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EQUIVARIANT COHOMOLOGY THEORIES ON G-CW COMPLEXES

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Introduction

G.Bredon developed the equivariant (generalized) cohomology theories in [3], in which he had to restrict himself to the case of finite groups. One of the purposes of this note is to generalize his theory by replacing G-complexes with G-CW complexes. Then, for example, the followings are still true for the case in which G is an arbitrary topological group. The E_2-term of the Atiyah-Hirzebruch spectral sequence associated to a G-cohomology theroy (in this note we frequently use 'G-' instead of 'equivariant') is a classical G-cohomology theory, which is easy to calculate (§1~§4). The G-obstruction theory works in a classical G-cohomology theory (§5). Moreover, for a G-cohomology theory we get a representation theorem of E.Brown (§6) and the Maunder's spectral sequence (§7).

As an application we study the equivariant K^*-theory in the last sestion (§8). The Atiyah-Hirzebruch spectral sequence for $K_G^*(X)$ collapses, if dim $X/G \leqq 2$ or X satisfies some other conditions. The E_2-term depends only on the orbit type decomposition of the orbit space, if X is a regular $O(n)$-manifold or the like. These facts enable us to calculate the equivariant K^*-group of Hirzebruch-Mayer $O(n)$-manifolds and Jänich knot $O(n)$-manifolds. Our spectral sequence for a differentiable G-manifold is similar to that of G.Segal which is defined by the equivariant nerve of his [13], but ours is easier to calculate the E_2-term.

In this note G denotes a fixed topological group. Terminologies and notation follow those of [3], [9], [10] in general, though σ denotes a closed cell which is the closure of an (open) cell in the definition of a G-CW complex in [10]. And $G\sigma$ denotes the G-orbit of σ and H_σ the unique isotropy subgroup at any interior point of σ. §0 is exposed for reference to the properties of G-CW complexes.

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0. Preliminaries about G-CW complexes

We summarize here the properties of G-CW complexes and G-CW complexes with base point (the base point in G-CW complex is always assumed to be a vertex which is left fixed by each element of G).

Proposition 0.1. (G-cellular approximation theorem) *Let $f: X \to Y$ be a G-map between G-CW complexes (with base point). Then f is (base point preserving) G-homotopic to a G-map, $f': X \to Y$ such that $f'(X^n) \subset Y^n$ for any n.*

This is Theorem 4.4 of [10]. Moreover, if f is G-cellular on a G-subcomplex A, then we may require $f'=f$ on A.

Proposition 0.2. (G-homotopy extension property) *Let $f_0: X \to Y$ be a given G-map of a G-CW complex X into an arbitrary G-space Y. Let $g_t: A \to Y$ be a G-homotopy of $g_0=f_0|A$, where A is a G-subcomplex of X. Then, there is a G-homotopy $f_t: X \to Y$, such that $f_t|A=g_t$.*

This is (J) of [10].

For a pair of G-CW complexes (X, A), collapsed A into a point, X/A forms a G-CW complex with a base point A/A (taken to be a disjoint point if $A=\phi$, in which case X^+ denotes X/ϕ). Let $i: A \to X$ be the inclusion. Consider the mapping cone $C_i=X \cup CA=(X \times \{1\} \cup A \times I)/A \times \{0\}$ with the obvious G-action, trivial on I. Then, by the G-homotopy extension property, we can prove that the collapsing map, $X \cup CA \to X \cup CA/CA=X/A$ is a G-homotopy equivalence. Therefore, we get

Proposition 0.3. *Let (X, A) be a pair of G-CW complexes (with base point) and let $i: A \to X$ be the natural inclusion. Then, in the following cofibering sequence, the vertical maps are G-homotopy equivalences:*

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[r, "j"] \arrow[dr] & C_i \arrow[r, "f"] \arrow[d, "\simeq G"] & C_j \arrow[r] \arrow[d, "\simeq G"] & C_f \arrow[d, "\simeq G"] \\ & & X/A \arrow[r] & SA \arrow[r] & SX \end{tikzcd}$$

Proposition 0.4. (Theorem of J.H.C.Whitehead) *Let $\varphi: (X, A) \to (Y, B)$ be a G-map between two pairs of G-CW complexes with base point. For each closed subgroup H which appears as an isotropy subgroup in X or Y, we assume that X^H, A^H, Y^H and B^H are arcwise connected, and the induced maps,*

$$\varphi_*: \pi_n(X^H, *) \to \pi_n(Y^H, *)$$

and

$$\varphi_*: \pi_n(A^H, *) \to \pi_n(B^H, *)$$

are bijective for $1 \leqq n \leqq$ max (dim X, dim Y). Then, $\varphi: (X, A) \to (Y, B)$ is a G-

homotopy equivalence.

This is a special case of *) Theorem 5.3 of [10].

Proposition 0.5. *Let G be a compact Lie group. Then any compact differentiable G-manifold has a G-finite G-CW complex structure.*

This comes from Proposition 4.4 of [9].

1. Definition of an equivariant cohomology theory on G-CW complexes

On the category of pairs of G-finite G-CW complexes and G-homotopy classes of G-maps, a *G-cohomology theory* is defined to be a sequence of contravariant functors $h_G^n(-\infty<n<\infty)$ into the category of abelian groups together with natural transformation δ^n: $h_G^n(A, \phi)\to h_G^{n+1}(X, A)$ such that the following axioms are satisfied (we put $h_G^n(X)=h_G^n(X, \phi)$):

(1) The inclusion $(X, X\cap A)\to(X\cup A, A)$ induces an isomorphism,

$$h_G^n(X\cup A, A)\xrightarrow{\cong} h_G^n(X, X\cap A).$$

(2) If (X, A) is a pair of G-finite G-CW complexes, the sequence,

$$\cdots\to h_G^n(X, A)\to h_G^n(X)\to h_G^n(A)\xrightarrow{\delta^n} h_G^{n+1}(X, A)\to\cdots$$

is exact.

Standard argument can be used to prove the exactness of Mayer-Vietoris sequence and the long sequence of triples.

Lemma 1.1. *For a pair of G-finite G-CW complexes (X,A), the collapsing map, $(X, A)\to(X/A, A/A)$, induces an isomorphism,*

$$h_G^n(X/A, A/A)\xrightarrow{\cong} h_G^n(X, A)$$

Proof. By the proposition 0.3 the collapsing map, $X\cup CA\to X\cup CA/CA =X/A$ is a G-homotopy equivalence. Moreover, $CA\to *$ is an G-homotopy equivalence, and $(X, A)\to(X\cup CA, CA)$ is an exision map. Hence, we get the commutative diagram (the homomorphisms are induced by the canonical G-maps),

$$\begin{tikzcd} h_G^n(X\cup CA, *) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h_G^n(X\cup CA, CA) \arrow[dl] \\ h_G^n(X/A, A/A) \arrow[r] & h_G^n(X, A) \arrow[u, "\cong"'] \end{tikzcd}$$

q.e.d.

) The footnote at p. 371 of [10] is inadequate. ') $\pi_k(X,Y)$ vainshes' should read "$\pi_k(X,Y,y)$ vanishes for every point y of Y" and also 'φ_*: *) $\pi_k(X)\to$*) $\pi_k(Y)$ is bijective or surjective' should read "φ_*: $\pi_k(X, x)\to\pi_k(Y, \varphi(x))$ is bijective or surjective for every point x of X". Then, the statements and proofs in [10] are true in the context except Theorem 5.2. In Theorem 5.2 we should add the assumption that each arcwise connected component of X or Y is n-simple for every $n\geq1$.

