gamma_3 = \sum \cup \sum' \),

\[
x = \frac{|\Gamma_f^* \circ \Gamma_f^*(\alpha) \cap \Gamma_f^*(\delta')|}{|\Gamma_f^* \circ \Gamma_f^*(\alpha) \cap \Gamma_f^*(\delta')|} \quad \text{for} \ (\alpha, \delta') \in \sum', \quad \text{and}
\]

\[
t = \frac{|\Gamma_3(\gamma_1) \cap \sum(\alpha)|}{|\Gamma_3(\gamma_1) \cap \sum(\alpha)|} \quad \text{for} \ (\alpha, \gamma_1) \in \Gamma_f^*.
\]

Since \( \Gamma_1 \circ \Gamma_f^* = \Gamma_3 \circ \Gamma_f^* \) and \( x_3 = 1 \), there exist quadrilaterals \( (\alpha, \gamma_1, \delta', \gamma_2) \), with \( \gamma_1 \neq \gamma_2 \) and \( (\alpha, \delta') \in \sum' \), whose edges are successively \( \Gamma_f^*, \Gamma_3, \Gamma_f^* \) and \( \Gamma_1 \). Count all of them in two ways then we have

\[
|\Omega| = \frac{v(v-1)}{k} = \frac{v}{x} x(x-1),
\]

so

\[
x-1 = \frac{(v-1)k}{v-v-1} = \frac{t(k+1)k}{t(k+1)+1-t} = \frac{tk(k+1)}{tk+1}.
\]

Therefore \( t = 1 \), and hence, \( v = k+2 \). This is impossible by Lemma 3. Thus we have \( x_2 > 1 \) and \( x_3 > 1 \).

Now we shall show that \( \lambda > 1 \). If \( \lambda = 1 \), by (1) we have

\[
v(v-1) = |\sum(\alpha)| x_2 x_3
\]

Since \( x_3 > 1 \), there exist quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \), with \( \gamma_1 \neq \gamma_2 \) and \( (\alpha, \delta) \in \sum \), whose edges are successively \( \Gamma_f^*, \Gamma_2, \Gamma_f^* \) and \( \Gamma_1 \). Count all of them in two ways then we have

\[
|\Omega| = \frac{v(v-1)}{k} \lambda_2 = |\Omega| |\sum(\alpha)| x_2(x_2-1),
\]

\[
(\lambda_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \sum(\alpha)| \quad \text{for} \ (\gamma_1, \gamma_2) \in \Gamma_f^*(\alpha))
\]

so
\[ \lambda_2 = \frac{\sum (\alpha) |x_2(x_2-1)}{v(v-1)}, \]

and by (1),
\[ \lambda_2 = \frac{x_2-1}{x_3}. \]

Thus \( \frac{x_2-1}{x_3} \) is a positive integer. Since \( x_2 > 1 \), in the same way, we have that \( \frac{x_2-1}{x_2} \) is a positive integer. This is impossible. Thus we have i) of Lemma.

**Lemma 23.** If \( \Gamma_1 \circ \Gamma_f = \Gamma_2 \circ \Gamma_f \) and \( \pi_1 \neq \pi_2 \), then for any \( \Gamma_t, \Gamma_t'(\neq) \), \( \Gamma_2 \circ \Gamma_t' \supset \Gamma_1 \circ \Gamma_t' \).

**Proof.** Assume \( \Gamma_1 \circ \Gamma_f \supset \Gamma_1 \circ \Gamma_t' \). Note that \( |v_1 - v_2| \geq 2 \) by Lemma 13, and hence, \( \pi_2 = \pi_2 \). If \( \{ \Gamma_t, \Gamma_t' \} = \{ \Gamma_1, \Gamma_2 \} \), then since \( \Gamma_1 \circ \Gamma_f \) is a \( G \)-orbit, \( \Gamma_1 \circ \Gamma_t' = \Gamma_1 \circ \Gamma_t' = \Gamma_2 \circ \Gamma_f \). This is a contrary to Lemma 14. Therefore we can assume that \( \Gamma_t \neq \Gamma_t' \). If \( \Gamma_t \neg \Gamma_t' \), then \( \Gamma_2 \circ \Gamma_t \in \{ \Gamma_1, \Gamma_t' \} \). By Lemma 21 have \( v_2 = v_1 - 1 \). This is a contradiction. Thus we have \( \{ \Gamma_t, \Gamma_t' \} \in \{ \Gamma_1, \Gamma_t' \} \).

From \( v_1 = v_2 \), we may assume \( v_1 = v_1 \). Since \( \Gamma_1 \circ \Gamma_t \in \{ \Gamma_1, \Gamma_1' \} \), \( v_1 = v_1 - 1 \) by Lemma 21. On the other hand, from \( |v_1 - v_2| \geq 2 \), \( v_1 \neq v_2 \). Since \( \Gamma_2 \circ \Gamma_t \in \{ \Gamma_1, \Gamma_t' \} \), \( v_1 = v_2 - 1 \). This is a contradiction.

**Lemma 24.** If \( \Gamma_1 \circ \Gamma_f = \Gamma_2 \circ \Gamma_f = \Gamma_3 \circ \Gamma_f = \Delta \), \( \pi_1 = \pi_2 = \pi_3 \) and \( |\Gamma_1(\alpha)| > 3 \), then \( \Gamma_1 \circ \Gamma_f \supset \Delta \) or \( \Gamma_1 \circ \Gamma_f \neq \Delta \).

