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## A COMPLEX OF CURVES AND A PRESENTATION FOR THE MAPPING CLASS GROUP OF A SURFACE

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### 1. Introduction

Let  $\Sigma_{g,n}$  be an oriented surface of genus  $g$  ( $\geq 2$ ) with  $n$  ( $\geq 0$ ) boundary components and denote by  $\mathcal{M}_{g,n}$  its mapping class group, that is to say, the group of orientation preserving diffeomorphisms of  $\Sigma_{g,n}$  which are the identity on  $\partial\Sigma_{g,n}$  modulo isotopy. For a simple closed curve  $a$  in  $\Sigma_{g,n}$ , we define the Dehn twist along  $a$  as indicated in Fig. 1. We denote the isotopy class of Dehn twist along  $a$  by the same letter  $a$ .

It is known that  $\mathcal{M}_{g,n}$  is generated by Dehn twists [5], [16]. McCool [19] showed that  $\mathcal{M}_{g,n}$  is finitely presented. Hatcher and Thurston [7] defined a simply connected complex whose vertices are isotopy classes of “cut systems” and introduced a method of giving a presentation for  $\mathcal{M}_{g,n}$  by making use of this complex. Harer [8] reduced the member of the 2-simplices of this complex, and Wajnryb [20] gave a simple presentation for  $\mathcal{M}_{g,1}$  and  $\mathcal{M}_{g,0}$ . Following Wajnryb’s presentation, Gervais [6] gave a symmetric presentation for  $\mathcal{M}_{g,n}$ . We set some notations indicating circles on  $\Sigma_{g,n}$  as in Fig. 2. A triple of integers  $(i, j, k) \in \{1, \dots, 2g+n-3\}^3$  will be said to be *good* when:

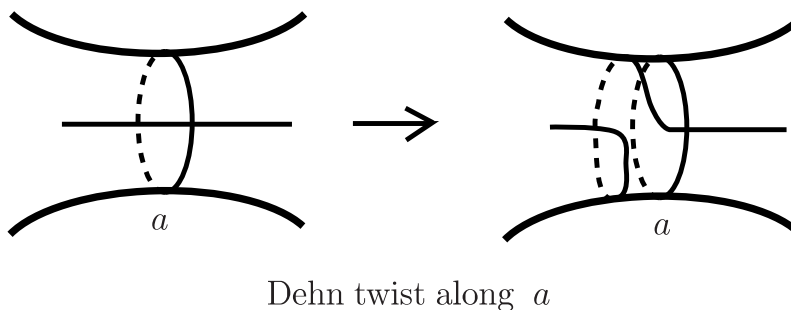


Fig. 1.

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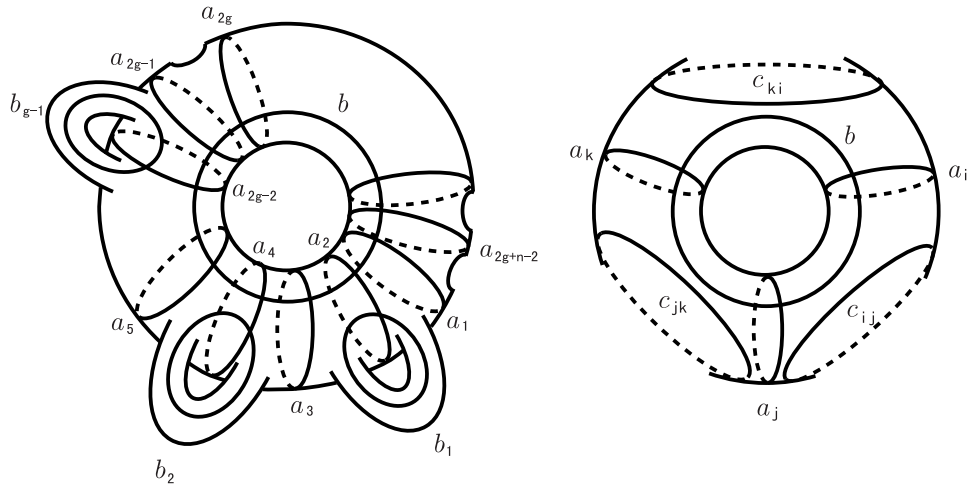


Fig. 2.

- i)  $(i, j, k) \notin \{(x, x, x) \mid x \in \{1, \dots, 2g + n - 2\}\}$ ,
- ii)  $i \leq j \leq k$  or  $j \leq k \leq i$  or  $k \leq i \leq j$ .

Gervais' symmetric presentation is as follows,

**Theorem 1.1** ([6]). *If  $g \geq 2, n \geq 0$ , then  $\mathcal{M}_{g,n}$  is generated by  $b, b_1, \dots, b_{g-1}, a_1, \dots, a_{2g+n-2}, c_{i,j}$ , and its defining relations are*

- (A) "HANDLES":  $c_{2i,2i+1} = c_{2i-1,2i}$  for all  $i, 1 \leq i \leq g - 1$ ,
- (B) "BRAIDS": for all  $x, y$  among the generators,  $xy = yx$  if the associated curves are disjoint and  $xyx = yxy$  if the associated curves intersect transversely in a single point,
- (C) "STARS":  $c_{ij}c_{jk}c_{ki} = (a_i a_j a_k b)^3$  for all good triples  $i, j, k$ , where  $c_{ii} = 1$ .

Let  $G_{g,n}$  denote the group with presentation given by Theorem 1.1.

On the other hand, Harvey [10] introduced a complex of curves for  $\Sigma_{g,n}$ , whose vertices are isotopy classes of essential (neither homotopic to a point nor any boundary component) simple closed curves and simplices are the set of vertices which are represented by disjoint and non-isotopic curves. Harer [9] showed the higher connectivity of this complex and, by using this complex, proved the stability of the cohomology group of mapping class groups. McCullough [18] defined a disk complex of a handle body (an oriented 3-dimensional manifold obtained from 3-ball by attaching 1-handles), which is defined from a complex of curves by replacing "curves" with "meridian disks". He showed that the disk complex is contractible. The author [12] gave a presentation for the mapping class group of a handle body by investigating the action of the mapping class group on this complex. The aim of this paper is to

give a Gervais' symmetric presentation for  $\mathcal{M}_{g,n}$  with the same method as above, that is to say, by investigating the action of  $\mathcal{M}_{g,n}$  on the complex of curves for  $\Sigma_{g,n}$ . We remark here that our method introduced in this paper does not use Wajnryb's simple presentation. This fact means that we do not need to use Hatcher-Thurston's complex to give a presentation for  $\mathcal{M}_{g,n}$ . In [21], Wajnryb proved simple connectedness of Hatcher-Thurston's complex without using Cerf Theory, and use this to give his simple presentation for  $\mathcal{M}_{g,0}$  and  $\mathcal{M}_{g,1}$ . On the other hand, Ivanov [13] gave an elementary proof of the simple connectivity of Harvey's complex, and Hatcher [11] gave an elementary proof of the higher connectivity of this complex. Therefore, our method introduced in this paper is another elementary approach to the mapping class group of a surface.

Recently, S. Benvenuti (Pisa Univ.) [1] showed a similar result, independently, using different "complex of curves", which includes separating curves. We remark that Matsumoto [17] gave a beautiful presentation for the mapping class groups of surfaces in terms of Artin groups.

We set notations and conventions used in this paper. Composition of elements of  $\mathcal{M}_{g,n}$  will be written from right to left. We will denote by  $\bar{x}$  the inverse of  $x$  and  $y(x)$  the conjugate  $yx\bar{y}$  of  $x$  by  $y$ . The notation  $\rightleftharpoons$  means "commute with". For example, for two elements  $x, y$  of  $\mathcal{M}_{g,n}$ ,  $x \rightleftharpoons y$  means  $xy = yx$ . We use braid relations and handle relations very often. We indicate the place to use a braid relation (resp. handle relation) by an underline together with the letter "braid" (resp. "handle") below it. For example, if  $x, y, z_1, z_2$  are loops on  $\Sigma_{g,n}$  and if  $x$  and  $y$  intersect transversely in a single point and  $z_1$  and  $z_2$  are disjoint, then

$$\cdots \underbrace{xyx}_{\text{braid}} \cdots \underbrace{z_1 z_2}_{\text{braid}} \cdots = \cdots yxy \cdots z_2 z_1 \cdots .$$

**2. A presentation for  $\mathcal{M}_{2,0}$**

Birman and Hilden [4] showed:

**Theorem 2.1** ([4]).  *$\mathcal{M}_{2,0}$  admits the presentation:*

*generators:*  $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5,$

*defining relations:*

- (i)  $\tau_i \tau_j = \tau_j \tau_i$ , if  $|i - j| \geq 2, 1 \leq i, j \leq 5,$
- (ii)  $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad 1 \leq i \leq 4,$
- (iii)  $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5)^6 = 1,$
- (iv)  $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1)^2 = 1,$
- (v)  $\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1 \rightleftharpoons \tau_i \quad 1 \leq i \leq 5.$

As we defined previously,  $G_{2,0}$  is a group with the following presentation:  
 generators:  $a_1, b, a_2, b_1, c_{1,2},$   
 defining relations:

- (1)  $a_1ba_1 = ba_1b, a_2ba_2 = ba_2b, a_2b_1a_2 = b_1a_2b_1, b_1c_{1,2}b_1 = c_{1,2}b_1c_{1,2}$ , every other pair of generators commutes,
- (2)  $(a_1a_1a_2b)^3 = c_{1,2}^2$ .

Let  $\psi_{2,0}: G_{2,0} \rightarrow \mathcal{M}_{2,0}$  be an epimorphism defined by  $\psi_{2,0}(a_1) = \tau_1, \psi_{2,0}(b) = \tau_2, \psi_{2,0}(a_2) = \tau_3, \psi_{2,0}(b_1) = \tau_4$  and  $\psi_{2,0}(c_{1,2}) = \tau_5$ . We want to prove  $\psi_{2,0}$  is an isomorphism. We shall construct an inverse map  $\phi_{2,0}: \mathcal{M}_{2,0} \rightarrow G_{2,0}$ . For each generators of  $G_{2,0}$ , we define  $\phi_{2,0}(\tau_1) = a_1, \phi_{2,0}(\tau_2) = b, \phi_{2,0}(\tau_3) = a_2, \phi_{2,0}(\tau_4) = b_1$ , and  $\phi_{2,0}(\tau_5) = c_{1,2}$ . If the relations (i)–(v) are mapped by  $\phi_{2,0}$  onto relations in  $G_{2,0}$ , then  $\phi_{2,0}$  extends to a homomorphism. Then, we can show  $\psi_{2,0} \circ \phi_{2,0} = \text{Id}_{\mathcal{M}_{2,0}}$  and  $\phi_{2,0}$  is an epimorphism, hence,  $\psi_{2,0}$  is an isomorphism. Therefore, in order to prove  $\phi_{2,0}$  is an isomorphism, it is enough to show that the defining relations (i)–(v) are satisfied in  $G_{2,0}$ .

Relations (i) and (ii) are nothing but the relations (1) for  $G_{2,0}$ . In  $G_{2,0}$ , the right hand side of relation (v) is  $a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1$ , hence we need to show

$$a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1 \stackrel{\Leftrightarrow}{=} a_1, b, a_2, b_1, c_{1,2}.$$

For short, we denote  $E = a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1$ . Using the relations (1), we can show  $E(b) = b, E(a_2) = a_2, E(b_1) = b_1, E(c_{1,2}) = c_{1,2}$ . In order to show  $E(a_1) = a_1$ , we have to give another presentation for  $E$ .

