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<td>Author(s)</td>
<td>Matumoto, Takao</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 10(1) P.51-P.68</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1973</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/11621">https://doi.org/10.18910/11621</a></td>
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<td>DOI</td>
<td>10.18910/11621</td>
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Osaka University
EQUIVARIANT COHOMOLOGY THEORIES
ON G-CW COMPLEXES

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(Received December 10, 1971)

Introduction

G.Bredon developed the equivariant (generalized) cohomology theories in [3], in which he had to restrict himself to the case of finite groups. One of the purposes of this note is to generalize his theory by replacing $G$-complexes with $G$-CW complexes. Then, for example, the followings are still true for the case in which $G$ is an arbitrary topological group. The $E_2$-term of the Atiyah-Hirzebruch spectral sequence associated to a $G$-cohomology theory (in this note we frequently use 'G-' instead of 'equivariant') is a classical $G$-cohomology theory, which is easy to calculate ($\S 1 \sim \S 4$). The $G$-obstruction theory works in a classical $G$-cohomology theory ($\S 5$). Moreover, for a $G$-cohomology theory we get a representation theorem of E.Brown ($\S 6$) and the Maunder's spectral sequence ($\S 7$).

As an application we study the equivariant $K$*-theory in the last section ($\S 8$). The Atiyah-Hirzebruch spectral sequence for $K_{2k}(X)$ collapses, if $\dim X/G \leq 2$ or $X$ satisfies some other conditions. The $E_2$-term depends only on the orbit type decomposition of the orbit space, if $X$ is a regular $O(n)$-manifold or the like. These facts enable us to calculate the equivariant $K$*-group of Hirzebruch-Mayer $O(n)$-manifolds and Janich knot $O(n)$-manifolds. Our spectral sequence for a differentiable $G$-manifold is similar to that of G.Segal which is defined by the equivariant nerve of his [13], but ours is easier to calculate the $E_2$-term.

In this note $G$ denotes a fixed topological group. Terminologies and notation follow those of [3], [9], [10] in general, though $\sigma$ denotes a closed cell which is the closure of an (open) cell in the definition of a $G$-CW complex in [10]. And $G\sigma$ denotes the $G$-orbit of $\sigma$ and $H_\sigma$ the unique isotropy subgroup at any interior point of $\sigma$. $\S 0$ is exposed for reference to the properties of $G$-CW complexes.

The author wishes to thank Professors Shôrô Araki and Akio Hattori for their criticisms and encouragements.

*) Supported in part by the Sakkokai Foundation.
0. Preliminaries about $G$-CW complexes

We summarize here the properties of $G$-CW complexes and $G$-CW complexes with base point (the base point in $G$-CW complex is always assumed to be a vertex which is left fixed by each element of $G$).

**Proposition 0.1.** (*G-cellular approximation theorem*) Let $f : X \to Y$ be a $G$-map between $G$-CW complexes (with base point). Then $f$ is (base point preserving) $G$-homotopic to a $G$-map $f' : X \to Y$ such that $f'(X^n) \subset Y^n$ for any $n$.

This is Theorem 4.4 of [10]. Moreover, if $f$ is $G$-cellular on a $G$-subcomplex $A$, then we may require $f' = f$ on $A$.

**Proposition 0.2.** (*G-homotopy extension property*) Let $f_0 : X \to Y$ be a given $G$-map of a $G$-CW complex $X$ into an arbitrary $G$-space $Y$. Let $g_t : A \to Y$ be a $G$-homotopy of $g_0 = f_0|A$, where $A$ is a $G$-subcomplex of $X$. Then, there is a $G$-homotopy $f_t : X \to Y$, such that $f_t|A = g_t$.

This is (J) of [10].

For a pair of $G$-CW complexes $(X, A)$, collapsed $A$ into a point, $X/A$ forms a $G$-CW complex with a base point $A/A$ (taken to be a disjoint point if $A = \phi$, in which case $X^+$ denotes $X/\phi$). Let $i : A \to X$ be the inclusion. Consider the mapping cone $C_f = X \cup CA = (X \times \{1\} \cup A \times I) / A \times \{0\}$ with the obvious $G$-action, trivial on $I$. Then, by the $G$-homotopy extension property, we can prove that the collapsing map, $X \cup CA \to X \cup CA / CA = X/A$ is a $G$-homotopy equivalence. Therefore, we get

**Proposition 0.3.** Let $(X, A)$ be a pair of $G$-CW complexes (with base point) and let $i : A \to X$ be the natural inclusion. Then, in the following cofibering sequence, the vertical maps are $G$-homotopy equivalences:

\[
\begin{array}{ccc}
A & \to & X \\
\downarrow & & \downarrow \\
C_i & \to & C_j \\
\downarrow & & \downarrow \\
X/A & \to & SA \\
\end{array}
\]

**Proposition 0.4.** (*Theorem of J.H.C.Whitehead*) Let $\varphi : (X, A) \to (Y, B)$ be a $G$-map between two pairs of $G$-CW complexes with base point. For each closed subgroup $H$ which appears as an isotropy subgroup in $X$ or $Y$, we assume that $X^H$, $A^H$, $Y^H$ and $B^H$ are arcwise connected, and the induced maps,

\[
\varphi_* : \pi_n(X^H, *) \to \pi_n(Y^H, *)
\]

and

\[
\tilde{\varphi}_* : \pi_n(A^H, *) \to \pi_n(B^H, *)
\]

are bijective for $1 \leq n \leq \max(\dim X, \dim Y)$. Then, $\varphi : (X, A) \to (Y, B)$ is a $G$-
Proposition 0.5. Let $G$ be a compact Lie group. Then any compact differentiable $G$-manifold has a $G$-finite $G$-CW complex structure.

This comes from Proposition 4.4 of [9].

1. Definition of an equivariant cohomology theory on $G$-CW complexes

On the category of pairs of $G$-finite $G$-CW complexes and $G$-homotopy classes of $G$-maps, a $G$-cohomology theory is defined to be a sequence of contravariant functors $h^n_G(-, \infty) \to \text{Ab}$ together with natural transformation $\delta^n: h^n_G(A, \phi) \to h^{n+1}_G(X, A)$ such that the following axioms are satisfied (we put $h^i_G(X) = h^i_G(X, \Phi)$):

1. The inclusion $(X, X \cap A) \to (X \cup A, A)$ induces an isomorphism,

$$h^n_G(X \cup A, A) \cong h^n_G(X, X \cap A).$$

2. If $(X, A)$ is a pair of $G$-finite $G$-CW complexes, the sequence,

$$\cdots \to h^n_G(X, A) \to h^n_G(X) \to h^n_G(A) \to h^{n+1}_G(X, A) \to \cdots$$

is exact.

Standard argument can be used to prove the exactness of Mayer-Vietoris sequence and the long sequence of triples.

Lemma 1.1. For a pair of $G$-finite $G$-CW complexes $(X, A)$, the collapsing map, $(X, A) \to (X/A, A/A)$, induces an isomorphism,

$$h^n_G(X/A, A/A) \cong h^n_G(X, A).$$

Proof. By the proposition 0.3 the collapsing map, $X \cup CA \to X \cup CA/CA = X/A$ is a $G$-homotopy equivalence. Moreover, $CA \to \ast$ is an $G$-homotopy equivalence, and $(X, A) \to (X \cup CA, CA)$ is an exision map. Hence, we get the commutative diagram (the homomorphisms are induced by the canonical $G$-maps),

$$
\begin{array}{ccc}
\vdots & \to & \vdots \\
\cong & & \cong \\
h^n_G(X \cup CA, \ast) & \cong & h^n_G(X \cup CA, CA) \\
\downarrow & & \downarrow \\
h^n_G(X/A, A/A) & \to & h^n_G(X, A)
\end{array}
$$

q.e.d.

