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A COMPLEX OF CURVES AND A PRESENTATION FOR THE MAPPING CLASS GROUP OF A SURFACE

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1. Introduction

Let $\Sigma_{g,n}$ be an oriented surface of genus g (≥ 2) with n (≥ 0) boundary components and denote by $\mathcal{M}_{g,n}$ its mapping class group, that is to say, the group of orientation preserving diffeomorphisms of $\Sigma_{g,n}$ which are the identity on $\partial\Sigma_{g,n}$ modulo isotopy. For a simple closed curve a in $\Sigma_{g,n}$, we define the Dehn twist along a as indicated in Fig. 1. We denote the isotopy class of Dehn twist along a by the same letter a.

It is known that $\mathcal{M}_{g,n}$ is generated by Dehn twists [5], [16]. McCool [19] showed that $\mathcal{M}_{g,n}$ is finitely presented. Hatcher and Thurston [7] defined a simply connected complex whose vertices are isotopy classes of "cut systems" and introduced a method of giving a presentation for $\mathcal{M}_{g,n}$ by making use of this complex. Harer [8] reduced the member of the 2-simplices of this complex, and Wajnryb [20] gave a simple presentation for $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,0}$. Following Wajnryb's presentation, Gervais [6] gave a symmetric presentation for $\mathcal{M}_{g,n}$. We set some notations indicating circles on $\Sigma_{g,n}$ as in Fig. 2. A triple of integers $(i, j, k) \in \{1, \dots, 2g+n-3\}^3$ will be said to be *good* when:

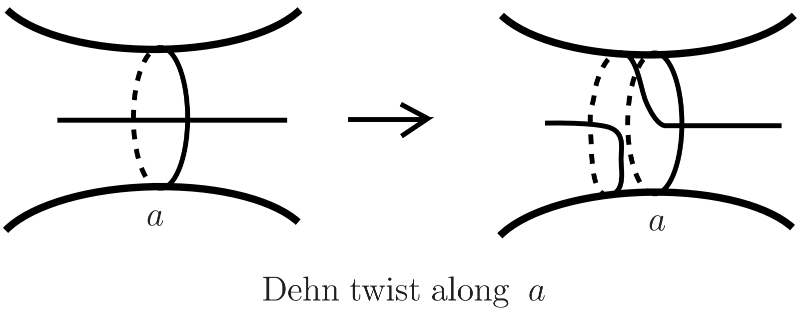

Fig. 1.

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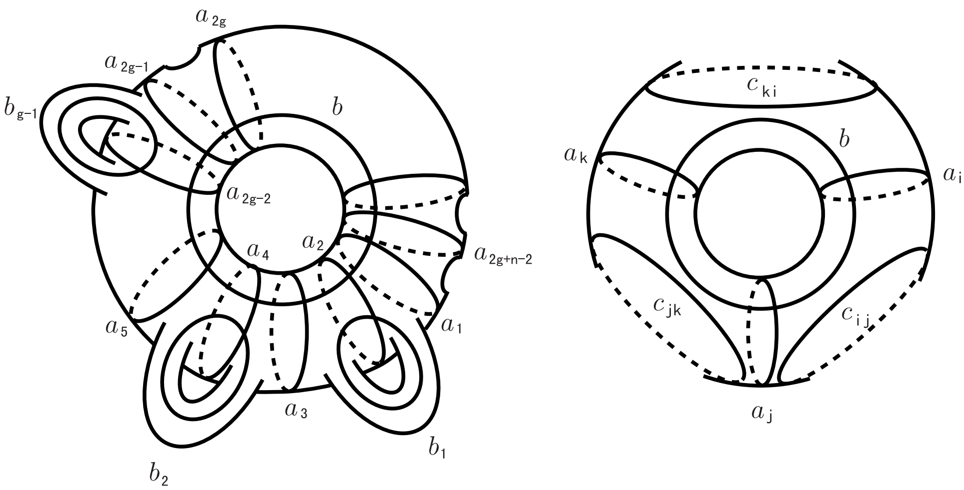

Fig. 2.

i) $(i, j, k) \notin \{(x, x, x) \mid x \in \{1, \ldots, 2g + n - 2\}\}$,
ii) $i \leq j \leq k$ or $j \leq k \leq i$ or $k \leq i \leq j$.
Gervais' symmetric presentation is as follows,

Theorem 1.1 ([6]). *If $g \geq 2$, $n \geq 0$, then $\mathcal{M}_{g,n}$ is generated by b, b_1, $\ldots$, b_{g-1}, a_1, $\ldots$, a_{2g+n-2}, $c_{i,j}$, and its defining relations are*
(A) *"HANDLES"*: $c_{2i,2i+1} = c_{2i-1,2i}$ *for all* i, $1 \leq i \leq g - 1$,
(B) *"BRAIDS": for all x, y among the generators, $xy = yx$ if the associated curves are disjoint and $xyx = yxy$ if the associated curves intersect transversely in a single point,*
(C) *"STARS"*: $c_{ij}c_{jk}c_{ki} = (a_i a_j a_k b)^3$ *for all good triples i, j, k, where* $c_{ll} = 1$.

Let $G_{g,n}$ denote the group with presentation given by Theorem 1.1.

On the other hand, Harvey [10] introduced a complex of curves for $\Sigma_{g,n}$, whose vertices are isotopy classes of essential (neither homotopic to a point nor any boundary component) simple closed curves and simplices are the set of vertices which are represented by disjoint and non-isotopic curves. Harer [9] showed the higher connectivity of this complex and, by using this complex, proved the stability of the cohomology group of mapping class groups. McCullough [18] defined a disk complex of a handle body (an oriented 3-dimensional manifold obtainted from 3-ball by attaching 1-handles), which is defined from a complex of curves by replacing "curves" with "meridian disks". He showed that the disk complex is contractible. The author [12] gave a presentation for the mapping class group of a handle body by investigating the action of the mapping class group on this complex. The aim of this paper is to give a Gervais' symmetric presentation for $\mathcal{M}_{g,n}$ with the same method as above, that is to say, by investigating the action of $\mathcal{M}_{g,n}$ on the complex of curves for $\Sigma_{g,n}$. We remark here that our method introduced in this paper does not use Wajnryb's simple presenation. This fact means that we do not need to use Hatcher-Thurston's complex to give a presentation for $\mathcal{M}_{g,n}$. In [21], Wajnryb proved simple connectedness of Hatcher-Thurston's complex without using Cerf Theory, and use this to give his simple presenatation for $\mathcal{M}_{g,0}$ and $\mathcal{M}_{g,1}$. On the other hand, Ivanov [13] gave an elementary proof of the simple connectivity of Harvey's complex, and Hatcher [11] gave an elementary proof of the higher connectivity of this complex. Therefore, our method introduced in this paper is another elementary approach to the mapping class group of a surface.

Recently, S. Benvenuti (Pisa Univ.) [1] showed a similar result, independently, using different "complex of curves", which includes separating curves. We remark that Matsumoto [17] gave a beautiful presentation for the mapping class groups of surfaces in terms of Artin groups.

We set notations and conventions used in this paper. Composition of elements of $\mathcal{M}_{g,n}$ will be written from right to left. We will denote by $\overline{x}$ the inverse of x and $y(x)$ the conjugate $yx\overline{y}$ of x by y. The notation $\rightleftarrows$ means "commute with". For example, for two elements x, y of $\mathcal{M}_{g,n}$, $x \rightleftarrows y$ means $xy = yx$. We use braid relations and handle relations very often. We indicate the place to use a braid relation (resp. handle relation) by an underline together with the letter "*braid*" (resp. "*handle*") below it. For example, if x, y, z_1, z_2 are loops on $\Sigma_{g,n}$ and if x and y intersect transversely in a single point and z_1 and z_2 are disjoint, then

$$\cdots \underbrace{xyx}_{braid} \cdots \underbrace{z_1 z_2}_{braid} \cdots = \cdots yxy \cdots z_2 z_1 \cdots .$$

2. A presentation for $\mathcal{M}_{2,0}$

Birman and Hilden [4] showed:

Theorem 2.1 ([4]). *$\mathcal{M}_{2,0}$ admits the presentation*:
generators: $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$,
defining relations:
(i) $\tau_i \tau_j = \tau_j \tau_i$, *if* $|i - j| \geq 2$, $1 \leq i, j \leq 5$,
(ii) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ $\quad 1 \leq i \leq 4$,
(iii) $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5)^6 = 1$,
(iv) $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1)^2 = 1$,
(v) $\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1 \rightleftarrows \tau_i$ $\quad 1 \leq i \leq 5$.

As we defined previously, $G_{2,0}$ is a group with the following presentation:
generators: $a_1, b, a_2, b_1, c_{1,2}$,
defining relations:

(1) $a_1ba_1 = ba_1b$, $a_2ba_2 = ba_2b$, $a_2b_1a_2 = b_1a_2b_1$, $b_1c_{1,2}b_1 = c_{1,2}b_1c_{1,2}$, every other pair of generators commutes,
(2) $(a_1a_1a_2b)^3 = c_{1,2}^2$.

Let $\psi_{2,0}\colon G_{2,0} \to \mathcal{M}_{2,0}$ be an epimorphism defined by $\psi_{2,0}(a_1) = \tau_1$, $\psi_{2,0}(b) = \tau_2$, $\psi_{2,0}(a_2) = \tau_3$, $\psi_{2,0}(b_1) = \tau_4$ and $\psi_{2,0}(c_{1,2}) = \tau_5$. We want to prove $\psi_{2,0}$ is an isomorphism. We shall construct an inverse map $\phi_{2,0}\colon \mathcal{M}_{2,0} \to G_{2,0}$. For each generators of $G_{2,0}$, we define $\phi_{2,0}(\tau_1) = a_1$, $\phi_{2,0}(\tau_2) = b$, $\phi_{2,0}(\tau_3) = a_2$, $\phi_{2,0}(\tau_4) = b_1$, and $\phi_{2,0}(\tau_5) = c_{1,2}$. If the relations (i)–(v) are mapped by $\phi_{2,0}$ onto relations in $G_{2,0}$, then $\phi_{2,0}$ extends to a homomorphism. Then, we can show $\psi_{2,0} \circ \phi_{2,0} = \mathrm{Id}_{\mathcal{M}_{2,0}}$ and $\phi_{2,0}$ is an epimorphism, hence, $\psi_{2,0}$ is an isomorphism. Therefore, in order to prove $\phi_{2,0}$ is an isomorphism, it is enough to show that the defining relations (i)–(v) are satisfied in $G_{2,0}$.

Relations (i) and (ii) are nothing but the relations (1) for $G_{2,0}$. In $G_{2,0}$, the right hand side of relation (v) is $a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1$, hence we need to show

$$a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1 \rightleftarrows a_1, b, a_2, b_1, c_{1,2}.$$

For short, we denote $E = a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1$. Using the relations (1), we can show $E(b) = b$, $E(a_2) = a_2$, $E(b_1) = b_1$, $E(c_{1,2}) = c_{1,2}$. In order to show $E(a_1) = a_1$, we have to give another presentation for E.

Lemma 2.2. $(c_{1,2}c_{1,2}a_2b_1)^3 = a_1a_1$.

Proof. We introduce an element $D = a_1ba_2b_1c_{1,2}a_1ba_2b_1a_1ba_2a_1ba_1$ of $\mathcal{M}_{2,0}$. By using the relations (1), we can show $D(a_1) = c_{1,2}$, $D(b) = b_1$, $D(a_2) = a_2$, $D(b_1) = b$, and $D(c_{1,2}) = a_1$. We take a conjugation of the relation (2) by D, then we get the equation we need. $\square$

Lemma 2.3. $E = a_1a_1ba_1a_1b\bar{c}_{1,2}\bar{c}_{1,2}\ \bar{b}_2\ \bar{c}_{1,2}\ \bar{c}_{1,2}\ \bar{b}_2$.