**Lemma 2.2.**  $(c_{1,2}c_{1,2}a_2b_1)^3 = a_1a_1$ .

Proof. We introduce an element  $D = a_1ba_2b_1c_{1,2}a_1ba_2b_1a_1ba_2a_1ba_1$  of  $\mathcal{M}_{2,0}$ . By using the relations (1), we can show  $D(a_1) = c_{1,2}, D(b) = b_1, D(a_2) = a_2, D(b_1) = b$ , and  $D(c_{1,2}) = a_1$ . We take a conjugation of the relation (2) by  $D$ , then we get the equation we need. □

**Lemma 2.3.**  $E = a_1a_1ba_1a_1b\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_2\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_2$ .

Proof. By the relations (1), we can show,

$$\begin{aligned} c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1 \underbrace{c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} &= c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2} \underbrace{a_2b_1a_2c_{1,2}}_{\text{braid}}c_{1,2}b_1 \\ &= c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}b_1a_2b_1c_{1,2}c_{1,2}b_1. \end{aligned}$$

We have shown  $a_1a_1 = (c_{1,2}c_{1,2}a_2b_1)^3$ , in Lemma 2.2. Therefore,

$$a_1a_1 = c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}b_1a_2b_1c_{1,2}c_{1,2}b_1.$$

From this equation,

$$a_2b_1c_{1,2}c_{1,2}b_1a_2 = \underbrace{\bar{c}_{1,2}\bar{c}_{1,2}a_1a_1}_{\text{braid}}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1 = a_1a_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1,$$

and hence we can show,

$$\begin{aligned}
 E &= a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 = a_1 b a_1 a_1 \underbrace{\bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1}_{\text{braid}} b a_1 && \text{by the above equation} \\
 &= a_1 b a_1 \underbrace{a_1 b a_1}_{\text{braid}} \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 = a_1 \underbrace{b a_1 b a_1}_{\text{braid}} b \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \\
 &= a_1 a_1 b a_1 a_1 b \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1. && \square
 \end{aligned}$$

We can show  $E(a_1) = a_1$  by using the above Lemma and the relations (1). The relation (iv) is interpreted as  $E^2 = 1$  in  $G_{2,0}$ . By Lemma 2.3,

$$\begin{aligned}
 E^2 &= a_1 a_1 b a_1 a_1 b \underbrace{\bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1}_{\text{braid}} a_1 a_1 b a_1 a_1 b \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \\
 &= a_1 a_1 b a_1 a_1 b a_1 a_1 b \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1 \bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1.
 \end{aligned}$$

If we can show  $(a_1 a_1 b)^4 = (c_{1,2} c_{1,2} b_1)^4$ , then  $E^2 = (c_{1,2} c_{1,2} b_1)^4 (\bar{c}_{1,2} \bar{c}_{1,2} \bar{b}_1)^4$ . Since we can show  $(c_{1,2} c_{1,2} b_1)^4 = (b_1 c_{1,2} c_{1,2})^4$  by the relations (1), we get  $E^2 = 1$ . Therefore it is enough to show:

**Lemma 2.4.**  $(a_1 a_1 b)^4 = (c_{1,2} c_{1,2} b_1)^4$

Proof. We denote  $r_1 = a_1 a_1 b a_1 a_1 b$ ,  $r_2 = c_{1,2} c_{1,2} b_1 c_{1,2} c_{1,2} b_1$  for short. We can show,

$$\begin{aligned}
 r_1 a_2 r_1 a_2 &= a_1 a_1 b a_1 a_1 b a_2 a_1 a_1 b a_1 a_1 b a_2 = a_1 a_1 b a_1 a_1 b a_2 a_1 \underbrace{b a_1 b a_1}_{\text{braid}} b a_2 \\
 &= a_1 a_1 b a_1 a_1 \underbrace{b a_2 b a_1}_{\text{braid}} \underbrace{b b a_1}_{\text{braid}} b a_2 = a_1 a_1 b a_1 a_1 a_2 \underbrace{b a_2 a_1}_{\text{braid}} \underbrace{b b a_1}_{\text{braid}} b a_2 \\
 &= a_1 a_1 b a_2 a_1 a_1 \underbrace{b a_1 a_2}_{\text{braid}} \underbrace{b b a_1}_{\text{braid}} b a_2 = a_1 a_1 b a_2 a_1 \underbrace{b a_1 b a_2}_{\text{braid}} b b a_1 b a_2 \\
 &= a_1 a_1 \underbrace{b a_2 b a_1}_{\text{braid}} b b a_2 b b a_1 b a_2 = a_1 a_1 a_2 b a_2 a_1 \underbrace{b b a_2 b b a_1}_{\text{braid}} b a_2 \\
 &= a_1 a_1 a_2 b a_2 a_1 \underbrace{b b a_2 b a_1}_{\text{braid}} b a_1 a_2 = a_1 a_1 a_2 b a_2 a_1 \underbrace{b b a_2 a_1}_{\text{braid}} \underbrace{b a_1 a_1 a_2}_{\text{braid}} \\
 &= a_1 a_1 a_2 b a_2 a_1 \underbrace{b b a_1 a_2}_{\text{braid}} b a_2 a_1 a_1 = a_1 a_1 a_2 b a_2 a_1 \underbrace{b b a_1}_{\text{braid}} \underbrace{b a_2 b a_1}_{\text{braid}} a_1 \\
 &= a_1 a_1 a_2 b a_2 a_1 \underbrace{b a_1 b a_1 a_2}_{\text{braid}} b a_1 a_1 = a_1 a_1 a_2 \underbrace{b a_2 a_1 a_1}_{\text{braid}} b a_1 a_1 a_2 b a_1 a_1 \\
 &= (a_1 a_1 a_2 b)^3 a_1 a_1
 \end{aligned}$$

and, by the relation (2),  $(a_1 a_1 a_2 b)^3 a_1 a_1 = c_{1,2} c_{1,2} a_1 a_1$ . Hence  $r_1 a_2 r_1 a_2 = c_{1,2} c_{1,2} a_1 a_1$ . From the last equation, we can show  $r_1^2 = r_1 \bar{a}_2 \bar{r}_1 c_{1,2} c_{1,2} a_1 a_1 \bar{a}_2$ . In the same way as above, but using Lemma 2.2 in place of relation (2), we can show  $r_2^2 =$

$r_2\bar{a}_2\bar{r}_2c_{1,2}c_{1,2}a_1\bar{a}_2$ . If we can show  $r_1(a_2) = r_2(a_2)$ , then we get  $r_1^2 = r_2^2$ . In fact,

$$\begin{aligned}
r_2(a_2) &= c_{1,2}\underbrace{c_{1,2}b_1c_{1,2}c_{1,2}b_1}_{\text{braid}}(a_2) = \underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}\underbrace{b_1c_{1,2}b_1}_{\text{braid}}(a_2) \\
&= b_1c_{1,2}\underbrace{b_1c_{1,2}b_1c_{1,2}}_{\text{braid}}(a_2) = b_1c_{1,2}c_{1,2}\underbrace{b_1c_{1,2}c_{1,2}}_{\text{braid}}(a_2) \\
&= b_1c_{1,2}c_{1,2}b_1(a_2) = b_1(a_1a_1a_2b)^3b_1(a_2) \quad \text{by the relation (2)} \\
&= b_1a_1a_1a_2ba_1a_1a_2ba_1a_1a_2\underbrace{bb_1}_{\text{braid}}(a_2) = b_1a_1a_1a_2ba_1a_1a_2ba_1a_1\underbrace{a_2b\bar{a}_2}_{\text{braid}}(b_1) \\
&= b_1a_1a_1a_2ba_1a_1a_2ba_1a_1\underbrace{\bar{b}a_2b}_{\text{braid}}(b_1) = b_1a_1a_1a_2\underbrace{ba_1a_1a_2ba_1a_1}_{\text{braid}}\bar{b}a_2(b_1) \\
&= b_1a_1a_1a_2ba_2a_1\underbrace{a_1ba_1a_1}_{\text{braid}}\bar{b}a_2(b_1) = b_1a_1a_1a_2ba_2a_1ba_1\underbrace{ba_1\bar{b}a_2}_{\text{braid}}(b_1) \\
&= b_1a_1a_1\underbrace{a_2ba_2a_1ba_1\bar{a}_1}_{\text{braid}}\underbrace{ba_1a_2}_{\text{braid}}(b_1) = b_1a_1a_1ba_2ba_1\underbrace{bba_2a_1}_{\text{braid}}(b_1) \\
&= b_1a_1a_1ba_2\underbrace{ba_1bba_2}_{\text{braid}}(b_1) = b_1a_1a_1ba_2a_1\underbrace{ba_1ba_2}_{\text{braid}}(b_1) = b_1a_1a_1ba_2a_1a_1\underbrace{ba_1a_2}_{\text{braid}}(b_1) \\
&= b_1a_1a_1ba_2a_1a_1\underbrace{ba_2a_1}_{\text{braid}}(b_1) = b_1a_1a_1\underbrace{ba_2a_1a_1ba_2}_{\text{braid}}(b_1) = b_1a_1a_1ba_1a_1\underbrace{a_2ba_2}_{\text{braid}}(b_1) \\
&= b_1a_1a_1ba_1a_1\underbrace{ba_2b}_{\text{braid}}(b_1) = b_1a_1a_1ba_1a_1\underbrace{ba_2}_{\text{braid}}(b_1) = b_1\underbrace{a_1a_1ba_1a_1b\bar{b}_1}_{\text{braid}}(a_2) \\
&= b_1\bar{b}_1a_1a_1ba_1a_1b(a_2) = a_1a_1ba_1a_1b(a_2) = r_1(a_2) \quad \square
\end{aligned}$$

The relation (iii) is interpreted as  $(a_1ba_2b_1c_{1,2})^6 = 1$ . If we regard  $a_1, b, a_2, b_1, c_{1,2}$  as generators of the 6-string braid group, namely,  $a_1$  as an interchange of the 1st and the 2nd string,  $b$  as an interchange of the 2nd and the 3rd string and so on, then  $(a_1ba_2b_1c_{1,2})^6$  is a full twist. By investigating a figure of a 6-string full twist, or repeatedly applying the relations (1), we can show

$$(a_1ba_2b_1c_{1,2})^6 = (a_1ba_2b_1c_{1,2})^2b_1a_2ba_1c_{1,2}b_1a_2b(a_2b_1c_{1,2})^4.$$