*) The footnote at p. 371 of [10] is inadequate. "*) $\pi_s(X, Y)$ vanishes' should read "$\pi_s(X, Y, y)$ vanishes for every point $y$ of $Y$" and also '*) $\pi_s(X) \to \ast$ $\pi_s(Y)$ is bijective or surjective' should read "$\varphi_s: \pi_s(X, x) \to \pi_s(Y, \varphi(x))$ is bijective or surjective for every point $x$ of $X$". Then, the statements and proofs in [10] are true in the context except Theorem 5.2. In Theorem 5.2 we should add the assumption that each arcwise connected component of $X$ or $Y$ is $n$-simple for every $n \geq 1$. \n
For a $G$-CW complex with base point $X$, $SX = S \wedge X$ (with obvious $G$-action, trivial on the "circle factor" $S$) denotes the reduced suspension of $X$. A reduced $G$-cohomology theory on the category of $G$-finite $G$-CW complexes with base point and base point preserving $G$-homotopy classes of base point preserving $G$-maps is a sequence of contravariant functors $\tilde{h}^n_G(-, \infty < n < \infty)$ into the category of abelian groups, together with natural transformations $\sigma^n: \tilde{h}^n_G(X) \rightarrow \tilde{h}^{n+1}_G(SX)$ satisfying the following axioms:

1. $\sigma^n$ is an isomorphism for each $n$ and $X$.
2. The short sequence, $\tilde{h}^n(X/A) \rightarrow \tilde{h}^n(X) \rightarrow \tilde{h}^n(A)$ is exact.

REMARK 1.2. By Proposition 0.3 and Axioms (1)', (2)' we get the long exact sequence for $Ag(\cdot)$.

Let $h^n_G$ be a $G$-cohomology theory. Define $\tilde{h}^n_G(X)$ by $h^n_G(X, \ast)$. Then $h^n_G$ is a reduced $G$-cohomology theory by Lemma 1.1. Conversely let $\tilde{h}^n_G$ be a reduced $G$-cohomology theory. Define $h^n_G(X, A)$ by $\tilde{h}^n_G(X/A)$. Then $\tilde{h}^n_G$ is a $G$-cohomology theory by Remark 1.2. This is a canonical one-to-one correspondence. Afterwards we identify $h^n_G(X, A)$ and $\tilde{h}^n_G(X/A)$.

We enclose this section after giving some examples.

EXAMPLES 1.3. of $G$-COHOMOLOGY THEORIES:

(i) $h^n_G(X) = H^n(X; G; \mathbb{Z})$.
(ii) $h^n_G(X) = K^n_G(X)$ when $G$ is a compact Lie group.
(iii) $h^n_G(X) = h^n(X \times_G E_G)$ where $E_G$ is a universal $G$-principal bundle and $h^n$ a cohomology theory for spaces.

2. On classification of $G$-maps between $G$-cells of the same dimension up to $G$-homotopy classes

Let $H$ be a closed subgroup of $G$. Suppose that $X$ is a space and $G/H \times X$ is a $G$-space with the obvious $G$-action, trivial on $X$. Let $Y$ be a $G$-space and $f: G/H \times X \rightarrow Y$ be a $G$-map. Since $f$ is $G$-equivariant, we get, $f(H/H \times X) \subset Y^H$ where $Y^H$ is the $H$-pointwise fixed subspace of $Y$. Therefore, we may define a map, $\bar{f}: X \rightarrow Y^H$, by $\bar{f}(x) = f(H/H \times x)$.

Lemma 2.1. In the above situation, the correspondence, $f \mapsto \bar{f}$, yields an isomorphism of sets,

$$G\text{-maps (}G/H \times X, Y) \xrightarrow{\sim} Maps (X, Y^H).$$

Moreover, the isomorphism induces another isomorphism,

$$[G/H \times X; Y]_G \xrightarrow{\sim} [X; Y^H]$$
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where \([\cdot, \cdot]_G\) stands for the set of G-homotopy classes of G-maps.

Proof. Let \(f: X \to Y^H\) be a map. Define a map, \(\tilde{f}: G/H \times X \to Y\), by \(\tilde{f}(gH, x) = g \cdot f(x)\) for any \(g \in G\), and any \(x \in X\). If \(gH = g'H\), then \(g = g' \cdot h\) for some \(h \in H\), so that \(g \cdot f(x) = g' \cdot f(x)\) (since \(f(x)\) is fixed by \(H\)), which shows that this definition is valid. By this definition \(\tilde{f}\) is certainly \(G\)-equivariant, and conversely if we assume that a map \(f: G/H \times X \to Y\) is \(G\)-equivariant, we get \(f(gH \times x) = g \cdot f(H \times x)\).

Therefore, the correspondence, \(f \mapsto \tilde{f}\), is the converse to the correspondence, \(\tilde{f} \mapsto f\). This proves the first isomorphism. The second isomorphism is induced, because the \(G\)-homotopy \(f_t(0 \leq t \leq 1)\) and homotopy \(\tilde{f}_t(0 \leq t \leq 1)\) correspond each other in the same way.

q.e.d.

Assume that \(X\) has a distinguished closed subspace \(\tilde{A}\) and \(Y\) has a base point \(y_0\) (the base point is left fixed by \(G\)).

Lemma 2.1'. The correspondence, \(f \mapsto \tilde{f}\), yields an isomorphism,

\[
\text{G-maps} \left(\left(\left(G/H \times X\right)\right) \setminus \left(\left(G/H \times \tilde{A}\right)\right), Y/y_0\right) \cong \text{Map} (X/\tilde{A}, Y^H/y_0).
\]

Moreover, the isomorphism induces another isomorphism,

\[
\left[\left(\left(G/H \times X\right)\right) \setminus \left(\left(G/H \times \tilde{A}\right)\right)\right], Y/y_0}_G, \cong \left[\left(X/\tilde{A}\right) \setminus \left(Y^H/y_0\right)\right],
\]

where \([\cdot, \cdot]_G, \) stands for the set of base point preserving \(G\)-homotopy classes of base point preserving \(G\)-maps.

Proof. The correspondence \(f \mapsto \tilde{f}\), is also defined in the same way as in Lemma 2.1.

q.e.d.

Therefore, we get

Corollary 2.2. Let \(H\) and \(K\) be two closed subgroups of \(G\) and \(n \geq 0\) be a fixed integer. Then, "the restriction" yields the following isomorphisms,

(i) \([G/H; G/K]_G \cong \pi_n((G/K)^H),\)
(ii) \([\left(G/H \times \Delta^n\right) \setminus \left(G/H \times \tilde{\Delta}^n\right); \left(G/K \times \Delta^n\right) \setminus \left(G/K \times \tilde{\Delta}^n\right)\]_G, \cong \pi_n((G/K)^H \times \Delta^n)\).

Here \(\pi_n(\cdot)\) stands for the set of arcwise connected components and \(*\) is the base point \((G/K)^H \times \tilde{\Delta}^n\).

Now let \(Y\) be a space and \(n \geq 1\) be an integer.