Proof. By the relations (1), we can show,

$$\begin{aligned} c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1\underbrace{c_{1,2}c_{1,2}a_2b_1}_{braid} &= c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}\underbrace{a_2b_1a_2}_{braid}c_{1,2}c_{1,2}b_1 \\ &= c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}b_1a_2b_1c_{1,2}c_{1,2}b_1. \end{aligned}$$

We have shown $a_1a_1 = (c_{1,2}c_{1,2}a_2b_1)^3$, in Lemma 2.2. Therefore,

$$a_1a_1 = c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}b_1a_2b_1c_{1,2}c_{1,2}b_1.$$

From this equation,

$$a_2b_1c_{1,2}c_{1,2}b_1a_2 = \underbrace{\bar{c}_{1,2}\bar{c}_{1,2}a_1a_1}_{braid}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1 = a_1a_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1,$$

and hence we can show,

$$
\begin{aligned}
E &= a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1 = a_1ba_1a_1\underbrace{\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1ba_1}_{\text{braid}} \qquad \text{by the above equation} \\
&= a_1ba_1\underbrace{a_1ba_1}_{\text{braid}}\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1 = a_1\underbrace{ba_1b}_{\text{braid}}a_1b\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1 \\
&= a_1a_1ba_1a_1b\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1. \qquad \square
\end{aligned}
$$

We can show $E(a_1) = a_1$ by using the above Lemma and the relations (1).

The relation (iv) is interpreted as $E^2 = 1$ in $G_{2,0}$. By Lemma 2.3,

$$
\begin{aligned}
E^2 &= a_1a_1ba_1a_1b\underbrace{\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1a_1a_1ba_1a_1}_{\text{braid}}b\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1 \\
&= a_1a_1ba_1a_1ba_1a_1ba_1a_1b\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_1.
\end{aligned}
$$

If we can show $(a_1a_1b)^4 = (c_{1,2}c_{1,2}b_1)^4$, then $E^2 = (c_{1,2}c_{1,2}b_1)^4(\bar{c}_{1,2}\bar{c}_{1,2}\bar{b}_2)^4$. Since we can show $(c_{1,2}c_{1,2}b_1)^4 = (b_1c_{1,2}c_{1,2})^4$ by the relations (1), we get $E^2 = 1$. Therefore it is enough to show:

Lemma 2.4. $(a_1a_1b)^4 = (c_{1,2}c_{1,2}b_1)^4$

Proof. We denote $r_1 = a_1a_1ba_1a_1b$, $r_2 = c_{1,2}c_{1,2}b_1c_{1,2}c_{1,2}b_1$ for short. We can show,

$$
\begin{aligned}
r_1a_2r_1a_2 &= a_1a_1ba_1a_1ba_2a_1\underbrace{a_1ba_1}_{\text{braid}}a_1ba_2 = a_1a_1ba_1a_1ba_2\underbrace{a_1ba_1}_{\text{braid}}ba_1ba_2 \\
&= a_1a_1ba_1a_1\underbrace{ba_2b}_{\text{braid}}a_1bba_1ba_2 = a_1a_1b\underbrace{a_1a_1a_2}_{\text{braid}}b\underbrace{a_2a_1}_{\text{braid}}bba_1ba_2 \\
&= a_1a_1ba_2a_1\underbrace{a_1ba_1}_{\text{braid}}a_2bba_1ba_2 = a_1a_1ba_2\underbrace{a_1ba_1}_{\text{braid}}ba_2bba_1ba_2 \\
&= a_1a_1\underbrace{ba_2b}_{\text{braid}}a_1bba_2bba_1ba_2 = a_1a_1a_2ba_2a_1bba_2b\underbrace{ba_1b}_{\text{braid}}a_2 \\
&= a_1a_1a_2ba_2a_1bba_2\underbrace{ba_1b}_{\text{braid}}a_1a_2 = a_1a_1a_2ba_2a_1bb\underbrace{a_2a_1}_{\text{braid}}b\underbrace{a_1a_1a_2}_{\text{braid}} \\
&= a_1a_1a_2ba_2a_1bba_1\underbrace{a_2ba_2}_{\text{braid}}a_1a_1 = a_1a_1a_2ba_2a_1b\underbrace{ba_1b}_{\text{braid}}a_2ba_1a_1 \\
&= a_1a_1a_2ba_2a_1\underbrace{ba_1b}_{\text{braid}}a_1a_2ba_1a_1 = a_1a_1a_2b\underbrace{a_2a_1a_1}_{\text{braid}}ba_1a_1a_2ba_1a_1 \\
&= (a_1a_1a_2b)^3a_1a_1
\end{aligned}
$$

and, by the relation (2), $(a_1a_1a_2b)^3a_1a_1 = c_{1,2}c_{1,2}a_1a_1$. Hence $r_1a_2r_1a_2 = c_{1,2}c_{1,2}a_1a_1$. From the last equation, we can show $r_1^2 = r_1\bar{a}_2\bar{r}_1c_{1,2}c_{1,2}a_1a_1\bar{a}_2$. In the same way as above, but using Lemma 2.2 in place of relation (2), we can show $r_2^2 =$

$r_2\bar{a}_2\bar{r}_2c_{1,2}c_{1,2}a_1a_1\bar{a}_2$. If we can show $r_1(a_2) = r_2(a_2)$, then we get $r_1^2 = r_2^2$. In fact,

$$
\begin{aligned}
r_2(a_2) &= c_{1,2}\underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}c_{1,2}b_1(a_2) = \underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}\ \underbrace{b_1c_{1,2}b_1}_{\text{braid}}(a_2)\\
&= b_1c_{1,2}\underbrace{b_1c_{1,2}b_1}_{\text{braid}}c_{1,2}(a_2) = b_1c_{1,2}c_{1,2}b_1\underbrace{c_{1,2}c_{1,2}(a_2)}_{\text{braid}}\\
&= b_1c_{1,2}c_{1,2}b_1(a_2) = b_1(a_1a_1a_2b)^3b_1(a_2) \quad \text{by the relation (2)}\\
&= b_1a_1a_1a_2ba_1a_1a_2ba_1a_1a_2b\underbrace{b_1(a_2)}_{\text{braid}} = b_1a_1a_1a_2ba_1a_1a_2ba_1a_1\underbrace{a_2b\bar{a}_2}_{\text{braid}}(b_1)\\
&= b_1a_1a_1a_2ba_1a_1a_2ba_1a_1\bar{b}a_2\underbrace{b(b_1)}_{\text{braid}} = b_1a_1a_1a_2b\underbrace{a_1a_1a_2}_{\text{braid}}ba_1a_1\bar{b}a_2(b_1)\\
&= b_1a_1a_1a_2ba_2a_1\underbrace{a_1ba_1}_{\text{braid}}a_1\bar{b}a_2(b_1) = b_1a_1a_1a_2ba_2a_1ba_1\underbrace{ba_1\bar{b}}_{\text{braid}}a_2(b_1)\\
&= b_1a_1a_1\underbrace{a_2ba_2}_{\text{braid}}a_1ba_1\bar{a}_1\underbrace{ba_1a_2}_{\text{braid}}(b_1) = b_1a_1a_1ba_2ba_1bba_2\underbrace{a_1(b_1)}_{\text{braid}}\\
&= b_1a_1a_1ba_2\underbrace{ba_1b}_{\text{braid}}ba_2(b_1) = b_1a_1a_1ba_2a_1\underbrace{ba_1b}_{\text{braid}}a_2(b_1) = b_1a_1a_1ba_2a_1a_1b\underbrace{a_1a_2}_{\text{braid}}(b_1)\\
&= b_1a_1a_1ba_2a_1a_1ba_2\underbrace{a_1(b_1)}_{\text{braid}} = b_1a_1a_1b\underbrace{a_2a_1a_1}_{\text{braid}}ba_2(b_1) = b_1a_1a_1ba_1a_1\underbrace{a_2ba_2}_{\text{braid}}(b_1)\\
&= b_1a_1a_1ba_1a_1ba_2\underbrace{b(b_1)}_{\text{braid}} = b_1a_1a_1ba_1a_1b\underbrace{a_2(b_1)}_{\text{braid}} = b_1\underbrace{a_1a_1ba_1a_1b\bar{b}_1}_{\text{braid}}(a_2)\\
&= b_1\bar{b}_1a_1a_1ba_1a_1b(a_2) = a_1a_1ba_1a_1b(a_2) = r_1(a_2) \qquad \square
\end{aligned}
$$

The relation (iii) is interpreted as $(a_1ba_2b_1c_{1,2})^6 = 1$. If we regard a_1, b, a_2, b_1, $c_{1,2}$ as generators of the 6-string braid group, namely, a_1 as an interchange of the 1st and the 2nd string, b as an interchange of the 2nd and the 3rd string and so on, then $(a_1ba_2b_1c_{1,2})^6$ is a full twist. By investigating a figure of a 6-string full twist, or repeatedly applying the relations (1), we can show

$$(a_1ba_2b_1c_{1,2})^6 = (a_1ba_2b_1c_{1,2})^2b_1a_2ba_1c_{1,2}b_1a_2b(a_2b_1c_{1,2})^4.$$

By Lemma 2.2,

$$
\begin{aligned}
a_1a_1 &= (c_{1,2}c_{1,2}a_2b_1)^3\\
&= \underbrace{c_{1,2}c_{1,2}a_2}_{\text{braid}}b_1c_{1,2}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1 = a_2c_{1,2}\underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1\\
&= a_2\underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}b_1c_{1,2}a_2b_1c_{1,2}c_{1,2}a_2b_1 = a_2b_1c_{1,2}b_1b_1\underbrace{c_{1,2}a_2}_{\text{braid}}b_1c_{1,2}c_{1,2}a_2b_1\\
&= a_2b_1c_{1,2}b_1b_1a_2\underbrace{c_{1,2}b_1c_{1,2}}_{\text{braid}}c_{1,2}a_2b_1 = a_2b_1c_{1,2}b_1\underbrace{b_1a_2b_1}_{\text{braid}}c_{1,2}b_1c_{1,2}a_2b_1\\
&= a_2b_1c_{1,2}\underbrace{b_1a_2b_1}_{\text{braid}}a_2c_{1,2}b_1c_{1,2}a_2b_1 = a_2b_1c_{1,2}a_2b_1\underbrace{a_2a_2c_{1,2}}_{\text{braid}}b_1\underbrace{c_{1,2}a_2}_{\text{braid}}b_1
\end{aligned}
$$

$$
\begin{aligned}
&= a_2b_1c_{1,2}a_2b_1c_{1,2}a_2\underbrace{a_2b_1a_2}_{braid}c_{1,2}b_1 = a_2b_1c_{1,2}a_2b_1c_{1,2}a_2b_1a_2\underbrace{b_1c_{1,2}b_1}_{braid}\\
&= a_2b_1c_{1,2}a_2b_1c_{1,2}a_2b_1\underbrace{a_2c_{1,2}}_{braid}b_1c_{1,2} = (a_2b_1c_{1,2})^4.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(a_1ba_2b_1c_{1,2})^6 &= (a_1ba_2b_1c_{1,2})^2b_1a_2ba_1c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}a_1ba_2\underbrace{b_1c_{1,2}b_1}_{braid}a_2ba_1c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}\underbrace{a_1ba_2c_{1,2}}_{braid}b_1\underbrace{c_{1,2}a_2ba_1}_{braid}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}a_1b\underbrace{a_2b_1a_2}_{braid}ba_1c_{1,2}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}\underbrace{a_1bb_1}_{braid}a_2\underbrace{b_1ba_1}_{braid}c_{1,2}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}b_1a_1\underbrace{ba_2b}_{braid}a_1b_1c_{1,2}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}b_1\underbrace{a_1a_2}_{braid}b\underbrace{a_2a_1}_{braid}b_1c_{1,2}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2\underbrace{a_1ba_1}_{braid}a_2b_1c_{1,2}c_{1,2}b_1a_2ba_1a_1\\
&= a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2b(a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1)a_1.
\end{aligned}
$$

We have already shown that $a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1 = E \rightleftarrows a_1$. Hence, $(a_1ba_2b_1c_{1,2})^6 = (a_1ba_2b_1c_{1,2}c_{1,2}b_1a_2ba_1)^2 = E^2 = 1$.

3. Elementary relations

In this section, we assume $g \geq 3$ or $g = 2$, $n \geq 1$. We shall prove some relations in $G_{g,n}$ which are frequently used in the following sections. The first one is known as the "lantern relation", which is proved in [6, Lemma 3]. So we omit the proof here:

Lemma 3.1. *For all good triples (i, j, k), one has in $G_{g,n}$ the relation,*

$$(L_{i,j,k}) : a_ic_{i,j}c_{j,k}a_k = c_{i,k}a_jXa_j\overline{X} = c_{i,k}\overline{X}a_jXa_j,$$

where $X = ba_ia_kb$.