**Proof.** Assume \( \Gamma_1 \circ \Gamma_f \supset \Delta \) and \( \Gamma_1 \circ \Gamma_f \supset \Delta \). We put \( v = v_1 = v_2 = v_3 \) and \( k = k_1 = k_2 = k_3 \). Since \( \pi_1 = \pi_2 = \pi_3 \), we have \( \pi_1 = \pi_2 = \pi_3 \) by Lemma 21. We shall show that \( \Gamma_f \circ \Gamma_t \in \{ \Gamma_1, \Gamma_t' \} \). If \( \Gamma_1 \circ \Gamma_f \neq \Gamma_2 \circ \Gamma_f \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 22, ii). Since \( |\Gamma_1(\alpha)| = |\Gamma_1(\alpha)| = |\Delta(\alpha)| = v(v-1) \), we have \( \Gamma_2 \circ \Gamma_f \supset \Gamma_3 \circ \Gamma_f \) by Lemma 8, iv). If \( \Gamma_f \circ \Gamma_t = \Gamma_f \circ \Gamma_t' = \Gamma_3 \circ \Gamma_f \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 22, ii). This is impossible. Thus we can conclude that \( k > 1 \). If \( k = 1 \), \( |\Gamma_1(\alpha)| = \frac{v(v-1)}{k} = v(v-1) \). Since \( \Gamma_f \circ \Gamma_3 \supset \Gamma_f \circ \Gamma_3, \Gamma_2 \circ \Gamma_f \supset \Gamma_3 \circ \Gamma_f \) by Lemma 8, iv). This is contrary to the assumption. Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_f \); then we have
\[
| \Omega | \frac{v(v-1)}{k} x_2 x_3,
\]
Here we put \( x_2 = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| \), \( x_3 = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| \) for \((\alpha, \delta) \in \Delta\) and \( x = \Gamma_\phi(\gamma_1) \cap \Gamma_\phi(\gamma_2) \cap \Delta(\alpha) | \) for \( \gamma_1, \gamma_2 (\neq) \in \Gamma_1(\alpha) \).

We shall show that \( x, x_2 \) and \( x_3 \) are smaller than \( k \). If \( x_2 \geq k \), then for \((\alpha, \gamma) \in \Gamma_1, \ |\Delta(\alpha) \cap \Gamma_\phi(\gamma)| \geq v-1 \). Of course, \( |\Delta(\alpha) \cap \Gamma_\phi(\gamma)| \leq v-1 \), and hence, \( |\Delta(\alpha) \cap \Gamma_\phi(\gamma)| = v-1 \). By Lemma 10, ii), we have \(|\Delta(\alpha)| = \frac{v(v-1)}{k+1}\), which is a contradiction. We can prove in the same way that \( x_3 < k \). Then, (1) yields

\[
x < x_2, \quad x_3 < k.
\]

Now

\[
(C_1 C_\phi C_3) = C_1(x D' + y S'),
\]

\[
(C_1 C_\phi) C_3 = (x D + y_2 S) C_3 = x_3 (v-1) C_3 + \text{terms not involving } C_3.
\]

\[
(\Delta' = \Gamma_\phi \circ \Gamma_1, \ \Gamma_1 \circ \Gamma_\phi = \Delta \cup \Sigma, \ \Gamma_\phi \circ \Gamma_3 = \Delta' \cup \Sigma', \ \ D = C(\Delta), \ D' = C(\Delta'), \ S = C(\Sigma) \text{ and } S' = C(\Sigma'))
\]

Since \( x_2 > x \) and the coefficient of \( C_3 \) contained in \( C_1 D' \) is at most \( v-1 \), \( C_3 \) is contained in \( C_1 S' \), that is, \( \Gamma_\phi \circ \Gamma_3 \supset \Sigma' \). On the other hand, since \( \Gamma_1 \circ \Gamma_\phi \supset \Delta \), there exists the following figure.

\[
\begin{array}{c}
\Gamma_1 \\
\Delta \\
\Gamma_3
\end{array}
\]

Therefore \( \Gamma_\phi \circ \Gamma_3 \supset \Delta' \). Thus \( \Gamma_\phi \circ \Gamma_3 = \Delta' \cap \Sigma' = \Gamma_\phi \circ \Gamma_3 \). By Lemma 10, i) we have \( C_1 C_3 = C_\phi C_3 \). So, \( \pi_1 = \pi_3 \) by Lemma 19, i). This is contrary to the hypothesis of this lemma.

**Lemma 25.** *If \( v_1, v_2, v_3 \) and \( v_4 > 3 \), then the following figures don't exist.*

Proof. For each figure above, we assume its existence and show that it implies a contradiction.

Non-existence of Fig. 1.
Case I. $\pi_1 \pm \pi_2, \pi_3, \pi_4$.

By Lemma 18 and Lemma 19, $v_1 = v_2 + 1 = v_3 + 1 = v_4 + 1$, $|\Gamma_1 \circ \Gamma_f^*(\alpha)| = \frac{v_2(v_2 - 1)}{2}$, $|\Gamma_2(\alpha) \cap \Gamma_3(\delta)| = |\Gamma_2(\alpha) \cap \Gamma_4(\delta)| = 1$ for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_f^*$ and $\pi_f^* = \pi_f^* = \pi_i^*$. Now let us consider the following figure.

$$\begin{array}{c}
\Gamma_2 \\
\Gamma_1 \circ \Gamma_f^* \\
\Gamma_3 \\
\Gamma_4
\end{array}$$

Then by Lemma 22, i) and iii), we have

$$|\Gamma_1 \circ \Gamma_f^*(\alpha)| = v_2(v_2 - 1) = (v_1 - 1)(v_1 - 2).$$

Thus,

$$|\Gamma_1 \circ \Gamma_f^*(\alpha)| = \frac{v_2(v_2 - 1)}{2} = (v_1 - 1)(v_1 - 2),$$

so

$$v_1 = 4, \quad v_2 = v_3 = v_4 = 3.$$  

This is contrary to the hypothesis of this lemma.

Case II. $\pi_1 = \pi_2 = \pi_3, \pi_4$.

By Lemma 21, $v_1 = v_2 = v_3 + 1 = v_4 + 1$ and $\pi_f^* = \pi_f^* = \pi_f^*$. But considering the following figure,

$$\begin{array}{c}
\Gamma_2 \\
\Gamma_1 \circ \Gamma_f^* \\
\Gamma_3 \\
\Gamma_4
\end{array}$$

we have $v_3 = v_2 + 1$ by Lemma 20. This is impossible.

Case III. $\pi_1 = \pi_2 = \pi_3 = \pi_4$.

By Lemma 20, $v_1 = v_2 = v_3 = v_4 + 1$. But since there exists the following figure,

$$\begin{array}{c}
\Gamma_4 \\
\Gamma_1 \circ \Gamma_f^* \\
\Gamma_2 \\
\Gamma_3
\end{array}$$
we have $v_4 = v_3 + 1 = v_2 + 1$ by Lemma 21, which is a contradiction.