For a G-CW complex with base point X, $SX=S\wedge X$ (with obvious G-action, trivial on the "circle factor" S) denotes the reduced suspension of X. A *reduced G-cohomology theory* on the category of G-finite G-CW complexes with base point and base point preserving G-homotopy classes of base point preserving G-maps is a sequence of contravariant functors $\tilde{h}_G^n(-\infty<n<\infty)$ into the category of abelian groups, together with natural transformations σ^n: $\tilde{h}_G^n(X)\rightarrow\tilde{h}_G^{n+1}(SX)$ satisfying the following axioms:

(1)′ σ^n is an isomorphism for each n and X.

(2)′ The short sequence,

$$\tilde{h}^n(X/A)\rightarrow\tilde{h}_G^n(X)\rightarrow\tilde{h}_G^n(A)$$

is exact.

Remark 1.2. By Proposition 0.3 and Axioms (1)′, (2)′ we get the long exact sequence for $\tilde{h}_G^*(\cdot)$.

Let h_G^* be a G-cohomology theory. Define $\tilde{h}_G^*(X)$ by $h_G^*(X, *)$. Then h_G^* is a reduced G-cohomology theory by Lemma 1.1. Conversely let $\tilde{h}_G^*$ be a reduced G-cohomology theory. Define $h_G^*(X, A)$ by $\tilde{h}_G^*(X/A)$. Then $\tilde{h}_G^*$ is a G-cohomology theory by Remark 1.2. This is a canonical one-to-one correspondence. Afterwards we identify $h_G^n(X, A)$ and $\tilde{h}_G^n(X/A)$.

We enclose this section after giving some examples.

Examples 1.3. of G-cohomology theories:

(i) $h_G^n(X)=H^n(X/G; \boldsymbol{Z})$.

(ii) $h_G^n(X)=K_G^n(X)$ when G is a compact Lie group.

(iii) $h_G^n(X)=h^n(X\times_G E_G)$ where E_G is a universal G-principal bundle and h^n a cohomology theory for spaces.

2. On classification of G-maps between G-cells of the same dimension up to G-homotopy classes

Let H be a closed subgroup of G. Suppose that $\bar{X}$ is a space and $G/H\times\bar{X}$ is a G-space with the obvious G-action, trivial on $\bar{X}$. Let Y be a G-space and $f\colon G/H\times\bar{X}\rightarrow Y$ be a G-map. Since f is G-equivariant, we get, $f(H/H\times\bar{X})\subset Y^H$ where Y^H is the H-pointwise fixed subspace of Y. Therefore, we may define a map, $\bar{f}\colon\bar{X}\rightarrow Y^H$, by $\bar{f}(x)=f(H/H\times x)$.

Lemma 2.1. *In the above situation, the correspondence, $f\mapsto\bar{f}$, yields an isomorphism of sets,*

$$G\text{-}maps\ (G/H\times\bar{X},\ Y)\xrightarrow{\cong}Maps\ (\bar{X},\ Y^H).$$

Moreover, the isomorphism induces another isomorphism,

$$[G/H\times\bar{X};\ Y]_G\xrightarrow{\cong}[\bar{X};\ Y^H]$$

where $[\cdot\,;\cdot]_G$ *stands for the set of G-homotopy classes of G-maps.*

Proof. Let $\bar{f}\colon \bar{X}\to Y^H$ be a map. Define a map, $f\colon G/H\times\bar{X}\to Y$, by $f(gH/H\times x)=g\cdot\bar{f}(x)$ for any $g\in G$, and any $x\in X$. If $gH/H=g'H/H$, then $g'=g\cdot h$ for some $h\in H$, so that $g\cdot\bar{f}(x)=g'\cdot\bar{f}(x)$ (since $\bar{f}(x)$ is fixed by H), which shows that this definition is valid. By this definition f is certainly G-equivariant, and conversely if we assume that a map $f\colon G/H\times\bar{X}\to Y$ is G-equivariant, we get $f(gH/H\times x)=g\cdot f(H/H\times x)$.

Therefore, the correspondence, $\bar{f}\mapsto f$, is the converse to the correspondence, $f\mapsto\bar{f}$. This proves the first isomorphism. The second isomorphism is induced, because the G-homotopy $f_t(0\leqq t\leqq 1)$ and homotopy $\bar{f}_t(0\leqq t\leqq 1)$ correspond each other in the same way.

q.e.d.

Assume that $\bar{X}$ has a distinguished closed subspace $\bar{A}$ and Y has a base point y_0 (the base point is left fixed by G).

Lemma 2.1′. *The correspondence,* $f\mapsto\bar{f}$, *yields an isomorphism,*

$$G\text{-}maps\ ((G/H\times\bar{X})/(G/H\times\bar{A}),\ Y/y_0)_0 \overset{\cong}{\to} Map\ (\bar{X}/\bar{A},\ Y^H/y_0)_0\,.$$

Moreover, the isomorphism induces another isomorphism,

$$[(G/H\times\bar{X})/(G/H\times\bar{A});\ Y/y_0]_{G,0} \overset{\cong}{\to} [\bar{X}/\bar{A};\ Y^H/y_0]_0\,,$$

where $[\cdot\,,\cdot]_{G,0}$ *stands for the set of base point preserving G-homotopy classes of base point preserving G-maps.*

Proof. The correspondence $\bar{f}\mapsto f$, is also defined in the same way as in Lemma 2.1.

q.e.d.

Therefore, we get

Corollary 2.2. *Let H and K be two closed subgroups of G and* $n\geqq 0$ *be a fixed integer. Then, "the restriction" yields the following isomorphisms,*

(i) $[G/H;\ G/K]_G \overset{\cong}{\to} \pi_0((G/K)^H)$,

(ii) $[(G/H\times\Delta^n)/(G/H\times\partial\Delta^n);\ (G/K\times\Delta^n)/(G/K\times\partial\Delta^n)]_{G,0}$

$$\overset{\cong}{\to} \pi_n((G/K)^H\times\Delta^n)/((G/K)^H\times\partial\Delta^n,\ *).$$

Here $\pi_0(\cdot)$ *stands for the set of arcwise connected components and* $*$ *is the base point* $((G/K)^H\times\partial\Delta^n)/(G/K)^H\times\partial\Delta^n)$.

Now let Y be a space and $n\geqq 1$ be an integer.

Lemma 2.3. $Y\times\Delta^n/Y\times\partial\Delta^n$ *is* $(n-1)$*-connected, and there are natural isomorphisms,*

$$\pi_n'(Y\times\Delta^n/Y\times\partial\Delta^n,\ *) \overset{\cong}{\to} H_n(Y\times\Delta^n/Y\times\partial\Delta^n;\ \boldsymbol{Z}) \overset{\cong}{\to} H_0(Y;\ \boldsymbol{Z})$$

Here $\pi_n'(\cdot)=\pi_n(\cdot)$ for $n\geqq 2$ and $\pi_1'(\cdot)$ is the abelianized group of $\pi_1(\cdot)$ and $H_n(\cdot\,;\boldsymbol{Z})$ is the singular homology group.