The next one is:

Lemma 3.2. *If $i \neq 2k$, one has in $G_{g,n}$ the relation,*

$$(X_{i,2k}) :(1)\ \overline{b_ka_{2k}c_{2k-1,2k}b_k}(c_{i,2k}) = ba_ia_{2k}b(a_{2k-1}),$$

(2) $b_k a_{2k} c_{2k-1,2k} b_k(c_{i,2k}) = \overline{b a_i a_{2k} b}(a_{2k-1})$,
(3) $\overline{b_k a_{2k} c_{2k-1,2k} b_k}(c_{2k,i}) = b a_i a_{2k} b(a_{2k+1})$,
(4) $b_k a_{2k} c_{2k-1,2k} b_k(c_{2k,i}) = \overline{b a_i a_{2k} b}(a_{2k+1})$.

Proof. We will prove (1). Other relations are proved in the same way. We write $X_1 = b_k a_{2k} c_{2k-1,2k} b_k$, $X_2 = b a_i a_{2k} b$ for short. Then,

$$\begin{aligned}\overline{X_2}\ \overline{X_1}(c_{i,2k}) &= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar c_{2k-1,2k}\bar a_{2k}\underbrace{\bar b_k(c_{i,2k})}_{\text{braid}}\\ &= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar c_{2k-1,2k}\bar a_{2k} c_{i,2k}(b_k).\end{aligned}$$

The lantern relation $L_{i,2k-1,2k}$ says $c_{i,2k} = a_{2k} c_{2k-1,2k} c_{i,2k-1} a_i \bar a_{2k-1}\overline{X_2}\bar a_{2k-1} X_2$. Therefore,

$$\begin{aligned}\overline{X_2}\ \overline{X_1}(c_{i,2k}) &= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar c_{2k-1,2k}\bar a_{2k} a_{2k} c_{2k-1,2k} c_{i,2k-1} a_i\bar a_{2k-1}\overline{X_2}\bar a_{2k-1} X_2(b_k)\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k c_{i,2k-1} a_i\bar a_{2k-1}\bar b\bar a_{2k}\bar a_i\bar b\bar a_{2k-1} b a_i a_{2k}\underbrace{b(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\underbrace{c_{i,2k-1} a_i\bar a_{2k-1}\bar b\bar a_{2k}\bar a_i\bar b\bar a_{2k-1} b a_i a_{2k}}_{\text{braid}}(b_k)\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k a_i\bar a_{2k-1}\bar b\bar a_{2k}\bar a_i\bar b\bar a_{2k-1} b\underbrace{a_i a_{2k}}_{\text{braid}}\underbrace{c_{i,2k-1}(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k a_i\bar a_{2k-1}\bar b\bar a_{2k}\bar a_i\underbrace{\bar b\bar a_{2k-1}}_{\text{braid}} b a_{2k}\underbrace{a_i(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\underbrace{a_i\bar a_{2k-1}}_{\text{braid}}\bar b\underbrace{\bar a_{2k}\bar a_i a_{2k-1}}_{\text{braid}}\bar b\underbrace{\bar a_{2k-1} a_{2k}}_{\text{braid}}(b_k)\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar a_{2k-1}\underbrace{a_i\bar b\bar a_i}_{\text{braid}} a_{2k-1}\underbrace{\bar a_{2k}\bar b a_{2k}}_{\text{braid}}\underbrace{\bar a_{2k-1}(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar a_{2k-1}\bar b\bar a_i\underbrace{b a_{2k-1}}_{\text{braid}} b\bar a_{2k}\underbrace{\bar b(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar a_{2k-1}\bar b\bar a_i a_{2k-1} b\underbrace{a_{2k-1}\bar a_{2k}}_{\text{braid}}(b_k)\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar a_{2k-1}\bar b\bar a_i a_{2k-1} b\bar a_{2k}\underbrace{a_{2k-1}(b_k)}_{\text{braid}} = \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\bar a_{2k-1}\bar b\bar a_i a_{2k-1} b\underbrace{\bar a_{2k}(b_k)}_{\text{braid}}\\
&= \bar b\bar a_{2k}\bar a_i\bar b\bar b_k\underbrace{\bar a_{2k-1}\bar b\bar a_i a_{2k-1} b b_k}_{\text{braid}}(a_{2k}) = \bar b\bar a_{2k}\bar a_i\bar b\bar b_k b_k\bar a_{2k-1}\bar b\underbrace{\bar a_i a_{2k-1}}_{\text{braid}} b(a_{2k})\\
&= \bar b\bar a_{2k}\bar a_i\bar b\underbrace{\bar a_{2k-1}\bar b a_{2k-1}}_{\text{braid}}\bar a_i b(a_{2k}) = \bar b\bar a_{2k}\bar a_i\bar b b\bar a_{2k-1}\underbrace{\bar b\bar a_i b}_{\text{braid}}(a_{2k})\\
&= \bar b\bar a_{2k}\bar a_i\underbrace{\bar a_{2k-1} a_i}_{\text{braid}}\bar b\underbrace{\bar a_i(a_{2k})}_{\text{braid}} = \bar b\bar a_{2k}\bar a_i a_i\bar a_{2k-1}\underbrace{\bar b(a_{2k})}_{\text{braid}} = \bar b\bar a_{2k}\underbrace{\bar a_{2k-1} a_{2k}}_{\text{braid}}(b)\\
&= \bar b\bar a_{2k} a_{2k}\underbrace{\bar a_{2k-1}(b)}_{\text{braid}} = \bar b b(a_{2k-1}) = a_{2k-1}.\end{aligned}$$

□

The third one is known as the “chain relation”:

Lemma 3.3. *One has in $G_{g,n}$ the relation*:

$$\{(c_{2g-2,2g-1})^2 a_{2g-2} b_{g-1}\}^3 = a_{2g-3} a_{2g-1}.$$

Proof. We write

$$\begin{aligned} D = {} & c_{2g-2,2g-1} b_{g-1} a_{2g-2} b a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2} b c_{2g-2,2g-1} b_{g-1} a_{2g-2} \\ & \times c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} \end{aligned}$$

for short. By using braid relations, we can show $D(c_{2g-2,2g-1}) = a_{2g-3}$, $D(b_{g-1}) = b$, $D(a_{2g-2}) = a_{2g-2}$, $D(a_{2g-3}) = c_{2g-2,2g-1}$. For $D(a_{2g-1})$,

$$\begin{aligned} & D(a_{2g-1}) \\ & = c_{2g-2,2g-1} b_{g-1} a_{2g-2} b a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2} b \\ & \quad \times \underbrace{c_{2g-2,2g-1} b_{g-1} a_{2g-2} c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1} (a_{2g-1})}_{braid} \\ & = c_{2g-2,2g-1} b_{g-1} a_{2g-2} b \underbrace{a_{2g-3} c_{2g-2,2g-1} b_{g-1} a_{2g-2}}_{braid} b (a_{2g-1}) \\ & = c_{2g-2,2g-1} b_{g-1} \underbrace{a_{2g-2} b c_{2g-2,2g-1}}_{braid} b_{g-1} a_{2g-2} a_{2g-3} b (a_{2g-1}) \\ & = \underbrace{c_{2g-2,2g-1} b_{g-1} c_{2g-2,2g-1}}_{braid} a_{2g-2} \underbrace{b b_{g-1}}_{braid} a_{2g-2} a_{2g-3} b (a_{2g-1}) \\ & = b_{g-1} c_{2g-2,2g-1} \underbrace{b_{g-1} a_{2g-2} b_{g-1}}_{braid} b a_{2g-2} a_{2g-3} b (a_{2g-1}) \\ & = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \underbrace{a_{2g-2} b a_{2g-2}}_{braid} a_{2g-3} b (a_{2g-1}) \\ & = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} b a_{2g-2} \underbrace{b a_{2g-3} b}_{braid} (a_{2g-1}) \\ & = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} b \underbrace{a_{2g-2} a_{2g-3}}_{braid} b \underbrace{a_{2g-3} (a_{2g-1})}_{braid} \\ & = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} b a_{2g-3} a_{2g-2} b (a_{2g-1}) \\ & = b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1} \overline{b_{g-1} c_{2g-2,2g-1} a_{2g-2} b_{g-1}} (c_{2g-2,2g-3}) \quad \text{by } X_{2g-3,2g-2}(3) \\ & = c_{2g-2,2g-3}. \end{aligned}$$

The star relation $E_{2g-3,2g-3,2g-2}$ of $G_{g,n}$ says:

$$\begin{aligned} \{(a_{2g-3})^2 a_{2g-2} b\}^4 & = \underbrace{c_{2g-3,2g-2}}_{handle} c_{2g-2,2g-3} \\ & = c_{2g-2,2g-1} c_{2g-2,2g-3}. \end{aligned}$$

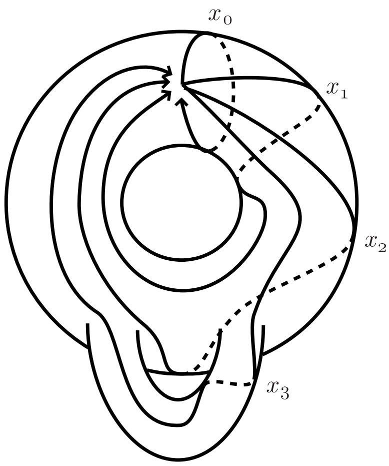

Fig. 3.

We take a conjugation of this equation by $\overline{D}$, then we get the equation which we need. $\square$

4. A presentation for $\mathcal{M}_{2,1}$

In this section, we give a presentation for $\mathcal{M}_{2,1}$ and show that $\mathcal{M}_{2,1} \cong G_{2,1}$. For this purpose, it is enough to show that all the relations for $\mathcal{M}_{2,1}$ are satisfied in $G_{2,1}$ by the same reason as Section 2.

Let p_1 be a point on Σ_2. We give a presentation for $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$ along the way of [3]. Let α be a surjection from $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$ to $\pi_0(\mathrm{Diff}^+(\Sigma_2))$ defined by forgetting the point p_1. We define a homomorphism β from $\pi_1(\Sigma_2, p_1)$ to $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$ as follows: The homotopy classes of loops indicated in Fig. 3 generate $\pi_1(\Sigma_2, p_1)$. For a loop l corresponding to one of these generators, we take a regular neighborhood A of this loop in Σ_2. Since this A is an annulus, its boundary has two connected components. With regard to the orientation for l, we denote by A_1 the right hand side of these components, and denote by A_2 the left hand side of them. We define β (which is an element of $\pi_1(\Sigma_2, p_1)$ corresponding to l) to be equal to $A_1\overline{A_2}$. For short, we write $x_i = \beta(x_i)$ $(i = 0, 1, 2, 3)$. For these homomorphisms α, β, there is a short exact sequence:

$$\text{(S1)} \qquad 0 \longrightarrow \pi_1(\Sigma_2, p_1) \xrightarrow{\beta} \pi_0(\mathrm{Diff}^+(\Sigma_2, p_1)) \xrightarrow{\alpha} \pi_0(\mathrm{Diff}^+(\Sigma_2)) \longrightarrow 0.$$

There is a natural surjection from $\pi_0(\mathrm{Diff}^+(\Sigma_{2,1}, \mathrm{rel}\,\partial\Sigma_{2,1}))$ to $\pi_0(\mathrm{Diff}^+(\Sigma_{2,1}/\partial\Sigma_{2,1}, \partial\Sigma_{2,1}/\partial\Sigma_{2,1}))$ and the latter one is isomorphic to $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$. Hence there is a surjection γ from $\pi_0(\mathrm{Diff}^+(\Sigma_{2,1}, \mathrm{rel}\,\partial\Sigma_{2,1})) \cong \mathcal{M}_{2,1}$ to $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$. The kernel of γ is an infinite cyclic group $\mathbb{Z}$ generated by the Dehn twist along the loop $\partial\Sigma_{2,1}$, which we denote by $c_{3,1}$. Hence, there is a short exact sequence:

(S2) $$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{2,1} \xrightarrow{\gamma} \pi_0(\mathrm{Diff}^+(\Sigma_2, p_1)) \longrightarrow 0$$