Lemma 2.3. \(Y \times \Delta^n/Y \times \partial \Delta^n\) is \((n-1)\)-connected, and there are natural isomorphisms,

\[
\pi_n(Y \times \Delta^n/Y \times \partial \Delta^n, *) \cong H_n(Y \times \Delta^n/Y \times \partial \Delta^n; Z), \quad \cong H_n(Y; Z)
\]
Here \( \pi_n'(\cdot) = \pi_n(\cdot) \) for \( n \geq 2 \) and \( \pi_1'(\cdot) \) is the abelianized group of \( \pi_1(\cdot) \) and \( H_n'(\cdot; \mathbb{Z}) \) is the singular homology group.

Proof. By the definition, \( Y \times \Delta^n/Y \times \partial \Delta^n \) is homeomorphic with the smash product \( Y^+ \wedge \Delta^n/\partial \Delta^n \). Hence \( Y \times \Delta^n/Y \times \partial \Delta^n \) is \((n-1)\)-connected. If we use the Hurwitz theorem, the rest is easily proved.

q.e.d.

Let \( \{ Y_{\lambda} : \lambda \in \Lambda \} \) be the family of all the arcwise connected components of \( Y \). Take an element \( y_{\lambda} \in Y_{\lambda} \) for each \( \lambda \). Then each element of \( H_0(Y; \mathbb{Z}) \) has \( \sum n_{\lambda} \cdot y_{\lambda}(n_{\lambda} = 0 \text{ except the finite } \lambda's) \) as its representative. Also any map: \( (\Delta^n, \partial \Delta^n) \rightarrow (Y \times \Delta^n/Y \times \partial \Delta^n, \ast) \) determines \( n_{\lambda} \) uniquely.

Now let \( H \) and \( K \) be closed subgroups of \( G \). Recall that for any element \( g \in N(H, K) = \{ g \in G, Hg \subset gK \} \), \( \hat{g} : G/H \rightarrow G/K \) is defined by \( \hat{g}(ah) = agK \), and this correspondence, \( g \mapsto \hat{g} \), induces an isomorphism,

\[
N(H, K)/K = (G/K)^H \xrightarrow{\cong} \text{G-maps } (G/H, G/K).
\]

Suppose that \( \{ g_{\lambda} \in G \} \) is the family of representatives of all arcwise connected components of \( N(H, K)/K = (G/K)^H \). Then any base point preserving \( G \)-map,

\[
f : (G/H \times \Delta^n)/(G/H \times \partial \Delta^n) \rightarrow (G/K \times \Delta^n)/(G/K \times \partial \Delta^n),
\]

determines \( n_{\lambda}(f) \) such that \( f \) is equal to \( \Sigma n_{\lambda}(f) \cdot g_{\lambda} \) in \( \pi_n'((G/K)^H \times \Delta^n)/(G/K)^H \times \partial \Delta^n), \ast) \approx H_0((G/K)^H; \mathbb{Z}) \).

Let \( L \) be another closed subgroup of \( G \). Suppose that \( g_{\lambda} \in N(H, K) \) and \( g_{\mu} \in N(K, L) \), then we get

\[
g_{\lambda} \cdot g_{\mu} \in N(H, L) \text{ (not } g_{\mu} \cdot g_{\lambda}!), \text{ and } (g_{\lambda} \cdot g_{\mu})' = \hat{g}_{\mu} \circ \hat{g}_{\lambda}.
\]

From this we get

**Proposition 2.4.** Let \( H, K \) and \( L \) be closed subgroup of \( G \). Suppose that \( \{ g_{\lambda} \in G \}, \{ g_{\mu} \in G \} \) and \( \{ g \in G \} \) are the families of representatives of all arcwise connected components of \( N(H, K)/K, N(K, L)/L \text{ and } N(H, L)/L \) respectively. Let \( f : (G/H \times \Delta^n)/(G/H \times \partial \Delta^n) \rightarrow (G/K \times \Delta^n)/(G/K \times \partial \Delta^n) \) and \( g : (G/K \times \Delta^n)/(G/K \times \partial \Delta^n) \rightarrow (G/L \times \Delta^n)/(G/L \times \partial \Delta^n), \) be base point preserving \( G \)-maps. Then,

\[
n_{\lambda}(g \circ f) = \Sigma n_{\lambda}(g)n_{\lambda}(f).
\]

Here the summation is taken over the pairs \((\lambda, \mu)\) such that \( g_{\lambda} \cdot g_{\mu} \) and \( g_{\nu} \) are in the same arcwise connected component of \( N(H, L)/L \).

### 3. Classical \( G \)-cohomology theory on \( G \)-CW complexes

We shall define a classical \( G \)-cohomology theory with coefficients in a (generic) \( G \)-coefficient system. In §4 the classical \( G \)-cohomology theory will be characterized as the \( G \)-cohomology theory which satisfies also the dimension
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DEFINITION 3.1. A (generic) $G$-coefficient system is a contravariant functor $M_G$ of the category of the left coset spaces of $G$ by closed subgroups, $G/H$, and $G$-homotopy classes of $G$-maps (equivariant with respect to left translation), $G/H \to G/K$, into the category of abelian groups.

REMARK. When $G$ is a discrete group, any two distinct $G$-maps between $G$-coset spaces cannot be $G$-homotopic and hence this definition coincides with the generic equivariant coefficient system of Bredon in [3].

EXAMPLES 3.2. OF $G$-COEFFICIENT SYSTEMS:

(i) $M_G = h^g$.  
(ii) $M_G = \mathbb{Z}$ with a trivial $G$-action.  
(iii) $M_G = \omega_n(Y)(n \geq 2)$, where $Y$ is a $G$-space with a base point $y_0$ and $\omega_n(Y)(G/H) = \pi_n(Y^H, y_0) \cong [(G/H \times \Delta^n)/(G/H \times \partial \Delta^n), Y/y_0]_{G,o}$.  

Let $M_G$ be a $G$-coefficient system. The $n$-dimensional $G$-cochain group of a pair of $G$-CW complexes $(X, A)$ with coefficients in $M_G$, denoted by $C^n_G(X, A ; M_G)$, is defined to be the group of all $G$-equivariant functions $\varphi$ on the $n$-cells of $(X, A)$ with $\varphi(\sigma) \in M_G(G/H_\sigma)$ and $M_G(\hat{g}) \varphi(\sigma) = \varphi(g \sigma)$ for a right translation $\hat{g} : G/H_\sigma \ni a \mapsto ag \in G/H_\sigma$. (If $\sigma$ is an $n$-cell of $A$ or a $p$-cell ($p \neq n$), then $\varphi(\sigma) = 0$.)

By the definition of the $G$-cochain group, $C^n_G(X, A ; M_G)$ is canonically isomorphic with $C^n_G(X^n/X^{n-1} \cup A ; M_G)$. Moreover, since $X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma)$ where $\sigma$ range over the representatives of all $n$-dimensional $G$-cells of $(X, A)$,

$C^n_G(X^n/X^{n-1} \cup A ; M_G) = C^n_G(\vee(G\sigma/G\partial\sigma) ; M_G) = \prod C^n_G(G\sigma/G\partial\sigma ; M_G).$

Let $f : (X, A) \to (Y, B)$ be a $G$-cellular map between pairs of $G$-CW complexes. Then, for every $n$, $f$ induces a $G$-map,

$f^n : X^n/X^{n-1} \cup A \to Y^n/Y^{n-1} \cup B$.