By Lemma 2.2,

$$\begin{aligned}
a_1a_1 &= (c_{1,2}c_{1,2}a_2b_1)^3 \\
&= \underbrace{c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} = a_2c_{1,2}\underbrace{c_{1,2}b_1c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} \\
&= \underbrace{a_2c_{1,2}b_1c_{1,2}b_1c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} = a_2b_1c_{1,2}\underbrace{b_1b_1c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} \\
&= a_2b_1c_{1,2}b_1b_1\underbrace{a_2c_{1,2}b_1c_{1,2}c_{1,2}a_2b_1}_{\text{braid}} = a_2b_1c_{1,2}b_1\underbrace{b_1a_2b_1c_{1,2}b_1c_{1,2}a_2b_1}_{\text{braid}} \\
&= a_2b_1c_{1,2}b_1\underbrace{a_2b_1a_2c_{1,2}b_1c_{1,2}a_2b_1}_{\text{braid}} = a_2b_1c_{1,2}a_2b_1\underbrace{a_2a_2c_{1,2}b_1c_{1,2}a_2b_1}_{\text{braid}}
\end{aligned}$$

$$\begin{aligned}
 &= a_2 b_1 c_{1,2} a_2 b_1 c_{1,2} \underbrace{a_2 b_1 a_2}_{\text{braid}} c_{1,2} b_1 = a_2 b_1 c_{1,2} a_2 b_1 c_{1,2} a_2 b_1 a_2 \underbrace{b_1 c_{1,2} b_1}_{\text{braid}} \\
 &= a_2 b_1 c_{1,2} a_2 b_1 c_{1,2} \underbrace{a_2 b_1 a_2}_{\text{braid}} c_{1,2} b_1 c_{1,2} = (a_2 b_1 c_{1,2})^4.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (a_1 b a_2 b_1 c_{1,2})^6 &= (a_1 b a_2 b_1 c_{1,2})^2 b_1 a_2 b a_1 c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} a_1 \underbrace{b a_2 b_1 c_{1,2} b_1 a_2}_{\text{braid}} b a_1 c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} \underbrace{a_1 b a_2 c_{1,2} b_1}_{\text{braid}} \underbrace{c_{1,2} a_2 b a_1}_{\text{braid}} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} a_1 \underbrace{b a_2 b_1 a_2}_{\text{braid}} b a_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} \underbrace{a_1 b b_1 a_2}_{\text{braid}} \underbrace{b_1 b a_1}_{\text{braid}} c_{1,2} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_1 \underbrace{b a_2 b a_1}_{\text{braid}} b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 \underbrace{a_1 a_2}_{\text{braid}} \underbrace{b a_2 a_1}_{\text{braid}} b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 \underbrace{a_1 b a_1}_{\text{braid}} a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 a_1 \\
 &= a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b (a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1) a_1.
 \end{aligned}$$

We have already shown that  $a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1 = E \iff a_1$ . Hence,  $(a_1 b a_2 b_1 c_{1,2})^6 = (a_1 b a_2 b_1 c_{1,2} c_{1,2} b_1 a_2 b a_1)^2 = E^2 = 1$ .

### 3. Elementary relations

In this section, we assume  $g \geq 3$  or  $g = 2, n \geq 1$ . We shall prove some relations in  $G_{g,n}$  which are frequently used in the following sections. The first one is known as the ‘‘lantern relation’’, which is proved in [6, Lemma 3]. So we omit the proof here:

**Lemma 3.1.** *For all good triples  $(i, j, k)$ , one has in  $G_{g,n}$  the relation,*

$$(L_{i,j,k}) : a_i c_{i,j} c_{j,k} a_k = c_{i,k} a_j X a_j \bar{X} = c_{i,k} \bar{X} a_j X a_j,$$

where  $X = b a_i a_k b$ .

The next one is:

**Lemma 3.2.** *If  $i \neq 2k$ , one has in  $G_{g,n}$  the relation,*

$$(X_{i,2k}) : (1) \overline{b_k a_{2k} c_{2k-1,2k} b_k} (c_{i,2k}) = b a_i a_{2k} b (a_{2k-1}),$$



$$(2) b_k a_{2k} c_{2k-1, 2k} b_k(c_{i, 2k}) = \overline{b a_i a_{2k} b}(a_{2k-1}),$$

$$(3) \overline{b_k a_{2k} c_{2k-1, 2k} b_k}(c_{2k, i}) = b a_i a_{2k} b(a_{2k+1}),$$

$$(4) b_k a_{2k} c_{2k-1, 2k} b_k(c_{2k, i}) = \overline{b a_i a_{2k} b}(a_{2k+1}).$$

Proof. We will prove (1). Other relations are proved in the same way. We write  $X_1 = b_k a_{2k} c_{2k-1, 2k} b_k$ ,  $X_2 = b a_i a_{2k} b$  for short. Then,

$$\begin{aligned} \overline{X_2} \overline{X_1}(c_{i, 2k}) &= \overline{b a_{2k} a_i \overline{b b_k} c_{2k-1, 2k} a_{2k} \overline{b_k}(c_{i, 2k})} \\ &= \overline{b a_{2k} a_i \overline{b b_k} c_{2k-1, 2k} a_{2k} c_{i, 2k}(b_k)}. \end{aligned}$$

The lantern relation  $L_{i, 2k-1, 2k}$  says  $c_{i, 2k} = a_{2k} c_{2k-1, 2k} c_{i, 2k-1} a_i \overline{a_{2k-1}} \overline{X_2} a_{2k-1} X_2$ . Therefore,

$$\begin{aligned} \overline{X_2} \overline{X_1}(c_{i, 2k}) &= \overline{b a_{2k} a_i \overline{b b_k} c_{2k-1, 2k} a_{2k} a_{2k} c_{2k-1, 2k} c_{i, 2k-1} a_i \overline{a_{2k-1}} \overline{X_2} a_{2k-1} X_2}(b_k) \\ &= \overline{b a_{2k} a_i \overline{b b_k} c_{i, 2k-1} a_i \overline{a_{2k-1}} \overline{b a_{2k} a_i \overline{b a_{2k-1}} b a_i a_{2k} b}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} c_{i, 2k-1} a_i \overline{a_{2k-1}} \overline{b a_{2k} a_i \overline{b a_{2k-1}} b a_i a_{2k}}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_i \overline{a_{2k-1}} \overline{b a_{2k} a_i \overline{b a_{2k-1}} b a_i a_{2k} c_{i, 2k-1}}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_i \overline{a_{2k-1}} \overline{b a_{2k} a_i \overline{b a_{2k-1}} b a_{2k} a_i}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_i \overline{a_{2k-1}} \overline{b a_{2k} a_i a_{2k-1}} \overline{b a_{2k-1}} a_{2k}}(b_k) \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} a_i \overline{b a_i} a_{2k-1} \overline{a_{2k} b a_{2k} a_{2k-1}}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} \overline{b a_i} \overline{b a_{2k-1}} \overline{b a_{2k} b}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} \overline{b a_i} a_{2k-1} \overline{b a_{2k-1}} \overline{a_{2k}}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} \overline{b a_i} a_{2k-1} \overline{b a_{2k} a_{2k-1}}(b_k)} = \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} \overline{b a_i} a_{2k-1} \overline{b a_{2k}}(b_k)} \\ &= \overline{b a_{2k} a_i \overline{b b_k} a_{2k-1} \overline{b a_i} a_{2k-1} \overline{b b_k}(a_{2k})} = \overline{b a_{2k} a_i \overline{b b_k} b_k a_{2k-1} \overline{b a_i} a_{2k-1} b}(a_{2k}) \\ &= \overline{b a_{2k} a_i \overline{b a_{2k-1}} \overline{b a_{2k-1}} a_i b}(a_{2k}) = \overline{b a_{2k} a_i \overline{b b a_{2k-1}} \overline{b a_i} b}(a_{2k}) \\ &= \overline{b a_{2k} a_i \overline{a_{2k-1} a_i \overline{b a_i}}(a_{2k})} = \overline{b a_{2k} a_i a_i \overline{a_{2k-1}} \overline{b}(a_{2k})} = \overline{b a_{2k} a_{2k-1} a_{2k}}(b) \\ &= \overline{b a_{2k} a_{2k} a_{2k-1}}(b) = \overline{b b}(a_{2k-1}) = a_{2k-1}. \end{aligned}$$

□

The third one is known as the “chain relation”:

**Lemma 3.3.** *One has in  $G_{g,n}$  the relation:*

$$\{(c_{2g-2,2g-1})^2 a_{2g-2} b_{g-1}\}^3 = a_{2g-3} a_{2g-1}.$$

Proof. We write

$$D = c_{2g-2,2g-1} b_{g-1} a_{2g-2} b a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2} b c_{2g-2,2g-1} b_{g-1} a_{2g-2} \\ \times c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1}$$

for short. By using braid relations, we can show  $D(c_{2g-2,2g-1}) = a_{2g-3}$ ,  $D(b_{g-1}) = b$ ,  $D(a_{2g-2}) = a_{2g-2}$ ,  $D(a_{2g-3}) = c_{2g-2,2g-1}$ . For  $D(a_{2g-1})$ ,

$$D(a_{2g-1}) \\ = c_{2g-2,2g-1} b_{g-1} a_{2g-2} b a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2} b \\ \times \frac{c_{2g-2,2g-1} b_{g-1} a_{2g-2} c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} (a_{2g-1})}{\text{braid}} \\ = c_{2g-2,2g-1} b_{g-1} a_{2g-2} \frac{b a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2} b (a_{2g-1})}{\text{braid}} \\ = c_{2g-2,2g-1} b_{g-1} \frac{a_{2g-2} b c_{2g-2,2g-1} b_{g-1} a_{2g-2} a_{2g-3} b (a_{2g-1})}{\text{braid}} \\ = \frac{c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} a_{2g-2} \frac{b b_{g-1} a_{2g-2} a_{2g-3} b (a_{2g-1})}{\text{braid}}}{\text{braid}} \\ = b_{g-1} c_{2g-2,2g-1} \frac{b_{g-1} a_{2g-2} b_{g-1} b a_{2g-2} a_{2g-3} b (a_{2g-1})}{\text{braid}} \\ = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \frac{a_{2g-2} b a_{2g-2} a_{2g-3} b (a_{2g-1})}{\text{braid}} \\ = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} b a_{2g-2} \frac{b a_{2g-3} b (a_{2g-1})}{\text{braid}} \\ = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \frac{b a_{2g-2} a_{2g-3} b a_{2g-3} (a_{2g-1})}{\text{braid} \quad \text{braid}} \\ = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} b a_{2g-3} a_{2g-2} b (a_{2g-1}) \\ = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \overline{b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} (c_{2g-2,2g-3})} \text{ by } X_{2g-3,2g-2}(3) \\ = c_{2g-2,2g-3}.$$

The star relation  $E_{2g-3,2g-3,2g-2}$  of  $G_{g,n}$  says:

$$\{(a_{2g-3})^2 a_{2g-2} b\}^4 = \frac{c_{2g-3,2g-2} c_{2g-2,2g-3}}{\text{handle}} \\ = c_{2g-2,2g-1} c_{2g-2,2g-3}.$$

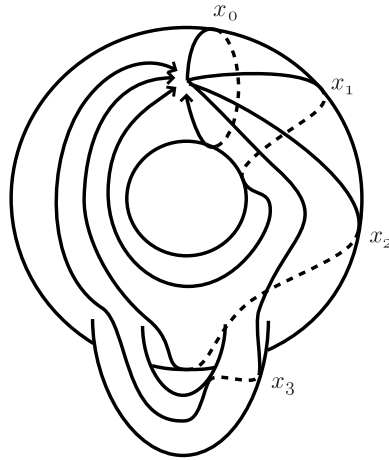


Fig. 3.

We take a conjugation of this equation by  $\bar{D}$ , then we get the equation which we need. □

#### 4. A presentation for $\mathcal{M}_{2,1}$

In this section, we give a presentation for  $\mathcal{M}_{2,1}$  and show that  $\mathcal{M}_{2,1} \cong G_{2,1}$ . For this purpose, it is enough to show that all the relations for  $\mathcal{M}_{2,1}$  are satisfied in  $G_{2,1}$  by the same reason as Section 2.