Case IV. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_0 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_3^* = \Gamma_0 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$.

Existence of the following figure is contrary to Lemma 24.

Case V. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_0 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_3^* = \Gamma_0 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$.

Since $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_0 \circ \Gamma_3^*$, we have by Lemma 22, i) $|\Gamma_0(\gamma_1) \cap \Gamma_0(\gamma_2)| > 1$ for $(\gamma_1, \gamma_2) \subseteq \Gamma_0 \circ \Gamma_1^*$, and hence, $\Gamma_0 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_0 \circ \Gamma_3$. So, we have $|\Gamma_0 \circ \Gamma_1^*(\alpha)| < v_1(v_1 - 1)$ by Lemma 8, iv). On the other hand, since $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_0 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$ we have by Lemma 22, ii) $|\Gamma_0(\gamma_1) \cap \Gamma_0(\gamma_2)| = |\Gamma_0(\gamma_1) \cap \Gamma_0(\gamma_2)| = 1$ for $(\gamma_1, \gamma_2) \subseteq \Gamma_0 \circ \Gamma_1^*$. Then from the existence of the following figure,

we have $|\Gamma_0 \circ \Gamma_1^*(\alpha)| = v_1(v_1 - 1)$ by Lemma 22, which is a contradiction.

Case VI. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_0 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_3^* = \Gamma_0 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$.

There exist the following figures, where $\Sigma$ is a $G$-orbit.

From Fig. a, we have $|\Sigma(\alpha)| = v_1(v_1 - 1)$ by Lemma 22, iii). On the other hand, from Fig. b, we have $|\Sigma(\alpha)| = \frac{v_1(v_1 - 1)}{k_1 + 1}$ by Lemma 22, ii), which is a contradiction.

Case VII. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_0 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_3^* = \Gamma_0 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$.

From $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_0 \circ \Gamma_3^*$, we have $|\Gamma_0(\gamma_1) \cap \Gamma_0(\gamma_2)| = 1$ for $\gamma_1, \gamma_2 (\neq) \subseteq \Gamma_1^*(\alpha)$, by Lemma 22, iii). Similarly from $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_4 \circ \Gamma_4^*$, we have
\[ |\Gamma_{1}(\gamma_{1}) \cap \Gamma_{4}(\gamma_{2})| = 1 \text{ for } \gamma_{1}, \gamma_{2}(\neq) \in \Gamma^{*}(\alpha). \]  
From \( \Gamma_{2} \circ \Gamma^{*} \cap \Gamma_{3} \circ \Gamma^{*} \supseteq \Gamma_{4} \circ \Gamma^{*} \), we have by Lemma 22

\[ |\Gamma_{1} \circ \Gamma^{*}(\alpha)| = v_{1}(v_{1} - 1). \]  
(1)

By Lemma 21, \( \pi_{2}^{*} = \pi_{3}^{*} = \pi_{4}^{*} \). Therefore we have by Lemma 8, iv)

\[ \Gamma^{*}_{1} \circ \Gamma_{2} \neq \Gamma^{*}_{3} \circ \Gamma_{3}, \quad \Gamma^{*}_{1} \circ \Gamma_{3} = \Gamma^{*}_{4} \circ \Gamma_{4} \quad \text{and} \quad \Gamma_{1} \circ \Gamma^{*}_{4} \quad (2 \leq i, j(\neq) \leq 4) \text{ contains some } \Gamma_{k}. \]  
(2)

We put

\[ v = v_{1} = v_{2} = v_{3} = v_{4}, \quad \Gamma_{1} \circ \Gamma^{*}_{1} = \Delta_{1}, \quad \Gamma^{*}_{1} \circ \Gamma_{2} = \Delta_{2}, \]
\[ \Gamma_{2} \circ \Gamma^{*}_{2} = \Delta_{1} \cup \Gamma_{1}, \quad \Gamma^{*}_{1} \circ \Gamma_{3} = \Delta_{2} \cup \Sigma_{1}, \quad \text{and} \quad D_{1} = C(\Delta_{1}), \]
\[ D_{2} = C(\Delta_{2}), \quad S_{1} = C(\Sigma_{1}) \quad \text{and} \quad s_{1} = |\Sigma(\alpha)|. \]

Now,

\[ (C_{2} C^{*}_{1}) C_{4} = (D_{1} + C_{1}) C_{4} = (v - 1) C_{3} + \cdots. \]

The coefficient of \( C_{3} \) of the above equation is \( v - 1 \) or \( v \) by (2). Next,

\[ C_{2}(C^{*}_{1} C_{4}) = C_{2}(D_{2} + x S_{1}), \]

so

\[ v^{2} = \frac{v(v - 1) + x}{k_{2}}. \]

By Lemma 8, i), \( s_{1} \geq v \), so

\[ x \leq v - \frac{v - 1}{k_{2}} \leq v - 2. \]  
(3)

We shall show that \( \Gamma^{*}_{1} \circ \Gamma_{4} \neq \Sigma_{1} \). If \( \Gamma^{*}_{1} \circ \Gamma_{4} = \Sigma_{1} \), there exists the following figure.

\[ \begin{align*}
\Gamma_{3} & \quad \left\uparrow \rightleftharpoons \left\downarrow \right \quad \Gamma_{4} \\
\Sigma_{1} & \\
\Gamma_{4} & \quad \left\downarrow \rightleftharpoons \left\uparrow \right \quad \Gamma_{4}
\end{align*} \]

Since \( \Gamma_{3} \circ \Gamma^{*}_{1} = \Delta_{1} \cup \Gamma_{1} \), we have \( \Gamma_{3} \circ \Gamma^{*}_{1} = \Delta_{1} = \Gamma_{1} \circ \Gamma^{*}_{1} \). This is contrary to the assumption of this case. From \( \Gamma_{2} \circ \Gamma^{*}_{1} \cap \Gamma_{3} \circ \Gamma^{*}_{1} \supseteq \Delta_{1} \) and (2), for \( \gamma_{1}, \gamma_{2}(\neq) \in \Gamma_{4}(\alpha) \) we have by Lemma 22, iii)

\[ |\Gamma^{*}_{1}(\gamma_{1}) \cap \Gamma^{*}_{1}(\gamma_{2})| = 1. \]  
(4)