Proof. By the definition, $Y\times\Delta^n/Y\times\partial\Delta^n$ is homeomorphic with the smash product $Y^+\wedge\Delta^n/\partial\Delta^n$. Hence $Y\times\Delta^n/Y\times\partial\Delta^n$ is $(n-1)$connected. If we use the Hurwicz theorem, the rest is easily proved.

q.e.d.

Let $\{Y_\lambda\colon\lambda\in\Lambda\}$ be the family of all the arcwise connected components of Y. Take an element $y_\lambda\in Y_\lambda$ for each λ. Then each element of $H_0(Y;\boldsymbol{Z})$ has $\Sigma n_\lambda\cdot y_\lambda(n_\lambda=0$ except the finite λ's) as its representative. Also any map: $(\Delta^n,\partial\Delta^n)\to(Y\times\Delta^n/Y\times\partial\Delta^n, *)$ determines n_λ uniquely.

Now let H and K be closed subgroups of G. Recall that for any element $g\in N(H, K)=\{g\in G, Hg\subset gK\}$, $\hat{g}\colon G/H\to G/K$ is defined by $\hat{g}(aH)=agK$, and this correspondence, $g\mapsto\hat{g}$, induces an isomorphism,

$$N(H, K)/K=(G/K)^H\overset{\cong}{\to}G\text{-maps }(G/H, G/K).$$

Suppose that $\{g_\lambda\in G\}$ is the family of representatives of all arcwise connected components of $N(H, K)/K=(G/K)^H$. Then any base point preserving G-map,

$$f\colon(G/H\times\Delta^n)/(G/H\times\partial\Delta^n)\to(G/K\times\Delta^n)/(G/K\times\partial\Delta^n),$$

determines $n_\lambda(f)$ such that f is equal to $\Sigma n_\lambda(f)\cdot g_\lambda$ in $\pi_n'(((G/K)^H\times\Delta^n)/((G/K)^H\times\partial\Delta^n)), *)\cong H_0((G/K)^H;\boldsymbol{Z})$.

Let L be another closed subgroup of G. Suppose that $g_\lambda\in N(H, K)$ and $g_\mu\in N(K, L)$, then we get

$$g_\lambda\cdot g_\mu\in N(H, L)\ (\text{not } g_\mu\cdot g_\lambda!),\ \text{and } (g_\lambda\cdot g_\mu)^\wedge=\hat{g}_\mu\circ\hat{g}_\lambda\,.$$

From this we get

Proposition 2.4. *Let H, K and L be closed subgroup of G. Suppose that $\{g_\lambda\in G\}$, $\{g_\mu\in G\}$ and $\{g_\nu\in G\}$ are the families of representatives of all arcwise connected components of $N(H, K)/K$, $N(K, L)/L$ and $N(H, L)/L$ respectively. Let $f\colon(G/H)\times\Delta^n)/(G/H\times\partial\Delta^n)\to(G/K\times\Delta^n)/(G/K\times\partial\Delta^n)$ and $g\colon(G/K\times\Delta^n)/(G/K\times\partial\Delta^n)\to(G/L\times\Delta^n)/(G/L\times\partial\Delta^n)$, be base point preserving G-maps. Then,*

$$n_\nu(g\circ f)=\Sigma n_\mu(g)n_\lambda(f)\,.$$

Here the summation is taken over the pairs (λ,μ) such that $g_\lambda\cdot g_\mu$ and g_ν are in the same arcwise connected component of $N(H, L)/L$.

3. Classical G-cohomology theory on G-CW complexes

We shall define a classical G-cohomology theory with coefficients in a (generic) G-coefficient system. In §4 the classical G-cohomology theory will be characterized as the G-cohomology theory which satisfies also the dimension axiom.

DEFINITION 3.1. A (generic) *G-coefficient system* is a contravariant functor M_G of the category of the left coset spaces of G by closed subgroups, G/H, and G-homotopy classes of G-maps (equivariant with respect to left translation), $G/H \to G/K$, into the category of abelian groups.

REMARK. When G is a discrete group, any two distinct G-maps between G-coset spaces cannot be G-homotopic and hence this definition coincides with the generic equivariant coefficient system of Bredon in [3].

EXAMPLES 3.2. OF G-COEFFICIENT SYSTEMS:
(i) $M_G = h_G^q$.
(ii) $M_G = \boldsymbol{Z}$ with a trivial G-action.
(iii) $M_G = \omega_n(Y)(n \geqq 2)$, where Y is a G-space with a base point y_0 and $\omega_n(Y)(G/H) = \pi_n(Y^H, y_0) \cong [(G/H \times \Delta^n)(G/H \times \partial\Delta^n),\ Y/y_0]_{G,0}$.

Let M_G be a G-coefficient system. The n-dimensional G-cochain group of a pair of G-CW complexes (X, A) with coefficients in M_G, denoted by $C_G^n(X, A; M_G)$, is defined to be the group of all G-equivariant functions φ on the n-cells of (X, A) with $\varphi(\sigma) \in M_G(G/H_\sigma)$ and $M_G(\hat{g})\varphi(\sigma) = \varphi(g\sigma)$ for a right translation $\hat{g}$: $G/H_{g\sigma} \ni aH_{g\sigma} = ag(H_\sigma)g^{-1} \mapsto agH_\sigma \in G/H_\sigma$. (If σ is an n-cell of A or a p-cell $(p \neq n)$, then $\varphi(\sigma) = 0$.)

By the definition of the G-cochain group, $C_G^n(X, A; M_G)$ is canonically isomorphic with $C_G^n(X^n/X^{n-1} \cup A;\ M_G)$. Moreover, since $X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma)$ where σ range over the representatives of all n-dimensional G-cells of (X, A),

$$C_G^n(X^n/X^{n-1} \cup A;\ M_G) = C_G^n(\vee(G\sigma/G\partial\sigma);\ M_G) = \prod C_G^n(G\sigma/G\partial\sigma;\ M_G).$$

Let f: $(X, A) \to (Y, \mathrm{B})$ be a G-cellular map between pairs of G-CW complexes. Then, for every n, f induces a G-map,

$$f^n\colon X^n/X^{n-1} \cup A \to Y^n/Y^{n-1} \cup B\,.$$

Suppose that σ and τ are representatives of all G-n-cells of (X, A) and (Y, B) respectively. Then we can define a G-map $f_{\sigma\tau}$ (between G-cells of the same dimension n) by $f_{\sigma\tau} = c \circ f^n \circ i$ in the following diagram:

$$\begin{array}{ccccc}
X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma) & \underset{i}{\overset{c}{\rightleftarrows}} & G\sigma/G\partial\sigma & = & (G/H_\sigma \times \Delta^n)/(G/H_\sigma \times \partial\Delta^n) \\
\big\downarrow f^n & & \big\downarrow f_{\sigma\tau} & & \big\downarrow f_{\sigma\tau} \\
Y^n/Y^{n-1} \cup B = \vee(G\tau/G\partial\tau) & \underset{i}{\overset{c}{\rightleftarrows}} & G\tau/G\partial\tau & = & (G/H_\tau \times \Delta^n)/(G/H_\tau \times \partial\Delta^n)
\end{array}$$

where i is the inclusion and c is the collapsing of the other factors.