In general, if there is a short exact sequence,

$$0 \longrightarrow L \xrightarrow{\phi} G \xrightarrow{\psi} R \longrightarrow 0,$$

and L and R are finitely presented, then a finite presentation for G is given as follows (see, for example, Chapter 10 of [14]). Let $l_1, \dots, l_m$ be the generators of L and, $r_1, \dots, r_n$ be the generators of R. For each $1 \le i \le m$, we denote by $\tilde{l_i}$ the image of l_i under ϕ, and for each $1 \le j \le n$, we fix one of the preimages of r_j by ψ and denote this $\tilde{r_j}$. Then G is generated by $\tilde{l_1}, \dots, \tilde{l_m}$ and $\tilde{r_1}, \dots, \tilde{r_n}$, and there are the following three types of relations for G.
(1) For each $1 \le i \le m$, $1 \le j \le n$, $\tilde{r_j}\tilde{l_i}\tilde{r_j}^{-1}$ is an element of $\phi(L)$. The equation

$$\tilde{r_j}\tilde{l_i}\tilde{r_j}^{-1} = \text{a presentaion of } \tilde{r_j}\tilde{l_i}\tilde{r_j}^{-1} \text{ in terms of } \tilde{l_1}, \dots \tilde{l_m}$$

is a relation for G,
(2) Each relation for R is presented by a word $w(r_1, \dots, r_n)$. The element $w(\tilde{r_1}, \dots, \tilde{r_n})$ is in the kernel of ψ and hence it is an element of $\phi(L)$. The equation

$$w(\tilde{r_1}, \dots, \tilde{r_n}) = \text{a presentation of } w(\tilde{r_1}, \dots, \tilde{r_n}) \text{ in terms of } \tilde{l_1}, \dots \tilde{l_m}$$

is a relation for G,
(3) For each relation for L, the equation obtained from this relation by replacing l_i with $\tilde{l_i}$ is also a relation for G
We apply this method to the above short exact sequences (S1) and (S2). For (S1), by observing that a_1, b, a_2, b_1, $c_{1,2}$ in $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$ are mapped, by α, to the elements of $\pi_0(\mathrm{Diff}^+(\Sigma_2))$ denoted by the same letters, we can see that $\pi_0(\mathrm{Diff}^+(\Sigma_2, p_1))$ is generated by x_0, x_1, x_2, x_3, a_1, b, a_2, b_1, $c_{1,2}$ and its defining relations are:

(1-a_1) $\quad a_1(x_0) = x_0,\ a_1(x_1) = x_1\overline{x}_0,\ a_1(x_2) = x_2\overline{x}_0,\ a_1(x_3) = x_3\overline{x}_0,$

(1-b) $\quad b(x_0) = x_1,\ b(x_1) = x_1\overline{x}_0x_1,\ b(x_2) = x_2,\ b(x_3) = x_3,$

(1-a_2) $\quad a_2(x_0) = x_0,\ a_2(x_1) = x_2,\ a_2(x_2) = x_2\overline{x}_1x_2,\ a_2(x_3) = x_3,$

(1-b_1) $\quad b_1(x_0) = x_0,\ b_1(x_1) = x_1,\ b_1(x_2) = x_2,\ b_1(x_3) = x_3\overline{x}_2x_3,$

(1-$c_{1,2}$) $\quad c_{1,2}(x_0) = x_0,\ c_{1,2}(x_1) = x_1,\ c_{1,2}(x_2) = x_2,\ c_{1,2}(x_3) = x_3\overline{x}_2x_1\overline{x}_0,$

(2-1) $\quad a_1ba_1 = ba_1b,\ a_2ba_2 = ba_2b,\ a_2b_1a_2 = b_1a_2b_1,\ b_1c_{1,2}b_1 = c_{1,2}b_1c_{1,2},$
other pairs of $\{a_1,\ b,\ a_2,\ b_1,\ c_{1,2}\}$ commute each other,

(2-2) $\quad (a_1a_1a_2b)^3\overline{c}_{1,2}^2 \in \beta(\pi_1(\Sigma_2, p_1)),$

(3) $\quad x_3\bar{x}_2x_1\bar{x}_0\bar{x}_3x_2\bar{x}_1x_0 = 1.$

Among the above relations, (1-a_1) to (1-$c_{1,2}$) can be checked by drawing figures of actions of a_1, b, a_2, b_1, $c_{1,2}$ on $\pi_1(\Sigma_2, p_1)$, (2-1) and (2-2) come from the relation (1) and (2), introduced in Section 2, for $\mathcal{M}_{2,0} \cong G_{2,0}$, and (3) is a relation for $\pi_1(\Sigma_2, p_1)$ which is obtained by reading the word on the boundary of an octahedron which is obtained by cutting Σ_2 along x_0, x_1, x_2, x_3. By making use of (S2), we can show that $\mathcal{M}_{2,1}$ is generated by x_0, x_1, x_2, x_3, a_1, b, a_2, b_1, $c_{1,2}$, $c_{3,1}$, and the defining relations are the relations (1-a_1) to (3) up to the powers of $c_{3,1}$. On the other hand, we can see $x_0 = a_1\bar{a}_3$, $x_1 = b(x_0)$, $x_2 = a_2(x_1)$, $x_3 = b_1(x_2)$ and hence, $\mathcal{M}_{2,1}$ is generated by a_1, a_2, a_3, b, b_1, $c_{1,2}$, $c_{3,1}$. We can now derive the defining relations for $\mathcal{M}_{2,1}$ from the raltions for $G_{2,1}$ as follows.

(1) It is shown, in the proof of Lemma 9 in [6], that all the relations (1-a_1) to (1-$c_{1,2}$) up to the powers of $c_{3,1}$ are derived from the relations for $G_{2,1}$. We remark that

$$c_{1,2}(x_3) = x_3\bar{x}_2x_1\bar{x}_0c_{3,1},$$

which will be used later.

(2-1) These relations are nothing but braid relations.

(2-2) The lantern relation $L_{2,3,1}$ says

$$a_2c_{2,3}c_{3,1}a_1 = c_{2,1}a_3Xa_3\bar{X} = c_{2,1}\bar{X}a_3Xa_3,$$
$$\text{where } X = ba_2a_1b,$$

that is to say,

$$c_{2,1}\bar{c}_{2,3} = a_2c_{3,1}a_1X\bar{a}_3\bar{X}\bar{a}_3 \qquad \cdots\cdots(\alpha)$$
$$a_1c_{2,3}\bar{a}_3 = \bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{X}a_3X \qquad \cdots\cdots(\beta).$$

The star relation $E_{1,1,2}$ says $(a_1a_1a_2b)^3 = c_{1,2}c_{2,1}$, so that, $(a_1a_1a_2b)^3(\bar{c}_{1,2})^2 = c_{2,1}\bar{c}_{1,2}$. For the right hand of the last equation, we can show,

$$\begin{aligned}
c_{2,1}\underbrace{\bar{c}_{1,2}}_{\textit{handle}} &= c_{2,1}\bar{c}_{2,3} = \underbrace{a_2c_{3,1}}_{\textit{braid}}a_1X\bar{a}_3\bar{X}\bar{a}_3 \qquad \text{by } (\alpha)\\
&= c_{3,1}a_2a_1X\bar{a}_3\bar{X}\bar{a}_3 = c_{3,1}a_2\underbrace{a_1ba_1}_{\textit{braid}}a_2b\bar{a}_3\bar{b}\bar{a}_2\bar{a}_1\bar{b}\bar{a}_3\\
&= c_{3,1}a_2ba_1\underbrace{ba_2b}_{\textit{braid}}\bar{a}_3\bar{b}\bar{a}_2\bar{a}_1\bar{b}\bar{a}_3 = c_{3,1}a_2ba_1a_2b\underbrace{a_2\bar{a}_3}_{\textit{braid}}\bar{b}\bar{a}_2\bar{a}_1\bar{b}\bar{a}_3\\
&= c_{3,1}a_2ba_1a_2b\bar{a}_3\underbrace{a_2\bar{b}\bar{a}_2}_{\textit{braid}}\bar{a}_1\bar{b}\bar{a}_3 = c_{3,1}a_2ba_1a_2\underbrace{b\bar{a}_3\bar{b}}_{\textit{braid}}\bar{a}_2\underbrace{b\bar{a}_1\bar{b}}_{\textit{braid}}\bar{a}_3\\
&= c_{3,1}a_2ba_1\underbrace{a_2\bar{a}_3}_{\textit{braid}}\bar{b}\underbrace{a_3\bar{a}_2\bar{a}_1}_{\textit{braid}}\bar{b}a_1\bar{a}_3 = c_{3,1}a_2ba_1\bar{a}_3\underbrace{a_2\bar{b}\bar{a}_2}_{\textit{braid}}\bar{a}_1a_3ba_1\bar{a}_3\\
&= c_{3,1}a_2ba_1\bar{a}_3\bar{b}\bar{a}_2b\bar{a}_1a_3ba_1\bar{a}_3 = c_{3,1}x_2\bar{x}_1x_0.
\end{aligned}$$

This shows $c_{2,1}\bar{c}_{1,2} \in \beta(\pi_1(\Sigma_2, p_1)) \times \mathbb{Z}$. Therefore, $(a_1a_1a_2b)^3(\bar{c}_{1,2})^2 \in \beta(\pi_1(\Sigma_2, p_1)) \times \mathbb{Z}$.

(3) Using the lantern relation $L_{2,3,1}$ and the braid relations, we can show,

$$\begin{aligned}
x_3(c_{2,3}) &= b_1a_2ba_1\bar{a}_3\bar{b}\bar{a}_2\underbrace{\bar{b}_1(c_{2,3})}_{\textit{braid}} = b_1a_2ba_1\underbrace{\bar{a}_3\bar{b}\bar{a}_2c_{2,3}}_{\textit{braid}}(b_1)\\
&= b_1a_2ba_1c_{2,3}\bar{a}_3\bar{b}\bar{a}_2(b_1)\\
&= b_1a_2b\bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{X}a_3X\bar{b}\bar{a}_2(b_1) \qquad \text{by } (\beta)\\
&= b_1a_2b\bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{b}\underbrace{\bar{a}_1\bar{a}_2}_{\textit{braid}}\bar{b}a_3b\underbrace{a_2a_1}_{\textit{braid}}b\bar{b}\bar{a}_2(b_1)\\
&= b_1a_2b\bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{b}\bar{a}_2\bar{a}_1\bar{b}a_3ba_1a_2\bar{a}_2(b_1)\\
&= b_1a_2b\bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{b}\bar{a}_2\underbrace{\bar{a}_1\bar{b}a_3ba_1(b_1)}_{\textit{braid}}\\
&= b_1a_2b\underbrace{\bar{c}_{3,1}\bar{a}_2c_{2,1}\bar{b}\bar{a}_2}_{\textit{braid}}(b_1) = b_1a_2b\bar{a}_2c_{2,1}\bar{b}\bar{a}_2\underbrace{\bar{c}_{3,1}(b_1)}_{\textit{braid}}\\
&= b_1\underbrace{a_2b\bar{a}_2c_{2,1}}_{\textit{braid}}\bar{b}\bar{a}_2(b_1) = b_1c_{2,1}\underbrace{a_2b\bar{a}_2}_{\textit{braid}}\bar{b}\bar{a}_2(b_1)\\
&= b_1c_{2,1}\bar{b}a_2b\bar{b}\bar{a}_2(b_1) = b_1c_{2,1}\underbrace{\bar{b}(b_1)}_{\textit{braid}} = b_1\underbrace{c_{2,1}(b_1)}_{\textit{braid}} = b_1\bar{b}_1(c_{2,1}) = c_{2,1}.
\end{aligned}$$