If \( \Gamma_{2} \circ \Sigma_{1} \) contains \( \Gamma_{1} \), then we have \( \Gamma^{*}_{1} \circ \Gamma_{3} = \Gamma^{*}_{1} \circ \Gamma_{4} \cap \Sigma_{1} \), and by (4)

\[ C_{2} S_{1} = \left(v - \frac{v - 1}{k_{4}}\right) C_{3} + \text{terms not involving } C_{3}. \]
When \( k_4 = 1 \), \( v - \frac{v - 1}{k_4} = 1 \). So \( \Gamma_2 \circ \Delta_2 \) contains \( \Gamma_3 \), by (3). When \( k_4 > 1 \), \( v - 1 > v - \frac{v - 1}{k_4} \). So, \( x = 1 \), and hence \( \Gamma_2 \circ \Delta_2 \) contains \( \Gamma_3 \).

In all cases, we can conclude that \( \Gamma_2 \circ \Delta_2 \) contains \( \Gamma_3 \), and hence, \( \Gamma_2^* \circ \Gamma_3 \supset \Delta_2 \).

Thus, we have the following figure.

So, \( \Gamma_1 \circ \Gamma_3^* = \Gamma_2 \circ \Gamma_3^* \). This is contrary to the assumption.

Non-existence of Fig. 2.

Case I. \( \pi_1 = \pi_2, \pi_3 \).

From \( \Gamma_1^* \circ \Gamma_2 \cap \Gamma_3^* \circ \Gamma_3 = \emptyset \) and \( \pi_1 = \pi_2, \pi_3 \), we have \( |\Gamma_1 \circ \Gamma_3^*(\alpha)| = \frac{v_1(v_1 - 1)}{2} \), \( v_1 = v_2 + 1 \) and \( \Gamma_1 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_3^* \) by Lemma 21 and Lemma 19. On the other hand, \( |\Gamma_1 \circ \Gamma_1^*(\alpha)| = |\Gamma_1 \circ \Gamma_2(\alpha)| = |\Gamma_2 \circ \Gamma_3^*(\alpha)| = |\Gamma_3^*(\alpha)| \cdot |\Gamma_3(\alpha)| = v_1(v_1 - 1) \). This is impossible.

Case II. \( \pi_1 = \pi_2 = \pi_3 \).

By Lemma 20, \( v_1 = v_2 = v_3 + 1 \). On the other hand from the existence of the following figure,

we have \( v_3 = v_2 + 1 = v_1 + 1 \) by Lemma 21, iii). This is impossible.

Case III. \( \pi_1 = \pi_2 = \pi_3, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \).

By Lemma 22, for \( (\alpha, \delta) \in \Gamma_1^* \circ \Gamma_1, 1 < |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| \) and \( 1 < |\Gamma_2^*(\alpha) \cap \Gamma_3^*(\delta)| \). The counting arguments show that \( |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| = |\Gamma_1(\gamma_1) \cap \Gamma_3(\gamma_3)| \) and \( |\Gamma_2^*(\alpha) \cap \Gamma_3^*(\delta)| = |\Gamma_1(\gamma_1) \cap \Gamma_3(\gamma_2)| \) for \( (\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^* \). Therefore, \( \Gamma_1^* \circ \Gamma_2 = \Gamma_2^* \circ \Gamma_3 \). Now \( \Gamma_1^* \circ \Gamma_2 \supset \Gamma_1^* \circ \Gamma_1 \) and \( \Gamma_1^* \circ \Gamma_3 \supset \Gamma_1^* \circ \Gamma_1 \). Since we can show that \( \pi_1^* = \pi_2^* = \pi_3^* \) by Lemma 21, we have a contradiction by Lemma 24.

Case IV. \( \pi_1 = \pi_2 = \pi_3, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \).

From \( \Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 \supset \Gamma_1^* \circ \Gamma_1 \), we have \( |\Gamma_1^* \circ \Gamma_1(\alpha)| = \frac{v(v - 1)}{k_1 + 1} \) by Lemma 22. This is impossible.
Case V. \( \pi_1 = \pi_2 = \pi_3, \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^* \).

By Lemma 21, we have \( \pi_1^* = \pi_2^* = \pi_3^* \). By Lemma 22, iii), \( |\Gamma_1 \circ \Gamma_1^*(\alpha)| = v(v-1) \), and by Lemma 8, iv), \( \Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2 \).

From the existence of the above figures, we have \( \Gamma_1^* \circ \Gamma_3 = \Gamma_2^* \circ \Gamma_1 \cup \Gamma_3^* \circ \Gamma_2 \).

Therefore,

\[
\left| \Gamma_1^* \circ \Gamma_1 \right| \cdot \left| \Gamma_3^* \circ \Gamma_2 \right| = \left| \Gamma_1^* \circ \Gamma_1 \right| + \left| \Gamma_3^* \circ \Gamma_2 \right| = v(v-1) + \frac{v(v-1)}{k_2}.
\]

This is impossible.

Non-existence of Fig. 3.

For the above figure, if \( \Sigma_1 = \Sigma_2 \) then there exists the following figure.

This is contrary to non-existence of Fig. 1. Thus we have \( \Sigma_1 = \Sigma_2, \pi_1^* = \pi_2^* \), \( \Gamma_1 \circ \Gamma_1^* = \Sigma_1 \cup \Sigma_2 \) and \( G_\alpha \) is not doubly transitive on \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \) by Lemma 12. So, by Lemma 20 we have \( \pi_1^* = \pi_2^* = \pi_3^* = \pi_4^* \). Also \( \Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_3 = \Gamma_3^* \circ \Gamma_4 \) by Lemma 22. From \( \Gamma_2^* \circ \Gamma_1 \cap \Gamma_3^* \circ \Gamma_4 = \Gamma_1^* \circ \Gamma_2 \), this is contrary to Lemma 24.

Non-existence of Fig. 4.

There exist the following figures.
Case I. $\pi^* = \pi^*_2$.

By Lemma 21, we have $v_1 = v_2 + 1$ from Fig. a, and $v_2 = v_3 + 1$ from Fig. b. This is impossible.

