Kikuchi, K.
Osaka J. Math.
24 (1987), 605–650

STEADY-STATE WAVE PROPAGATION PROBLEM IN INHOMOGENEOUS ANISOTROPIC MEDIA INCLUDING CRYSTALS

Koji KIKUCHI

(Received December 18, 1985)

1. Introduction

The steady-state wave propagation problems in homogeneous media are governed by first order systems of the form

$$(\Lambda^0-\lambda)v=g\,,$$

where

$$\Lambda^0=\sum_{j=1}^{n}A_jD_j$$

($D_j=(1/i)\partial/\partial x_j$, A_j's are $m\times m$ hermitian matrices).

Let Λ be a perturbed system of Λ^0 having the following form:

$$\Lambda=E(x)^{-1}\{\sum_{j=1}^{n}A_j(x)D_j+B(x)\}\qquad(x\in\Omega)\,, \tag{1.1}$$

where Ω is an exterior domain with smooth boundary, that is, $R^n\backslash\Omega$ is compact. In this paper we consider a steady-state wave propagation problem in inhomogeneous media. The steady-state wave propagation problem treated here is the following:

$$\begin{cases}(\Lambda-\lambda)v=g & \text{for}\quad x\in\Omega\ (g\text{: given})\\ v \text{ satisfies } + \text{ (or } -\text{) radiation condition}\\ v(x)\in N(x) & \text{for}\quad x\in\partial\Omega\,,\end{cases} \tag{1.2}$$

where $N(x)$ is a prescribed vector subspaces of C^m depending smoothly on $x\in\partial\Omega$. The coefficients and $N(x)$ satisfy the conditions given later, which includes the wave propagation in crystals. Our purpose here is to prove Rellich uniqueness theorem and to establish the limiting absorption principle. The most difficult point is how to define the radiation condition, because the existence of the singularities and the parabolic points of the slowness surface complicates the behavior at infinity of the solution. However in our previous paper [4] we obtain the expansion formula at infinity of the Green function of

$\Lambda^0-\lambda$ under some conditions which includes the wave propagation problem in crystals. We use the result of [4] crucially.

In [4] we assume that the space dimension n is odd. This is also assumed here. The conditions for Λ are the following:

Ai) $E(x)$ is hermitian, positive definit, bounded and measurable in $x\in\Omega$, $A_j(x)$'s are hermitian and continuously differentiable with respect to x, $B(x)$ is continuous with respect to x.

Aii) The operator Λ is formally self-adjoint, that is,

$$\sum_{j=1}^{n}\frac{\partial A_j(x)}{\partial x_j}=i\{B(x)-B(x)^*\} \qquad \text{for} \quad x\in\Omega .$$

Aiii) Outside of a sufficiently large ball, say for $|x|>R$, $E(x)=I$, $A_j(x)=A_j$ (constant) and $B(x)=0$.

Note that, by Ai), A_j's are hermitian.

Aiv) $N(x)$ is maximally conservative, that is,

$$\left(\sum_{j=1}^{n}A_j(x)n_j(x)\zeta\right)\bar{\zeta}=0 \qquad \text{for any} \quad \zeta\in N(x)$$

and $N(x)$ is not properly contained in any other subspace of $\boldsymbol{C}^m$ having this property. ($n(x)=(n_1(x),\cdots,n_n(x))$ is the outer unit normal of $\partial\Omega$ at x).

Av) Λ^0 satisfies the following condition:

1) Λ^0 is strongly propagative.
2) The symbol of Λ^0 satisfies Si)$\sim$Svi) of [4, section 1]*).
3) If $n>3$, we assume

$$|K(s)|\geqq \text{Const.}\ \text{dist}_S(s, Z_S) .$$

($K(s)$ is the Gaussian curvature of S. Z_S is the set which containes all algebraic singularities and all parabolic points of S. The exact definition is given in [4, page 579]).

Avi) $$\text{rank}\,\Lambda(x, n(x))=\text{Const.} \qquad \text{near} \quad \partial\Omega .$$

Avii) For any $R_0<r'<r$ and any $u\in\mathcal{D}(\Lambda)\ominus\mathcal{N}(\Lambda)$

$$\sum_{j=1}^{n}\|D_ju\|_{\Omega_{r'}}\leqq \text{Const.}\{\|u\|_{\Omega_r}+\|\Lambda u\|_{\Omega_r}\} ,$$

where $\Omega_r=\{|x|<r\}\cap\Omega$, $\|u\|_{\Omega_r}=\left(\int_{\Omega_r}u(x)^*E(x)u(x)dx\right)^{1/2}$ and $\mathcal{N}(\Lambda)$ is the nullspace of Λ.

---

*) Here we correct the assumption Si) in page 579. "$(n-d)$-dimensional" must be replaced with "$(n-1-d)$-dimensional".