Hence we get:

$$\begin{aligned}
c_{2,3}(\bar{x}_3) &= c_{2,3}\bar{x}_3\bar{c}_{2,3} = \bar{x}_3x_3c_{2,3}\bar{x}_3\bar{c}_{2,3}\\
&= \bar{x}_3c_{2,1}\bar{c}_{2,3} \qquad \text{from the above equation } x_3(c_{2,3}) = c_{2,1}\\
&= \bar{x}_3a_2c_{3,1}a_1X\bar{a}_3\bar{X}\bar{a}_3 \qquad \text{by } (\alpha)\\
&= \bar{x}_3a_2\underbrace{c_{3,1}a_1ba_2a_1b\bar{a}_3\bar{b}\bar{a}_1\bar{a}_2\bar{b}\bar{a}_3}_{\textit{braid}} = \bar{x}_3a_2a_1b\underbrace{a_2a_1}_{\textit{braid}}b\bar{a}_3\bar{b}\bar{a}_1\bar{a}_2\bar{b}\bar{a}_3c_{3,1}\\
&= \bar{x}_3a_2\underbrace{a_1ba_1}_{\textit{braid}}a_2\underbrace{b\bar{a}_3}_{\textit{braid}}\bar{b}\bar{a}_1\bar{a}_2\bar{b}\bar{a}_3c_{3,1} = \bar{x}_3a_2ba_1b\underbrace{a_2\bar{a}_3}_{\textit{braid}}\bar{b}\underbrace{a_3\bar{a}_1\bar{a}_2}_{\textit{braid}}\bar{b}\bar{a}_3c_{3,1}\\
&= \bar{x}_3a_2ba_1b\bar{a}_3\underbrace{a_2\bar{b}\bar{a}_2}_{\textit{braid}}a_3\bar{a}_1\bar{b}\bar{a}_3c_{3,1} = \bar{x}_3a_2ba_1\underbrace{b\bar{a}_3\bar{b}}_{\textit{braid}}\bar{a}_2ba_3\bar{a}_1\bar{b}\bar{a}_3c_{3,1}\\
&= \bar{x}_3a_2ba_1\bar{a}_3\bar{b}\underbrace{a_3\bar{a}_2}_{\textit{braid}}ba_3\bar{a}_1\bar{b}\bar{a}_3c_{3,1} = \bar{x}_3a_2ba_1\bar{a}_3\bar{b}\bar{a}_2\underbrace{a_3ba_3}_{\textit{braid}}\bar{a}_1\bar{b}\bar{a}_3c_{3,1}\\
&= \bar{x}_3a_2ba_1\bar{a}_3\bar{b}\bar{a}_2ba_3\underbrace{b\bar{a}_1\bar{b}}_{\textit{braid}}\bar{a}_3c_{3,1} = \bar{x}_3a_2ba_1\bar{a}_3\bar{b}\bar{a}_2ba_3\bar{a}_1\bar{b}a_1\bar{a}_3c_{3,1}\\
&= \bar{x}_3x_2\bar{x}_1x_0c_{3,1}.
\end{aligned}$$

Previously, we remarked that $c_{1,2}(x_3) = x_3\bar{x}_2x_1\bar{x}_0c_{3,1}$. Hence,

$$\begin{aligned}
x_3\bar{x}_2x_1\bar{x}_0\bar{x}_3x_2\bar{x}_1x_0 &= c_{1,2}x_3\bar{c}_{1,2}\underbrace{\bar{c}_{3,1}c_{2,3}\bar{x}_3\bar{c}_{2,3}\bar{c}_{3,1}}_{\textit{braid}}\\
&= c_{1,2}x_3\bar{c}_{1,2}\ \underbrace{c_{2,3}}_{\textit{handle}}\ \bar{x}_3\ \underbrace{\bar{c}_{2,3}}_{\textit{handle}}\ (\bar{c}_{3,1})^2
\end{aligned}$$

$$= c_{1,2}x_3\bar{c}_{1,2}c_{1,2}\bar{x}_3\bar{c}_{1,2}(\bar{c}_{3,1})^2 = (\bar{c}_{3,1})^2.$$

This shows that, modulo powers of $c_{3,1}$, $x_3\bar{x}_2x_1\bar{x}_0\bar{x}_3x_2\bar{x}_1x_0 = 1$ is derived from relations for $G_{2,1}$.

From the above results, we can now conclude:

Proposition 4.1. $\mathcal{M}_{2,1} \cong G_{2,1}$.

5. Action of $\mathcal{M}_{g,n}$ on $X(\Sigma_{g,n})$ and a presentation for $\mathcal{M}_{g,n}$

In this section, we assume $g \geq 3$, and $n \geq 1$. We call a simple closed curve on $\Sigma_{g,n}$ *non-separating*, if its complement is connected. Define a simplicial complex $X(\Sigma_{g,n})$ of dimension $g-1$, whose vertices (0-simplices) are the isotopy classes of non-separating simple closed curves on $\Sigma_{g,n}$, and whose simplices are determined by the rule that a collection of $k+1$ distinct vertices spans a k-simplex if and only if it admits a collection of representative which are pairwise disjoint and the complement of their disjoint union is connected. This complex $X(\Sigma_{g,n})$ is defined by Harer [9]. In the same paper, he showed the following Theorem:

Theorem 5.1 ([9, Theorem 1.1]). *$X(\Sigma_{g,n})$ is homotopy equivalent to a wedge of $(g-1)$-dimensional spheres.*

Especially, if $g \geq 3$, $X(\Sigma_{g,n})$ is simply connected.

For each element ϕ of $\mathcal{M}_{g,n}$ and a simplex $([C_0], \dots, [C_n])$ of $X(\Sigma_{g,n})$, $([\phi(C_0)], \dots, [\phi(C_n)])$ is also a simplex of $X(\Sigma_{g,n})$. Hence, we can define an action of $\mathcal{M}_{g,n}$ on $X(\Sigma_{g,n})$ by $\phi([C_0], \dots, [C_n]) = ([\phi(C_0)], \dots, [\phi(C_n)])$. We can see that, each of $\{$2-simplices of $X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$, $\{$1-simplices of $X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$ and $\{$vertices of $X(\Sigma_{g,n})\}/\mathcal{M}_{g,n}$ consists of one element, each of which is represented by $([C_0],[C_1],[C_2])$, $([C_0],[C_1])$, and $([C_0])$, where $C_0 = c_{2g-2,2g-1}$, $C_1 = a_{2g-2}$, $C_2 = a_{2g-4}$. If the stabilizer of each vertex is finitely presented, and if that of each 1-simplex is finitely generated, we can obtain a presentation for $\mathcal{M}_{g,n}$ as in the way of [15], [20]. Here, we shall recall this method.

We fix a vertex v_0 of $X(\Sigma_{g,n})$, fix an edge (= a 1-simplex with orientation) e_0 of $X(\Sigma_{g,n})$ which emanates from v_0, and fix a 2-simplex f_0 of $X(\Sigma_{g,n})$ which contains v_0. Let C_0, C_1 and C_2 be non-separating simple closed curves defined as above, and we set $v_0 = [C_0]$, $e_0 = ([C_0],[C_1])$ and $f_0 = ([C_0],[C_1],[C_2])$. We choose an element t_1 of $\mathcal{M}_{g,n}$ which switches the vertices of e_0. In our situation, we set $t_1 = b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}$. By this notation, we see $e_0 = (v_0, t_1(v_0))$. We denote by $(\mathcal{M}_{g,n})_{v_0}$ the stabilizer of v_0, by $(\mathcal{M}_{g,n})_{e_0}$ that of e_0, and by $\langle t_1 \rangle$ an infinite cyclic group generated by t_1. The free product $(\mathcal{M}_{g,n})_{v_0} * \langle t_1 \rangle$ with the following three types of relations defines a presentation for $\mathcal{M}_{g,n}$. (In Subsection 5.1, we give a set of generators for $(\mathcal{M}_{g,n})_{v_0}$. In the following statements, "a presentation of s as an element

of $(\mathcal{M}_{g,n})_{v_0}$" means a presentation of s as a word of elements of this set of generators.)
(Y1) t_1^2 = a presentation of t_1^2 as an element of $(\mathcal{M}_{g,n})_{v_0}$.
(Y2) For each generator s of $(\mathcal{M}_{g,n})_{e_0}$,

$$\begin{aligned} & t_1(\text{a presentation of } s \text{ as an element of } (\mathcal{M}_{g,n})_{v_0})\overline{t_1} \\ & \quad = \text{ a presentation of } t_1 s \overline{t_1} \text{ as an element of } (\mathcal{M}_{g,n})_{v_0}. \end{aligned}$$

(Y3) For the loop ∂f_0 in $X(\Sigma_{g,n})$, we define an element W_{f_0} of $(\mathcal{M}_{g,n})_{v_0} * \langle t_1 \rangle$ in the following manner. The loop ∂f_0 consists of three vertices v_0, v_1, v_2 and three edges e_1, e_2, e_3 such that $e_1 = (v_0, v_1)$, $e_2 = (v_1, v_2)$, $e_3 = (v_2, v_0)$. There is an element h_1 of $(\mathcal{M}_{g,n})_{v_0}$ such that $h_1(e_0) = e_1$ i.e. $e_1 = (v_0, h_1 t_1(v_0))$, then $\overline{h_1 t_1}(e_2)$ is an edge emanating from v_0. Hence, there is an element h_2 of $(\mathcal{M}_{g,n})_{v_0}$ such that $h_2(e_0) = \overline{h_1 t_1}(e_2)$ i.e. $e_2 = (h_1 t_1(v_0), h_1 t_1 h_2 t_1(v_0))$, then $\overline{h_1 t_1 h_2 t_1}(e_3)$ is an edge emanating from v_0. So, there is an element h_3 of $(\mathcal{M}_{g,n})_{v_0}$ such that $h_3(e_0) = \overline{h_1 t_1 h_2 t_1}(e_3)$ i.e. $e_3 = (h_1 t_1 h_2 t_1(v_0), h_1 t_1 h_2 t_1 h_3 t_1(v_0))$. We define $W_{f_0} = h_1 t_1 h_2 t_1 h_3 t_1$. This element W_{f_0} fixes v_0, so the following is a relation for $\mathcal{M}_{g,n}$:

$$W_{f_0} = \text{ a presentation of } W_{f_0} \text{ as an element of } (\mathcal{M}_{g,n})_{v_0}.$$

Under the assumption that $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$, if we can show all the relations for $(\mathcal{M}_{g,n})_{v_0}$ and the relations of the above three types (Y1) (Y2) (Y3) are satisfied in $G_{g,n}$, then we can show the following theorem by the same reason as Section 2.

Theorem 5.2. *If $g \geq 3$, $n \geq 1$ and $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$, then $\mathcal{M}_{g,n} \cong G_{g,n}$.*

In the previous section, we have shown $\mathcal{M}_{2,1} \cong G_{2,1}$ (Proposition 4.1), therefore, $\mathcal{M}_{g,1} \cong G_{g,1}$ for any $g \geq 2$. On the other hand, Gervais showed the following theorem in §3 of [6]:

Theorem 5.3. *If $g \geq 1$, $n \geq 1$ and $\mathcal{M}_{g,n} \cong G_{g,n}$, then $\mathcal{M}_{g,n+1} \cong G_{g,n+1}$, $\mathcal{M}_{g,n-1} \cong G_{g,n-1}$.*

Theorem 1.1 is proved by Theorem 5.2 and Theorem 5.3. We remark that Theorem 5.3 was proved without using Wajnryb's simple presentation [20]. In the following subsections, we show all relations for $(\mathcal{M}_{2,1})_{v_0}$ (Subsection 5.1), relations of type (Y1) and (Y2) (Subsection 5.2), and a relation of type (Y3) (Subsection 5.3) are satisfied in $G_{g,n}$.