Case II. $\pi^* = \pi^*_2 = \pi^*_3$.

By Lemma 20, we have $v_1 = v_2 = v_3 + 1$ and $\Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$ from Fig. b. On the other hand, $\Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \supset \Gamma_2 \circ \Gamma_1 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \supset \Gamma_2 \circ \Gamma_1 \circ \Gamma_2^*$. This is impossible.

Case III. $\pi^* = \pi^*_2 = \pi^*_3$.

By assumption, $\Gamma_2 \circ \Gamma_2^* \supset \Gamma_3 \circ \Gamma_3^* \supset \Gamma_2 \circ \Gamma_3 \circ \Gamma_3^*$. This is impossible.

Case IV. $\pi^* = \pi^*_2 = \pi^*_3$.

From Fig. a, $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^* \supset \Gamma_2 \circ \Gamma_3$ for some $\Gamma_i$ by Lemma 22. So, $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^* \supset \Gamma_2 \circ \Gamma_3$. This is impossible.

Case V. $\pi^* = \pi^*_2 = \pi^*_3$.

We put $\Sigma_1 = \Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^*$. By Lemma 22, $|\Sigma_1(\alpha)| = \frac{v(v-1)}{k_i + 1}$.

From that $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^*$, we have $\Gamma_1 \circ \Gamma_2 = \sum \cup \Gamma_1 \circ \Gamma_2^*$. So $v^2 = \frac{v(v-1)}{k_i + 1} + \frac{v(v-1)}{k_i}$. Therefore $k_i = 1$ and $v - 1 = k_i + 1 = 2$. This is contrary to the hypothesis of this lemma.

Case VI. $\pi^* = \pi^*_2 = \pi^*_3$.

We put $\Sigma = \Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^*$. By Lemma 22, we have $\Gamma_1 \circ \Gamma_2 \supset \sum \cup \Gamma_i$, $\Gamma_1 \circ \Gamma_2^* = \sum \cup \Gamma_j$ from some $\Gamma_i, \Gamma_j$ and $|\Sigma(\alpha)| = v(v-1)$. Since $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_2 \circ \Gamma_2^*$, we have $\Gamma_1 \circ \Gamma_2 = \sum = \Gamma_2 \circ \Gamma_2^*$. On the other hand, since $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^* \supset \Gamma_2 \circ \Gamma_2^*$, we have $\Gamma_1 \circ \Gamma_2 = \sum = \Gamma_2 \circ \Gamma_2^*$. This is impossible.

Lemma 26. For $\Gamma_1, \Gamma_2$ and $\Gamma_3$, suppose that $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1 \circ \Gamma_2^*$ contains a $G$-orbit $\sum$ in $\Omega \times \Omega$, and $v_1, v_2, v_3 > 3$. Then, there does not exist $\Gamma_i$ such that $\Gamma_1 \circ \Gamma_i = \sum$. 

Proof. From non-existences of Fig. 2, Fig. 3, Fig. 4 of Lemma 24, we have this assertion.

Lemma 27. (P. J. Cameron [3], Prop.)
If \( \Gamma^* \neq \Gamma_i \) and \( \Gamma_i \circ \Gamma_i \equiv \Gamma_i \cup \Gamma_i^* \cup (\Gamma_i \cup \Gamma_i)^* \cup (\Gamma_i \circ \Gamma_i) \), then \( G \) has rank 4.

3. Proof of Theorem 1

We put

\[ x_i = \# \{ \Gamma_j | \Delta_i = \Gamma_j \circ \Gamma_i^* \} \]
\[ y_i = \# \{ (\Gamma_k, \Gamma_l) | \Gamma_k \circ \Gamma_l \supset \Delta_i \} \]

and assume that \( x_1 \geq \cdots \geq x_r > x_{r+1} = \cdots = x_r = 0 \). Counting in two ways triliteralas \((\Gamma, \Gamma, \Delta)\) such that \( \Gamma \circ \Gamma \supset \Delta \), we have by Lemma 9 and 11

\[ s^2 \leq \sum_{i=1}^r y_i. \]

The equality means that, for any \( \Gamma_i \) and \( \Gamma_j \), we cannot have \( \Gamma_i \circ \Gamma_j = \Delta_k \cup \Delta_l \), \( \Delta_k \neq \Delta_l \).

When \( x_i > 0 \), by Lemma 26 \( y_i \leq x_i + s \). When \( x_i = 0 \), by non-existence of Fig. 1 of Lemma 25 \( y_i \leq 2s \). Therefore

\[ s^2 \leq \sum_{i=1}^r y_i \leq \sum_{i=1}^r (x_i + s) + 2(t-r)s, \]

so

\[ s^2 \leq (r+1)s + (t-r)s, \]
\[ s \leq 2t-r+1. \tag{1} \]

Now, let \( \Delta_i = \Gamma_i \circ \Gamma_i^* \) and we put

\[ A = \{ \{ \Gamma_i, \Gamma_j \} : \text{unordered pair } | \Gamma_i \circ \Gamma_j \supset \Delta_i, \Gamma_i \neq \Gamma_j \} \],
\[ B = \{ \Gamma_i | \{ \Gamma_i, \Gamma_j \} \in A \} \].

For \( \{ \Gamma_i, \Gamma_j \}, \{ \Gamma_k, \Gamma_l \} (\neq) \in A \), \( \{ \Gamma_i, \Gamma_j \} \cap \{ \Gamma_k, \Gamma_l \} = \emptyset \) by Lemma 26. Therefore \( |B| = 2|A| \). Furthermore, for \( \{ \Gamma_i, \Gamma_j \}, \{ \Gamma_k, \Gamma_l \} (\neq) \in A \), and for \( \Gamma_m, \Gamma_n (\neq) \in B \), \( \Gamma_m \circ \Gamma_l \cap \Gamma_m \circ \Gamma_l \), \( \Gamma_m \circ \Gamma_l \cap \Gamma_m \circ \Gamma_l \), \( \Gamma_m \circ \Gamma_m, \Gamma_m \circ \Gamma_n \) are disjoint to each other by non-existence of Fig. 1 of Lemma 25. Thus we have

\[ |A| + (s - |B|) = s - |A| \leq t, \tag{2} \]

and by Lemma 26

\[ |A| - 1 \leq t - r. \tag{3} \]

Assume \( s = 2t - r + 1 \). Since the equality of (1) hold \( y_i = x_i + s \), and hence
\[ |A| = \frac{s}{2} \quad \text{and} \quad \frac{s}{2} - 1 \leq t - r \quad \text{by (3), and hence,} \quad 2t - r + 1 = s \leq 2t - 2r + 2. \]

So \( r = 1 \). Therefore, if \( r > 1 \), we conclude that \( s \leq 2t - r \).