Suppose that $\sigma$ and $\tau$ are representatives of all $G$-$n$-cells of $(X, A)$ and $(Y, B)$ respectively. Then we can define a $G$-map $f_{\sigma\tau}$ (between $G$-cells of the same dimension $n$) by $f_{\sigma\tau} = c \cdot f^n \circ i$ in the following diagram:

\[
\begin{array}{ccc}
X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma) & \xrightarrow{c} & G\sigma/G\partial\sigma = (G/H_\sigma \times \Delta^n)/(G/H_\sigma \times \partial \Delta^n) \\
\downarrow f^n & & \downarrow f_{\sigma\tau} \\
Y^n/Y^{n-1} \cup B = \vee(G\tau/G\partial\tau) & \xrightarrow{i} & G\tau/G\partial\tau = (G/H_\tau \times \Delta^n)/(G/H_\tau \times \partial \Delta^n)
\end{array}
\]

where $i$ is the inclusion and $c$ is the collapsing of the other factors.
Let \( \{g_{h(\sigma, \tau)} \in G\} \) be the family of representatives of all arcwise connected components of \((G/H)^{n_\sigma}\) as in §2.

Define \( f^* = C^*_G(f; M_G) : C^*_G(Y, B; M_G) \to C^*_G(X, A; M_G) \) by

\[
(f^* \varphi)(\sigma) = \sum_{\lambda, (\sigma, \tau)} n_{\lambda(\sigma, \tau)}(f_{\tau}) M_G(\lambda G_{\sigma, \tau}) \varphi(\tau)
\]

where \( \tau \) ranges over the representatives of all \( G \)-\( n \)-cells of \((Y, B)\). The sum is finite because \( n_{\lambda(f_{\tau})} = 0 \) except the finite \( \lambda \)'s.

**Proposition 3.3.** Let \( M_G \) be a \( G \)-coefficient system. Then, \( C^*_G(\cdot; M_G) \) is a contravariant functor from the category of pairs of \( G \)-CW complexes and \( G \)-cellular maps into the category of abelian groups.

Proof. If we fix the representatives, \((g \circ f)^* = f^* \circ g^*\) by Proposition 2.4. It is easily seen that \( f^* \) is determined independent of the representatives. Remark that \( f^* \) depends only on the \( G \)-homotopy class of the \( G \)-map \( f^* \).

Now recall that \( X^n/X^{n-1} \cup A \) has the same \( G \)-homotopy type with \( X^n \cup C(X^{n-1} \cup A) \) canonically. As a special case of Proposition 0.3, we have a Puppe sequence (the horizontal sequence),

\[
\begin{array}{ccc}
S(X^{n-1}/X^{n-3} \cup A) & \rightarrow & X^{n-1}/X^{n-2} \cup A \rightarrow X^n/X^{n-1} \cup A \rightarrow S(X^{n-1}/X^{n-3} \cup A) \\
 & \downarrow \delta & \downarrow \delta \\
X^n/X^{n-3} \cup A & \rightarrow & S(X^{n-2}/X^{n-3} \cup A)
\end{array}
\]

Since both the vertical and oblique sequences are cofiberings, we get that \( S(\delta) \circ \delta \) is \( G \)-homotopic to the trivial map. On the other hand we have a canonical isomorphism,

\[
\sigma : C^*_G(X^{n-1}/X^{n-2} \cup A; M_G) \cong C^*_G(S(X^{n-1}/X^{n-2} \cup A); M_G). 
\]

Define the coboundary homomorphism

\[
\delta : C^*_G(X, A; M_G) \to C^*_G(X, A; M_G)
\]

by \( \delta = C^*_G(\delta) \circ \sigma \). Then, because \( S(\delta) \circ \delta = 0 \), we get \( \delta \circ \delta = 0 \).

**Definition 3.4.** The classical \( G \)-cohomology theory on a pair of \( G \)-CW complexes \((X, A)\) with the coefficients in a \( G \)-coefficient system \( M_G \), denoted by \( H^*_G(X, A; M_G) \), is defined by \( H^*_G(X, A; M_G) = H^*(C^*_G(X, A; M_G), \delta) \).

**Remark 3.5.** Let \( \sigma \) and \( \tau \) be \( n \)-cell and \((n-1)\)-cell of \((X, A)\). We write \([\sigma, g_{h(\sigma, \tau)}]\) for \( n_{h(\sigma, \tau)}(\partial_{\sigma}) \) where \( \partial_{\sigma} : G\sigma/G\partial\sigma \to S(G\tau/G\partial\tau) \). Then, we get the formula,
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\[ (\delta \varphi)(\sigma) = \sum \sum \left[ \sigma, g_{s(\sigma, \tau)} \right] M_{\sigma} \varphi(g_{s(\sigma, \tau)}) \]

where \( \tau \) ranges over the representatives of all \( G -(n-1) \)-cells and \( g_{s(\sigma, \tau)} \) ranges over the representatives of all arcwise connected components of \( N(H_{\sigma}, H_{\tau})/H_{\tau} \).

**Theorem 3.6.** The classical \( G \)-cohomology theory \( H^{p}_{G}(\cdot; M_{G}) \) is a \( G \)-cohomology theory in the sense of \( \S 1 \).

Proof. We prove here only the \( G \)-homotopy axiom. The exision axiom and the exactness axiom is trivially satisfied. Let \( f: (X, A) \rightarrow (Y, B) \) be a \( G \)-map between pairs of \( G \)-CW complexes. By a \( G \)-cellular approximation theorem we may assume that \( f \) is \( G \)-cellular. The induced map \( f^*: C_{*}^{G}(Y, B; M_{G}) \rightarrow C_{*}^{G}(X, A; M) \) commutes with \( \delta \), in fact, \( f^{*} \circ \delta = C_{*}^{G}(f) \circ C_{*}^{G}(\delta) \circ \sigma = C_{*}^{G}(\delta \circ f) \circ \sigma = C_{*}^{G}(S(f) \circ \delta) \circ \sigma = C_{*}^{G}(\delta) \circ C_{*}^{G}(f) \circ \sigma = C_{*}^{G}(\delta) \circ \sigma \circ C_{*}^{G}^{-1}(f) = \delta \circ f^{*} \). This gives an induced map \( f^*: H_{G}^{p}(Y, B; M_{G}) \rightarrow H_{G}^{p}(X, A; M_{G}) \).

If \( f \) is \( G \)-homotopic to \( g \), we may assume that not only \( f \) and \( g \) are \( G \)-cellular but \( G \)-homotopy \( F: (X, I, A \times I) \rightarrow (Y, B) \) with \( F|X \times \{0\} = f, F|X \times \{1\} = g \) is also \( G \)-cellular. Then, \( F \) gives a homotopy connecting the chain maps, \( f^* \) and \( g^* \): \( C_{*}^{G}(Y, B; M_{G}) \rightarrow C_{*}^{G}(X, A; M_{G}) \) and hence \( f^* \circ g^* = h_{G}^{p}(Y, B; M_{G}) \rightarrow H_{G}^{p}(X, A; M_{G}) \). Therefore, even if \( f \) is not a \( G \)-cellular map the induced map \( f^*: H_{G}^{p}(Y, B; M_{G}) \rightarrow H_{G}^{p}(X, A; M_{G}) \) is well-defined and satisfies the \( G \)-homotopy axiom.

q.e.d.