5.1. A presentation for $(\mathcal{M}_{g,n})_{v_0}$.

We assume that $\mathcal{M}_{g-1,n} \cong G_{g-1,n}$, and $n \geq 1$. Let $\mathrm{Diff}^+(\Sigma_{g,n})$ denote the group of orientation preserving diffeomorphisms of $\Sigma_{g,n}$. For subsets $A_1, \ldots, A_m$ and B of $\Sigma_{g,n}$, we define $\mathrm{Diff}^+(\Sigma_{g,n}, A_1, \ldots, A_m, \mathrm{rel}\, B) =$

$\{\phi \in \mathrm{Diff}^+(\Sigma_{g,n}) \mid \phi(A_1) = A_1, \ldots, \phi(A_m) = A_m, \ \phi|_B = \mathrm{id}_B\}$. In this subsection, we give a presentation for $(\mathcal{M}_{g,n})_{v_0} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, C_0, \mathrm{rel}\,\partial\Sigma_{g,n}))$. Let $\Sigma'_{g,n}$ be a surface obtained from $\Sigma_{g,n}$ by cutting along C_0, and let E_1, E_2 be connected components of $\partial\Sigma'_{g,n}$ which appeared as a result of cutting. Let α be a natural surjection from $\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \mathrm{rel}\,\partial\Sigma_{g,n}))$ to $\mathbb{Z}_2$ which is a permutation group of E_1 and E_2, and β be an inclusion of $\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, \mathrm{rel}\,\partial\Sigma'_{g,n}))$ into $\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \mathrm{rel}\,\partial\Sigma_{g,n}))$. Then, there is a short exact sequence:

$$\begin{aligned}0 \longrightarrow \pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, \mathrm{rel}\,\partial\Sigma'_{g,n})) &\xrightarrow{\beta} \pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \mathrm{rel}\,\partial\Sigma_{g,n}))\\ &\xrightarrow{\alpha} \mathbb{Z}_2 \longrightarrow 0\end{aligned}$$

We can see that

$$\pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, C_0, \mathrm{rel}\,\partial\Sigma_{g,n})) \cong \frac{\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \mathrm{rel}\,\partial\Sigma_{g,n}))}{c_{2g-2,2g-1} = c_{2g-3,2g-2}},$$

and

$$\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, \mathrm{rel}\,\partial\Sigma'_{g,n})) \cong \mathcal{M}_{g-1,n+2}.$$

By Theorem 5.3, $\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, \mathrm{rel}\,\partial\Sigma'_{g,n})) \cong G_{g-1,n+2}$. Let $r_{g-1} = \{(c_{2g-3,2g-2})^2 b_{g-1}\}^2$. Then $r_{g-1} \in \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, C_0, \mathrm{rel}\,\partial\Sigma_{g,n}))$, that is to say, we can regard r_{g-1} as an element of $\pi_0(\mathrm{Diff}^+(\Sigma'_{g,n}, E_1 \cup E_2, \mathrm{rel}\,\partial\Sigma_{g,n}))$. Then $\alpha(r_{g-1})$ generates $\mathbb{Z}_2$. From the above observations, we can see:
$\pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, C_0, \mathrm{rel}\,\partial\Sigma_{g,n}))$ is isomorphic to $G_{g-1,n+2} * \langle r_{g-1}\rangle$ with the following relations:
(A1) $c_{2g-2,2g-1} = c_{2g-3,2g-2}$,
(A2) For each generator t of $G_{g-1,n+2}$,

$$r_{g-1} t \bar{r}_{g-1} = \quad \text{a presentation of } r_g t \bar{r}_g \quad \text{as an element of} \quad G_{g-1,n+2},$$

(A3) $r_{g-1}^2 = c_{2g-3,2g-1}$.

We need to show that these relations are derived from relations for $G_{g,n}$.
(1) The relation (A1) is nothing but a handle relation.
(2) By repeatedly applying star relations, we can show $G_{g-1,n+2}$ is generated by $\mathcal{E} = \{b, a_i \ (1 \le i \le 2g+n-2), c_{2j-1,2j} \ (1 \le j \le g-2), c_{k-1,k} \ (2g-2 \le k \le 2g+n-2), \ c_{2g+n-2,1}\}$. Here, we remark that $(\mathcal{M}_{g,n})_{v_0}$ is generated by $\mathcal{E} \cup \{r_{g-1}\}$. By drawing figures, we can show:

$$\begin{aligned}&r_{g-1}(b) = b, \ \ r_{g-1}(a_i) = a_i \quad \text{if } i \neq 2g-2, \ \ 1 \le i \le 2g+n-2\\ &r_{g-1}(c_{2j-1,2j}) = c_{2j-1,2j} \quad \text{if } 1 \le j \le g-2\\ &r_{g-1}(c_{2g-3,2g-2}) = c_{2g-2,2g-1}, \ \ r_{g-1}(c_{2g-2,2g-1}) = c_{2g-3,2g-2}\end{aligned}$$

$$r_{g-1}(c_{k-1,k}) = c_{k-1,k} \quad \text{if } 2g \le k \le 2g+n-2$$
$$r_{g-1}(c_{2g+n-2,1}) = c_{2g+n-2,1}$$
$$r_{g-1}a_{2g-2}\overline{r}_{g-1}c_{2g-3,2g-1}a_{2g-2} = a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2 \quad \cdots\cdots(*)$$

The above equations except $(*)$ are derived from braid relation. We shall show that the equation $(*)$ is satisfied in $G_{g,n}$.

$$\begin{aligned}
r_{g-1}(a_{2g-2}) &= c_{2g-3,2g-2}\underbrace{c_{2g-3,2g-2}b_{g-1}c_{2g-3,2g-2}}_{braid}c_{2g-3,2g-2}b_{g-1}(a_{2g-2})\\
&= \underbrace{c_{2g-3,2g-2}b_{g-1}c_{2g-3,2g-2}}_{braid}\ \underbrace{b_{g-1}c_{2g-3,2g-2}b_{g-1}}_{braid}(a_{2g-2})\\
&= b_{g-1}c_{2g-3,2g-2}\underbrace{b_{g-1}c_{2g-3,2g-2}b_{g-1}}_{braid}c_{2g-3,2g-2}(a_{2g-2})\\
&= b_{g-1}c_{2g-3,2g-2}c_{2g-3,2g-2}b_{g-1}\underbrace{c_{2g-3,2g-2}c_{2g-3,2g-2}(a_{2g-2})}_{braid}\\
&= b_{g-1}c_{2g-3,2g-2}c_{2g-3,2g-2}b_{g-1}(a_{2g-2}).
\end{aligned}$$

By a star relation $E_{2g-3,2g-2,2g-1}$ and a handle relation $c_{2g-3,2g-2} = c_{2g-2,2g-1}$,

$$c_{2g-3,2g-2}c_{2g-3,2g-2}c_{2g-1,2g-3} = (a_{2g-3}a_{2g-2}a_{2g-1}b)^3.$$

Therefore we have,

$$\begin{aligned}
r_{g-1}(a_{2g-2}) &= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^3\underbrace{\overline{c}_{2g-1,2g-3}b_{g-1}}_{braid}(a_{2g-2})\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^3b_{g-1}\underbrace{\overline{c}_{2g-1,2g-3}(a_{2g-2})}_{braid}\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^3\underbrace{b_{g-1}(a_{2g-2})}_{braid}\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^2a_{2g-3}\underbrace{a_{2g-2}a_{2g-1}}_{braid}b\overline{a}_{2g-2}(b_{g-1})\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^2a_{2g-3}a_{2g-1}\underbrace{a_{2g-2}b\overline{a}_{2g-2}}_{braid}(b_{g-1})\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^2a_{2g-3}a_{2g-1}\overline{b}a_{2g-2}\underbrace{b(b_{g-1})}_{braid}\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^2a_{2g-3}a_{2g-1}\overline{b}\underbrace{a_{2g-2}(b_{g-1})}_{braid}\\
&= b_{g-1}(a_{2g-3}a_{2g-2}a_{2g-1}b)^2\underbrace{a_{2g-3}a_{2g-1}\overline{b}\overline{b}_{g-1}}_{braid}(a_{2g-2})\\
&= b_{g-1}(a_{2g-3}\underbrace{a_{2g-2}a_{2g-1}b}_{braid})^2\overline{b}_{g-1}a_{2g-3}a_{2g-1}\overline{b}(a_{2g-2})\\
&= (\underbrace{b_{g-1}a_{2g-3}a_{2g-1}}_{braid}a_{2g-2}\underbrace{b\overline{b}_{g-1}}_{braid})^2a_{2g-3}a_{2g-1}\overline{b}(a_{2g-2})
\end{aligned}$$

$$
\begin{aligned}
&= (a_{2g-3}a_{2g-1}\underbrace{b_{g-1}a_{2g-2}\bar{b}_{g-1}}_{\text{braid}}b)^2 a_{2g-3}a_{2g-1}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}b_{g-1}a_{2g-2}b\underbrace{a_{2g-3}a_{2g-1}\bar{a}_{2g-2}}_{\text{braid}}b_{g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}b_{g-1}\underbrace{a_{2g-2}b\bar{a}_{2g-2}}_{\text{braid}}a_{2g-3}a_{2g-1}b_{g-1}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\underbrace{b_{g-1}\bar{b}}_{\text{braid}}a_{2g-2}\underbrace{ba_{2g-3}a_{2g-1}b_{g-1}}_{\text{braid}}a_{2g-2}ba_{2g-3}a_{2g-1}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}ba_{2g-3}\underbrace{a_{2g-1}a_{2g-2}}_{\text{braid}}b\underbrace{a_{2g-3}a_{2g-1}}_{\text{braid}}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}ba_{2g-3}a_{2g-2}\underbrace{a_{2g-1}ba_{2g-1}}_{\text{braid}}a_{2g-3}\bar{b}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}b_{g-1}a_{2g-2}b_{g-1}ba_{2g-3}a_{2g-2}ba_{2g-1}\underbrace{ba_{2g-3}\bar{b}}_{\text{braid}}(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}\bar{a}_{2g-2}\bar{b}\underbrace{b_{g-1}a_{2g-2}b_{g-1}}_{\text{braid}}b\underbrace{a_{2g-3}a_{2g-2}}_{\text{braid}}b\underbrace{a_{2g-1}\bar{a}_{2g-3}}_{\text{braid}}b\underbrace{a_{2g-3}(a_{2g-2})}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}\underbrace{\bar{a}_{2g-2}\bar{b}a_{2g-2}}_{\text{braid}}b_{g-1}\underbrace{a_{2g-2}ba_{2g-2}}_{\text{braid}}\ \underbrace{a_{2g-3}b\bar{a}_{2g-3}}_{\text{braid}}a_{2g-1}b(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}\underbrace{\bar{b}b_{g-1}}_{\text{braid}}ba_{2g-2}b\bar{b}a_{2g-3}\underbrace{ba_{2g-1}}_{\text{braid}}b(a_{2g-2}) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}a_{2g-3}a_{2g-1}b\underbrace{a_{2g-1}(a_{2g-2})}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}a_{2g-3}a_{2g-1}\underbrace{b(a_{2g-2})}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}\underbrace{a_{2g-3}a_{2g-1}\bar{a}_{2g-2}}_{\text{braid}}(b) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}b_{g-1}a_{2g-2}\bar{a}_{2g-2}a_{2g-3}a_{2g-1}(b) \\
&= a_{2g-3}a_{2g-1}b\bar{a}_{2g-2}\underbrace{b_{g-1}a_{2g-3}a_{2g-1}}_{\text{braid}}(b) \\
&= a_{2g-3}a_{2g-1}b\underbrace{\bar{a}_{2g-2}a_{2g-3}a_{2g-1}}_{\text{braid}}\ \underbrace{b_{g-1}(b)}_{\text{braid}} \\
&= a_{2g-3}a_{2g-1}ba_{2g-3}a_{2g-1}\underbrace{\bar{a}_{2g-2}(b)}_{\text{braid}} = \underbrace{a_{2g-3}a_{2g-1}}_{\text{braid}}ba_{2g-3}a_{2g-1}b(a_{2g-2}) \\
&= a_{2g-1}\underbrace{a_{2g-3}ba_{2g-3}}_{\text{braid}}a_{2g-1}b(a_{2g-2}) = a_{2g-1}ba_{2g-3}\underbrace{ba_{2g-1}}_{\text{braid}}b(a_{2g-2}) \\
&= a_{2g-1}b\underbrace{a_{2g-3}a_{2g-1}}_{\text{braid}}b\underbrace{a_{2g-1}(a_{2g-2})}_{\text{braid}} = \underbrace{a_{2g-1}ba_{2g-1}}_{\text{braid}}a_{2g-3}b(a_{2g-2}) \\
&= ba_{2g-1}\underbrace{ba_{2g-3}b}_{\text{braid}}(a_{2g-2}) = b\underbrace{a_{2g-1}a_{2g-3}}_{\text{braid}}b\underbrace{a_{2g-3}(a_{2g-2})}_{\text{braid}} = ba_{2g-3}a_{2g-1}b(a_{2g-2}).
\end{aligned}
$$

The lantern relation $L_{2g-3,2g-2,2g-1}$ says,

$$c_{2g-3,2g-1}a_{2g-2}ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b} = a_{2g-3}c_{2g-3,2g-2}c_{2g-2,2g-1}a_{2g-1}.$$

Then,

$$\begin{aligned}
ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-1}\bar{b} &= \underbrace{\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1}a_{2g-3}c_{2g-3,2g-2}c_{2g-2,2g-1}a_{2g-1}}_{braid} \\
&= a_{2g-1}a_{2g-3}c_{2g-3,2g-2}\underbrace{c_{2g-2,2g-1}}_{handle}\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1} \\
&= a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2\bar{a}_{2g-2}\bar{c}_{2g-3,2g-1}.
\end{aligned}$$

Therefore,

$$ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b}c_{2g-3,2g-1}a_{2g-2} = a_{2g-1}a_{2g-3}(c_{2g-3,2g-2})^2.$$