We shall show that when \( r = 1 \), \( s \leq 2t - 2 \). Assume \( r = 1 \) and \( 2t \geq s \geq 2t - 1 \), and put \( \Delta = \Gamma_i \circ \Gamma^*_j \), \( 1 \leq i \leq s \). If \( \pi_i = \pi_j \) for some \( \Gamma_i \) and \( \Gamma_j \), then by Lemma 23, \( \Delta \subseteq \Gamma_i \circ \Gamma^*_j \) for any \( \Gamma_i, \Gamma_j (\neq) \), and hence, \( \Gamma^*_j \circ \Gamma^*_i \cap \Gamma^*_i \circ \Gamma^*_j = \emptyset \). So \( s \leq t \).

This is contrary to the assumption that \( t \geq 2 \). Thus, it holds that \( \pi_i = \pi_j = \cdots = \pi_s \).

Now, Suppose \( \Gamma_i \circ \Gamma_j = \Delta \cup \Gamma^*_j \) for some \( \Gamma_i, \Gamma_j \) and \( \Gamma_k \), and put \( D = C(\Delta), \Gamma_i \circ \Gamma_k = \Delta' \cup \Gamma^*_j, \Gamma' = C(\Delta'), t = |\Gamma_i(\alpha) \cap \Gamma^*_j(\beta)| \) for \( (\alpha, \beta) \in \Gamma^*_j, x = |\Gamma_i(\alpha) \cap \Gamma^*_j(\delta)| \) for \( (\alpha, \delta) \in \Delta, v = v_i = v_2 = \cdots, k = k_1 = k_2 = \cdots \). Then we have
\[
(C_i C_j)C_k = (tC^*_k + xD)C_k = tvI + tkD + xDC_k, \quad C_i(C_j C_k) = C_i(t'C^*_k + x'D') = t'vI + t'kD + x'C_D'.
\]

\[
(t' = |\Gamma_i(\alpha) \cap \Gamma^*_j(\beta)| \) for \( (\alpha, \beta) \in \Gamma^*_j, x' = |\Gamma_i(\alpha) \cap \Gamma^*_j(\delta)| \) for \( (\alpha, \delta) \in \Delta' \).\]

We have \( t = t' \) by counting in two ways triplilaterals \((\beta, \alpha, \gamma)\) whose edges are successively \( \Gamma_i, \Gamma_j \) and \( \Gamma_k \), and have \( |\Delta(\alpha)| = |\Delta'(\alpha)| \) and \( x = x' \) by Lemma 10. So,
\[
(C_i C_j)C_k = (tC^*_k + xD)C_k = (v - 1)C_k + \cdots.
\]

If \( C_i = C_k \), \( |\Delta'(\alpha)| = \frac{v(v - 1)}{k + 1} \) by Lemma 10. This is impossible. Thus \( C_i \neq C_k \).

Similarly, \( C_j = C_k \).

When \( S = 2t \), then the equality of (1) holds. Therefore, for any \( \Gamma_i \), there exists \( \Gamma_j \) such that \( \Gamma_i \circ \Gamma_j = \Delta \cup \Gamma^*_j \) for some \( \Gamma^*_j \). So, as is shown above, \( \Gamma_i = \Gamma_j = \Gamma_k \). Therefore we have any \( \Gamma_i \).

\[
\Gamma_i \neq \Gamma^*_j, \Gamma_i \circ \Gamma_i = \Delta \cup \Gamma^*_j \quad \text{and} \quad \Gamma_i \circ \Gamma^*_i \cap \Gamma_i \circ \Gamma^*_k = \emptyset \quad \text{for} \quad \Gamma_j \neq \Gamma_i, \Gamma_k.
\]

When \( s = 2t - 1 \), then \( |A| \leq t - 1 \), and from (2) \( s - |A| \leq t \). So \( |A| = t - 1 \). Therefore, there is a unique \( \Gamma_i \) such that for any \( \Gamma_i (\neq \Gamma_i), \Gamma_i \circ \Gamma^*_i \subseteq \Delta \). We shall show that for any \( \Gamma_i, \Gamma_j (\neq), \Gamma_i \circ \Gamma^*_i \) contains some \( \Gamma_k \). Assume \( \Gamma_i \circ \Gamma^*_i = \Delta_k \cup \Delta_i \) for some \( \Gamma_k, \Gamma_j (\neq) \). Count in two ways the paired \( (\Gamma_m, \Delta_n) \) such that \( \Gamma_i \circ \Gamma^*_i \) contains \( \Delta_n \), then by Lemma 25, we have
\[
2t = s + 1 \leq |\{ (\Gamma_m, \Delta_n) | \Gamma_i \circ \Gamma^*_i \supset \Delta_n \}| \leq 2t.
\]

So, equality holds. Thus for any \( \Delta_n \), there exist \( \Gamma_k \) and \( \Gamma_q (\neq) \) such that \( \Gamma_i \circ \Gamma^*_i \) and \( \Gamma_i \circ \Gamma^*_i \) contains \( \Delta_k \). Therefore we may choose \( \Gamma_q \) such that \( \Gamma_i \circ \Gamma^*_i \cap \Gamma_i \circ \Gamma^*_i = \emptyset \) and \( \Gamma_i \neq \Gamma_k \). Then \( \Gamma_i \circ \Gamma^*_i \supset \Gamma_i \circ \Gamma^*_i = \Delta \). This is impossible. Thus, again as is shown above, we can conclude that for any \( \Gamma_i (\neq \Gamma_n) \),
\[ \Gamma_i \oplus \Gamma^*_i, \Gamma_i \circ \Gamma_i = \Delta \cup \Gamma_i^* \text{ and} \]
\[ \Gamma_i \circ \Gamma_i^* \cap \Gamma_i \circ \Gamma_i^* = \emptyset \text{ for } \Gamma_i \neq \Gamma_i^*. \]

Thus if \( s \geq 2t - 1 \), there exists \( \Gamma_i \) such that
\[ \Gamma_i \oplus \Gamma^*_i \text{ and } \Gamma_i \circ \Gamma_i = \Gamma_i \circ \Gamma_i^* \cup \Gamma_i^*. \]

By Lemma 27, this show that \( G \) has rank 4. This is impossible for \( s \geq 2t - 1 \) and \( t \geq 2 \).