Let $\{g_{\lambda(\sigma,\tau)} \in G\}$ be the family of representatives of all arcwise connected components of $(G/H_\tau)^{H_\sigma}$ as in §2.
Define $f^* = C_G^n(f; M_G)\colon C_G^n(Y, B; M_G) \to C_G^n(X, A; M_G)$ by

$$(f^*\varphi)(\sigma) = \sum_\tau \sum_{\lambda(\sigma,\tau)} n_{\lambda(\sigma,\tau)}(f_{\sigma\tau}) M_G(\hat{g}_{\lambda(\sigma,\tau)}) \varphi(\tau)$$

where τ ranges over the representatives of all G-n-cells of (Y, B). The sum is finite because $n_\lambda(f_{\sigma\tau}) = 0$ except the finite λ's.)

Proposition 3.3. *Let M_G be a G-coefficient system. Then, $C_G^n(\cdot\,; M_G)$ is a contravariant functor from the category of pairs of G-*CW *complexes and G-cellular maps into the category of abelian groups.*

Proof. If we fix the representatives, $(g \circ f)^* = f^* \circ g^*$ by Proposition 2.4. It is easily seen that f^* is determined independent of the representatives. Remark that f^* depends only on the G-homotopy class of the G-map f^n.

q.e.d.

Now recall that $X^n/X^{n-1} \cup A$ has the same G-homotopy type with $X^n \cup C(X^{n-1} \cup A)$ canonically. As a special case of Proposition 0.3, we have a Puppe sequence (the horizontal sequence),

$$\begin{tikzcd}[column sep=small]
 & & & S(X^{n-1}/X^{n-3} \cup A) \arrow[d] \\
X^{n-1}/X^{n-2} \cup A \arrow[r] & X^n/X^{n-2} \cup A \arrow[r] \arrow[d] & X^n/X^{n-1} \cup A \arrow[r, "\partial"'] \arrow[ur] & S(X^{n-1}/X^{n-2} \cup A) \arrow[d, "S(\partial)"] \\
 & X^n/X^{n-3} \cup A \arrow[ur] & & S\,S(X^{n-2}/X^{n-3} \cup A)
\end{tikzcd}$$

Since both the vertical and oblique sequences are cofiberings, we get that $S(\partial) \circ \partial$ is G-homotopic to the trivial map. On the other hand we have a canonical isomorphism,

$$\sigma\colon C_G^{n-1}(X^{n-1}/X^{n-2} \cup A; M_G) \xrightarrow{\cong} C_G^n(S(X^{n-1}/X^{n-2} \cup A); M_G)\,.$$

Define the coboundary homomorphism

$$\delta\colon C_G^{n-1}(X, A; M_G) \to C_G^n(X, A; M_G)$$

by $\delta = C_G^n(\partial) \circ \sigma$. Then , because $S(\partial) \circ \partial \simeq_G 0$, we get $\delta \circ \delta = 0$.

Definition 3.4. The *classical G-cohomology theory* on a pair of G-CW complexes (X, A) with the coefficients in a G-coefficient system M_G, denoted by $H_G^*(X, A; M_G)$, is defined by $H_G^n(X, A; M_G) = H^n(C_G^*(X, A; M_G), \delta)$.

Remark 3.5. Let σ and τ be n-cell and $(n-1)$-cell of (X, A). We write $[\sigma, g_{\lambda(\sigma,\tau)}\tau]$ for $n_{\lambda(\sigma,\tau)}(\partial_{\sigma\tau})$ where $\partial_{\sigma\tau}\colon G\sigma/G\partial\sigma \to S(G\tau/G\partial\tau)$. Then, we get the formula,

$$(\delta\varphi)(\sigma)=\sum_{\tau}\sum_{\lambda(\sigma,\tau)}[\sigma, g_{\lambda(\sigma,\tau)}\tau]M_G(\hat{g}_{\lambda(\sigma,\tau)})\varphi(\tau)$$

where τ ranges over the representatives of all G-$(n-1)$-cells and $g_{\lambda(\sigma,\tau)}$ ranges over the representatives of all arcwise connceted components of $N(H_\sigma, H_\tau)/\mathrm{H}_\tau$.

Theorem 3.6. *The classical G-cohomology theory $H_G^*(\cdot\,; M_G)$ is a G-cohomology theory in the sense of* §1.

Proof. We prove here only the G-homotopy axiom. The exision axiom and the exactness axiom is trivially satisfied. Let $f\colon (X, A)\to(Y, B)$ be a G-map between pairs of G-CW complexes. By a G-cellular approximation theorem we may assume that f is G-cellular. The induced map $f^*\colon C_G^n(Y, B; M_G)\to C_G^n(X, A; M)$ commutes with δ, in fact, $f^*\circ\delta=C_G^n(f)\circ C_G^n(\partial)\circ\sigma=C_G^n(\partial\circ f)\circ\sigma=C_G^n(S(f)\circ(\partial)\circ\sigma=C_G^n(\partial)\circ C_G^n(f)\circ\sigma=C_G^n(\partial)\circ\sigma\circ C_G^{n-1}(f)=\delta\circ f^*$. This gives an induced map $f^*\colon H_G^*(Y, B; M_G)\to H_G^*(X, A; M_G)$. If f is G-homotopic to g, we may assume that not only f and g are G-cellular but G-homotopy $F\colon (X\times I, A\times I)\to(Y, B)$ with $F|X\times\{0\}=f$, $F|X\times\{1\}=g$ is also G-cellular. Then, F gives a homotopy connecting the chain maps, f^* and $g^*\colon C_G^*(Y, B; M_G)\to C_G^*(X, A; M_G)$ and hence $f^*=g^*\colon H_G^*(Y, B; M_G)\to H_G^*(X, A; M_G)$. Therefore, even if f is not a G-cellular map the induced map $f^*\colon H_G^*(Y, B; M_G)\to H_G^*(X, A; M_G)$ is well-defined and satisfies the G-homotopy axiom.

q.e.d.

4. Spectral sequence of Atiyah-Hirzebruch type

Suppose that (X, A) is a fixed pair of G-finite G-CW complexes. Put $H(p, q)=\Sigma h_G^n(X^{q-1}, X^{p-1}\cup A)$. Then, the collection of $H(p, q)$'s satisfies the axioms (S.P. 1)-(S.P. 5) of Cartan-Eilenberg [5. p.334] and hence induces a spectral sequence resulting to $h_G^*(X, A)$. The E_1-term and the 1st differential of the spectral sequence are easily calculated as follows:

$$E_1^{p,q}=h_G^{p+q}(X^p, X^{p-1}\cup A)$$
$$d_1=\delta\colon h_G^{p+q}(X^p, X^{p-1}\cup A)\to h_G^{p+q+1}(X^{p+1}, X^p\cup A)$$

where δ is the coboundary homomorphism.

Lemma 4.1.

(i) *$h_G^{p+q}(X^p, X^{p-1}\cup A)=\tilde{h}_G^{p+q}(X^p/X^{p-1}\cup A)$ is decomposed into the direct product $\prod\tilde{h}_G^{p+q}(G\sigma/G\partial\sigma)$, where σ ranges over representatives of all p-dimensional G-cells of X/A.*

(ii) *And for each direct factor, there are isomorphisms, $\tilde{h}_G^{p+q}(G\sigma/G\partial\sigma)=h_G^{p+q}(G\sigma, G\partial\sigma)\cong h_G^{p+q}(G/H_\sigma\times\Delta^p, G/H_\sigma\times\partial\Delta^p)\cong h_G^q(G/H_\sigma)$.*

Proof of (i). Since $X^p/X^{p-1}\cup A=\vee(G\sigma/G\partial\sigma)$ is the one point union of finite $(G\sigma/G\partial\sigma)$'s we get the decomposition by the usual argument.