REMARK If Λ has the form

$$\Lambda = E(x)^{-1} \sum_{j=1}^{n} A_j D_j ,$$

$\Omega = \boldsymbol{R}^n$ and $E(x) \in C^1(\boldsymbol{R}^n)$, then the assumption Avii) follows from Ai)$\sim$Av). (See J.R. Schulenberger and C.H. Wilcox [12]).

The main Theorem of this paper is as follows:

Theorem 1.1. *Let* $\lambda \in \boldsymbol{R}^1 \setminus \{0\}$. *If* $v_\pm \in L^2_{loc}(\Omega)$ *satisfies*

$$(1.3) \qquad \begin{cases} (\Lambda - \lambda) v_\pm = 0 & \textit{for} \quad x \in \Omega \\ v_\pm \textit{ satisfies } \pm \textit{ radiation condition} \\ v_\pm \in N(x) & \textit{for} \quad x \in \partial\Omega , \end{cases}$$

then $v_\pm \in L^2(\Omega)$.

Especially $v_\pm \equiv 0$ *if* $\lambda \notin \sigma_p(\Lambda)$.

We shall apply Theorem 1.1 and establish the limiting absorption principle (Theorem 5.1), which assures the existence of the solution of the steady-state wave propagation problem (1.2). Then, the eigenfunction expansion theorem for Λ (Theorem 6.1) can be derived from Theorem 5.1. Here the generalized eigenfunction will be obtained as the unique solution of the following steady-state wave propagation problem.

$$(1.4) \qquad \begin{cases} (\Lambda - \lambda_j(\xi)) \Phi_j^\pm(\xi) = (\lambda_j(\xi) - \lambda) E(x)^{-1/2} (e^{ix\xi} \hat{P}_j(\xi)) \\ \Phi_j^\pm \text{ satisfies } \pm \text{ radiation condition} \\ \Phi_j^\pm \in N(x) \qquad \text{for} \quad x \in \partial\Omega \end{cases}$$

($\lambda_j(\xi)$'s are eigenvalues of $\Lambda^0(\xi)$ and $\hat{P}_j(\xi)$'s are projections to the eigenspaces. See [4, section 2]).

The theory of eigenfunction expansion is developed mainly for the Schrödinger equation and the d'Alembert equation (see, for example T. Ikebe [2], C.H. Wilcox [15]). For example in the case of the d'Alembert equation the steady-state wave propagation problem is to solve the Helmholtz equation under the Sommerfeld radiation condition. So the generalized eigenfunction is characterized as the unique solution of the Helmholtz equation satisfying the Sommerfeld radiation condition. For first order systems K. Mochizuki [6] treated isotropic systems, that is, the systems whose slowness surface (defined by [4, (0.7)]) consists of some concentric spheres. J.R. Schulenberger and C.H. Wilcox [8] treated the systems whose slowness surface is smooth and strictly convex. They found radiation conditions attached the steady-state wave propagation problem and obtained the generalized eigenfunctions as the unique solutions of steady-state wave propagation problems. But there are many important systems in physics which are not included in their theories. The wave propagation problem in crystals is one of such systems. There are some literatures related to the wave propagation in crystals (H. Tamura [13], R. Weder [14]). In these papers they show the limiting absorption principle, and their theories assure the existence of the generalized eigenfunctions. Concerning their results it is conjectured that these generalized eigenfunctions can be characterized as the unique solutions of steady-state wave propagation problems under suitable radiation conditions. But this fact has not yet been shown. Our Theorem 1.1 shows that the above conjecture holds.

The paper is organized as follows. In section 2 some fundamental facts related to the unperturbed system are shown. In section 3 the radiation condition will be defined and the Rellich uniqueness theorem for the unperturbed system $\Lambda^0-\lambda$ will be proved. Section 4 is devoted to prove Theorem 1.1, that is, the Rellich uniqueness theorem for $\Lambda-\lambda$. In section 5 the limiting absorption principle will be established. In section 6 the eigenfunction expansion theorem will be stated briefly.

The author would like to express his sincere gratitude to Professor M. Ikawa for his kind suggestions and constant help.

2. Some fact related to the unperturbed (homogeneous) system

First we shall consider the equation

$$(\Lambda^0-\zeta)u=f, \tag{2.1}$$

where $\zeta\in \boldsymbol{C}\backslash\{0\}$ and $f\in C_0^\infty(\boldsymbol{R}^n)$. In this section some properties which play an important role in the followring sections are prepared.

The Green function G and $G_\pm$ are defined in [4, (0.5) and (0.6)]. They satisfy

$$\begin{cases}(\Lambda^0-\zeta)G(x,\zeta)=\delta(x)I\\ (\Lambda^0-\lambda)G_\pm(x,\lambda)=\delta(x)I\end{cases}\quad \text{in}\quad \boldsymbol{R}^n. \tag{2.2}$$

Let D be a bounded open set of $\boldsymbol{R}^n$, and let U be an open ball in $\boldsymbol{R}^n$ with center at origin. Bounded domains X and W are taken with

$$W\pm\bar{U}\subset X \quad \text{and} \quad X\subset\subset D.$$

Let ϕ be a smooth function with

$$\phi=\begin{cases}1 & \text{for}\quad x\in X\\ 0 & \text{for}\quad x\in \boldsymbol{R}^n\backslash D\end{cases}$$

and $\tilde{u}=\phi u$. $U'\subsetneqq U$ is another open ball with center at origin and we put

$$\psi(x)=\begin{cases}1 & \text{for} \quad x\in U' \\ 0 & \text{for} \quad x\in \boldsymbol{R}^n\setminus U.\end{cases}$$

Then the following theorem holds (J.R. Schulenberger [7]).