4. Spectral sequence of Atiyah-Hirzebruch type

Suppose that \( (X, A) \) is a fixed pair of \( G \)-finite \( G \)-CW complexes. Put \( H(p, q) = \Sigma h_{G}^{p}(X^{p-1}, X^{p-1} \cup A) \). Then, the collection of \( H(p, q) \)'s satisfies the axioms (S.P. 1)-(S.P. 5) of Cartan-Eilenberg [5. p.334] and hence induces a spectral sequence resulting to \( h_{G}^{p}(X, A) \). The \( E_1 \)-term and the 1st differential of the spectral sequence are easily calculated as follows:

\[ E_{1}^{p, q} = h_{G}^{p+q}(X^{p}, X^{p-1} \cup A) \]

\[ d_1 = \delta: h_{G}^{p+q}(X^{p}, X^{p-1} \cup A) \rightarrow h_{G}^{p+q+1}(X^{p+1}, X^{p} \cup A) \]

where \( \delta \) is the coboundary homomorphism.

**Lemma 4.1.**

(i) \( h_{G}^{p+q}(X^{p}, X^{p-1} \cup A) = \prod h_{G}^{p+q}(G\sigma, G\sigma \circ G\delta\sigma) \), where \( \sigma \) ranges over representatives of all \( p \)-dimensional \( G \)-cells of \( X \).

(ii) And for each direct factor, there are isomorphisms, \( h_{G}^{p+q}(G\sigma, G\sigma \circ G\delta\sigma) = h_{G}^{p+q}(G\sigma, G\sigma) \sim h_{G}^{p+q}(G|H_{\sigma} \times \Delta^{p}, G|H_{\delta} \times \delta\Delta^{p}) \sim h_{G}^{p}(G|H_{\delta}) \).

Proof of (i). Since \( X^{p}/X^{p-1} \cup A = \vee(G\sigma, G\delta\sigma) \) is the one point union of finite \( G\sigma, G\delta\sigma \)'s we get the decomposition by the usual argument.
Proof of (ii). The 2nd isomorphism is induced by the $G$-characteristic map, $G_f: (G/H \times \Delta^p, G/H \times \partial \Delta^p) \to (G\sigma, G\partial \sigma)$, which is a relative $G$-homeomorphism. Now we shall prove the last isomorphism. Put $H = H_\sigma$. Since the inclusion, $G/H \times (\partial \Delta^p - \Delta^{p-1}) \to G/H \times \Delta^p$, has a $G$-equivariant deformation retraction, we get the isomorphism,

$$h^g_\ast (G/H \times \Delta^p, G/H \times \partial \Delta^p) \cong h^g_\ast (G/H \times (\partial \Delta^p - \Delta^{p-1}))$$

in the exact sequence of a triple $(G/H \times \Delta^p, G/H \times \partial \Delta^p, G/H \times (\partial \Delta^p - \Delta^{p-1}))$. By the exision axiom, we get the isomorphism,

$$h^g_\ast (G/H \times \Delta^p, G/H \times (\partial \Delta^p - \Delta^{p-1})) \cong h^g_\ast (G/H \times \Delta^p, G/H \times (\partial \Delta^p - \Delta^{p-1}))$$

Combining the isomorphisms of these two types repeatedly, we get

$$h^g_\ast (G/H \times \Delta^p, G/H \times \partial \Delta^p) \cong h^g_\ast (G/H \times \Delta^p, G/H \times (\partial \Delta^p - \Delta^{p-1}))$$

$$\cdots \cong h^g_\ast (G/H \times \Delta^p, G/H \times \partial \Delta^p) = h^g_\ast (G/H).$$

q.e.d.

We shall consider the difference of taking another representative $g\sigma$ instead of $\sigma$, as a representative of a $p$-dimensional $G$-cell $G\sigma$. Put $H = H_\sigma$. Then $gHG^{-1} = H_\sigma$. Since we may identify $agH$-orbit of $\sigma$ with $agHg^{-1}$-orbit of $g\sigma$ in $G\sigma$, a canonical right translation $\hat{g}: G/gHg^{-1} \to agHg^{-1} \subset G/H$ induces a required isomorphism, $h^g_\ast (\hat{g}): h^g_\ast (G/H_\sigma) \to h^g_\ast (G/Hg\sigma)$. This shows that $h^g_\ast (X^p, X^{p-1} \cup A) \cong C^g_\ast (X^p, X^{p-1} \cup A; h^g_\ast)$.

**Theorem 4.2.** The $E_2^{p,q}$-term of the Atiyah-Hirzebruch spectral sequence for a $G$-cohomology theory, $h^g_\ast$, on $G$-finite $G$-CW complexes, is a classical $G$-cohomology theory with coefficients in $h^g_\ast$.

Proof. By the result above we can identify $E_2^{p,q} = h^g_\ast (X^p, X^{p-1} \cup A)$ with $C^g_\ast (X, A; h^g_\ast)$. And the coboundary homomorphisms are induced from $\partial$ in the Puppe sequence in both cases.

q.e.d.

Assume that the $G$-cohomology theory $h^g_\ast (\cdot)$ is defined also on (not $G$-finite) $G$-CW complexes, and satisfies the additivity axiom:

(3) The inclusions, $i_\ast: X_\ast \to \Pi X_\ast$, induce an isomorphism,

$$\Pi h^g_\ast (i_\ast): \Pi h^g_\ast (X_\ast) \cong h^g_\ast (\Pi X_\ast)$$

Then, Lemma 4.1 and Theorem 4.2 are also valid for a pair of (not $G$-finite) $G$-CW complexes.

The classical $G$-cohomology theory is defined on $G$-CW complexes and satisfies the additivity axiom. Therefore, we get as usual

**Theorem 4.3.** The classical $G$-cohomology theory is characterized to be
the $G$-cohomology theory defined on $G$-CW complexes which satisfies also the additivity axiom and the dimension axiom.

Here we mean by dimension axiom,

$$h^G_0(G/H) = 0$$

for $n \neq 0$ and all closed subgroup $H$ of $G$.

The additivity axiom and the dimension axiom are as follows, for the reduced $G$-cohomology theory.

(3)' The inclusions, $i_a: X_a \to \sqcup X_a$, induce an isomorphism,

$$\prod h^G_0(i_a): \prod h^G_0(X_a) \cong h^G_0(\sqcup X_a).$$

(4)' $h^G_0(G/H)^* = 0$ for $n \neq 0$ and all $H$.

5. $G$-obstruction theory

Let $Y$ be a $G$-space with a base point. Then in the classical $G$-cohomology group $h^G_0(\cdot; \omega_n(Y))$, we can make a $G$-obstruction theory similar to that of Bredon [3].

Let $n \geq 1$ be a fixed integer and $A$ be a $G$-subcomplex of a $G$-CW complex $X$. We shall assume, for simplicity, that the pointwise fixed subspace $Y^H$ of $Y$ by $H$ is non-empty, arcwise connected and $n$-simple for each closed subgroup $H$ of $G$ which appears as an isotropy subgroup at a point of $X$.

Assume that we are given a $G$-map $\varphi: X^n \cup A \to Y$. Let $\sigma$ be an $(n+1)$-cell of $X$ and let $f_\sigma: \partial \Delta^{n+1} \to X^n$ be the characteristic attaching map of $\sigma$ and $H_{\sigma} = H$. Because the image of $\partial \Delta^{n+1}$ by $\varphi \circ f$ is pointwise fixed by $H$, we get a map: $\partial \Delta^{n+1} \to Y^H$. We define $c_\varphi(\sigma) \in \pi_n(Y^H, \ast) = \omega_n(Y)(G/H)$ to be the unique base point preserving homotopy class which is free homotopic to the above map $(\pi_n(Y^H, \ast) \cong [S^n, Y^H])$ because $Y^H$ is $n$-simple. Since $\varphi$ is a $G$-map, we get $c_\varphi(g\sigma) = g \cdot c_\varphi(\sigma) \in \pi_n(Y^{gH\sigma^{-1}}, \ast) = \omega_n(Y)(G/GHg^{-1})$ and hence $c_\varphi \in C^{n+1}_G(X, A; \omega_n(Y))$.