Proof of (ii). The 2nd isomorphism is induced by the G-characteristic map, Gf_σ: $(G/H_\sigma\times\Delta^p,\ G/H_\sigma\times\partial\Delta^p)\to(G\sigma,\ G\partial\sigma)$, which is a relative G-homeomorphism. Now we shall prove the last isomorphism. Put $H=H_\sigma$. Since the inclusion, $G/H\times(\partial\Delta^p-\Delta^{p-1})\to G/H\times\Delta^p$, has a G-equivariant deformation retraction, we get the isomorphism,

$$h_G^{p+q}(G/H\times\Delta^p,\ G/H\times\partial\Delta^p)\overset{\delta}{\underset{\cong}{\leftarrow}} h_G^{p+q-1}(G/H\times\partial\Delta^p,\ G/H\times(\partial\Delta^p-\Delta^{p-1}))$$

in the exact sequence of a triple $(G/H\times\Delta^p,\ G/H\times\partial\Delta^p,\ G/H\times(\partial\Delta^p-\Delta^{p-1}))$. By the exision axiom, we get the isomorphism,

$$h_G^{p+q-1}(G/H\times\partial\Delta^p,\ G/H\times(\partial\Delta^p-\Delta^{p-1}))\overset{\cong}{\leftarrow} h_G^{p+q-1}(G/H\times\Delta^{p-1},\ G/H\times\partial\Delta^{p-1})\,.$$

Combining the isomorphisms of these two types repeatedly, we get

$$\begin{aligned}h_G^{p+q}(G/H\times\Delta^p,\ G/H\times\partial\Delta^p)&\overset{\cong}{\leftarrow} h_G^{p+q-1}(G/H\times\Delta^{p-1},\ G/H\times\partial\Delta^{p-1})\\ &\cdots\overset{\cong}{\leftarrow} h_G^q(G/H\times\Delta^0,\ G/H\times\partial\Delta^0)=h_G^q(G/H)\,.\end{aligned}$$

q.e.d.

We shall consider the difference of taking another representative $g\sigma$ instead of σ, as a representative of a p-dimensional G-cell $G\sigma$. Put $H=H_\sigma$. Then $gHg^{-1}=H_{g\sigma}$. Since we may identify agH-orbit of σ with $agHg^{-1}$-orbit of $g\sigma$ in $G\sigma$, a canonical right translation $\hat{g}$: $G/gHg^{-1}\ni agHg^{-1}\mapsto agH\in G/H$ induces a required isomorphism, $h_G^q(\hat{g})$: $h_G^q(G/H_\sigma)\to h_G^q(G/H_{g\sigma})$. This shows that $h_G^{p+q}(X^p,\ X^{p-1}\cup A)\cong C_G^p(X^p,\ X^{p-1}\cup A;\ h_G^q)$.

Theorem 4.2. *The $E_2^{*,q}$-term of the Atiyah-Hirzebruch spectral sequence for a G-cohomology theory, h_G^*, on G-finite G-CW complexes, is a classical G-cohomology theory with coefficients in h_G^q.*

Proof. By the result above we can identify $E_1^{p,q}=h_G^{p+q}(X^p,\ X^{p-1}\cup A)$ with $C_G^p(X,\ A;\ h_G^q)$. And the coboundary homomorphisms are induced from ∂ in the Puppe sequence in both cases.

q.e.d.

Assume that the G-cohomology theory $h_G^n(\cdot)$ is defined also on (not G-finite) G-CW complexes, and satisfies the additivity axiom:

(3) The inclusions, i_α: $X_\alpha\to\coprod X_\alpha$, induce an isomorphism,

$$\Pi h_G^n(i_\alpha)\colon\ \Pi h_G^n(X_\alpha)\overset{\cong}{\leftarrow} h_G^n(\coprod X_\alpha)$$

Then, Lemma 4.1 and Theorem 4.2 are also valid for a pair of (not G-finite) G-CW complexes.

The classical G-cohomology theory is defined on G-CW complexes and satisfies the additivity axiom. Therefore, we get as usual

Theorem 4.3. *The classical G-cohomology theory is characterized to be* *the G-cohomology theory defined on G-CW complexes which satisfies also the additivity axiom and the dimension axiom.*

Here we mean by dimension axiom,

(4) $h^n_G(G/H)=0$ for $n\neq 0$ and all closed subgroup H of G.

The aditivity axiom and the dimension axiom are as follows, for the reduced G-cohomology theory.

(3)′ The inclusions, $i_\alpha\colon X_\alpha\to\vee X_\alpha$, induce an isomorphism,

$$\prod\tilde{h}^n_G(i_\alpha)\colon \prod\tilde{h}^n_G(X_\alpha)\overset{\cong}{\leftarrow}\tilde{h}^n_G(\vee X_\alpha)\,.$$

(4)′ $\tilde{h}^n_G((G/H)^+)=0$ for $n\neq 0$ and all H.

5. G-obstruction theory

Let Y be a G-space with a base point. Then in the classical G-cohomology group $H^*_G(\cdot\,;\,\omega_n(Y))$, we can make a G-obstruction theory similar to that of Bredon [3].

Let $n\geqq 1$ be a fixed integer and A be a G-subcomplex of a G-CW complex X. We shall assume, for simplicity, that the pointwise fixed subspace Y^H of Y by H is non-empty, arcwise connected and n-simple for each closed subgroup H of G which appears as an isotropy subgroup at a point of X.

Assume that we are given a G-map $\varphi\colon X^n\cup A\to Y$. Let σ be an $(n+1)$-cell of X and let $f_\sigma\colon\partial\Delta^{n+1}\to X^n$ be the characteristic attaching map of σ and $H_\sigma=H$. Because the image of $\partial\Delta^{n+1}$ by $\varphi\circ f$ is pointwise fixed by H, we get a map: $\partial\Delta^{n+1}\to Y^H$. We define $c_\varphi(\sigma)\in\pi_n(Y^H,\,*)=\omega_n(Y)(G/H)$ to be the unique base point preserving homotopy class which is free homotopic to the above map ($\pi_n(Y^H,*)\cong[S^n;\,Y^H]$ because Y^H is n-simple). Since φ is a G-map, we get $c_\varphi(g\sigma)=g\cdot c_\varphi(\sigma)\in\pi_n(Y^{gHg^{-1}},\,*)=\omega_n(Y)(G/gHg^{-1})$ and hence $c_\varphi\in C^{n+1}_G(X,A;\,\omega_n(Y))$.

Lemma 5.1. $\delta c_\varphi=0\in C^{n+2}_G(X,A;\,\omega_n(Y))$.