Let  $p_1$  be a point on  $\Sigma_2$ . We give a presentation for  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$  along the way of [3]. Let  $\alpha$  be a surjection from  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$  to  $\pi_0(\text{Diff}^+(\Sigma_2))$  defined by forgetting the point  $p_1$ . We define a homomorphism  $\beta$  from  $\pi_1(\Sigma_2, p_1)$  to  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$  as follows: The homotopy classes of loops indicated in Fig. 3 generate  $\pi_1(\Sigma_2, p_1)$ . For a loop  $l$  corresponding to one of these generators, we take a regular neighborhood  $A$  of this loop in  $\Sigma_2$ . Since this  $A$  is an annulus, its boundary has two connected components. With regard to the orientation for  $l$ , we denote by  $A_1$  the right hand side of these components, and denote by  $A_2$  the left hand side of them. We define  $\beta$  (which is an element of  $\pi_1(\Sigma_2, p_1)$  corresponding to  $l$ ) to be equal to  $A_1\bar{A}_2$ . For short, we write  $x_i = \beta(x_i)$  ( $i = 0, 1, 2, 3$ ). For these homomorphisms  $\alpha, \beta$ , there is a short exact sequence:

$$(S1) \quad 0 \longrightarrow \pi_1(\Sigma_2, p_1) \xrightarrow{\beta} \pi_0(\text{Diff}^+(\Sigma_2, p_1)) \xrightarrow{\alpha} \pi_0(\text{Diff}^+(\Sigma_2)) \longrightarrow 0.$$

There is a natural surjection from  $\pi_0(\text{Diff}^+(\Sigma_{2,1}, \text{rel } \partial\Sigma_{2,1}))$  to  $\pi_0(\text{Diff}^+(\Sigma_{2,1}/\partial\Sigma_{2,1}, \partial\Sigma_{2,1}/\partial\Sigma_{2,1}))$  and the latter one is isomorphic to  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$ . Hence there is a surjection  $\gamma$  from  $\pi_0(\text{Diff}^+(\Sigma_{2,1}, \text{rel } \partial\Sigma_{2,1})) \cong \mathcal{M}_{2,1}$  to  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$ . The kernel of  $\gamma$  is an infinite cyclic group  $\mathbb{Z}$  generated by

the Dehn twist along the loop  $\partial\Sigma_{2,1}$ , which we denote by  $c_{3,1}$ . Hence, there is a short exact sequence:

$$(S2) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{2,1} \xrightarrow{\gamma} \pi_0(\text{Diff}^+(\Sigma_2, p_1)) \longrightarrow 0$$

In general, if there is a short exact sequence,

$$0 \longrightarrow L \xrightarrow{\phi} G \xrightarrow{\psi} R \longrightarrow 0,$$

and  $L$  and  $R$  are finitely presented, then a finite presentation for  $G$  is given as follows (see, for example, Chapter 10 of [14]). Let  $l_1, \dots, l_m$  be the generators of  $L$  and,  $r_1, \dots, r_n$  be the generators of  $R$ . For each  $1 \leq i \leq m$ , we denote by  $\tilde{l}_i$  the image of  $l_i$  under  $\phi$ , and for each  $1 \leq j \leq n$ , we fix one of the preimages of  $r_j$  by  $\psi$  and denote this  $\tilde{r}_j$ . Then  $G$  is generated by  $\tilde{l}_1, \dots, \tilde{l}_m$  and  $\tilde{r}_1, \dots, \tilde{r}_n$ , and there are the following three types of relations for  $G$ .

(1) For each  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ ,  $\tilde{r}_j \tilde{l}_i \tilde{r}_j^{-1}$  is an element of  $\phi(L)$ . The equation

$$\tilde{r}_j \tilde{l}_i \tilde{r}_j^{-1} = \text{a presentation of } \tilde{r}_j \tilde{l}_i \tilde{r}_j^{-1} \text{ in terms of } \tilde{l}_1, \dots, \tilde{l}_m$$

is a relation for  $G$ ,

(2) Each relation for  $R$  is presented by a word  $w(r_1, \dots, r_n)$ . The element  $w(\tilde{r}_1, \dots, \tilde{r}_n)$  is in the kernel of  $\psi$  and hence it is an element of  $\phi(L)$ . The equation

$$w(\tilde{r}_1, \dots, \tilde{r}_n) = \text{a presentation of } w(\tilde{r}_1, \dots, \tilde{r}_n) \text{ in terms of } \tilde{l}_1, \dots, \tilde{l}_m$$

is a relation for  $G$ ,

(3) For each relation for  $L$ , the equation obtained from this relation by replacing  $l_i$  with  $\tilde{l}_i$  is also a relation for  $G$

We apply this method to the above short exact sequences (S1) and (S2). For (S1), by observing that  $a_1, b, a_2, b_1, c_{1,2}$  in  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$  are mapped, by  $\alpha$ , to the elements of  $\pi_0(\text{Diff}^+(\Sigma_2))$  denoted by the same letters, we can see that  $\pi_0(\text{Diff}^+(\Sigma_2, p_1))$  is generated by  $x_0, x_1, x_2, x_3, a_1, b, a_2, b_1, c_{1,2}$  and its defining relations are:

- (1-a<sub>1</sub>)  $a_1(x_0) = x_0, a_1(x_1) = x_1\bar{x}_0, a_1(x_2) = x_2\bar{x}_0, a_1(x_3) = x_3\bar{x}_0,$
- (1-b)  $b(x_0) = x_1, b(x_1) = x_1\bar{x}_0x_1, b(x_2) = x_2, b(x_3) = x_3,$
- (1-a<sub>2</sub>)  $a_2(x_0) = x_0, a_2(x_1) = x_2, a_2(x_2) = x_2\bar{x}_1x_2, a_2(x_3) = x_3,$
- (1-b<sub>1</sub>)  $b_1(x_0) = x_0, b_1(x_1) = x_1, b_1(x_2) = x_2, b_1(x_3) = x_3\bar{x}_2x_3,$
- (1-c<sub>1,2</sub>)  $c_{1,2}(x_0) = x_0, c_{1,2}(x_1) = x_1, c_{1,2}(x_2) = x_2, c_{1,2}(x_3) = x_3\bar{x}_2x_1\bar{x}_0,$
- (2-1)  $a_1ba_1 = ba_1b, a_2ba_2 = ba_2b, a_2b_1a_2 = b_1a_2b_1, b_1c_{1,2}b_1 = c_{1,2}b_1c_{1,2},$   
other pairs of  $\{a_1, b, a_2, b_1, c_{1,2}\}$  commute each other,
- (2-2)  $(a_1a_1a_2b)^3 \bar{c}_{1,2}^2 \in \beta(\pi_1(\Sigma_2, p_1)),$

$$(3) \quad x_3 \bar{x}_2 x_1 \bar{x}_0 \bar{x}_3 x_2 \bar{x}_1 x_0 = 1.$$

Among the above relations, (1- $a_1$ ) to (1- $c_{1,2}$ ) can be checked by drawing figures of actions of  $a_1, b, a_2, b_1, c_{1,2}$  on  $\pi_1(\Sigma_2, p_1)$ , (2-1) and (2-2) come from the relation (1) and (2), introduced in Section 2, for  $\mathcal{M}_{2,0} \cong G_{2,0}$ , and (3) is a relation for  $\pi_1(\Sigma_2, p_1)$  which is obtained by reading the word on the boundary of an octahedron which is obtained by cutting  $\Sigma_2$  along  $x_0, x_1, x_2, x_3$ . By making use of (S2), we can show that  $\mathcal{M}_{2,1}$  is generated by  $x_0, x_1, x_2, x_3, a_1, b, a_2, b_1, c_{1,2}, c_{3,1}$ , and the defining relations are the relations (1- $a_1$ ) to (3) up to the powers of  $c_{3,1}$ . On the other hand, we can see  $x_0 = a_1 \bar{a}_3, x_1 = b(x_0), x_2 = a_2(x_1), x_3 = b_1(x_2)$  and hence,  $\mathcal{M}_{2,1}$  is generated by  $a_1, a_2, a_3, b, b_1, c_{1,2}, c_{3,1}$ . We can now derive the defining relations for  $\mathcal{M}_{2,1}$  from the relations for  $G_{2,1}$  as follows.

(1) It is shown, in the proof of Lemma 9 in [6], that all the relations (1- $a_1$ ) to (1- $c_{1,2}$ ) up to the powers of  $c_{3,1}$  are derived from the relations for  $G_{2,1}$ . We remark that

$$c_{1,2}(x_3) = x_3 \bar{x}_2 x_1 \bar{x}_0 c_{3,1},$$

which will be used later.

(2-1) These relations are nothing but braid relations.

(2-2) The lantern relation  $L_{2,3,1}$  says

$$a_2 c_{2,3} c_{3,1} a_1 = c_{2,1} a_3 X a_3 \bar{X} = c_{2,1} \bar{X} a_3 X a_3,$$

where  $X = b a_2 a_1 b$ ,

that is to say,

$$\begin{aligned} c_{2,1} \bar{c}_{2,3} &= a_2 c_{3,1} a_1 X \bar{a}_3 \bar{X} \bar{a}_3 && \dots\dots (\alpha) \\ a_1 c_{2,3} \bar{a}_3 &= \bar{c}_{3,1} \bar{a}_2 c_{2,1} \bar{X} a_3 X && \dots\dots (\beta). \end{aligned}$$

The star relation  $E_{1,1,2}$  says  $(a_1 a_1 a_2 b)^3 = c_{1,2} c_{2,1}$ , so that,  $(a_1 a_1 a_2 b)^3 (\bar{c}_{1,2})^2 = c_{2,1} \bar{c}_{1,2}$ . For the right hand of the last equation, we can show,

$$\begin{aligned} c_{2,1} \underbrace{\bar{c}_{1,2}}_{\text{handle}} &= c_{2,1} \bar{c}_{2,3} = \underbrace{a_2 c_{3,1} a_1 X \bar{a}_3 \bar{X} \bar{a}_3}_{\text{braid}} && \text{by } (\alpha) \\ &= c_{3,1} a_2 a_1 X \bar{a}_3 \bar{X} \bar{a}_3 = c_{3,1} a_2 \underbrace{a_1 b a_1 a_2 b \bar{a}_3 \bar{b} \bar{a}_2 \bar{a}_1 \bar{b} \bar{a}_3}_{\text{braid}} \\ &= c_{3,1} a_2 b a_1 \underbrace{b a_2 \bar{b} \bar{a}_3 \bar{b} \bar{a}_2 \bar{a}_1 \bar{b} \bar{a}_3}_{\text{braid}} = c_{3,1} a_2 b a_1 a_2 \underbrace{b a_2 \bar{a}_3 \bar{b} \bar{a}_2 \bar{a}_1 \bar{b} \bar{a}_3}_{\text{braid}} \\ &= c_{3,1} a_2 b a_1 a_2 \underbrace{b \bar{a}_3 a_2 \bar{b} \bar{a}_2 \bar{a}_1 \bar{b} \bar{a}_3}_{\text{braid}} = c_{3,1} a_2 b a_1 a_2 \underbrace{b \bar{a}_3 \bar{b} \bar{a}_2}_{\text{braid}} \underbrace{\bar{a}_1 \bar{b} \bar{a}_3}_{\text{braid}} \\ &= c_{3,1} a_2 b a_1 \underbrace{a_2 \bar{a}_3 \bar{b} \bar{a}_3 \bar{a}_2 \bar{a}_1}_{\text{braid}} \underbrace{\bar{b} a_1 \bar{a}_3}_{\text{braid}} = c_{3,1} a_2 b a_1 \bar{a}_3 \underbrace{a_2 \bar{b} \bar{a}_2}_{\text{braid}} \bar{a}_1 a_3 b a_1 \bar{a}_3 \\ &= c_{3,1} a_2 b a_1 \bar{a}_3 \bar{b} \bar{a}_2 \bar{b} \bar{a}_1 a_3 b a_1 \bar{a}_3 = c_{3,1} x_2 \bar{x}_1 x_0. \end{aligned}$$



$$= c_{1,2}x_3\bar{c}_{1,2}c_{1,2}\bar{x}_3\bar{c}_{1,2}(\bar{c}_{3,1})^2 = (\bar{c}_{3,1})^2.$$

This shows that, modulo powers of  $c_{3,1}, x_3\bar{x}_2x_1\bar{x}_0\bar{x}_3x_2\bar{x}_1x_0 = 1$  is derived from relations for  $G_{2,1}$ .