4. Proof of Theorem 2

When \( r = t \), we have \( s \leq t \) by Theorem 1. On the other hand, from \( s \geq r = t \), we conclude that \( s = t = r \).

We put \( \Gamma_i \circ \Gamma_i^* = \Delta_i, A_i = \{ \{ \Gamma_i; \Gamma_i \} \text{ unordered pair} | \Gamma_i \circ \Gamma_i^* \supset \Delta_i, \Gamma_i \neq \Gamma_i^* \} \).

Then \( |A_i| - 1 \leq t - r = 0 \), so \( |A_i| \leq 1 \).

Count in two ways triplilaterals \( (\Gamma_i, \Gamma_j, \Delta_k) \) such that \( \Gamma_i \circ \Gamma_i^* \supset \Delta_k \), we have
\[ s^2 \leq 3s, \]
so
\[ s \leq 3. \tag{1} \]

Case \( t = 2 \). If \( |\Gamma_i(\alpha)| \neq |\Gamma_j(\alpha)| \), by T. Ito [6], \( G \) is isomorphic to the small Janko simple group and \( G_{a_i} \) is isomorphic to \( \text{PSL}(2, 11) \). We shall prove that the case of \( |\Gamma_i(\alpha)| = |\Gamma_j(\alpha)| \) does not occur. We put \( |\Gamma_i(\alpha)| = |\Gamma_j(\alpha)| = \nu \). It is easy to prove that \( \pi_i = \pi_j \). We shall show that \( \Gamma_i \) and \( \Gamma_j \) are self paired. If not, then \( \Gamma_i \circ \Gamma_i^* = \Gamma_j^* \). Since \( \Gamma_i \circ \Gamma_i^* \oplus \Gamma_j \circ \Gamma_j^* = \Gamma_i \circ \Gamma_j \), we have that \( \Gamma_i \circ \Gamma_i^* \supset \Gamma_j \circ \Gamma_j^* \). By Lemma 7. By Lemma 11, there exists a \( G \)-orbit \( \Sigma \) in \( \Gamma_i \circ \Gamma_i^* \) such that \( G_i \) is not 2-transitive on \( \Sigma(\alpha) \), and \( \sum \neq \Delta_i, \Delta_j \). This is impossible for \( t = 2 \). Thus, we have \( \Gamma_i \circ \Gamma_j = \Delta_i \cup \Delta_j \). So, \( \nu^2 = |\Gamma_i \circ \Gamma_j(\alpha)| = |\Gamma_i \circ \Gamma_j(\alpha)| = \frac{\nu(\nu - 1)}{k_1} + \frac{\nu(\nu - 1)}{k_2} \). This is impossible.

Case \( t = 3 \). For this case, the equality of (1) holds. So we have \( |A_i| = 1 \) for \( 1 \leq i \leq 3 \). We shall show that if \( \Gamma_i \circ \Gamma_i^* \neq \Delta_i, \text{ then } \Gamma_i = \Gamma_i^* \) or \( \Gamma_i = \Gamma_i^* \). If \( \Gamma_i, \Gamma_j \neq \Gamma_i^* \), then since \( \Gamma_i \circ \Gamma_i^* \cap \Gamma_j \circ \Gamma_j^* = \emptyset \), there exists a \( G \)-orbit \( \Sigma \) in \( \Gamma_i \circ \Gamma_i^* \cap \Gamma_j \circ \Gamma_j^* \), such that \( G \) is not 2-transitive on \( \Sigma(\alpha) \) by Lemma 12, and for any \( \Gamma_i, \Gamma_j \circ \Gamma_i^* \neq \Sigma \) by Lemma 25. From \( r = t \), this is impossible. Thus we may assume that there exist the following figures.

![Figures a, b, c](https://via.placeholder.com/150)
If \( \pi_1 = \pi_2, \pi_3 \), then \( v_1v_2 = |\Gamma_1^+ \circ \Gamma_2(\alpha)| = |\Gamma_1^+ \circ \Gamma_3(\alpha)| = \frac{v_1(v_1-1)}{k_1} \) from Fig. a, so \( v_1 > v_3 \). Similarly, \( v_3 > v_1 \) from Fig. c. Therefore \( v_3 > v_2 \). On the other hand, \( v_2v_3 = \frac{v_2(v_2-1)}{k_2} \) from Fig. b, so \( v_2 > v_3 \). This is impossible. Thus we have \( \pi_1 = \pi_2 = \pi_3 \). By Lemma 7, \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are self-paired.

Thus \( \Gamma_1 \circ \Gamma_2 = \Gamma_3 \cup \Delta_1, \Gamma_2 \circ \Gamma_3 = \Gamma_1 \cup \Delta_2, \Gamma_3 \circ \Gamma_1 = \Gamma_2 \cup \Delta_3 \). Put \( |\Gamma_1(\alpha)| = v \), then by Lemma 8, iii) we have

\[ |\Delta_1(\alpha)| = |\Delta_2(\alpha)| = |\Delta_3(\alpha)| = v(v-1). \]

We put

\[ D_i = C(\Delta_i) \text{ and } C_i = C(\Gamma_i), \]
\[ 1 \leq i \leq 3; \]
\[ D_1C_3 = x_1D_1 + x_2D_2 + x_3D_3. \]

Then

\[ x_1 + x_2 + x_3 = v \]
\[ D_2C_3 = x_2D_1 + \text{terms not involving } D_1, \]
\[ D_2C_3 = x_3D_1 + \text{terms not involving } D_1. \]  \hspace{1cm} (2)