Theorem 2.1. *If a distribution u satisfies*

$$\Lambda^0 u-\zeta u=0 \qquad in \quad \boldsymbol{R}^n$$

for $\zeta\in \boldsymbol{C}\setminus\{0\}$, *then u satisfies for any* $x\in W$

$$\begin{aligned} u(x) &= u*(\Lambda^0-\zeta I)(1-\psi)G(x, \zeta) \\ &= [(\Lambda^0-\zeta I)\tilde{u}]*(1-\psi)G(x, \zeta).\end{aligned}$$

REMARK Theorem 2.1 holds for any G which satisfies (2.2).

Denote by Δ a subset of complex plane

$$\Delta=\{\zeta=\lambda+i\varepsilon \text{ (or } \lambda-i\varepsilon);\ \lambda\in[a, b] \text{ and } \varepsilon\in(0, \varepsilon_0]\}$$

for $[a, b]\subset \boldsymbol{R}^1\setminus\{0\}$ and $\varepsilon_0>0$. Let $\zeta\in\Delta$ and let $f\in C_0^\infty(\boldsymbol{R}^n)$. Then the solution $u=u(x, \zeta)$ of (2.1) has the following representations

$$\begin{aligned} u(x, \zeta) &= \mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta I)^{-1}\hat{f}(\cdot)] \\ &= u^0(x, \zeta)+\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\left(\int_S e^{irx\cdot s}\hat{P}(s)|T(s)|^{-1}\hat{f}(rs)\phi_1(rs)dS\right)dr, \end{aligned} \tag{2.3}$$

where $T(s)$ is the polar reciplocal map on S ([4, page 579]), $\hat{P}(s)=\hat{P}_k(s)$ (projection to the eigenspace associated with λ_k) if $s\in S_k$, ϕ_1 is a function given by (2.15) of [4] and $u^0(x, \zeta)$ is a function satisfying

$$|u^0(x, \zeta)|\leqq C|x|^{-n}$$

for some constant C independent of ζ and $\eta=x/|x|$. The proof of (2.3) is almost the same as that of (2.25) of [4] (the case of the Green function). The formula (2.3) implies the following lemma.

Lemma 2.2. *For any* $\sigma>0$ *it follows that*

$$|u(x, \zeta)|\leqq C_\sigma|x|^\sigma$$

with a constant C_σ *independent of* η *and* $\zeta\in\Delta$. *Moreover* $u(x, \lambda\pm i\varepsilon)$ *converges to a limit* $u(x, \lambda)$ *uniformly for* ζ *on* Δ. *Then it follows that*

$$|u_\pm(x, \lambda)|\leqq C_\sigma|x|^\sigma.$$

The proof of Lemma 2.2 is almost the same as that of the last part of Theorem 7.1 of [4].

The following theorem can be proved in the same way as the proof of Theorem 7.1 of [4].

Theorem 2.3. *Under the same assumption of Theorem* 7.1 *of* [4] *the limit*

$$u_{\pm}(x, \lambda) = \lim_{\varepsilon \to 0} u(x, \lambda \pm i\varepsilon)$$

exists and $u_{\pm}(x, \lambda)$ *satisfies*

$$(2.4) \quad \begin{aligned} u_{\pm}(x, \lambda) = & \sum_{\gamma=1}^{\rho(\pm\eta)} e^{i\lambda|x||T(s)|^{-1}} |x|^{-(n-1)/2} |\lambda|^{(n-1)/2} \\ & \cdot |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s) \psi_{\mathrm{sign}(\pm\lambda)}(s)|_{s=s^{(\gamma)}(\pm\eta)} + q_{\pm}(x, \lambda), \end{aligned}$$

where $q_{\pm}$ *satisfies that for any* p *with* $1 \leqq p < 1 + 1/l$ (l *of Theorem* 7.1 *of* [4]) *there exists* $\nu = \nu_p > 0$ *such that*

$$(2.5) \quad |q_{\pm}(x, \lambda)| \leqq C(\eta)|x|^{-(n-1)-\nu}$$

and

$$(2.6) \quad C(\eta) \in L^p(S^{n-1}).$$

($s^{(\gamma)}(\eta) = s^{\rho(\eta)\gamma}(\eta) = s^{\beta\gamma}(\eta)$ with $\beta = \rho(\eta)$. See [4, page 606~page 607]. ψ_+ and ψ_- are defined in [4, page 586]).