In the above equation, we exchange

$$ba_{2g-3}a_{2g-1}ba_{2g-2}\bar{b}\bar{a}_{2g-1}\bar{a}_{2g-3}\bar{b} = ba_{2g-3}a_{2g-1}b(a_{2g-2})$$

with $r_{g-1}(a_{2g-2})$, then we get $(*)$. Hence the relation (A2) is satisfied in $G_{g,n}$.
(3) At first, we can see:

$$\begin{aligned}
&r_{g-1}a_{2g-2}r_{g-1}\underbrace{a_{2g-2}(\bar{c}_{2g-2,2g-1})^2}_{braid} \\
&\quad= \{(\underbrace{c_{2g-3,2g-2}}_{handle})^2 b_{g-1}\}^2 a_{2g-2}\{(\underbrace{c_{2g-3,2g-2}}_{handle})^2 b_{g-1}\}^2 \underbrace{a_{2g-2}(\bar{c}_{2g-2,2g-1})^2}_{braid} \\
&\quad= \{(c_{2g-2,2g-1})^2 b_{g-1}\}^2 \\
&\qquad\times a_{2g-2}c_{2g-2,2g-1}\underbrace{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}}_{braid}c_{2g-2,2g-1}b_{g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2} \\
&\quad= \{(c_{2g-2,2g-1})^2 b_{g-1}\}^2 \\
&\qquad\times a_{2g-2}\underbrace{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}}_{braid}\ \underbrace{b_{g-1}c_{2g-2,2g-1}b_{g-1}}_{braid}(\bar{c}_{2g-2,2g-1})^2a_{2g-2} \\
&\quad= \{(c_{2g-2,2g-1})^2 b_{g-1}\}^2 \\
&\qquad\times a_{2g-2}b_{g-1}c_{2g-2,2g-1}\underbrace{b_{g-1}c_{2g-2,2g-1}b_{g-1}}_{braid}c_{2g-2,2g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2} \\
&\quad= \{(c_{2g-2,2g-1})^2 b_{g-1}\}^2 \\
&\qquad\times a_{2g-2}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}(\bar{c}_{2g-2,2g-1})^2a_{2g-2} \\
&\quad= (c_{2g-2,2g-1})^2b_{g-1}(c_{2g-2,2g-1})^2\underbrace{b_{g-1}a_{2g-2}b_{g-1}}_{braid}(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2} \\
&\quad= (c_{2g-2,2g-1})^2b_{g-1}\underbrace{(c_{2g-2,2g-1})^2a_{2g-2}}_{braid}b_{g-1}a_{2g-2}(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2} \\
&\quad= \{(c_{2g-2,2g-1})^2b_{g-1}a_{2g-2}\}^3 \\
&\quad= a_{2g-3}a_{2g-1} \qquad \text{by Lemma 3.3.}
\end{aligned}$$

Therefore,

$$r_{g-1}^2 = r_{g-1}\bar{a}_{2g-2}\bar{r}_{g-1}a_{2g-1}a_{2g-3}(c_{2g-2,2g-1})^2\bar{a}_{2g-2}.$$

From the above equation and $(*)$, we can see $r_{g-1}^2 = c_{2g-3,2g-1}$.

5.2. Generators of $(\mathcal{M}_{g,n})_{e_0}$, and relations of type (Y1) and (Y2). In this subsection, we give generators of

$$(\mathcal{M}_{g,n})_{e_0} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, c_{2g-2,2g-1}, a_{2g-4}, \mathrm{rel}\,\partial\Sigma_{g,n}))$$

and, by investigating the action of t_1 on these elements, we will give relations of type (Y2), and show that these relations and a relation of type (Y1) are satisfied in $G_{g,n}$.

At first, we show $t_1^2 \in (\mathcal{M}_{g,n})_{v_0}$. By Lemma 3.3 and braid relations,

$$\begin{aligned}
&a_{2g-3}a_{2g-1}\\
&= c_{2g-2,2g-1}c_{2g-2,2g-1}\\
&\qquad \times a_{2g-2}b_{g-1}\underbrace{c_{2g-2,2g-1}c_{2g-2,2g-1}a_{2g-2}}_{braid}b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\\
&= c_{2g-2,2g-1}c_{2g-2,2g-1}\\
&\qquad \times a_{2g-2}b_{g-1}a_{2g-2}c_{2g-2,2g-1}\underbrace{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}}_{braid}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\\
&= c_{2g-2,2g-1}c_{2g-2,2g-1}\\
&\qquad \times a_{2g-2}b_{g-1}a_{2g-2}\underbrace{c_{2g-2,2g-1}b_{g-1}c_{2g-2,2g-1}}_{braid}b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\\
&= c_{2g-2,2g-1}c_{2g-2,2g-1}a_{2g-2}\underbrace{b_{g-1}a_{2g-2}b_{g-1}}_{braid}c_{2g-2,2g-1}b_{g-1}b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\\
&= c_{2g-2,2g-1}c_{2g-2,2g-1}a_{2g-2}a_{2g-2}b_{g-1}\underbrace{a_{2g-2}c_{2g-2,2g-1}}_{braid}b_{g-1}b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\\
&= (c_{2g-2,2g-1})^2(a_{2g-2})^2t_1^2 \qquad \text{since } t_1 = b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}.
\end{aligned}$$

Therefore, $t_1^2 = (\bar{a}_{2g-2})^2(\bar{c}_{2g-2,2g-1})^2a_{2g-3}a_{2g-1} \in (\mathcal{M}_{g,n})_{v_0}$. This shows that the relation of type (Y1) is satisfied in $G_{g,n}$.

Let $\Sigma''_{g,n}$ be a surface obtained from $\Sigma_{g,n}$ by cutting along $C_0 = c_{2g-2,2g-1}$, $C_1 = a_{2g-2}$. As in Fig. 4, let C'_0 and C''_0 (resp. C'_1 and C''_1) be connected components of $\partial\Sigma''_{g,n}$ which appeared as a result of cutting along C_0 (resp. C_1). We denote the simple closed curve in the interior of $\Sigma''_{g,n}$ which is homotopic to C'_0 (resp. C''_0, C'_1, C''_1) and Dehn twist along this curve by the same letter. We can see that $G_{g-2,n+4} \cong \mathcal{M}_{g-2,n+4}$ is generated by $a_i(1 \leq i \leq 2(g-3)+(n+4))$, b, $b_j(1 \leq j \leq g-3)$, $c_{2k-1,2k}(1 \leq k \leq g-3)$, $c_{l,l+1}(2(g-3)+1 \leq l \leq 2(g-3)+(n+3))$, and $c_{2(g-3)+(n+4),1}$. There is a homomorphism γ from $\mathcal{M}_{g-2,n+4}$ to $\pi_0(\mathrm{Diff}^+(\Sigma''_{g,n}, \mathrm{rel}\,\partial\Sigma''_{g,n}))$ defined by

$$\gamma(a_i) = c_{i,2g-4} \quad \text{if } 1 \leq i \leq 2g-5,$$

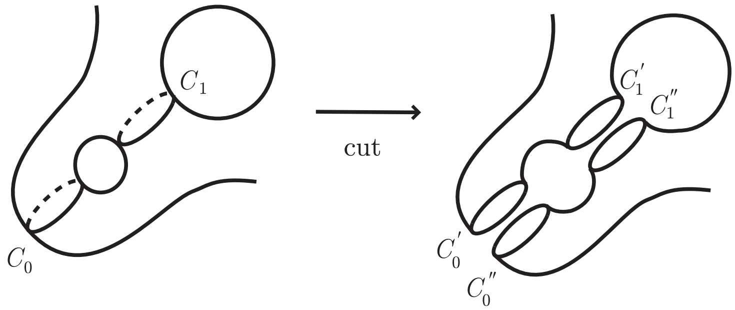

Fig. 4.

$$\begin{aligned}
\gamma(a_{2g-4}) &= c_{2g-4,2g-2},\\
\gamma(a_{2g-3}) &= a_{2g-4},\\
\gamma(a_i) &= c_{i,2g-4} \quad \text{if } 2g-2 \le i \le 2(g-3)+(n+4),\\
\gamma(b) &= b_{g-2},\\
\gamma(b_j) &= b_j \quad \text{if } 1 \le j \le g-3,\\
\gamma(c_{2k-1,2k}) &= c_{2k-1,2k} \quad \text{if } 1 \le k \le g-3,\\
\gamma(c_{2(g-3)+1,2(g-3)+2}) &= C_0'',\\
\gamma(c_{2(g-3)+2,2(g-3)+3}) &= C_1'',\\
\gamma(c_{2(g-3)+3,2(g-3)+4}) &= C_1',\\
\gamma(c_{2(g-3)+4,2(g-3)+5}) &= C_0',\\
\gamma(c_{l,l+1}) &= c_{l,l+1} \quad \text{if } 2(g-3)+5 \le l \le 2(g-3)+(n+3),\\
\gamma(c_{2(g-3)+(n+4),1}) &= a_{2g-2}.
\end{aligned}$$

This homomorphism is induced by a homeomorphism from $\Sigma_{g-2,n+4}$ to $\Sigma''_{g,n}$. Hence, γ is an isomorphism, and this fact means that the set

$$\mathcal{C}''_{g,n} = \left\{ \begin{array}{l|l} c_{i,2g-4},\ c_{2g-4,2g-2}, & 1 \le i \le 2g-5, \\ b_{g-2},\ b_j,\ c_{2j-1,2j}, & 2g-2 \le i \le 2(g-3)+(n+4), \\ C_0',\ C_0'',\ C_1',\ C_1'', & 1 \le j \le g-3, \\ c_{l,l+1},\ c_{2(g-3)+(n+4),1} & 2(g-3)+5 \le l \le 2(g-3)+(n+3) \end{array} \right\}$$

generates $\pi_0(\mathrm{Diff}^+(\Sigma''_{g,n}, \mathrm{rel}\,\partial\Sigma''_{g,n}))$. Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ denote the group, whose first factor is a permutation group of C_0' and C_0'' and the second factor is that of C_1' and C_1''. We denote by δ a natural homomorphism from $\pi_0(\mathrm{Diff}^+(\Sigma''_{g,n}, C_0' \cup C_0'', C_1' \cup C_1'', \mathrm{rel}\,\partial\Sigma_{g,n}))$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and ϵ an inclusion of $\pi_0(\mathrm{Diff}^+(\Sigma''_{g,n}, \mathrm{rel}\,\partial\Sigma''_{g,n}))$ into

$$\pi_0(\mathrm{Diff}^+(\Sigma''_{g,n}, C_0' \cup C_0'', C_1' \cup C_1'', \mathrm{rel}\,\partial\Sigma_{g,n})).$$

Then, there is a short exact sequence,

$$\begin{aligned} 0 &\longrightarrow \pi_0(\operatorname{Diff}^+(\Sigma''_{g,n}, \operatorname{rel}\partial\Sigma''_{g,n})) \\ &\xrightarrow{\epsilon} \pi_0(\operatorname{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \operatorname{rel}\partial\Sigma_{g,n})) \xrightarrow{\delta} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 0 \end{aligned}$$

Let $p = ba_{2g-2}a_{2g-2}b$, $p' = t_1 p \bar{t}_1$. Then, by drawing some figures, we can check that p and $p' \in (\mathcal{M}_{g,n})_{e_0}$ and p (resp. p') reverse the orientation of C_1 (resp. C_0). Hence, p induces a homeomorphism on $\Sigma''_{g,n}$ which exchanges C'_0 with C''_0 (resp. C'_1 with C''_1). On the other hand, there is an isomorphism

$$\begin{aligned} &\frac{\pi_0(\operatorname{Diff}^+(\Sigma''_{g,n}, C'_0 \cup C''_0, C'_1 \cup C''_1, \operatorname{rel}\partial\Sigma_{g,n}))}{(C'_0 = C''_0, C'_1 = C''_1)} \\ &\qquad \cong \pi_0(\operatorname{Diff}^+(\Sigma_{g,n}, c_{2g-2,2g-1}, a_{2g-2}, \operatorname{rel}\partial\Sigma_{g,n})), \end{aligned}$$

which maps $C'_0 = C''_0$ to $c_{2g-2,2g-1}$, $C'_1 = C''_1$ to a_{2g-2}. Therefore, we can show that $(\mathcal{M}_{g,n})_{e_0}$ is generated by $(\mathcal{C}''_{g,n} - \{C'_0, C''_0, C'_1, C''_1\}) \cup \{c_{2g-2,2g-1}, a_{2g-2}, p, p'\}$. For each element s of $\mathcal{C}''_{g,n} - \{c_{2g-2,2g-4}, c_{2g-4,2g-2}, C'_0, C''_0, C'_1, C''_1\}$, the associated curve of s is disjoint from those of b_{g-1}, a_{2g-2}, and $c_{2g-2,2g-1}$. Hence, by braid relations, $t_1 s \bar{t}_1 = s \in (\mathcal{M}_{g,n})_{v_0}$. This fact shows that, for the above element s, the relation of type (Y2) is satisfied in $G_{g,n}$.