From the above results, we can now conclude:

**Proposition 4.1.**  $\mathcal{M}_{2,1} \cong G_{2,1}$ .

**5. Action of  $\mathcal{M}_{g,n}$  on  $X(\Sigma_{g,n})$  and a presentation for  $\mathcal{M}_{g,n}$**

In this section, we assume  $g \geq 3$ , and  $n \geq 1$ . We call a simple closed curve on  $\Sigma_{g,n}$  *non-separating*, if its complement is connected. Define a simplicial complex  $X(\Sigma_{g,n})$  of dimension  $g - 1$ , whose vertices (0-simplices) are the isotopy classes of non-separating simple closed curves on  $\Sigma_{g,n}$ , and whose simplices are determined by the rule that a collection of  $k + 1$  distinct vertices spans a  $k$ -simplex if and only if it admits a collection of representative which are pairwise disjoint and the complement of their disjoint union is connected. This complex  $X(\Sigma_{g,n})$  is defined by Harer [9]. In the same paper, he showed the following Theorem:

**Theorem 5.1** ([9, Theorem 1.1]).  $X(\Sigma_{g,n})$  is homotopy equivalent to a wedge of  $(g - 1)$ -dimensional spheres.

Especially, if  $g \geq 3$ ,  $X(\Sigma_{g,n})$  is simply connected.

For each element  $\phi$  of  $\mathcal{M}_{g,n}$  and a simplex  $([C_0], \dots, [C_n])$  of  $X(\Sigma_{g,n})$ ,  $([\phi(C_0)], \dots, [\phi(C_n)])$  is also a simplex of  $X(\Sigma_{g,n})$ . Hence, we can define an action of  $\mathcal{M}_{g,n}$  on  $X(\Sigma_{g,n})$  by  $\phi([C_0], \dots, [C_n]) = ([\phi(C_0)], \dots, [\phi(C_n)])$ . We can see that, each of  $\{2\text{-simplices of } X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$ ,  $\{1\text{-simplices of } X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$  and  $\{\text{vertices of } X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$  consists of one element, each of which is represented by  $([C_0], [C_1], [C_2])$ ,  $([C_0], [C_1])$ , and  $([C_0])$ , where  $C_0 = c_{2g-2,2g-1}$ ,  $C_1 = a_{2g-2}$ ,  $C_2 = a_{2g-4}$ . If the stabilizer of each vertex is finitely presented, and if that of each 1-simplex is finitely generated, we can obtain a presentation for  $\mathcal{M}_{g,n}$  as in the way of [15], [20]. Here, we shall recall this method.

We fix a vertex  $v_0$  of  $X(\Sigma_{g,n})$ , fix an edge (= a 1-simplex with orientation)  $e_0$  of  $X(\Sigma_{g,n})$  which emanates from  $v_0$ , and fix a 2-simplex  $f_0$  of  $X(\Sigma_{g,n})$  which contains  $v_0$ . Let  $C_0, C_1$  and  $C_2$  be non-separating simple closed curves defined as above, and we set  $v_0 = [C_0]$ ,  $e_0 = ([C_0], [C_1])$  and  $f_0 = ([C_0], [C_1], [C_2])$ . We choose an element  $t_1$  of  $\mathcal{M}_{g,n}$  which switches the vertices of  $e_0$ . In our situation, we set  $t_1 = b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}$ . By this notation, we see  $e_0 = (v_0, t_1(v_0))$ . We denote by  $(\mathcal{M}_{g,n})_{v_0}$  the stabilizer of  $v_0$ , by  $(\mathcal{M}_{g,n})_{e_0}$  that of  $e_0$ , and by  $\langle t_1 \rangle$  an infinite cyclic group generated by  $t_1$ . The free product  $(\mathcal{M}_{g,n})_{v_0} * \langle t_1 \rangle$  with the following three types of relations defines a presentation for  $\mathcal{M}_{g,n}$ . (In Subsection 5.1, we give a set of generators for  $(\mathcal{M}_{g,n})_{v_0}$ . In the following statements, ‘‘a presentation of  $s$  as an element

of  $(\mathcal{M}_{g,n})_{v_0}$ ” means a presentation of  $s$  as a word of elements of this set of generators.)

(Y1)  $t_1^2 =$  a presentation of  $t_1^2$  as an element of  $(\mathcal{M}_{g,n})_{v_0}$ .

(Y2) For each generator  $s$  of  $(\mathcal{M}_{g,n})_{e_0}$ ,

$$\begin{aligned} & t_1(\text{a presentation of } s \text{ as an element of } (\mathcal{M}_{g,n})_{v_0})\overline{t_1} \\ & = \text{a presentation of } t_1 s \overline{t_1} \text{ as an element of } (\mathcal{M}_{g,n})_{v_0}. \end{aligned}$$

(Y3) For the loop  $\partial f_0$  in  $X(\Sigma_{g,n})$ , we define an element  $W_{f_0}$  of  $(\mathcal{M}_{g,n})_{v_0} * \langle t_1 \rangle$  in the following manner. The loop  $\partial f_0$  consists of three vertices  $v_0, v_1, v_2$  and three edges  $e_1, e_2, e_3$  such that  $e_1 = (v_0, v_1), e_2 = (v_1, v_2), e_3 = (v_2, v_0)$ . There is an element  $h_1$  of  $(\mathcal{M}_{g,n})_{v_0}$  such that  $h_1(e_0) = e_1$  i.e.  $e_1 = (v_0, h_1 t_1(v_0))$ , then  $\overline{h_1 t_1}(e_2)$  is an edge emanating from  $v_0$ . Hence, there is an element  $h_2$  of  $(\mathcal{M}_{g,n})_{v_0}$  such that  $h_2(e_0) = \overline{h_1 t_1}(e_2)$  i.e.  $e_2 = (h_1 t_1(v_0), h_1 t_1 h_2 t_1(v_0))$ , then  $\overline{h_1 t_1 h_2 t_1}(e_3)$  is an edge emanating from  $v_0$ . So, there is an element  $h_3$  of  $(\mathcal{M}_{g,n})_{v_0}$  such that  $h_3(e_0) = \overline{h_1 t_1 h_2 t_1}(e_3)$  i.e.  $e_3 = (h_1 t_1 h_2 t_1(v_0), h_1 t_1 h_2 t_1 h_3 t_1(v_0))$ . We define  $W_{f_0} = h_1 t_1 h_2 t_1 h_3 t_1$ . This element  $W_{f_0}$  fixes  $v_0$ , so the following is a relation for  $\mathcal{M}_{g,n}$ :

$$W_{f_0} = \text{a presentation of } W_{f_0} \text{ as an element of } (\mathcal{M}_{g,n})_{v_0}.$$

Under the assumption that  $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$ , if we can show all the relations for  $(\mathcal{M}_{g,n})_{v_0}$  and the relations of the above three types (Y1) (Y2) (Y3) are satisfied in  $G_{g,n}$ , then we can show the following theorem by the same reason as Section 2.

**Theorem 5.2.** *If  $g \geq 3, n \geq 1$  and  $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$ , then  $\mathcal{M}_{g,n} \cong G_{g,n}$ .*

In the previous section, we have shown  $\mathcal{M}_{2,1} \cong G_{2,1}$  (Proposition 4.1), therefore,  $\mathcal{M}_{g,1} \cong G_{g,1}$  for any  $g \geq 2$ . On the other hand, Gervais showed the following theorem in §3 of [6]:

**Theorem 5.3.** *If  $g \geq 1, n \geq 1$  and  $\mathcal{M}_{g,n} \cong G_{g,n}$ , then  $\mathcal{M}_{g,n+1} \cong G_{g,n+1}, \mathcal{M}_{g,n-1} \cong G_{g,n-1}$ .*

Theorem 1.1 is proved by Theorem 5.2 and Theorem 5.3. We remark that Theorem 5.3 was proved without using Wajnryb’s simple presentation [20]. In the following subsections, we show all relations for  $(\mathcal{M}_{2,1})_{v_0}$  (Subsection 5.1), relations of type (Y1) and (Y2) (Subsection 5.2), and a relation of type (Y3) (Subsection 5.3) are satisfied in  $G_{g,n}$ .

**5.1. A presentation for  $(\mathcal{M}_{g,n})_{v_0}$ .** We assume that  $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$ , and  $n \geq 1$ . Let  $\text{Diff}^+(\Sigma_{g,n})$  denote the group of orientation preserving diffeomorphisms of  $\Sigma_{g,n}$ . For subsets  $A_1, \dots, A_m$  and  $B$  of  $\Sigma_{g,n}$ , we define  $\text{Diff}^+(\Sigma_{g,n}, A_1, \dots, A_m, \text{rel } B) =$



$\{\phi \in \text{Diff}^+(\Sigma_{g,n}) \mid \phi(A_1) = A_1, \dots, \phi(A_m) = A_m, \phi|_B = \text{id}_B\}$ . In this subsection, we give a presentation for  $(\mathcal{M}_{g,n})_{v_0} = \pi_0(\text{Diff}^+(\Sigma_{g,n}, C_0, \text{rel } \partial\Sigma_{g,n}))$ . Let  $\Sigma'_{g,n}$  be a surface obtained from  $\Sigma_{g,n}$  by cutting along  $C_0$ , and let  $E_1, E_2$  be connected components of  $\partial\Sigma'_{g,n}$  which appeared as a result of cutting. Let  $\alpha$  be a natural surjection from  $\pi_0(\text{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \text{rel } \partial\Sigma_{g,n}))$  to  $\mathbb{Z}_2$  which is a permutation group of  $E_1$  and  $E_2$ , and  $\beta$  be an inclusion of  $\pi_0(\text{Diff}^+(\Sigma'_{g,n}, \text{rel } \partial\Sigma'_{g,n}))$  into  $\pi_0(\text{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \text{rel } \partial\Sigma_{g,n}))$ . Then, there is a short exact sequence:

$$0 \longrightarrow \pi_0(\text{Diff}^+(\Sigma'_{g,n}, \text{rel } \partial\Sigma'_{g,n})) \xrightarrow{\beta} \pi_0(\text{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \text{rel } \partial\Sigma_{g,n})) \xrightarrow{\alpha} \mathbb{Z}_2 \longrightarrow 0$$

We can see that

$$\pi_0(\text{Diff}^+(\Sigma_{g,n}, C_0, \text{rel } \partial\Sigma_{g,n})) \cong \frac{\pi_0(\text{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \text{rel } \partial\Sigma_{g,n}))}{c_{2g-2,2g-1} = c_{2g-3,2g-2}}$$

and

$$\pi_0(\text{Diff}^+(\Sigma'_{g,n}, \text{rel } \partial\Sigma'_{g,n})) \cong \mathcal{M}_{g-1,n+2}.$$

By Theorem 5.3,  $\pi_0(\text{Diff}^+(\Sigma'_{g,n}, \text{rel } \partial\Sigma'_{g,n})) \cong G_{g-1,n+2}$ . Let  $r_{g-1} = \{(c_{2g-3,2g-2})^2 b_{g-1}\}^2$ . Then  $r_{g-1} \in \pi_0(\text{Diff}^+(\Sigma_{g,n}, C_0, \text{rel } \partial\Sigma_{g,n}))$ , that is to say, we can regard  $r_{g-1}$  as an element of  $\pi_0(\text{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \text{rel } \partial\Sigma_{g,n}))$ . Then  $\alpha(r_{g-1})$  generates  $\mathbb{Z}_2$ . From the above observations, we can see:

$\pi_0(\text{Diff}^+(\Sigma_{g,n}, C_0, \text{rel } \partial\Sigma_{g,n}))$  is isomorphic to  $G_{g-1,n+2} * \langle r_{g-1} \rangle$  with the following relations:

- (A1)  $c_{2g-2,2g-1} = c_{2g-3,2g-2}$ ,
- (A2) For each generator  $t$  of  $G_{g-1,n+2}$ ,

$$r_{g-1} t \bar{r}_{g-1} = \text{ a presentation of } r_g t \bar{r}_g \text{ as an element of } G_{g-1,n+2},$$

(A3)  $r_{g-1}^2 = c_{2g-3,2g-1}$ .