3. The Rellich uniqueness theorem for unperturbed (homogeneous) system

To begin with we shall formulate the radiation condition, which is suitable for our problem.

In section 1 we introduced a set Z_S. Here we introduce the other sets. Let Z_W be the polar reciprocal image of Z_S (see [4, page 580]). Then we denote by $\bar{Z}_S$ and $\bar{Z}_W$ by

$$\bar{Z}_S = \{rs;\ r \in \boldsymbol{R},\ s \in Z_S\}$$

and

$$\bar{Z}_W = \{rw;\ r \in \boldsymbol{R},\ w \in Z_W\},$$

respectively. (In [4] we denote these sets by Z and $\bar{Z}$, respectively.)

Let $\mathcal{P}_{\pm}$ be a class of all complex-valued functions $R(\eta, \xi)$ defined on $(S^{n-1} \setminus \bar{Z}_W) \times \boldsymbol{R}^n$ with the following properties:

i) $R(\eta, \xi) = \sum a_{\alpha}(\eta)\xi^{\alpha}$ (a polynomial of ξ), where $a_{\alpha}(\eta) \in C^{\infty}(S^{n-1} \setminus \bar{Z}_W)$.
ii) $R(\eta, \lambda s^{(\gamma)}(\pm\eta)) = 0,\quad R(\eta, \lambda s^{(\gamma)}(\mp\eta)) \neq 0$

for $\gamma = 1, 2, \cdots, \rho(\pm\eta)$, $\eta \in S^{n-1} \setminus \bar{Z}_W$.

Remark that $\mathcal{P}_+$ (and $\mathcal{P}_-$) are not empty. In fact the function

$$R_\pm(\eta, \xi) = \prod_{\gamma=1}^{\rho(\pm\eta)} (\eta\xi - \lambda\eta s^{(\gamma)}(\pm\eta))$$

satisfies the conditions i) and ii) above.

Next we shall construct a function $\tilde{R}_\pm \in \mathcal{P}_\pm$ which satisfies

$$\tilde{R}_\pm(\eta, \xi) \in \mathcal{P}_\pm \quad \text{and} \quad 1-\tilde{R}_\pm(-\eta, \xi) \in \mathcal{P}_\pm . \tag{3.1}$$

Let $R_\pm(\eta, \xi) = \sum a_\nu(\eta)\xi^\nu$ be a function of $\mathcal{P}_\pm$ and $R_{\pm,1}(\eta, \xi) = \sum b_\mu(\eta)\xi^\mu$ be a function which satisfies

$$R_{\pm,1}(\eta, \lambda s^{(\gamma)}(\mp\eta)) = R_\pm(\eta, \lambda s^{(\gamma)}(\mp\eta))^{-1} .$$

$R_{\pm,1}$ can be constructed in the following way. Put

$$p_\gamma^\pm(\eta, \xi) = \sum_{l=1}^{n} (\xi_l - \lambda s_l^{(\gamma)}(\mp\eta))^2$$

and

$$\begin{aligned} l_\gamma^\pm(\eta, \xi) = {} & p_1^\pm(\eta, \xi) \cdots \hat{p}_\gamma^\pm(\eta, \xi) \cdots p_{\rho(\mp\eta)}^\pm(\eta, \xi) \\ & \cdot \{p_1^\pm(\eta, \lambda s^{(\gamma)}(\mp\eta)) \cdots \hat{p}_\gamma^\pm(\eta, \lambda s^{(\gamma)}(\mp\eta)) \\ & \qquad \cdots p_{\rho(\mp\eta)}^\pm(\eta, \lambda s^{(\gamma)}(\mp\eta))\}^{-1}, \end{aligned}$$

where $\hat{p}_\gamma^\pm$ denotes the omission of the factor $p_\gamma^\pm$. Then

$$l_\gamma^\pm(\eta, \lambda s^{(\delta)}(\mp\eta)) = \delta_{\gamma\delta} .$$

So put

$$R_{\pm,1}(\eta, \xi) = \sum_{\gamma=1}^{\rho(\mp\eta)} R_\pm(\eta, \lambda s^{(\gamma)}(\mp\eta))^{-1} l_\gamma^\pm(\eta, \xi) .$$

Define

$$\tilde{R}_\pm(\eta, \xi) = R_{\pm,1}(\eta, \xi) R_\pm(\eta, \xi) = \sum_{\mu,\nu} a_\nu(\eta) b_\mu(\eta) \xi^{\nu+\mu} . \tag{3.2}$$

Then $\tilde{R}_\pm$ satisfies (3.1) because $\tilde{R}_\pm$ satisfies

$$\tilde{R}_\pm(\eta, \lambda s^{(\gamma)}(\pm\eta)) = 0 , \quad \tilde{R}_\pm(\eta, \lambda s^{(\gamma)}(\mp\eta)) = 1$$

and

$$\tilde{R}_\pm(-\eta, \lambda s^{(\gamma)}(\pm\eta)) = \tilde{R}_\pm(-\eta, \lambda s^{(\gamma)})(\mp(-\eta))) = 1 .$$

Next we give the definition of $\pm$ radiation condition for the operator $\Lambda - \lambda$.