Proof. Let τ be an $(n+2)$-cell of (X,A) and $i\colon(G\tau,G\partial\tau)\to(X,A)$ be the inclusion. Then $i^*\delta c_\varphi=\delta i^*c_\varphi$ and $i^*C_\varphi\in C^{n+1}_G(G\tau,\,G\partial\tau;\,\omega_n(Y))$. According to our definition of $C^{n+1}_G(\cdot\,;\,\omega_n(Y))$ on G-CW complexes, $C^{n+1}_G(G\tau,\,G\partial\tau;\,\omega_n(Y))=0$. Therefore, $i^*c_\varphi=0$ and hence $i^*\delta c_\varphi=0$, that is, $c_\varphi(\tau)=0$ for any $(n+2)$-cell τ of (X,A).

q.e.d.

Now identifying the G-homotpy classes of G-maps: $G/H\times\partial\Delta^{n+1}\to Y$ and the homotopy classes of maps: $\partial\Delta^{n+1}\to Y^H$, we can reduce the proof of the following lemmas to the ordinary obstruction theory as Bredon did.

Lemma 5.2. *$c_\varphi=0$ if and only if φ is extendable equivariantly on $X^{n+1}\cup A$.*

Lemma 5.3. *Let $d\in C^n_G(X,A;\,\omega_n(Y))$. Then, there is a G-map $\theta\colon X^n\cup A\to Y$, coinciding with φ on $X^{n+1}\cup A$ such that $d_{\theta,\varphi}=d$.*

Here the difference cochain $d_{\theta,\varphi}$ is defined to be the class which corresponds to $c_{\theta*\varphi}$ by the isomorphism, $C_G^n(X, A; \omega_n(Y)) \to C_G^{n+1}(X\times I, A\times I \cup X\times \partial I; \omega_n(Y))$. $\theta*\varphi$ is a G-map: $(X\times I)^n \cup A\times I \to Y$ which is φ on $X^n\times\{0\} \cup X^{n-1}\times I$ and θ on $X^n\times\{1\}$.

Combining these three lemmas, we get

Theorem 5.4. *Let φ: $X^n \cup A \to Y$ be a G-map. Then $\varphi|X^{n-1}\cup A$ can be extended to G-map: $X^{n+1}\cup A \to Y$ if and only if the G-cohomology class of c_φ in $H_G^{n+1}(X, A; \omega_n(Y))$ vanishes.*

Also the argument of Bredon in 'primary obstructions' [3, II.5.2] is valid to this case. In particular, we get

Proposition 5.5. *Let $n\geqq 1$ be a fixed integer and let Y be a G-space with base point such that Y^H is non-empty, arcwise connected and n-simple for every closed subgroup H of G. Suppose that $\omega_k(Y)$ vanishes for $k\neq n$, then a primary obstruction map,*

$$\alpha\colon [X;\ Y]_G \overset{\cong}{\to} H_G^n(X;\ \omega_n(Y))$$

is an isomorphism for any G-CW complex X.

Proposition 5.5.' *Under the assumption above, a primary obstruction map,*

$$\alpha'\colon [X,\ Y]_{G,0} \overset{\cong}{\to} H_G^n(X;\ \omega_n(Y))$$

is an isomorphism for any G-CW complex X with base point.

6. Representation theorem of E. Brown

We shall prove the following representation theorem as an application of E.Brown's abstract homotopy theory [4].

Theorem 6.1. *If a reduced G-cohomology group $\tilde{h}_G^n$ on G-CW complexes with base point satisfies the additivity axiom, then $\tilde{h}_G^n$ is representable, that is, there is a G-space Y_n with base point and a natural transformation T: $[\cdot\,;\ Y_n]_{G,0} \to \tilde{h}_G^n(\cdot)$ such that T is an isomorphism for any G-CW complex with base point, where $[\cdot\,;\cdot]_{G,0}$ stands for the set of base point preserving G-homotopy classes of base point preserving G-maps.*

Let $\mathfrak{C}$ be the category of G-CW complexes with base point such that the H-stationary subspace is arcwise connected for each H, and base point preserving G-homotopy classes of base point preserving G-maps. In $\mathfrak{C}$ there is a (not unique) sequential direct limit by approximating G-maps by G-cellular maps and making their telescope. Also we get a (not unique) 'push out' as a double mapping cylinder in $\mathfrak{C}$. If we choose one representative for each class of conjugate closed subgroups, $\{(G/H\times\Delta^p)/(G/H\times\partial\Delta^p);\ H \text{ representative},\ 0<p<\infty\}$ is a small subcategory of $\mathcal{C}$.

Let $\mathcal{C}_0$ be a minimal subcategory which contains $(G/H\times\Delta^p)/(G/H\times\partial\Delta^p)$'s $(0<p<\infty)$ and their 'push out'. Then $\mathcal{C}_0$ is a small, full subcategory of $\mathcal{C}$ and also a subcategory of G-finite G-CW complexes with base point and we get

Proposition 6.2. *A pair* $(\mathcal{C}, \mathcal{C}_0)$ *is a homotopy category in the sense of E.Brown.*

Proof of Theorem 6.1. Since reduced G-cohomology theory has a Mayer-Vietoris exact sequence, $\tilde{h}_G^n$ (restricted on $\mathcal{C}$) with the additivity axiom is a homotopy functor in the sense of E.Brown. Moreover, we get $\overline{\mathcal{C}}_0=\mathcal{C}$ by an equivariant version of J.H.C.Whitehead's theorem. (See Proposition 0.4.). Therefore, by Theorem 2.8 of [4], we get a $Y_n'\in\mathcal{C}$ unique up to G-homotopy equivalence and a natural transformation T: $[\cdot\,;\,Y_n']_{G,0}\to\tilde{h}_G^n(\cdot)$ such that T is an isomorphism for each $X\in\mathcal{C}$.

Define $Y_n=\Omega Y_{n+1}'$. For any G-CW complex X with base point, $SX\in\mathcal{C}$. Therefore, we get

$$\begin{array}{ccc} [X,\ Y_n]_{G,0} & & \tilde{h}_G^n(X) \\ \cong\downarrow & & \downarrow\cong \\ [SX,\ Y_{n+1}']_{G,0} & \xrightarrow{\cong} & \tilde{h}_G^{n+1}(SX) \end{array}$$

q.e.d.

REMARK. Even when $\tilde{h}_G^n$ is defined only on G-finite G-CW complexes, by the method of Adams [2], we get a reduced G-cohomology theory on G-CW complexes which satisfies the additivity axiom and coincides with $\tilde{h}_G^n$ on G-finite G-CW complexes.

Let $Y_{n+1}'\in\mathcal{C}$ be a representing space of $\tilde{h}_G^n$ in the category of $\mathcal{C}$. Then, the isomorphism: $h_G^{n+1}(X)\xrightarrow{\cong}h_G^{n+2}(SX)$ induces a G-map h_{n+1}': Y_{n+1}': $\to\Omega Y_{n+2}'$ which is a weak G-homotopy equivalence, that is, $(h_{n+1}')_*$: $\pi_i((Y_{n+1}')^H)\xrightarrow{\cong}\pi_i((\Omega Y_{n+2}')^H)$ for any i and any H. Hence, taking their loop spaces, we get also a weak G-homotpy equivalence, h_n: $Y_n\to\Omega Y_{n+1}$. Then, $Y=\{Y_n,\ h_n;\ -\infty<n<\infty\}$ forms a weak Ω-spectrum for $\tilde{h}_G^*$. This fact is used in §7 to make a spectral sequence of C.Maunder.