We need to show that these relations are derived from relations for  $G_{g,n}$ .

- (1) The relation (A1) is nothing but a handle relation.
- (2) By repeatedly applying star relations, we can show  $G_{g-1,n+2}$  is generated by  $\mathcal{E} = \{b, a_i \ (1 \leq i \leq 2g+n-2), c_{2j-1,2j} \ (1 \leq j \leq g-2), c_{k-1,k} \ (2g-2 \leq k \leq 2g+n-2), c_{2g+n-2,1}\}$ . Here, we remark that  $(\mathcal{M}_{g,n})_{v_0}$  is generated by  $\mathcal{E} \cup \{r_{g-1}\}$ . By drawing figures, we can show:

$$\begin{aligned} r_{g-1}(b) &= b, \quad r_{g-1}(a_i) = a_i \quad \text{if } i \neq 2g-2, \ 1 \leq i \leq 2g+n-2 \\ r_{g-1}(c_{2j-1,2j}) &= c_{2j-1,2j} \quad \text{if } 1 \leq j \leq g-2 \\ r_{g-1}(c_{2g-3,2g-2}) &= c_{2g-2,2g-1}, \quad r_{g-1}(c_{2g-2,2g-1}) = c_{2g-3,2g-2} \end{aligned}$$

$$\begin{aligned}
 r_{g-1}(c_{k-1,k}) &= c_{k-1,k} \quad \text{if } 2g \leq k \leq 2g+n-2 \\
 r_{g-1}(c_{2g+n-2,1}) &= c_{2g+n-2,1} \\
 r_{g-1}a_{2g-2}\bar{r}_{g-1}c_{2g-3,2g-1}a_{2g-2} &= a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2 \cdots \cdots (*)
 \end{aligned}$$

The above equations except (\*) are derived from braid relation. We shall show that the equation (\*) is satisfied in  $G_{g,n}$ .

$$\begin{aligned}
 r_{g-1}(a_{2g-2}) &= c_{2g-3,2g-2} \underbrace{c_{2g-3,2g-2} b_{g-1} c_{2g-3,2g-2} c_{2g-3,2g-2} b_{g-1}}_{\text{braid}} (a_{2g-2}) \\
 &= \underbrace{c_{2g-3,2g-2} b_{g-1} c_{2g-3,2g-2}}_{\text{braid}} \underbrace{b_{g-1} c_{2g-3,2g-2} b_{g-1}}_{\text{braid}} (a_{2g-2}) \\
 &= b_{g-1} c_{2g-3,2g-2} \underbrace{b_{g-1} c_{2g-3,2g-2} b_{g-1}}_{\text{braid}} c_{2g-3,2g-2} (a_{2g-2}) \\
 &= b_{g-1} c_{2g-3,2g-2} c_{2g-3,2g-2} b_{g-1} \underbrace{c_{2g-3,2g-2} c_{2g-3,2g-2}}_{\text{braid}} (a_{2g-2}) \\
 &= b_{g-1} c_{2g-3,2g-2} c_{2g-3,2g-2} b_{g-1} (a_{2g-2}).
 \end{aligned}$$

By a star relation  $E_{2g-3,2g-2,2g-1}$  and a handle relation  $c_{2g-3,2g-2} = c_{2g-2,2g-1}$ ,

$$c_{2g-3,2g-2} c_{2g-3,2g-2} c_{2g-1,2g-3} = (a_{2g-3} a_{2g-2} a_{2g-1} b)^3.$$

Therefore we have,

$$\begin{aligned}
 r_{g-1}(a_{2g-2}) &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^3 \underbrace{\bar{c}_{2g-1,2g-3}}_{\text{braid}} b_{g-1} (a_{2g-2}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^3 b_{g-1} \underbrace{\bar{c}_{2g-1,2g-3}}_{\text{braid}} (a_{2g-2}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^3 \underbrace{b_{g-1}}_{\text{braid}} (a_{2g-2}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^2 a_{2g-3} \underbrace{a_{2g-2} a_{2g-1}}_{\text{braid}} \bar{b} a_{2g-2} (b_{g-1}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^2 a_{2g-3} a_{2g-1} \underbrace{a_{2g-2} \bar{b} a_{2g-2}}_{\text{braid}} (b_{g-1}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^2 a_{2g-3} a_{2g-1} \bar{b} a_{2g-2} \underbrace{b}_{\text{braid}} (b_{g-1}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^2 a_{2g-3} a_{2g-1} \underbrace{\bar{b} a_{2g-2}}_{\text{braid}} (b_{g-1}) \\
 &= b_{g-1} (a_{2g-3} a_{2g-2} a_{2g-1} b)^2 \underbrace{a_{2g-3} a_{2g-1} \bar{b} \bar{b}}_{\text{braid}} (a_{2g-2}) \\
 &= b_{g-1} (a_{2g-3} \underbrace{a_{2g-2} a_{2g-1}}_{\text{braid}} b)^2 \bar{b}_{g-1} a_{2g-3} a_{2g-1} \bar{b} (a_{2g-2}) \\
 &= \underbrace{(b_{g-1} a_{2g-3} a_{2g-1} a_{2g-2} \bar{b} \bar{b}_{g-1})}_{\text{braid}}^2 a_{2g-3} a_{2g-1} \bar{b} (a_{2g-2})
 \end{aligned}$$

$$\begin{aligned}
&= (a_{2g-3}a_{2g-1}\underbrace{b_{g-1}a_{2g-2}\bar{b}_{g-1}b}_{\text{braid}})^2 a_{2g-3}a_{2g-1}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}b_{g-1}a_{2g-2}\underbrace{ba_{2g-3}a_{2g-1}\bar{a}_{2g-2}b_{g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}b_{g-1}\underbrace{a_{2g-2}\bar{b}a_{2g-2}a_{2g-3}a_{2g-1}b_{g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\underbrace{b_{g-1}\bar{b}a_{2g-2}ba_{2g-3}a_{2g-1}b_{g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}\underbrace{ba_{2g-3}a_{2g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}\underbrace{ba_{2g-3}a_{2g-2}a_{2g-1}ba_{2g-3}a_{2g-1}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}\underbrace{ba_{2g-3}a_{2g-2}ba_{2g-1}ba_{2g-3}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}\underbrace{ba_{2g-3}a_{2g-2}ba_{2g-1}\bar{a}_{2g-3}ba_{2g-3}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}a_{2g-2}b_{g-1}\underbrace{a_{2g-2}ba_{2g-2}}_{\text{braid}}\underbrace{a_{2g-3}\bar{b}a_{2g-3}a_{2g-1}b}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}\bar{b}b_{g-1}\underbrace{ba_{2g-2}b\bar{b}a_{2g-3}ba_{2g-1}b}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}a_{2g-3}a_{2g-1}\underbrace{ba_{2g-1}(a_{2g-2})}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}a_{2g-3}a_{2g-1}\underbrace{b(a_{2g-2})}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}\underbrace{a_{2g-3}a_{2g-1}\bar{a}_{2g-2}(b)}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}\bar{a}_{2g-2}a_{2g-3}a_{2g-1}(b) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}\underbrace{b_{g-1}a_{2g-3}a_{2g-1}(b)}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}a_{2g-3}a_{2g-1}\underbrace{b_{g-1}(b)}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}ba_{2g-3}a_{2g-1}\underbrace{\bar{a}_{2g-2}(b)}_{\text{braid}} = \underbrace{a_{2g-3}a_{2g-1}ba_{2g-3}a_{2g-1}b}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-1}\underbrace{a_{2g-3}ba_{2g-3}a_{2g-1}b}_{\text{braid}}(a_{2g-2}) = a_{2g-1}\underbrace{ba_{2g-3}ba_{2g-1}b}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-1}\underbrace{ba_{2g-3}a_{2g-1}ba_{2g-1}(a_{2g-2})}_{\text{braid}} = \underbrace{a_{2g-1}ba_{2g-1}a_{2g-3}b}_{\text{braid}}(a_{2g-2}) \\
&= \underbrace{ba_{2g-1}ba_{2g-3}b}_{\text{braid}}(a_{2g-2}) = \underbrace{ba_{2g-1}a_{2g-3}ba_{2g-3}(a_{2g-2})}_{\text{braid}} = \underbrace{ba_{2g-3}a_{2g-1}b}_{\text{braid}}(a_{2g-2}).
\end{aligned}$$

The lantern relation  $L_{2g-3,2g-2,2g-1}$  says,

$$c_{2g-3,2g-1}a_{2g-2}ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b} = a_{2g-3}c_{2g-3,2g-2}c_{2g-2,2g-1}a_{2g-1}.$$

Then,

$$\begin{aligned} ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-1}\bar{b} &= \frac{\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1}a_{2g-3}c_{2g-3,2g-2}c_{2g-2,2g-1}a_{2g-1}}{\text{braid}} \\ &= a_{2g-1}a_{2g-3}c_{2g-3,2g-2}\frac{c_{2g-2,2g-1}\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1}}{\text{handle}} \\ &= a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1}. \end{aligned}$$

Therefore,

$$ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b}c_{2g-3,2g-1}a_{2g-2} = a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2.$$

In the above equation, we exchange

$$ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b} = ba_{2g-3}a_{2g-1}b(a_{2g-2})$$

with  $r_{g-1}(a_{2g-2})$ , then we get (\*). Hence the relation (A2) is satisfied in  $G_{g,n}$ .

(3) At first, we can see:

$$\begin{aligned} r_{g-1}a_{2g-2}r_{g-1}\frac{a_{2g-2}(\bar{c}_{2g-2,2g-1})^2}{\text{braid}} &= \frac{\{(c_{2g-3,2g-2})^2b_{g-1}\}^2a_{2g-2}\{(c_{2g-3,2g-2})^2b_{g-1}\}^2a_{2g-2}(\bar{c}_{2g-2,2g-1})^2}{\text{handle} \quad \text{handle} \quad \text{braid}} \\ &= \{(c_{2g-2,2g-1})^2b_{g-1}\}^2 \\ &\quad \times a_{2g-2}c_{2g-2,2g-1}\frac{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2}}{\text{braid}} \\ &= \{(c_{2g-2,2g-1})^2b_{g-1}\}^2 \\ &\quad \times a_{2g-2}\frac{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}}{\text{braid}}\frac{b_{g-1}c_{2g-2,2g-1}b_{g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2}}{\text{braid}} \\ &= \{(c_{2g-2,2g-1})^2b_{g-1}\}^2 \\ &\quad \times a_{2g-2}b_{g-1}c_{2g-2,2g-1}\frac{b_{g-1}c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2}}{\text{braid}} \\ &= \{(c_{2g-2,2g-1})^2b_{g-1}\}^2 \\ &\quad \times a_{2g-2}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2} \\ &= (c_{2g-2,2g-1})^2b_{g-1}(c_{2g-2,2g-1})^2\frac{b_{g-1}a_{2g-2}b_{g-1}}{\text{braid}}(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2} \\ &= (c_{2g-2,2g-1})^2b_{g-1}\frac{(c_{2g-2,2g-1})^2a_{2g-2}b_{g-1}a_{2g-2}(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2}}{\text{braid}} \\ &= \{(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2}\}^3 \\ &= a_{2g-3}a_{2g-1} \quad \text{by Lemma 3.3.} \end{aligned}$$

Therefore,

$$r_{g-1}^2 = r_{g-1} \bar{a}_{2g-2} \bar{r}_{g-1} a_{2g-1} a_{2g-3} (c_{2g-2,2g-1})^2 \bar{a}_{2g-2}.$$

From the above equation and (\*), we can see  $r_{g-1}^2 = c_{2g-3,2g-1}$ .