DEFINITION A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is said to satisfy $\pm$ radiation condition when u satisfies

Ri) u is smooth in $\{|x| > R_0\}$ for a large R_0 positive.

Rii) $|u(x)|\leqq C_\sigma\langle x\rangle^\sigma$ for any $\sigma>0$ and for some constant C_σ.

Riii) There exists a function $C(\eta)\in L^p(S^{n-1})$ for any $1\leqq p<2$ if $n<3$, or for some $p>1$ if $n=3$ such that

$$|u(x)|\leqq C(\eta)|x|^{-(n-1)/2}$$

Riv) Let $\alpha(\xi)\in C^\infty(\boldsymbol{R}^n)$ be a function which satisfies supp $\alpha\subset\boldsymbol{R}^n\backslash\bar{Z}_S$ and have polynomial order at infinity, and let $\beta(x)\in C^\infty(\boldsymbol{R}^n)$ be a function such that

$$\beta(x)=\begin{cases}0 & \text{if} \quad |x|\leqq R_0\\ 1 & \text{if} \quad |x|\geqq R_0+1\,.\end{cases} \tag{3.3}$$

Then there exists some constant $C_{\alpha,\beta}$ such that

$$|\alpha(D_x)(\beta u)|\leqq C_{\alpha,\beta}|x|^{-(n-1)/2}, \tag{3.4}$$

and it holds that for any $R\in\mathcal{P}_\pm$ and for any $\chi\in C_0^\infty(S^{n-1}\backslash\bar{Z}_W)$

$$|\chi(\eta)R(\eta, D_x)[\alpha(D_x)(\beta u)]|\leqq C_{\alpha,\beta,\chi}|x|^{-n/2}\,. \tag{3.5}$$

Remark 1 If there exists a function $\beta_0\in C^\infty(\boldsymbol{R}^n)$ with (3.3) which satisfies (3.4) and (3.5), then any β with (3.3) satisfies (3.4) and (3.5). In fact, since both β_0 and β have the property (3.3),

$$\beta u-\beta_0 u=0 \qquad \text{for} \quad |x|\leqq R_0 \text{ and } |x|\geqq R_0+1\,.$$

Then $\beta u-\beta_0 u\in C_0^\infty(\boldsymbol{R}^n)$ for any function u satisfying the radiation condition, and since $\alpha(D_x)$ is a pseudodifferential operator of finite order, $\alpha(D_x)[\beta u-\beta_0 u]\in\mathcal{S}$. Hence

$$\alpha(D_x)[\beta u]=\alpha(D_x)[\beta u-\beta_0 u]+\alpha(D_x)[\beta_0 u]$$

satisfies (3.4) and (3.5).

Remark 2 A solution u of

$$(\Lambda^0-\lambda)u=f \qquad \text{for} \quad f\in C_0^\infty(\boldsymbol{R}^n)$$

satisfies $\pm$ radiation condition if and only if u satisfies Ri)$\sim$Riii) for $R_0=0$ and (3.4) and (3.5) for $\beta\equiv 1$. This fact is clear from the hypoellipticity of $\Lambda^0-\lambda$([10]).

First we prove the existence theorem.

Theorem 3.1. *Put*

$$u(x,\zeta)=\mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta I)^{-1}\hat{f}(\cdot)] \qquad (\zeta\in\boldsymbol{C}\backslash\boldsymbol{R} \text{ and } f\in C_0^\infty(\boldsymbol{R}^n))$$

and

$$u_\pm(x, \lambda) = \lim_{\varepsilon \searrow 0} u(x, \lambda \pm i\varepsilon) .$$

Then $u_\pm(x, \lambda)$ is a solution of

$$(\Lambda^0 - \lambda)u = f$$

which satisfies $\pm$ radiation condition.

Proof. Clearly it holds that

$$(\Lambda^0 - \zeta)u(x, \zeta) = f .$$

The both sides of this equation converge as $\varepsilon \searrow 0$ in the sense of $\mathcal{S}'$, and this implies that $u_\pm(x, \lambda)$ is a solution.

As for the radiation condition $u_\pm(x, \lambda)$ satisfies Ri) clearly from the hypoellipticity of $\Lambda^0 - \lambda$, Rii) from (2.7) and Riii) from Theorem 2.3. Then it remains only to prove that $u_\pm$ satisfies Riv). From Remark 2 it is enough to prove the case of $\beta \equiv 1$. Operate $\alpha(D_x)$ to the both side of $(\Lambda^0 - \zeta)u(x, \zeta) = f$. Since both $\alpha(\xi)$and $\Lambda^0(\xi) - \lambda$ are independent of x, they commutes. Thus it follows that

$$(\Lambda^0 - \zeta)\alpha(D_x)u = \alpha(D_x)f .$$

In the same way as in the case of the Green function we can represent

$$\alpha(D_x)u(x, \zeta) = u_{\alpha,0}(x, \zeta) + \int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} \cdot \left(\int_S e^{irs\cdot x} \hat{P}(s) |T(s)|^{-1} \alpha(rs) \hat{f}(rs) \phi_1(rs) dS \right) dr ,$$

where

$$|u_{\alpha,0}(x, \zeta)| \leqq C_\alpha |x|^{-1}$$
(C_α is independemt of $\eta \in S^{n-1}$ and $\zeta \in \Delta$) .