**Lemma 5.1.** $\delta c_\varphi = 0 \in C^{n+2}_G(X, A; \omega_n(Y))$.

Proof. Let $\tau$ be an $(n+2)$-cell of $(X, A)$ and $i: (G\tau, G\partial \tau) \to (X, A)$ be the inclusion. Then $i^*\delta c_\varphi = \delta i^*c_\varphi$ and $i^*C_\varphi \in C^{n+1}_G(G\tau, G\partial \tau; \omega_n(Y))$. According to our definition of $C^{n+1}_G(\cdot; \omega_n(Y))$ on $G$-CW complexes, $C^{n+1}_G(G\tau, G\partial \tau; \omega_n(Y)) = 0$. Therefore, $i^*c_\varphi = 0$ and hence $i^*\delta c_\varphi = 0$, that is, $c_\varphi(\tau) = 0$ for any $(n+2)$-cell $\tau$ of $(X, A)$.

q.e.d.

Now identifying the $G$-homotopy classes of $G$-maps: $G/H \times \partial \Delta^{n+1} \to Y$ and the homotopy classes of maps: $\partial \Delta^{n+1} \to Y^H$, we can reduce the proof of the following lemmas to the ordinary obstruction theory as Bredon did.

**Lemma 5.2.** $c_\varphi = 0$ if and only if $\varphi$ is extendable equivariantly on $X^{n+1} \cup A$.

**Lemma 5.3.** Let $d \in C^n_0(X, A; \omega_n(Y))$. Then, there is a $G$-map $\theta: X^n \cup A \to Y$, coinciding with $\varphi$ on $X^{n+1} \cup A$ such that $d_{\theta, \varphi} = d$. 

Here the difference cochain $d_{\theta \phi}$ is defined to be the class which corresponds to $c_{\phi \phi}$ by the isomorphism, $C^n_G(X, A; \omega_m(Y)) \to C^{n+1}_G(X \times I, A \times I \cup X \times \delta I; \omega_m(Y))$.

$\theta \star \phi$ is a G-map: $(X \times I)^n \cup A \times I \to Y$ which is $\phi$ on $X^n \times \{0\} \cup X^{n-1} \times I$ and $\theta$ on $X^n \times \{1\}$.

Combining these three lemmas, we get

**Theorem 5.4.** Let $\varphi: X^n \cup A \to Y$ be a G-map. Then $\varphi|X^{n-1} \cup A$ can be extended to G-map: $X^{n+1} \cup A \to Y$ if and only if the G-cohomology class of $c_\varphi$ in $H^{n+1}_G(X, A; \omega_m(Y))$ vanishes.

Also the argument of Bredon in 'primary obstructions' [3, II.5.2] is valid to this case. In particular, we get

**Proposition 5.5.** Let $n \geq 1$ be a fixed integer and let $Y$ be a G-space with base point such that $Y^H$ is non-empty, arcwise connected and $n$-simple for every closed subgroup $H$ of $G$. Suppose that $\omega_k(Y)$ vanishes for $k \neq n$, then a primary obstruction map,

$$\sigma: [X; Y]_G \cong H^n_G(X; \omega_m(Y))$$

is an isomorphism for any G-CW complex $X$.

**Proposition 5.5.’** Under the assumption above, a primary obstruction map,

$$\sigma’: [X, Y]_{G, 0} \cong H^n_G(X; \omega_m(Y))$$

is an isomorphism for any G-CW complex $X$ with base point.

6. **Representation theorem of E. Brown**

We shall prove the following representation theorem as an application of E.Brown's abstract homotopy theory [4].

**Theorem 6.1.** If a reduced G-cohomology group $\tilde{h}^n_G$ on G-CW complexes with base point satisfies the additivity axiom, then $\tilde{h}^n_G$ is representable, that is, there is a G-space $Y^\infty$ with base point and a natural transformation $T: [\cdot; Y^\infty]_{G, 0} \to \tilde{h}^n_G(\cdot)$ such that $T$ is an isomorphism for any G-CW complex with base point, where $[\cdot; \cdot]_{G, 0}$ stands for the set of base point preserving G-homotopy classes of base point preserving G-maps.

Let $\mathcal{C}$ be the category of G-CW complexes with base point such that the $H$-stationary subspace is arcwise connected for each $H$, and base point preserving G-homotopy classes of base point preserving G-maps. In $\mathcal{C}$ there is a (not unique) sequential direct limit by approximating G-maps by G-cellular maps and making their telescope. Also we get a (not unique) 'push out' as a double mapping cylinder in $\mathcal{C}$. If we choose one representative for each class of conjugate closed subgroups, $\{(G/H \times \Delta^p)/(G/H \times \partial \Delta^p); H$ representative, $0 < p < \infty\}$ is a
small subcategory of \( \mathcal{C} \).

Let \( \mathcal{C}_0 \) be a minimal subcategory which contains \((G/H \times \Delta^p)/(G/H \times \partial \Delta^p)\)'s \((0 < p < \infty)\) and their 'push out'. Then \( \mathcal{C}_0 \) is a small, full subcategory of \( \mathcal{C} \) and also a subcategory of \( G \)-finite \( G \)-CW complexes with base point and we get

**Proposition 6.2.** A pair \((\mathcal{C}, \mathcal{C}_0)\) is a homotopy category in the sense of E.Brown.

Proof of Theorem 6.1. Since reduced \( G \)-cohomology theory has a Mayer-Vietoris exact sequence, \( \tilde{h}_G^0 \) (restricted on \( \mathcal{C} \)) with the additivity axiom is a homotopy functor in the sense of E.Brown. Moreover, we get \( \overline{\mathcal{C}} = \mathcal{C} \) by an equivariant version of J.H.C.Whitehead's theorem. (See Proposition 0.4.). Therefore, by Theorem 2.8 of [4], we get a \( Y_n' \in \mathcal{C} \) unique up to \( G \)-homotopy equivalence and a natural transformation \( T: [\cdot , Y_n']_{G,H} \rightarrow \tilde{h}_G^0(\cdot) \) such that \( T \) is an isomorphism for each \( X \in \mathcal{C} \).

Define \( Y_n = \Omega Y_{n+1} \). For any \( G \)-CW complex \( X \) with base point, \( SX \in \mathcal{C} \). Therefore, we get

\[
[X, Y_n]_{G_0} \xrightarrow{\tilde{h}_G^0(X)} \tilde{h}_G^0(X) \xrightarrow{\approx} [SX, Y_n']_{G_0} \xrightarrow{\approx} \tilde{h}_G^0(SX)
\]

q.e.d.

**Remark.** Even when \( \tilde{h}_G^n \) is defined only on \( G \)-finite \( G \)-CW complexes, by the method of Adams [2], we get a reduced \( G \)-cohomology theory on \( G \)-CW complexes which satisfies the additivity axiom and coincides with \( \tilde{h}_G^0 \) on \( G \)-finite \( G \)-CW complexes.