7. Killing the elements of the G-homotopy groups and C.Maunder's spectral sequence

Let Y be a G-space with base point y_0 such that Y^H is arcwise connected for each closed subgroup H of G. An element in the n-th homotopy group $\pi_n(Y^H,y_0)$ of H-stationary subspace Y^H is called to be an element of G-n-homotopy groups of Y. An element $[f]\in\pi_n(Y^H,\,y_0)$ with f: $S^n=\Delta^n/\partial\Delta^n\to Y^H$ is killed by attaching a G-$(n+1)$-cell represented by an $(n+1)$-cell σ which has f as its characteristic attaching map and H as its isotropy subgroup, that is, $H_\sigma=H$. If we fix n and kill all the elements of G-n-homotopy groups, we get a relative G-CW complex $\tilde{Y}$ such that $\tilde{Y}^{-1}=Y$. Then, $i_*\colon \pi_n(Y^H, y_0)\to\pi_n(\tilde{Y}^H, y_0)$ is a zero map for any closed subgroup H, where $i\colon Y^H\to\tilde{Y}^H$. On the other hand, by the G-cellular approximation theorem we get $\pi_k(\tilde{Y}^H, Y^H, y_0)$ vanishes for $k<n$ and any H, that is, $i_*\colon \pi_k(Y^H, y_0)\to\pi_k(\tilde{Y}^H, y_0)$ is an isomorphism for $k<n$ and a surjection for $k=n$. Therefore, $\pi_k(\tilde{Y}^H, y_0)$ is canonically isomorphic with $\pi_k(Y^H, y_0)$ for $k<n$ and vanishes for $k=n$. By this reason we call $\tilde{Y}$ a G-space obtained of Y by killing the elements of G-n-homotopy groups.

Let $Y(1, p)$ be a G-space obtained of Y by killing the elements of G-homotopy groups of dimensions $\geqq(p+1)$ one after the other. Then, $Y(1,p)$ is uniquely determined up to G-homotopy types rel. Y by the usual argument on (relative) G-CW complexes. For $p\leqq q$, $Y(p, q)$ denotes the mapping track of $i(p, q)$: $Y(1, q)\to Y(1, p-1)$. Moreover, let $Y^{(r)}(p, q)$ denote the mapping track of $i^{(r)}(p, q)\colon Y(r, q)\to Y(r, p-1)$ for $r<p\leqq q$. Then, it is easily seen that the natural G-map: $Y^{(r)}(p, q)\to Y(p, q)$ has a G-homotopy inverse. Therefore, by taking mapping tracks repeatedly, we get a following G-fibering sequence of G-spaces. (The G-spaces are determined up to G-homotopy types.)

$$\Omega Y(r, t) \to \Omega Y(r, s) \xrightarrow{\delta} Y(s+1, t) \to Y(r, t) \to Y(r, s), \qquad r\leqq s<t .$$

Here, that $X\to Y\to Z$ is a G-fibering stands for that $X^H\to Y^H\to Z^H$ is a fibering for any H. In particular, $\pi_k(Y(p, q)^H, y_0)$ is isomorphic with $\pi_k(Y^H, y_0)$ for $p\leqq k\leqq q$ and vanishes otherwise.

In §6 we have obtained a weak Ω-spectrum for a G-cohomology theory $\tilde{h}_G^*$. Let X be a G-finite G-CW complex and put $\bar{H}(p, q)=\sum_n[S(X^+); Y'_{n+1}(p+2, q)]_{G,0}$. Then, by the G-fibering sequence above, we get a spectral sequence resulting to $h_G^*(X)=\Sigma[S(X^+); Y'_{n+1}]_{G,0}$. The E_2-term, $\bar{E}_2^{p,q}=[S(X^+); Y'_{p+n+1}(p+1, p+1)]_{G,0}$ is isomorphic with $H_G^{p+1}(S(X^+); \pi_{p+1}(Y'_{p+q+1}))=H_G^p(X; h_G^q)$ by Proposition 5.5′. Moreover, since $[S((X^{p+1})^+); Y'_{p+q+1}(1, p+1)]_{G,0}\cong[S(X^+); Y'_{p+q+1}(1, p+1)]_{G,0}$ and $[S(X^p/X^{p-1}); Y'_{p+q+1}]_{G,0}\cong[S(X^p/X^{p-1}); Y'_{p+q+1}(1, p+1)]_{G,0}$, the Maunder's argument using exact couples [11] is also valid in this case. Hence, we get

Theorem 7.1 *Let h_G^* be G-cohomology theory. Then, the spectral sequence above is isomorphic with the Atiyah-Hirzebruch spectral sequence except the E_1-term for any G-finite G-CW complex X.*

Proposition 7.2. *The r-th differential $\bar{d}_r$; $\bar{E}_r^{p,q}\to\bar{E}_r^{p+r,q-r+1}$ in the Maunder's spectral sequence is induced from the 'higher cohomology operation' determined by the G-homotopy class of*

$$\delta_r = \delta\circ h'_{p+q+1}\colon Y'_{p+q+1}(p+1, (p+r-1)\xrightarrow{h'_{p+q+1}}\Omega Y'_{p+q+2}(p+2, p+r) \xrightarrow{\delta} Y'_{p+q+2}(p+r+1, p+r+1) .$$

Remark that $[\delta_r]\in H_G^{p+r+1}(Y'_{p+q+1}(p+1, p+r-1), \omega_{p+q+1}(Y'_{p+q+2}))$.

Corollary 7.3. *$E_r^{p,q}=\bar{E}_r^{p,q}(r\geqq 2)$ together with the differentials d_r are G-homotopy type invariant.*

This is also proved from Theorem 4.2 and comparison of spectral sequences.

8. Applications to the equivariant K^*-theory

In this section G denotes a compact Lie group. We shall applicate our results to K_G^*-theory.

Theorem 8.1. *Let X be a G-finite G-CW complex. There exists a spectral sequence $E_r^{p,q}(r\geqq 1, -\infty<p, q<\infty)$ with*

$$E_1^{p,q}\cong C_G^p(X, K_G^q)$$

d_1 being the coboundary homomorphism.

$$E_2^{p,q}\cong H_G^p(X, K_G^q),$$
$$E_\infty^{p,q}\cong G_pK_G^{p+q}(X) = K_{G,p}^{p+q}(X)/K_{G,p+1}^{p+q}(X)$$

where $K_{G,p}^n(X)=$Kernel $(K_G^n(X)\to K_G^n(X^{p-1}))$. The G-coefficient system, $K_G^q(G/H)$ is isomorphic with $K_G(G/H)$ for q even and vanishes for q odd (*See* [13]).

This is a special case of Theorem 4.2.

A. Collapsing theorems

If r is even, the r-th differential is a zero map, because d_r is a map of $E_r^{p,q}$ into $E_r^{p+r,q-r+1}$ where one of the domain or the image vanishes. Therefore, we get

Theorem 8.2. *If one of the following conditions is satisfied, then the above spectral sequence collapses:*

(i) *$H_G^p(X; K_G)$ vanishes for every odd p.*

(ii) *$H_G^p(X; K_G)$ vanishes for every $p\geqq 3$.*

For the reduced K_G^*-theory, we get

Theorem 8.2′. *If X has a base point, then the spectral sequence,*

$$\tilde{H}_G^p(X; K_G^q)\Rightarrow \tilde{K}_G^{p+q}(X)$$

collapses if:

(i) *$\tilde{H}_G^p(X; K_G)$ vanishes for every odd p or for every even p, or*

(ii) *$\tilde{H}_G^p(X; K_G)$ vanishes except $p=r, r+1, r+2$* for some r.