Note that supp $\alpha(\xi) \subset \boldsymbol{R}^n \backslash \bar{Z}_S$. Put

$$v(x, r) = \int_S e^{ix\cdot s} \hat{P}(s) |T(s)|^{-1} \alpha(rs) \hat{f}(rs) \phi_1(rs) dS .$$

It is an integral on a smooth surface. Then we apply the usual stationary phase method to this integral, and we have

$$\begin{aligned} v(x, r) = \sum_{\gamma=1}^{\rho(\eta)} & (2\pi)^{-(n-1)/2} e^{i|x||T(s)|^{-1}} |K(s)|^{-1/2} \\ & \cdot |T(s)|^{-1} \hat{P}(s) \phi_1(rs) \alpha(rs) \hat{f}(rs) \psi_+(s) |_{s=s^{(\gamma)}(\eta)} \\ & \cdot |x|^{-(n-1)/2} + \sum_{\gamma=1}^{\rho(-\eta)} (2\pi)^{-(n-1)/2} e^{i|x||T(s)|^{-1}} \\ & \cdot |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s) \phi_1(rs) \alpha(rs) \hat{f}(rs) \\ & \cdot \psi_-(s) |_{s=s^{(\gamma)}(-\eta)} |x|^{-(n-1)/2} + q_0(x, r) , \end{aligned}$$

where

$$|q_0(x, r)| \leqq C_\alpha |x|^{-(n+1)/2}.$$

Thus it holds that

$$\alpha(D_x)u(x, \zeta) = (\text{principal part}) + q(x, \zeta),$$

where

$$|q(x, \zeta)| \leqq C_\alpha |x|^{-n/2-\nu} \qquad \text{for any } \nu \text{ with } \quad 0 \leqq \nu < 1/2.$$

Moreover

$$\lim_{\varepsilon \searrow 0} \alpha(D_x)u(x, \lambda \pm i\varepsilon)$$

has the same property. The continuity of pseudo differential operators in $\mathcal{S}'$ implies

$$\lim_{\varepsilon \searrow 0} \alpha(D_x)u(x, \lambda \pm i\varepsilon) = \alpha(D_x)u_\pm(x, \lambda).$$

Then

$$\alpha(D_x)u_\pm(x, \lambda) = \sum_{\gamma=1}^{\rho(\pm\eta)} (2\pi)^{-(n-1)/2} e^{i\lambda|x||T(s)|^{-1}} \; |\lambda|^{(n-1)/2} \; |x|^{-(n-1)/2}$$
$$\cdot |K(s)|^{-1/2} |T(s)|^{-1} \hat{P}(s) \alpha(\lambda s) \hat{f}(\lambda s) \;\; \psi_{\text{sign}(\pm\lambda)}(s)|_{s=s^{(\gamma)}(\pm\eta)} + q_{\alpha,\pm}(x, \lambda),$$

where

$$|q_{\alpha,\pm}(x, \lambda)| \leqq C_{\alpha,\nu} |x|^{-n/2-\nu} \qquad \text{for any } \nu \text{ with } \quad 0 \leqq \nu < 1/2.$$

This shows that $u_\pm(x, \lambda)$ satisfies Riv). Q.E.D.

In order to prove the uniqueness theorem we prepare several lemmas (Lemma 3.2~3.8).

Lemma 3.2. *When* $|x| \geqq 2|y|$, *it holds that*

1) $$|D_x^\alpha G_\pm(x-y, \lambda)| \leqq C_y(\eta) |x|^{-(n-1)/2},$$

where $C_y(\eta) \in L^p(S^{n-1})$ *for some* $p > 1$.

2) *For any* $R(\eta, \xi) \in \mathcal{P}_\pm$ *there exists* $\nu > 0$ *such that for any* $\chi \in C_0^\infty(S^{n-1} \backslash \bar{Z}_W)$

$$|\chi(\eta) R(\eta, D_x) G_\pm(x-y, \lambda)| \leqq C_{y,\chi}(\eta) |x|^{-(n-1)/2-\nu},$$

where $C_{y,\chi}(\eta) \in L^p(S^{n-1})$ *for some* $p > 1$.