**5.2. Generators of  $(\mathcal{M}_{g,n})_{e_0}$ , and relations of type (Y1) and (Y2).** In this subsection, we give generators of

$$(\mathcal{M}_{g,n})_{e_0} = \pi_0(\text{Diff}^+(\Sigma_{g,n}, c_{2g-2,2g-1}, a_{2g-4}, \text{rel } \partial \Sigma_{g,n}))$$

and, by investigating the action of  $t_1$  on these elements, we will give relations of type (Y2), and show that these relations and a relation of type (Y1) are satisfied in  $G_{g,n}$ .

At first, we show  $t_1^2 \in (\mathcal{M}_{g,n})_{v_0}$ . By Lemma 3.3 and braid relations,

$$\begin{aligned} & a_{2g-3} a_{2g-1} \\ &= c_{2g-2,2g-1} c_{2g-2,2g-1} \\ & \quad \times a_{2g-2} b_{g-1} \underbrace{c_{2g-2,2g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} c_{2g-2,2g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}}_{\text{braid}} \\ &= c_{2g-2,2g-1} c_{2g-2,2g-1} \\ & \quad \times a_{2g-2} b_{g-1} a_{2g-2} c_{2g-2,2g-1} \underbrace{c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}}_{\text{braid}} \\ &= c_{2g-2,2g-1} c_{2g-2,2g-1} \\ & \quad \times a_{2g-2} b_{g-1} a_{2g-2} \underbrace{c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}}_{\text{braid}} \\ &= c_{2g-2,2g-1} c_{2g-2,2g-1} a_{2g-2} \underbrace{b_{g-1} a_{2g-2} b_{g-1} c_{2g-2,2g-1} b_{g-1} b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}}_{\text{braid}} \\ &= c_{2g-2,2g-1} c_{2g-2,2g-1} a_{2g-2} a_{2g-2} b_{g-1} \underbrace{a_{2g-2} c_{2g-2,2g-1} b_{g-1} b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}}_{\text{braid}} \\ &= (c_{2g-2,2g-1})^2 (a_{2g-2})^2 t_1^2 \quad \text{since } t_1 = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}. \end{aligned}$$

Therefore,  $t_1^2 = (\bar{a}_{2g-2})^2 (\bar{c}_{2g-2,2g-1})^2 a_{2g-3} a_{2g-1} \in (\mathcal{M}_{g,n})_{v_0}$ . This shows that the relation of type (Y1) is satisfied in  $G_{g,n}$ .

Let  $\Sigma''_{g,n}$  be a surface obtained from  $\Sigma_{g,n}$  by cutting along  $C_0 = c_{2g-2,2g-1}$ ,  $C_1 = a_{2g-2}$ . As in Fig. 4, let  $C'_0$  and  $C''_0$  (resp.  $C'_1$  and  $C''_1$ ) be connected components of  $\partial \Sigma''_{g,n}$  which appeared as a result of cutting along  $C_0$  (resp.  $C_1$ ). We denote the simple closed curve in the interior of  $\Sigma''_{g,n}$  which is homotopic to  $C'_0$  (resp.  $C''_0$ ,  $C'_1$ ,  $C''_1$ ) and Dehn twist along this curve by the same letter. We can see that  $G_{g-2,n+4} \cong \mathcal{M}_{g-2,n+4}$  is generated by  $a_i$  ( $1 \leq i \leq 2(g-3) + (n+4)$ ),  $b$ ,  $b_j$  ( $1 \leq j \leq g-3$ ),  $c_{2k-1,2k}$  ( $1 \leq k \leq g-3$ ),  $c_{l,l+1}$  ( $2(g-3) + 1 \leq l \leq 2(g-3) + (n+3)$ ), and  $c_{2(g-3)+(n+4),1}$ . There is a homomorphism  $\gamma$  from  $\mathcal{M}_{g-2,n+4}$  to  $\pi_0(\text{Diff}^+(\Sigma''_{g,n}, \text{rel } \partial \Sigma''_{g,n}))$  defined by

$$\gamma(a_i) = c_{i,2g-4} \quad \text{if } 1 \leq i \leq 2g-5,$$

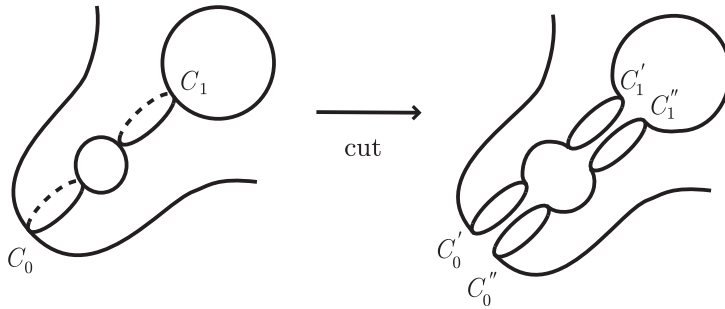


Fig. 4.

$$\begin{aligned}
 \gamma(a_{2g-4}) &= c_{2g-4,2g-2}, \\
 \gamma(a_{2g-3}) &= a_{2g-4}, \\
 \gamma(a_i) &= c_{i,2g-4} \quad \text{if } 2g-2 \leq i \leq 2(g-3) + (n+4), \\
 \gamma(b) &= b_{g-2}, \\
 \gamma(b_j) &= b_j \quad \text{if } 1 \leq j \leq g-3, \\
 \gamma(c_{2k-1,2k}) &= c_{2k-1,2k} \quad \text{if } 1 \leq k \leq g-3, \\
 \gamma(c_{2(g-3)+1,2(g-3)+2}) &= C''_0, \\
 \gamma(c_{2(g-3)+2,2(g-3)+3}) &= C''_1, \\
 \gamma(c_{2(g-3)+3,2(g-3)+4}) &= C'_1, \\
 \gamma(c_{2(g-3)+4,2(g-3)+5}) &= C'_0, \\
 \gamma(c_{l,l+1}) &= c_{l,l+1} \quad \text{if } 2(g-3) + 5 \leq l \leq 2(g-3) + (n+3), \\
 \gamma(c_{2(g-3)+(n+4),1}) &= a_{2g-2}.
 \end{aligned}$$

This homomorphism is induced by a homeomorphism from  $\Sigma_{g-2,n+4}$  to  $\Sigma''_{g,n}$ . Hence,  $\gamma$  is an isomorphism, and this fact means that the set

$$\mathcal{C}''_{g,n} = \left\{ \begin{array}{l|l} c_{i,2g-4}, c_{2g-4,2g-2}, & 1 \leq i \leq 2g-5, \\ b_{g-2}, b_j, c_{2j-1,2j}, & 2g-2 \leq i \leq 2(g-3) + (n+4), \\ C'_0, C''_0, C'_1, C''_1, & 1 \leq j \leq g-3, \\ c_{l,l+1}, c_{2(g-3)+(n+4),1} & 2(g-3) + 5 \leq l \leq 2(g-3) + (n+3) \end{array} \right\}$$

generates  $\pi_0(\text{Diff}^+(\Sigma''_{g,n}, \text{rel } \partial \Sigma''_{g,n}))$ . Let  $\mathbb{Z}_2 \times \mathbb{Z}_2$  denote the group, whose first factor is a permutation group of  $C'_0$  and  $C''_0$  and the second factor is that of  $C'_1$  and  $C''_1$ . We denote by  $\delta$  a natural homomorphism from  $\pi_0(\text{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \text{rel } \partial \Sigma''_{g,n}))$  to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , and  $\epsilon$  an inclusion of  $\pi_0(\text{Diff}^+(\Sigma''_{g,n}, \text{rel } \partial \Sigma''_{g,n}))$  into

$$\pi_0(\text{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \text{rel } \partial \Sigma''_{g,n})).$$

Then, there is a short exact sequence,

$$0 \longrightarrow \pi_0(\text{Diff}^+(\Sigma''_{g,n}, \text{rel } \partial\Sigma''_{g,n})) \\ \xrightarrow{\epsilon} \pi_0(\text{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \text{rel } \partial\Sigma_{g,n})) \xrightarrow{\delta} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 0$$

Let  $p = ba_{2g-2}a_{2g-2}b$ ,  $p' = t_1p\bar{t}_1$ . Then, by drawing some figures, we can check that  $p$  and  $p' \in (\mathcal{M}_{g,n})_{e_0}$  and  $p$  (resp.  $p'$ ) reverse the orientation of  $C_1$  (resp.  $C_0$ ). Hence,  $p$  induces a homeomorphism on  $\Sigma''_{g,n}$  which exchanges  $C'_0$  with  $C''_0$  (resp.  $C'_1$  with  $C''_1$ ). On the other hand, there is an isomorphism

$$\frac{\pi_0(\text{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \text{rel } \partial\Sigma_{g,n}))}{(C'_0 = C''_0, C'_1 = C''_1)} \\ \cong \pi_0(\text{Diff}^+(\Sigma_{g,n}, c_{2g-2,2g-1}, a_{2g-2}, \text{rel } \partial\Sigma_{g,n})),$$

which maps  $C'_0 = C''_0$  to  $c_{2g-2,2g-1}$ ,  $C'_1 = C''_1$  to  $a_{2g-2}$ . Therefore, we can show that  $(\mathcal{M}_{g,n})_{e_0}$  is generated by  $(C''_{g,n} - \{C'_0, C''_0, C'_1, C''_1\}) \cup \{c_{2g-2,2g-1}, a_{2g-2}, p, p'\}$ . For each element  $s$  of  $C''_{g,n} - \{c_{2g-2,2g-1}, c_{2g-4,2g-2}, C'_0, C''_0, C'_1, C''_1\}$ , the associated curve of  $s$  is disjoint from those of  $b_{g-1}$ ,  $a_{2g-2}$ , and  $c_{2g-2,2g-1}$ . Hence, by braid relations,  $t_1s\bar{t}_1 = s \in (\mathcal{M}_{g,n})_{v_0}$ . This fact shows that, for the above element  $s$ , the relation of type (Y2) is satisfied in  $G_{g,n}$ .