Let \( Y_{n+1} \in \mathcal{C} \) be a representing space of \( \tilde{h}_G^0 \) in the category of \( \mathcal{C} \). Then, the isomorphism: \( \tilde{h}_G^{n+1}(X) \xrightarrow{\approx} \tilde{h}_G^{n+2}(SX) \) induces a \( G \)-map \( h_{n+1}: Y_{n+1} \rightarrow \Omega Y_{n+2} \) which is a weak \( G \)-homotopy equivalence, that is, \((h_{n+1})_*: \pi_i(Y_{n+1}) \rightarrow \pi_i(\Omega Y_{n+2})\) for any \( i \) and any \( H \). Hence, taking their loop spaces, we get also a weak \( G \)-homotopy equivalence, \( h_n: Y_n \rightarrow \Omega Y_{n+1} \). Then, \( Y = \{ Y_n, h_n; -\infty < n < \infty \} \) forms a weak \( \Omega \)-spectrum for \( \tilde{h}_G^n \). This fact is used in §7 to make a spectral sequence of C.Maunder.

**7. Killing the elements of the \( G \)-homotopy groups and C.Maunder's spectral sequence**

Let \( Y \) be a \( G \)-space with base point \( y_0 \) such that \( Y^H \) is arcwise connected for each closed subgroup \( H \) of \( G \). An element in the \( n \)-th homotopy group \( \pi_n(Y^H, y_0) \) of \( H \)-stationary subspace \( Y^H \) is called to be an element of \( G \)-\( n \)-homotopy groups of \( Y \). An element \( [f] \in \pi_n(Y^H, y_0) \) with \( f: S^n = \Delta^0/\partial \Delta^0 \rightarrow Y^H \) is killed by attaching a \( G \)-\((n+1)\)-cell represented by an \((n+1)\)-cell \( \sigma \) which has \( f \) as its charac-
teristic attaching map and $H$ as its isotropy subgroup, that is, $H_ε=H$. If we fix $n$ and kill all the elements of $G$-$n$-homotopy groups, we get a relative $G$-CW complex $Y$ such that $Y^1=Y$. Then, $i*: π_n(Y^H, y_0)→π_n(Y^H, y_0)$ is a zero map for any closed subgroup $H$, where $i: Y^H→Y^H$. On the other hand, by the $G$-cellular approximation theorem we get $π_k(Y^H, Y^H, y_0)$ vanishes for $k<n$ and any $H$, that is, $i*: π_k(Y^H, y_0)→π_k(Y^H, y_0)$ is an isomorphism for $k<n$ and a surjection for $k=n$. Therefore, $π_k(Y^H, y_0)$ is canonically isomorphic with $π_k(Y^H, y_0)$ for $k<n$ and vanishes for $k=n$. By this reason we call $Y$ a $G$-space obtained of $Y$ by killing the elements of $G$-$n$-homotopy groups.

Let $Y(1, p)$ be a $G$-space obtained of $Y$ by killing the elements of $G$-homotopy groups of dimensions $≥(p+1)$ one after the other. Then, $Y(1, p)$ is uniquely determined up to $G$-homotopy types rel. $Y$ by the usual argument on (relative) $G$-CW complexes. For $p≤q$, $Y(p, q)$ denotes the mapping track of $i(p, q): Y(1, q)→Y(1, p−1)$. Moreover, let $Y^{cr}(p, q)$ denote the mapping track of $i^{cr}(p, q): Y(r, q)→Y(r, p−1)$ for $r<p≤q$. Then, it is easily seen that the natural $G$-map: $Y^{cr}(p, q)→Y(p, q)$ has a $G$-homotopy inverse. Therefore, by taking mapping tracks repeatedly, we get a following $G$-fibering sequence of $G$-spaces. (The $G$-spaces are determined up to $G$-homotopy types.)

$$ΩY(r, t)→ΩY(r, s)→Y(s+1, t)→Y(r, t)→Y(r, s), \quad r≤s<t.$$  

Here, that $X→Y→Z$ is a $G$-fibration stands for that $X^H→Y^H→Z^H$ is a fibering for any $H$. In particular, $π_k(Y(p, q)^H, y_0)$ is isomorphic with $π_k(Y^H, y_0)$ for $p≤k≤q$ and vanishes otherwise.

In §6 we have obtained a weak $Ω$-spectrum for a $G$-cohomology theory $h^g$. Let $X$ be a $G$-finite $G$-CW complex and put $H(p, q)=∑[S(X^+); Y_{n+1}(p+2, q)]_{G, 0}$. Then, by the $G$-fibering sequence above, we get a spectral sequence resulting to $h^g(X)=∑[S(X^+); Y_{n+1}]_{G, 0}$. The $E_2$ term, $E_2^{p,q}=S(X^+); Y_{p+1}(p+1, p+1)_{G, 0}$, is isomorphic with $H^g(S(X^+); π_{p+1}(Y_{p+1}))=H^g(S(X^+); h^g)$ by Proposition 5.5'. Moreover, since $[S((X^{p+1})^+); Y_{p+q+1}(1, p+1)]_{G, 0}=S(X^+); Y_{p+q+1}(1, p+1)_{G, 0}$ and $[S(X^p; X^{p+1})^+); Y_{p+q+1}(1, p+1)]_{G, 0}=S(X^p; X^{p+1}); Y_{p+q+1}(1, p+1)]_{G, 0}$, the Maunder's argument using exact couples [11] is also valid in this case. Hence, we get

Theorem 7.1. Let $h^g$ be $G$-cohomology theory. Then, the spectral sequence above is isomorphic with the Atiyah-Hirzebruch spectral sequence except the $E_1$-term for any $G$-finite $G$-CW complex $X$.

Proposition 7.2. The $r$-th differential $d_r: E_2^{p,q}→E_2^{p+r, q−r+1}$ in the Maunder's spectral sequence is induced from the 'higher cohomology operation' determined by the $G$-homotopy class of

$$δ_r=δ=δh^g_{p+q+1}: Y_{p+1}(p+1, p+r−1)→ΩY_{p+q+1}(p+2, p+r)→Y_{p+q+1}(p+r+1, p+q+1).$$
Remark that $[\delta_r] \in H^p_{G}(Y'_{p+q+1}(p+1, p+r-1), \omega_{p+q+1}(Y'_{p+q+1}))$.

**Corollary 7.3.** $E^p_{r,q} = E^r_{p,q}(r \geq 2)$ together with the differentials $d_r$ are $G$-homotopy type invariant.

This is also proved from Theorem 4.2 and comparison of spectral sequences.

8. Applications to the equivariant $K^*$-theory

In this section $G$ denotes a compact Lie group. We shall apply our results to $K^*_G$-theory.

**Theorem 8.1.** Let $X$ be a $G$-finite $G$-CW complex. There exists a spectral sequence $E^p_{r,q}(r \geq 1, -\infty < p, q < \infty)$ with

$$E^p_{r,q} \cong C^p_e(X, K^q_G)$$

$d_r$ being the coboundary homomorphism.

$$E^p_{0,q} \cong H^p_{G}(X, K^q_G),$$

$$E^p_{0,q} \cong G^p K^p_{G}(X) = K^p_{G}(G/H)$$

where $K^p_{G,H}(X) = \text{Kernel}(K^p_{G}(X) \to K^p_{G}(X^{G/H}))$. The $G$-coefficient system, $K^q_G(G/H)$ is isomorphic with $K^q_G(G/H)$ for $q$ even and vanishes for $q$ odd (See [13]).

This is a special case of Theorem 4.2.

A. Collapsing theorems

If $r$ is even, the $r$-th differential is a zero map, because $d_r$ is a map of $E^p_{r,q}$ into $E^p_{r+r,q-r+1}$ where one of the domain or the image vanishes. Therefore, we get

**Theorem 8.2.** If one of the following conditions is satisfied, then the above spectral sequence collapses:

(i) $H^p_{G}(X; K^q_G)$ vanishes for every odd $p$.