Proof.

$$G(x-y, \zeta) = \mathcal{F}^{-1}[e^{-iy\xi}(\Lambda^0(\xi) - \zeta I)^{-1}]$$
$$= \mathcal{F}^{-1}[\phi_1(\xi) e^{-iy\xi}(\Lambda^0(\xi) - \zeta I)] + \mathcal{F}^{-1}[\phi_2(\xi) e^{-iy\xi}(\Lambda^0(\xi) - \zeta I)^{-1}]$$

$$= \int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r - \zeta} \left(\int_S e^{irx\cdot s} \hat{P}(s) |T(s)|^{-1} \phi_1(rs) e^{-iy\cdot rs} dS \right) dr$$
$$+ G_2(x-y, \zeta) .$$

Then $|x| \geqq 2|y|$ implies

$$|G_2(x-y, \zeta)| \leqq C_l |x-y|^{-l} \leqq C_l((1/2)|x|)^{-l} .$$

Details are omitted since they are similar to the proof of Theorem 7.1 of [4].

Q.E.D.

Remark that, when $|x| \geqq 2|y|$, we have $|x| \sim |x-y|$ as $|x| \to \infty$.

Lemma 3.3. *Let $\psi \in C_0^\infty(\mathbf{R}^1)$ be a function such that*

$$\psi(t) = \begin{cases} 0 & \text{if} \quad |t| > 2 \\ 1 & \text{if} \quad |t| < 1 . \end{cases}$$

Fix $\varepsilon > 0$ and put

$$\Phi_\varepsilon^\pm(x) = (\Lambda^0 - \lambda)(1 - \psi(\varepsilon|x|)) G_\pm(x, \lambda)$$

and

$$k_\varepsilon^y = \{x;\ \varepsilon^{-1} < |x-y| < 2\varepsilon^{-1}\} .$$

Then we have

1) supp $\Phi_\varepsilon^\pm(x) \subset k_\varepsilon^0$.

2) $\Phi_\varepsilon^\pm(x-y) = -\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x-y|)) G_\pm(x-y, \lambda)$ *for* $x \in k_\varepsilon^y$.

3) $|\Phi_\varepsilon^\pm(x-y)| \leqq C_y(\eta) \varepsilon^{(n+1)/2}$,

where $C_y(\eta) \in L^p(S^{n-1})$ for some $p > 1$.

We often omit λ of $G_\pm$ for simplicity ($G_\pm(x) = G_\pm(x, \lambda)$).

Proof. 1) $\psi(\varepsilon|x|) = 1$ for $|x| < \varepsilon^{-1}$, so $\Phi_\varepsilon^\pm(x) = 0$ for $|x| < \varepsilon^{-1}$; $\psi(\varepsilon|x|) = 0$ for $|x| > 2\varepsilon^{-1}$ and $(\Lambda^0 - \lambda) G_\pm(x) = 0$ for $|x| > 2\varepsilon^{-1}$, so $\Phi_\varepsilon^\pm(x) = 0$ for $|x| > 2\varepsilon^{-1}$.

2) It is sufficient to prove the case of $y = 0$. For $x \in k_\varepsilon^0$, we have from the definition

$$\Phi_\varepsilon^\pm(x) = -(\Lambda^0 - \lambda)\psi(\varepsilon|x|) G_\pm(x) = -\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x|)) G_\pm(x) .$$

3) We have $D_x^\gamma \psi(\varepsilon|x-y|) = O(\varepsilon^{|\gamma|})$ uniformly for η and y. On k_ε^y it holds that $|x-y| \sim \varepsilon^{-1}$. So by the above remark $|x| \sim \varepsilon^{-1}$. Then Lemma 3.2 implies

$$|G_\pm(x-y)| \leqq C_y(\eta) |x|^{-(n-1)/2} \leqq C_y(\eta) \varepsilon^{(n-1)/2} .$$

Then the inequality of 3) follows from 2). Q.E.D.

Let $\tilde{R}_\pm$ be a function given in (3.2).

Lemma 3.4. *For $x\in k_\varepsilon^y$, $x=|x|\eta$, put*

$$\Psi_\varepsilon^{\pm,y}(x)=(1-\tilde{R}_\pm(-\eta,D_x))I\cdot\Phi_\varepsilon^\pm(x+y).$$

Then for any $\chi\in C_0^\infty(S^{n-1}\backslash\bar{Z}_W)$

$$|\chi(\eta)\Psi_\varepsilon^{\pm,y}(x)|\leqq C_{\chi,y}(\eta)\varepsilon^{(n+1)/2+\sigma},$$

where $\sigma>1$ and $C_{\chi,y}(\eta)\in L^p(S^{n-1})$ for some $p>1$.