B. On E_2-term

We consider the classical G-cohomology theory with coefficients in K_G.

$K_G(G/H)$ is canonically isomorphic with $R(H)$, where $R(H)$ is the Grothendieck group of the isomorphic classes of complex representations of H.

Remark that $K_G(\hat{g})$; $K_G(G/G) \to K_G(G/G)$ is an identity isomorphism for any $g \in G$, because any inner automorphism of G induces an identity isomorphism on $R(G)$. Therefore, if we assume that the restriction maps i^*: $R(G) \to R(H)$ is surjective, then $K_G(\hat{g}) = K_G(\hat{g}')$: $K_G(G/H) \to K_G(G/H')$ for any elements g, g' of $N(H', H)$. Hence, by Remark 3.5 we shall get

Proposition 8.3. *Let X be a G-finite G-CW complex whose isotropy subgroups satisfy the condition*:

($*$) *the restriction map*: $R(G) \to R(H)$ *is a surjection for any closed subgroup H which appears as an isotropy subgroup at a point of X.*

Then, $H_G^p(X: K_G)$ can be calculated by considering only the orbit type decomposition of the orbit space.

Proof. As we remark above, by the condition ($*$), $K_G(\hat{g})$: $K_G(G/H) \to K_G(G/H')$ is independent of the choice of $g \in N(H', H)$ for any isotropy subgroups H, H'. So, we may write this map by $K_G(H \to H')$.
Then, we get the formula:

$$(\delta\varphi)(\sigma) = \sum_{\tau} \sum_{\lambda(\sigma,\tau)} [\sigma, g_{\lambda(\sigma,\tau)}\tau] K_G(H_\sigma \leftarrow H_\tau)\varphi(\tau) .$$

On the other hand, it is easy to see that

$$\sum_{\lambda(\sigma,\tau)} [\sigma, g_{\lambda(\sigma,\tau)}\tau] = [\sigma/G, \tau/G] \in Z$$

where σ/G and τ/G are the induced cells on X/G.

q.e.d.

REMARK 8.4. We call an $O(n)$-manifold to be a regular $O(n)$-manifold if each isotropy subgroup is conjugate to $O(k)$ ($k \leqq n$). Then any regular $O(n)$-manifold satisfies the condition ($*$) above, because the restriction map ρ_n: $R(O(n)) \to R(O(n-1))$ is a surjection. This fact is easily checked by the classical representation theory as in [14], but we refer the reader to [12].

C. A conclusion

Combining these results with Proposition 0.5, we get

Proposition 8.5. *For a compact regular $O(n)$ manifold X, if dim $X/G \leqq 2$, then, $K_G^0(X)/K_{G,2}^0(X)$, $K_{G,2}^0(X)$ and $K_G^1(X)$ depend only on the orbit type decomposition of the orbit space.*

D. Examples

Now we shall calculate $K_G^*(X)$ for some regular $O(n)$-manifolds.

(i) *Hirzebruch-Mayer $O(n)$-manifold $W^{2n-1}(d)$ for $n \geqq 2$* [7]: The orbit space is a 2-disk D^2 the orbit type of whose interior is $(O(n-2))$ and the boundary $(O(n-1))$.

Define a presheaf $\mathfrak{F}$ on the orbit space D^2 by $\Gamma(U, \mathfrak{F})=\Gamma(U, U\times R(O(n-2)))$ if $U\subset \operatorname{Int} D^2$ and by $\Gamma(U, \mathfrak{F})=\Gamma(U, U\times R(O(n-1))$ if $U\cap \partial D^2\neq\phi$. Then, by Proposition 8.3, $H_G^*(W^{2n-1}(d); K_G)\cong H^*(D^2, \mathfrak{F})$. Remark that $\mathfrak{F}$ forms a sheaf. Define $\mathfrak{G}$ and $\mathfrak{H}$ by $\mathfrak{G}=$constant sheaf $\operatorname{Ker}\rho_{n-1}$ on ∂D^2 which is considered to be a sheaf over D^2 and $\mathfrak{H}=$constant sheaf $R(O(n-2))$ on whole D^2. Then, since ρ_{n-1}: $R(O(n-1))\to R(O(n-2))$ is surjective, we get an exact sequence of sheaves,

$$0\to\mathfrak{G}\to\mathfrak{F}\to\mathfrak{H}\to 0\,.$$

The following notation is simpler and reasonable to denote this exact sequence.

$$\begin{matrix} S^1 \\ \cap \\ D^2 \end{matrix} : 0\to\begin{pmatrix}\operatorname{Ker}\rho_{n-1} \\ 0\end{pmatrix}\to\begin{pmatrix}R(O(n-1)) \\ R(O(n-2))\end{pmatrix}\to\begin{pmatrix}R(O(n-2)) \\ R(O(n-2))\end{pmatrix}\to 0$$

From the associated long exact sequence, we get

$$H_G^0\cong R(O(n-1)),\ H_G^1\cong\operatorname{Ker}\rho_{n-1}\text{ and }H_G^2\cong\operatorname{Coker}\rho_{n-1}=0\,.$$

Therefore,

$$K_G^0\cong R(O(n-1)\text{ and }K_G^1\cong\operatorname{Ker}\rho_{n-1}\,.$$

(ii) *Jänich knot $O(n)$-manifold for $n\geqq 3$* [8]: Let $S^1\subset S^3$ be a knot. The orbit space is a 4-disk D^4 where the orbit type of each difference domain of $D^4\supset S^3\supset S^1$ is $(O(n-2))$, $(O(n-1))$, $(O(n))$ respectively.

As in (i), we consider the following exact sequence of sheaves.

$$\begin{matrix} S^1 \\ \cap \\ S^3 \\ \cap \\ D^4 \end{matrix} : 0\to\begin{pmatrix}\operatorname{Ker}\rho_n \\ 0 \\ 0\end{pmatrix}\to\mathfrak{F}'=\begin{pmatrix}R(O(n)) \\ R(O(n-1)) \\ R(O(n-2))\end{pmatrix}\to\begin{pmatrix}R(O(n-1)) \\ R(O(n-1)) \\ R(O(n-2))\end{pmatrix}\to 0$$

Then, $H_G^*=H_G^*(X; K)\cong H^*(D^4; \mathfrak{F}')$ is calculated as follows:

$$H_G^0\cong R(O(n)),\ H_G^1\cong\operatorname{Ker}\rho_n,\ H_G^2=0,\ H_G^3\cong\operatorname{Ker}\rho_{n-1}\text{ and }H_G^4=0\,.$$

In particular, if we consider that the $O(n)$-manifold has a base point, then $\tilde{H}_G^0=0$ and $\tilde{H}_G^*$ satisfies the condition (ii) of Theorem 8.2$'$. Therefore, we get

$$\tilde{K}_G^0=0,\text{ that is, }K_G^0\cong R(O(n))$$

and

$$0\to\operatorname{Ker}\rho_{n-1}\to K_G^1\to\operatorname{Ker}\rho_n\to 0$$

is exact.

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