In Subsection 5.1, we showed that  $(\mathcal{M}_{g,n})_{v_0}$  is generated by  $\mathcal{E} \cup \{r_{g-1}\}$ , so a presentation of some element as an element of  $(\mathcal{M}_{g,n})_{v_0}$  means a presentation of this elements as a word of  $\mathcal{E} \cup \{r_{g-1}\}$ . Here, we need to present  $p$  and  $p'$  as words of these elements. Since  $b, a_{2g-2} \in \mathcal{E}$ ,  $p$  is presented as an element of  $(\mathcal{M}_{g,n})_{v_0}$ . We shall present  $p'$  as an element of  $(\mathcal{M}_{g,n})_{v_0}$ .

$$a_{2g-2}bt_1(b) = a_{2g-2}bb_{g-1}\frac{c_{2g-2,2g-1}a_{2g-2}}{\text{braid}}\frac{b_{g-1}(b)}{\text{braid}} \\ = a_{2g-2}bb_{g-1}a_{2g-2}\frac{c_{2g-2,2g-1}(b)}{\text{braid}} = a_{2g-2}bb_{g-1}\frac{a_{2g-2}(b)}{\text{braid}} \\ = a_{2g-2}\frac{bb_{g-1}\bar{b}(a_{2g-2})}{\text{braid}} = a_{2g-2}\frac{b_{g-1}(a_{2g-2})}{\text{braid}} = a_{2g-2}\bar{a}_{2g-2}(b_{g-1}) = b_{g-1}, \\ a_{2g-2}bt_1(a_{2g-2}) = a_{2g-2}bb_{g-1}c_{2g-2,2g-1}a_{2g-2}\frac{b_{g-1}(a_{2g-2})}{\text{braid}} \\ = a_{2g-2}bb_{g-1}c_{2g-2,2g-1}a_{2g-2}\bar{a}_{2g-2}(b_{g-1}) \\ = a_{2g-2}bb_{g-1}\frac{c_{2g-2,2g-1}(b_{g-1})}{\text{braid}} = a_{2g-2}bb_{g-1}\bar{b}_{g-1}(c_{2g-2,2g-1}) \\ = \frac{a_{2g-2}b(c_{2g-2,2g-1})}{\text{braid}} = c_{2g-2,2g-1}.$$

Here, we remark that these equations show  $t_1(a_{2g-2}) \in (\mathcal{M}_{g,n})_{v_0}$ . From these equa-

tions, we can show,

$$\begin{aligned} a_{2g-2}bt_1p\bar{t}_1\bar{b}\bar{a}_{2g-2} &= a_{2g-2}bt_1ba_{2g-2}a_{2g-2}b\bar{t}_1\bar{b}\bar{a}_{2g-2} \\ &= b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{g-1} &= \left( \underbrace{(c_{2g-3,2g-2})^2}_{\text{handle}} b_{g-1} \right)^2 \\ &= (c_{2g-2,2g-1})^2 b_{g-1}^2 \end{aligned}$$

Hence,  $b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1} = (\bar{c}_{2g-1,2g-1})^2 r_{g-1}$ . From the above equations, we can show  $p' = t_1 p \bar{t}_1 = \bar{b}\bar{a}_{2g-1}(\bar{c}_{2g-2,2g-1})^2 r_{g-1} a_{2g-2} b$ . This gives a presentation of  $p'$  as an element of  $(\mathcal{M}_{g,n})_{v_0}$ . For  $p$ , the relation of type (Y2) is

$$t_1(ba_{2g-2}a_{2g-2}b)\bar{t}_1 = t_1 p \bar{t}_1 = \bar{b}\bar{a}_{2g-2}(\bar{c}_{2g-2,2g-1})^2 r_{g-1} a_{2g-2} b$$

This relation is satisfied in  $G_{g,n}$ . For  $p'$ , the relation of type (Y2) is,

$$t_1(\bar{b}\bar{a}_{2g-2}(\bar{c}_{2g-2,2g-1})^2 r_{g-1} a_{2g-2} b)\bar{t}_1 \in (\mathcal{M}_{g,n})_{v_0}.$$

We shall show that this equation is satisfied in  $G_{g,n}$ . Previously, we have shown  $(t_1)^2, p \in (\mathcal{M}_{g,n})_{v_0}$ . By the definition of  $p'$ , we can show,

$$t_1(\bar{b}\bar{a}_{2g-2}(\bar{c}_{2g-2,2g-1})^2 r_{g-1} a_{2g-2} b)\bar{t}_1 = t_1(t_1 p \bar{t}_1)\bar{t}_1 = t_1^2 p \bar{t}_1^2 \in (\mathcal{M}_{g,n})_{v_0}.$$

For  $c_{2g-2,2g-1}, a_{2g-4}$ , we can show  $t_1$  exchanges  $c_{2g-2,2g-1}$  and  $a_{2g-4}$ ,

$$\begin{aligned} t_1(c_{2g-2,2g-1}) &= b_{g-1} \frac{c_{2g-2,2g-1} a_{2g-1}}{\text{braid}} \frac{b_{g-1} (c_{2g-2,2g-1})}{\text{braid}} \\ &= b_{g-1} a_{2g-2} c_{2g-2,2g-1} \bar{c}_{2g-2,2g-1} (b_{g-1}) \\ &= b_{g-1} \frac{a_{2g-2} (b_{g-1})}{\text{braid}} = b_{g-1} \bar{b}_{g-1} (a_{2g-2}) = a_{2g-2}, \\ t_1(a_{2g-2}) &= b_{g-1} c_{2g-2,2g-1} a_{2g-2} \frac{b_{g-1} (a_{2g-2})}{\text{braid}} \\ &= b_{g-1} c_{2g-2,2g-1} \frac{a_{2g-2} \bar{a}_{2g-2} (b_{g-1})}{\text{braid}} \\ &= b_{g-1} \frac{c_{2g-2,2g-1} (b_{g-1})}{\text{braid}} = b_{g-1} \bar{b}_{g-1} (c_{2g-2,2g-1}) = c_{2g-2,2g-1}. \end{aligned}$$

This fact shows  $t_1 c_{2g-2,2g-1} \bar{t}_1, t_1 a_{2g-1} \bar{t}_1 \in (\mathcal{M}_{g,n})_{v_0}$ .

For  $c_{2g-2,2g-4}$ ,

$$t_1(c_{2g-2,2g-4}) = b_{g-1} \frac{c_{2g-2,2g-1} a_{2g-2} b_{g-1} (c_{2g-2,2g-4})}{\text{braid}}$$



$$\begin{aligned}
&= b_{g-1} a_{2g-2} \underbrace{c_{2g-2,2g-1}}_{\text{handle}} b_{g-1} (c_{2g-2,2g-4}) \\
&= b_{g-1} a_{2g-2} c_{2g-3,2g-2} b_{g-1} (c_{2g-2,2g-4}) \\
&= \overline{b a_{2g-4} a_{2g-2} b} (a_{2g-1}) \quad (\text{by } X_{2g-4,2g-2}(4)).
\end{aligned}$$

Since  $b, a_{2g-1}, a_{2g-2}, a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$ , this equation shows  $t_1 c_{2g-2,2g-4} \bar{t}_1 \in (\mathcal{M}_{g,n})_{v_0}$ .

For  $c_{2g-4,2g-2}$ , we do the same way as above,

$$\begin{aligned}
t_1 (c_{2g-4,2g-2}) &= b_{g-1} \underbrace{c_{2g-2,2g-1} a_{2g-2}}_{\text{braid}} b_{g-1} (a_{2g-4,2g-2}) \\
&= b_{g-1} a_{2g-2} \underbrace{c_{2g-2,2g-1}}_{\text{handle}} b_{g-1} (c_{2g-4,2g-2}) \\
&= b_{g-1} a_{2g-2} c_{2g-3,2g-2} b_{g-1} (c_{2g-4,2g-2}) \\
&= \overline{b a_{2g-4} a_{2g-2} b} (a_{2g-3}) \quad (\text{by } X_{2g-4,2g-2}(2)).
\end{aligned}$$

Since  $b, a_{2g-2}, a_{2g-3}, a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$ , this equation shows  $t_1 c_{2g-4,2g-2} \bar{t}_1 \in (\mathcal{M}_{g,n})_{v_0}$ .

Here, we conclude that all the relations of type (Y2) are satisfied in  $G_{g,n}$ .

**5.3. Relations of type (Y3).** We define  $t_2 = b a_{2g-2} a_{2g-4} b$ . For the notations used to present a relation of type (Y3), it is possible to set  $h_1 = 1$ ,  $h_2 = t_2$  and  $h_3 = t_2$ . Then,  $W_{f_0} = t_1 t_2 t_1 t_2 t_1$ . By braid relations, we can show  $t_1 t_2 t_1 = t_2 t_1 t_2$  as follows.

$$\begin{aligned}
t_1 t_2 (b_{g-1}) &= b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \underbrace{b a_{2g-2} a_{2g-4} b}_{\text{braid}} (b_{g-1}) \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \underbrace{b a_{2g-2}}_{\text{braid}} (b_{g-1}) \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \underbrace{b \bar{b}_{g-1}}_{\text{braid}} (a_{2g-2}) \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \bar{b}_{g-1} b (a_{2g-2}) \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} \underbrace{b (a_{2g-2})}_{\text{braid}} \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} \bar{a}_{2g-2} (b) = \underbrace{b_{g-1} c_{2g-2,2g-1} (b)}_{\text{braid}} = b, \\
t_1 t_2 (c_{2g-2,2g-1}) &= b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \underbrace{b a_{2g-2} a_{2g-4} b}_{\text{braid}} (c_{2g-2,2g-1}) \\
&= b_{g-1} c_{2g-2,2g-1} a_{2g-2} \underbrace{b_{g-1} (c_{2g-2,2g-1})}_{\text{braid}} \\
&= b_{g-1} c_{2g-2,2g-1} \underbrace{a_{2g-2} \bar{c}_{2g-2,2g-1}}_{\text{braid}} (b_{g-1}) \\
&= b_{g-1} c_{2g-2,2g-1} \bar{c}_{2g-2,2g-1} a_{2g-2} (b_{g-1})
\end{aligned}$$

$$\begin{aligned}
 &= b_{g-1} \underbrace{a_{2g-2}(b_{g-1})}_{\text{braid}} = b_{g-1} \bar{b}_{g-1}(a_{2g-2}) = a_{2g-2}, \\
 t_1 t_2(a_{2g-2}) &= b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1} b a_{2g-2} a_{2g-4} \underbrace{b(a_{2g-2})}_{\text{braid}} \\
 &= b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1} b a_{2g-2} \underbrace{a_{2g-4} \bar{a}_{2g-2}(b)}_{\text{braid}} \\
 &= b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1} b a_{2g-2} \bar{a}_{2g-2} a_{2g-4}(b) \\
 &= b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1} \underbrace{b a_{2g-4}(b)}_{\text{braid}} \\
 &= b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1} b \bar{b}(a_{2g-4}) \\
 &= \underbrace{b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1}(a_{2g-4})}_{\text{braid}} = a_{2g-4}.
 \end{aligned}$$

Therefore,  $t_1 t_2 t_1 \bar{t}_2 \bar{t}_1 = t_1 t_2 (b_{g-1} c_{2g-2, 2g-1} a_{2g-2} b_{g-1}) \bar{t}_2 \bar{t}_1 = b a_{2g-2} a_{2g-4} b = t_2$ , that is  $t_1 t_2 t_1 = t_2 t_1 t_2$ . Hence, we get  $W_{f_0} = t_1 t_2 t_1 t_2 t_1 = t_1^2 t_2 t_1^2$ . As we have shown in Subsection 5.2,  $t_1^2 \in (\mathcal{M}_{g,n})_{v_0}$ , and, since  $b, a_{2g-2}, a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$ , we can show  $t_2 \in (\mathcal{M}_{g,n})_{v_0}$ . By using these facts, we conclude that  $W_{f_0} \in (\mathcal{M}_{g,n})_{v_0}$  is satisfied in  $G_{g,n}$ .

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