Now

\[ (C_1C_2)C_3 = (D_1 + C_3)C_3 = vI + D_3 + D_1C_3, \]
\[ C_1(C_2C_3) = C_1(D_2 + C_1) = vI + D_1 + D_2C_1. \]

So

\[ D_2C_1 = D_2C_3 + D_3 - D_1 = (x_1 - 1)D_1 + x_2D_2 + (x_3 + 1)D_3. \]

Similarly

\[ D_2C_2 = D_2C_3 + D_3 - D_2 = x_1D_1 + (x_2 - 1)D_2 + (x_3 + 1)D_3. \]

Next

\[ (C_1C_2)C_3 = (vI + D_1)C_3 = vC_3 + D_1C_3, \]
\[ C_1(C_2C_3) = C_1(D_3 + C_2) = C_3 + D_1 + D_3C_1. \]

So

\[ D_3C_1 = D_1C_3 + (v-1)C_2 - D_1 \]
\[ = (x_1 - 1)D_1 + x_2D_2 + x_3D_3 + (v - 1)C_3. \]

Similarly

\[ D_1C_2 = D_2C_1 + (v-1)C_1 - D_2 \]
\[ = (x_1 - 1)D_1 + (x_2 - 1)D_2 + (x_3 + 1)D_3 + (v - 1)C_1, \]
\[ D_2C_3 = D_3C_2 + (v-1)C_2 - D_3 \]
\[ = x_1D_1 + (x_2 - 1)D_2 + x_3D_3 + (v - 1)C_2. \]  \hspace{1cm} (3)

Furthermore
\[(C_1C_2)C_2 = (vI + D_1)C_2 = vC_2 + D_1C_2,\]
\[C_1(C_3 + D_1) = C_3 + D_1C_1\]

So
\[D_1C_1 = D_1C_2 + (v-1)C_2 - D_3\]
\[= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_1 + (v-1)C_2.\]

Similarly
\[D_2C_2 = D_2C_3 + (v-1)C_3 - D_1\]
\[= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_2 + (v-1)C_3,\]
\[D_3C_3 = D_3C_1 + (v-1)C_1 - D_2\]
\[= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_3 + (v-1)C_1.\] (4)

Thus (2), (3) and (4) yield
\[x_1 = x_2, x_1 - 1 = x_3.\]

We put \(x_3 = x\), then
\[v = x_1 + x_2 + x_3 = (x+1) + (x+1) + x = 3x + 2.\] (5)

It is easy to show that the graph \((\Omega, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)\) is a strongly regular graph with parameters \(3v, 2, 3\).

From the conditions of the existence of the strongly regular graph, (see [1] p. 97) it holds that
\[(3-2)^2 + 4(3v-3) = 12v - 11 = d^2,\] (6)
\[(d \text{ is a positive integer})\]
\[m = \frac{3v}{2 \cdot 3 \cdot d} \cdot ((3v-1+3-2)(d+3-2)-2 \cdot 3) = \frac{3}{2}v^2 + \frac{3v(v-2)}{2d}.\] (7)
\[(m \text{ is a positive integer})\]

From (7), \(\frac{3v(v-2)}{d}\) is integer, and hence
\[12v - 11 = d^2 \text{ is a divisor of } v_0(v-2)^2.\]
So

\[ 12v - 11 \text{ is a divisor of } 11^2 \cdot 13^3. \]

From \( v = 3x + 2 \), we conclude

\[ v = 11. \]

Lastly, we shall prove that the primitive group satisfying these conditions does not exist. It is easy to prove that \( G_a \) acts faithfully on \( \Gamma_3(\alpha) \). We shall show that for \( \gamma_1, \gamma' \in \Gamma_3(\alpha) \), \( G_{a, \gamma_1, \gamma'} \) has the fixed points in \( \Gamma_3(\alpha) \setminus \{\gamma_1, \gamma'\} \).

For \( (\alpha, \gamma_1) \in \Gamma_3 \), put \( \{\gamma_2\} = \Gamma_3(\alpha) \cap \Gamma_3(\gamma_1) \) and \( \{\gamma_3\} = \Gamma_3(\alpha) \cap \Gamma_3(\gamma_3) \). Then, \( G_{a, \gamma_1} \) fix \( \gamma_2 \) and \( \gamma_3 \). So we must have that \( (\gamma_2, \gamma_3) \in \Gamma_3 \). Now for \( \gamma_1, \gamma' \in \Gamma_3(\alpha) \), put \( \{\delta_1\} = \Gamma_3(\gamma_1) \cap \Gamma_3(\gamma') \), \( \{\delta_2\} = \Gamma_3(\gamma_3) \cap \Gamma_3(\gamma') \). Then \( G_{a, \gamma_1, \gamma'} \) fix \( \delta_1 \) and \( \delta_2 \). Since \( (\gamma_1, \gamma') \not\supseteq \Gamma_3 \), we have \( (\delta_1, \delta_2) \not\supseteq \Gamma_3 \). Therefore \( \Gamma_3(\gamma_1) \cap \Gamma_3(\delta_2) = \{\delta\} + \{\delta\} \).

So, \( G_{a, \gamma_1, \gamma'} \) fix \( \delta_1 \) and \( \delta \). Since \( \Gamma_3(\gamma_1) \not\supseteq \alpha, \delta_1, \delta (\pm) \), in the same way, we obtain that \( G_{a, \gamma_1, \gamma'} \) has the fix points in \( \Gamma_3(\alpha) \setminus \{\gamma_1 \cup \gamma'\} \). The order of \( G_a \) is at most one million. If \( G_a \) is non-solvable, then the minimal normal subgroup of \( G_a \) is non-solvable simple. From [5], it is isomorphic to the Mathieu group \( M_{11} \) or the transitive extension of the alternating group \( A_5 \) act on ten points. These groups have not the representation such that it is doubly-transitive on eleven points and it's stabilizer of two points has the additional fixed point. Thus, we can conclude that \( G_a \) is solvable and the order of \( G_a \) is 110. So \( |G| = |\Omega| \cdot 11 \cdot 10 = 364 \cdot 11 \cdot 10 = 2^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \). \( G \) is non-solvable group and \( (|G|, 3) = 1 \). But there does not exist such group by M. Hall [5].

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**References**