(ii) $H^p_{G}(X; K^q_G)$ vanishes for every $p \geq 3$.

For the reduced $K^*_G$-theory, we get

**Theorem 8.2'.** If $X$ has a base point, then the spectral sequence,

$$H^p_{G}(X; K^q_G) \Rightarrow K^p_{G}(X)$$

collapses if:

(i) $H^p_{G}(X; K^q_G)$ vanishes for every odd $p$ or for every even $p$, or

(ii) $H^p_{G}(X; K^q_G)$ vanishes except $p=r, r+1, r+2$ for some $r$.

B. On $E_2$-term

We consider the classical $G$-cohomology theory with coefficients in $K^*_G$. 
$K_G(G/H)$ is canonically isomorphic with $R(H)$, where $R(H)$ is the Grothendieck group of the isomorphic classes of complex representations of $H$.

Remark that $K_G(\hat{g})$: $K_G(G/G)\to K_G(G/G)$ is an identity isomorphism for any $g\in G$, because any inner automorphism of $G$ induces an identity isomorphism on $R(G)$. Therefore, if we assume that the restriction maps $i^*$: $R(G)\to R(H)$ is surjective, then $K_G(\hat{g})=K_G(\hat{g}')$: $K_G(G|H)\to K_G(G|H')$ for any elements $g, g'$ of $N(H', H)$. Hence, by Remark 3.5 we shall get

**Proposition 8.3.** Let $X$ be a $G$-finite $G$-CW complex whose isotropy subgroups satisfy the condition:

\[ (\ast) \text{ the restriction map: } R(G)\to R(H) \text{ is a surjection for any closed subgroup } H \text{ which appears as an isotropy subgroup at a point of } X. \]

Then, $H^G(X)$ can be calculated by considering only the orbit type decomposition of the orbit space.

Proof. As we remark above, by the condition $(\ast)$, $K_G(\hat{g})$: $K_G(G|H)\to K_G(G|H')$ is independent of the choice of $g\in N(H', H)$ for any isotropy subgroups $H, H'$. So, we may write this map by $K_G(H^N(G))$. Then, we get the formula:

\[ (\delta \varphi)(\sigma) = \sum \sum [\sigma, g_{\lambda(\sigma, \tau)}] K_G(H_{(\sigma, \tau)} \to H \varphi(\tau)). \]

On the other hand, it is easy to see that

\[ \sum \lambda(\sigma, \tau) = [\sigma/G, \tau/G] \in Z \]

where $\sigma/G$ and $\tau/G$ are the induced cells on $X/G$.

q.e.d.

**Remark 8.4.** We call an $O(n)$-manifold to be a regular $O(n)$-manifold if each isotropy subgroup is conjugate to $O(k)$ ($k\leq n$). Then any regular $O(n)$-manifold satisfies the condition $(\ast)$ above, because the restriction map $\rho_*: R(O(n))\to R(O(n-1))$ is a surjection. This fact is easily checked by the classical representation theory as in [14], but we refer the reader to [12].

C. **A conclusion**

Combining these results with Proposition 0.5, we get

**Proposition 8.5.** For a compact regular $O(n)$ manifold $X$, if dim $X/G\leq 2$, then, $K^*_G(X)$, $K^*_G(X)$, $K^*_G(X)$ and $K^*_G(X)$ depend only on the orbit type decomposition of the orbit space.

D. **Examples**

Now we shall calculate $K^*_G(X)$ for some regular $O(n)$-manifolds.

(i) **Hirzebruch-Mayer $O(n)$-manifold $W^{\ast n-1}(d)$ for $n\geq 2$** [7]: The orbit
space is a 2-disk $D^2$ the orbit type of whose interior is $(O(n-2))$ and the boundary $(O(n-1))$.

Define a presheaf $\mathfrak{F}$ on the orbit space $D^2$ by $\Gamma(U, \mathfrak{F})=\Gamma(U, U \times R(O(n-2)))$ if $U \subset \text{Int } D^2$ and by $\Gamma(U, \mathfrak{F})=\Gamma(U, U \times R(O(n-1)))$ if $U \cap \partial D^2 \neq \emptyset$. Then, by Proposition 8.3, $H^*_G(W_{n-1}(d); K_G)\cong H^*(D^2, \mathfrak{F})$. Remark that $\mathfrak{F}$ forms a sheaf. Define $\mathfrak{G}$ and $\mathfrak{G}^\prime$ by $\mathfrak{G}={\text{constant sheaf Ker }}\rho_{n-1}$ on $\partial D^2$ which is considered to be a sheaf over $D^2$ and $\mathfrak{G}^\prime={\text{constant sheaf R(O(n-2))}}$ on whole $D^2$. Then, since $\rho_{n-1}: R(O(n-1)) \to R(O(n-2))$ is surjective, we get an exact sequence of sheaves,

$$0 \to \mathfrak{G} \to \mathfrak{F} \to \mathfrak{G}^\prime \to 0.$$

The following notation is simpler and reasonable to denote this exact sequence.

$$\begin{array}{ccc}
S^1 \\
\cap: 0 \to \begin{pmatrix}
\text{Ker }\rho_{n-1} \\
0
\end{pmatrix} \to \begin{pmatrix}
R(O(n-1)) \\
R(O(n-2))
\end{pmatrix} \to \begin{pmatrix}
R(O(n-2)) \\
R(O(n-2))
\end{pmatrix} \to 0
\end{array}$$

From the associated long exact sequence, we get

$$H^*_G \cong R(O(n-1)), H^1_G \cong \text{Ker }\rho_{n-1} \text{ and } H^2_G \cong \text{Coker }\rho_{n-1}=0.$$

Therefore,

$$K^0_G \cong R(O(n-1)) \text{ and } K^1_G \cong \text{Ker }\rho_{n-1}.$$

(ii) Jänich knot $O(n)$-manifold for $n \geq 3$ [8]: Let $S^1 \subset S^3$ be a knot. The orbit space is a 4-disk $D^4$ where the orbit type of each difference domain of $D^4 \supset S^1 \supset S^3$ is $(O(n-2)), (O(n-1)), (O(n))$ respectively.

As in (i), we consider the following exact sequence of sheaves.

$$\begin{array}{ccccc}
S^3 \\
\cap: 0 \to \begin{pmatrix}
\text{Ker }\rho_n \\
0
\end{pmatrix} \to \begin{pmatrix}
R(O(n)) \\
R(O(n-1))
\end{pmatrix} \to \begin{pmatrix}
R(O(n-1)) \\
R(O(n-2))
\end{pmatrix} \to 0
\end{array}$$

Then, $H^*_G=H^*_G(X; K) \cong H^*(D^4; \mathfrak{F}^\prime)$ is calculated as follows:

$$H^0_G \cong R(O(n)), H^1_G \cong \text{Ker }\rho_n, H^2_G=0, H^3_G \cong \text{Ker }\rho_{n-1} \text{ and } H^4_G=0.$$

In particular, if we consider that the $O(n)$-manifold has a base point, then $H^0_G=0$ and $H^*_G$ satisfies the condition (ii) of Theorem 8.2'. Therefore, we get

$$K^0_G=0,$$

that is, $K^0_G \cong R(O(n))$

and

$$0 \to \text{Ker }\rho_{n-1} \to K^1_G \to \text{Ker }\rho_n \to 0$$

is exact.
References