In Subsection 5.1, we showed that $(\mathcal{M}_{g,n})_{v_0}$ is generated by $\mathcal{E} \cup \{r_{g-1}\}$, so a presentation of some element as an element of $(\mathcal{M}_{g,n})_{v_0}$ means a presentation of this elements as a word of $\mathcal{E} \cup \{r_{g-1}\}$. Here, we need to present p and p' as words of these elements. Since $b, a_{2g-2} \in \mathcal{E}$, p is presented as an element of $(\mathcal{M}_{g,n})_{v_0}$. We shall present p' as an element of $(\mathcal{M}_{g,n})_{v_0}$.

$$\begin{aligned} a_{2g-2}bt_1(b) &= a_{2g-2}bb_{g-1}\underbrace{c_{2g-2,2g-1}a_{2g-2}}_{braid}\ \underbrace{b_{g-1}(b)}_{braid} \\ &= a_{2g-2}bb_{g-1}a_{2g-2}\underbrace{c_{2g-2,2g-1}(b)}_{braid} = a_{2g-2}bb_{g-1}\underbrace{a_{2g-2}(b)}_{braid} \\ &= a_{2g-2}\underbrace{bb_{g-1}\bar{b}}_{braid}(a_{2g-2}) = a_{2g-2}\underbrace{b_{g-1}(a_{2g-2})}_{braid} = a_{2g-2}\bar{a}_{2g-2}(b_{g-1}) = b_{g-1}, \end{aligned}$$

$$\begin{aligned} a_{2g-2}bt_1(a_{2g-2}) &= a_{2g-2}bb_{g-1}c_{2g-2,2g-1}a_{2g-2}\underbrace{b_{g-1}(a_{2g-2})}_{braid} \\ &= a_{2g-2}bb_{g-1}c_{2g-2,2g-1}a_{2g-2}\bar{a}_{2g-2}(b_{g-1}) \\ &= a_{2g-2}bb_{g-1}\underbrace{c_{2g-2,2g-1}(b_{g-1})}_{braid} = a_{2g-2}bb_{g-1}\bar{b}_{g-1}(c_{2g-2,2g-1}) \\ &= \underbrace{a_{2g-2}b(c_{2g-2,2g-1})}_{braid} = c_{2g-2,2g-1}. \end{aligned}$$

Here, we remark that these equations show $t_1(a_{2g-2}) \in (\mathcal{M}_{g,n})_{v_0}$. From these equations, we can show,

$$\begin{aligned} a_{2g-2}bt_1p\overline{t_1}\overline{b}\overline{a}_{2g-2} &= a_{2g-2}bt_1ba_{2g-2}a_{2g-2}b\overline{t_1}\overline{b}\overline{a}_{2g-2} \\ &= b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{g-1} &= ((\underbrace{c_{2g-3,2g-2}}_{handle})^2 b_{g-1})^2 \\ &= ((c_{2g-2,2g-1})^2 b_{g-1})^2 \end{aligned}$$

Hence, $b_{g-1}c_{2g-2,2g-1}c_{2g-2,2g-1}b_{g-1} = (\overline{c}_{2g-1,2g-1})^2 r_{g-1}$. From the above equations, we can show $p' = t_1p\overline{t_1} = \overline{b}\overline{a}_{2g-1}(\overline{c}_{2g-2,2g-1})^2 r_{g-1}a_{2g-2}b$. This gives a presentation of p' as an element of $(\mathcal{M}_{g,n})_{v_0}$. For p, the relation of type (Y2) is

$$t_1(ba_{2g-2}a_{2g-2}b)\overline{t_1} = t_1p\overline{t_1} = \overline{b}\overline{a}_{2g-2}(\overline{c}_{2g-2,2g-1})^2 r_{g-1}a_{2g-2}b$$

This relation is satisfied in $G_{g,n}$. For p', the relation of type (Y2) is,

$$t_1(\overline{b}\overline{a}_{2g-2}(\overline{c}_{2g-2,2g-1})^2 r_{g-1}a_{2g-2}b)\overline{t_1} \in (\mathcal{M}_{g,n})_{v_0}.$$

We shall show that this equation is satisfied in $G_{g,n}$. Previously, we have shown $(t_1)^2$, $p \in (\mathcal{M}_{g,n})_{v_0}$. By the definition of p', we can show,

$$t_1(\overline{b}\overline{a}_{2g-2}(\overline{c}_{2g-2,2g-1})^2 r_{g-1}a_{2g-2}b)\overline{t_1} = t_1(t_1p\overline{t_1})\overline{t_1} = t_1^2 p\overline{t_1}^2 \in (\mathcal{M}_{g,n})_{v_0}.$$

For $c_{2g-2,2g-1}$, a_{2g-4}, we can show t_1 exchanges $c_{2g-2,2g-1}$ and a_{2g-4},

$$\begin{aligned} t_1(c_{2g-2,2g-1}) &= b_{g-1}\underbrace{c_{2g-2,2g-1}a_{2g-1}}_{braid}\ \underbrace{b_{g-1}(c_{2g-2,2g-1})}_{braid} \\ &= b_{g-1}a_{2g-2}c_{2g-2,2g-1}\overline{c}_{2g-2,2g-1}(b_{g-1}) \\ &= b_{g-1}\underbrace{a_{2g-2}(b_{g-1})}_{braid} = b_{g-1}\overline{b}_{g-1}(a_{2g-2}) = a_{2g-2}, \end{aligned}$$

$$\begin{aligned} t_1(a_{2g-2}) &= b_{g-1}c_{2g-2,2g-1}a_{2g-2}\underbrace{b_{g-1}(a_{2g-2})}_{braid} \\ &= b_{g-1}c_{2g-2,2g-1}\underbrace{a_{2g-2}\overline{a}_{2g-2}}_{braid}(b_{g-1}) \\ &= b_{g-1}\underbrace{c_{2g-2,2g-1}(b_{g-1})}_{braid} = b_{g-1}\overline{b}_{g-1}(c_{2g-2,2g-1}) = c_{2g-2,2g-1}. \end{aligned}$$

This fact shows $t_1c_{2g-2,2g-1}\overline{t_1}$, $t_1a_{2g-1}\overline{t_1} \in (\mathcal{M}_{g,n})_{v_0}$.

For $c_{2g-2,2g-4}$,

$$t_1(c_{2g-2,2g-4}) = b_{g-1}\underbrace{c_{2g-2,2g-1}a_{2g-2}}_{braid}b_{g-1}(c_{2g-2,2g-4})$$

$$
\begin{aligned}
&= b_{g-1}a_{2g-2}\underbrace{c_{2g-2,2g-1}}_{\textit{handle}}b_{g-1}(c_{2g-2,2g-4})\\
&= b_{g-1}a_{2g-2}c_{2g-3,2g-2}b_{g-1}(c_{2g-2,2g-4})\\
&= \overline{ba_{2g-4}a_{2g-2}b}(a_{2g-1}) \qquad (\text{by } X_{2g-4,2g-2}(4)).
\end{aligned}
$$

Since $b,\ a_{2g-1},\ a_{2g-2},\ a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$, this equation shows $t_1c_{2g-2,2g-4}\overline{t_1} \in (\mathcal{M}_{g,n})_{v_0}$.

For $c_{2g-4,2g-2}$, we do the same way as above,

$$
\begin{aligned}
t_1(c_{2g-4,2g-2}) &= b_{g-1}\underbrace{c_{2g-2,2g-1}a_{2g-2}}_{\textit{braid}}b_{g-1}(a_{2g-4,2g-2})\\
&= b_{g-1}a_{2g-2}\underbrace{c_{2g-2,2g-1}}_{\textit{handle}}b_{g-1}(c_{2g-4,2g-2})\\
&= b_{g-1}a_{2g-2}c_{2g-3,2g-2}b_{g-1}(c_{2g-4,2g-2})\\
&= \overline{ba_{2g-4}a_{2g-2}b}(a_{2g-3}) \qquad (\text{by } X_{2g-4,2g-2}(2)).
\end{aligned}
$$

Since $b,\ a_{2g-2},\ a_{2g-3},\ a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$, this equation shows $t_1c_{2g-4,2g-2}\overline{t_1} \in (\mathcal{M}_{g,n})_{v_0}$.

Here, we conclude that all the relations of type (Y2) are satisfied in $G_{g,n}$.

5.3. Relations of type (Y3). We define $t_2 = ba_{2g-2}a_{2g-4}b$. For the notations used to present a relation of type (Y3), it is possible to set $h_1 = 1$, $h_2 = t_2$ and $h_3 = t_2$. Then, $W_{f_0} = t_1t_2t_1t_2t_1$. By braid relations, we can show $t_1t_2t_1 = t_2t_1t_2$ as follows.

$$
\begin{aligned}
t_1t_2(b_{g-1}) &= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}ba_{2g-2}\underbrace{a_{2g-4}b(b_{g-1})}_{\textit{braid}}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}b\underbrace{a_{2g-2}(b_{g-1})}_{\textit{braid}}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\underbrace{b\overline{b}_{g-1}}_{\textit{braid}}(a_{2g-2})\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\overline{b}_{g-1}b(a_{2g-2})\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}\underbrace{b(a_{2g-2})}_{\textit{braid}}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}\overline{a}_{2g-2}(b) = \underbrace{b_{g-1}c_{2g-2,2g-1}(b)}_{\textit{braid}} = b,\\
t_1t_2(c_{2g-2,2g-1}) &= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}\underbrace{ba_{2g-2}a_{2g-4}b(c_{2g-2,2g-1})}_{\textit{braid}}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}\underbrace{b_{g-1}(c_{2g-2,2g-1})}_{\textit{braid}}\\
&= b_{g-1}c_{2g-2,2g-1}\underbrace{a_{2g-2}\overline{c}_{2g-2,2g-1}}_{\textit{braid}}(b_{g-1})\\
&= b_{g-1}c_{2g-2,2g-1}\overline{c}_{2g-2,2g-1}a_{2g-2}(b_{g-1})
\end{aligned}
$$

$$
\begin{aligned}
&= b_{g-1}\underbrace{a_{2g-2}(b_{g-1})}_{braid} = b_{g-1}\bar{b}_{g-1}(a_{2g-2}) = a_{2g-2},\\
t_1t_2(a_{2g-2}) &= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}ba_{2g-2}a_{2g-4}\underbrace{b(a_{2g-2})}_{braid}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}ba_{2g-2}\underbrace{a_{2g-4}\bar{a}_{2g-2}(b)}_{braid}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}ba_{2g-2}\bar{a}_{2g-2}a_{2g-4}(b)\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}b\underbrace{a_{2g-4}(b)}_{braid}\\
&= b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}b\bar{b}(a_{2g-4})\\
&= \underbrace{b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1}(a_{2g-4})}_{braid} = a_{2g-4}.
\end{aligned}
$$

Therefore, $t_1t_2t_1\bar{t}_2\bar{t}_1 = t_1t_2(b_{g-1}c_{2g-2,2g-1}a_{2g-2}b_{g-1})\bar{t}_2\bar{t}_1 = ba_{2g-2}a_{2g-4}b = t_2$, that is $t_1t_2t_1 = t_2t_1t_2$. Hence, we get $W_{f_0} = t_1t_2t_1t_2t_1 = t_1^2t_2t_1^2$. As we have shown in Subsection 5.2, $t_1^2 \in (\mathcal{M}_{g,n})_{v_0}$, and, since b, a_{2g-2}, $a_{2g-4} \in (\mathcal{M}_{g,n})_{v_0}$, we can show $t_2 \in (\mathcal{M}_{g,n})_{v_0}$. By using these facts, we conclude that $W_{f_0} \in (\mathcal{M}_{g,n})_{v_0}$ is satisfied in $G_{g,n}$.

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