Proof. Let $x\in k_\varepsilon^y$. By Lemma 3.3 and (3.2)

$$\begin{aligned}\Psi_\varepsilon^{\pm,y}(x)&=(1-\tilde{R}_\pm(-\eta,D_x))\,I\cdot\Phi_\varepsilon^\pm(x+y)\\&=\Phi_\varepsilon^\pm(x+y)+\sum_{\nu,\mu}a_\nu(-\eta)b_\mu(-\eta)D_x^{\nu+\mu}(\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x+y|))G_\pm(x+y))\\&=\Phi_\varepsilon^\pm(x+y)+(\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x+y|)))(\sum_{\nu,\mu}a_\nu(-\eta)b_\mu(-\eta)\\&\qquad\cdot D^{\nu+\mu}G_\pm(x+y))+Q_0\\&=\Phi_\varepsilon^\pm(x+y)+\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x+y|))\cdot\tilde{R}(-\eta,D_x)G_\pm(x+y)+Q_0\\&=\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x+y|))G_\pm(x+y)+\sum_{j=1}^n A_j(D_j\psi(\varepsilon|x+y|)\\&\qquad\cdot\tilde{R}(-\eta,D_x)G_\pm(x+y)+Q_0,\end{aligned}$$

where

$$Q_0=\sum_{j=1}^n A_j\sum_{|\delta|\geqq 2}C_\delta(D_x^\delta\chi)(\varepsilon|x+y|)\sum_{\nu,\mu}a_\nu(-\eta)b_\mu(-\eta)D^{\nu+\mu-\delta+e_j}G_\pm(x+y).$$

From Lemma 3.2 1)

$$|\chi(\eta)Q_0|\leqq C_{\chi,y}(\eta)\varepsilon^{2+(n-1)/2}=C_{\chi,y}(\eta)\,\varepsilon^{(n+3)/2}$$

for $C_{\chi,y}(\eta)\in L^p(S^{n-1})$, $p>1$. On the other hand, since $1-\tilde{R}_\pm(-\eta,\xi)\in\mathcal{P}_\pm$, Lemma 3.2 2) implies

$$|\chi(\eta)(1-R(-\eta,D_x))G_\pm(x+y)|\leqq C_{\chi,y}(\eta)\varepsilon^{(n+1)/2+\nu}.$$

This gives the conclusion. Q.E.D.

Since ε is sufficiently small, we may assume that u is smooth in k_ε^y. From the remark after Lemma 3.2

(3.6) $$k_\varepsilon^y\subset\tilde{k}_\varepsilon=\{x;\,(2/3)\varepsilon^{-1}<|x|<4\varepsilon^{-1}\}$$

follows if $|x|>2|y|$.

Lemma 3.5. *Let $u_\pm$ be a solution of $(\Lambda^0-\lambda)u=0$ which satisfies $\pm$ radiation condition. For any function $\alpha(\xi)$ with supp $\alpha\subset\boldsymbol{R}^n\backslash\bar{Z}_S$ and for any $\chi\in C^\infty(S^{n-1})$ we have*

$$\lim_{\varepsilon\searrow 0}\int_{k_\varepsilon^y}\chi(\eta)(\tilde{R}_\pm(\eta, D_x)\alpha(D_x)u_\pm(x))\Phi_\varepsilon^\pm(x-y)dx = 0\,.$$

Proof. $\tilde{R}_\pm(\eta, \xi)\in\mathcal{P}_\pm$, and so for $x\in k_\varepsilon^y$

$$|\tilde{R}_\pm(\eta, D_x)\alpha(D_x)u_\pm(x)|\leqq C\,|x|^{-n/2-\nu}$$

for any $\nu<1/2$. By Lemma 3.3 it holds that

$$|\Phi_\varepsilon^\pm(y-x)|\leqq C_y(\eta)\varepsilon^{(n+1)/2} \qquad \text{for} \quad C_y(\eta)\in L^1(S^{n-1})\,.$$

Hence we have

$$|\int_{k_\varepsilon^y}\chi(\eta)(\tilde{R}_\pm(\eta, D_x)\alpha(D_x)u_\pm(x))\Phi_\varepsilon^\pm(y-x)dx|$$
$$\leqq \text{Const.}\int_{(2/3)\varepsilon^{-1}}^{4\varepsilon^{-1}}\varepsilon^{(n-1)/2+\nu}\varepsilon^{(n+1)/2}\varepsilon^{-n+1}d\,|x|\int_{S^{n-1}}|\chi(\eta)C_y(\eta)|\,d\eta$$
$$= \text{Const.}\,\varepsilon^{1+\nu}\varepsilon^{-1} = \text{Const.}\,\varepsilon^\nu\,.$$

Q.E.D.

Lemma 3.6. *Put*

$$\hat{u}_\varepsilon^\pm(y) = (\alpha(D_x)u_\pm)*\Phi_\varepsilon^\pm(y)\,.$$

Then $\lim_{\varepsilon\searrow 0}\hat{u}_\varepsilon^\pm(y)=0$ *for any* $y\in\boldsymbol{R}^n$.

Proof. Let $y\in\boldsymbol{R}^n$ be fixed, and δ be any positive number. Since $C_{\Phi,y}(\eta)\in L^1(S^{n-1})$, there exist two open sets U and V with the properties

$$\bar{Z}_W\cap S^{n-1}\subset U\subset\subset V\subset S^{n-1}$$

and

$$\int_V C_{\alpha u}\cdot C_{\Phi,y}(\eta)dy<\delta\,. \tag{3.7}$$

Put $\chi\in C_0^\infty(S^{n-1}\backslash\bar{Z}_W)$ as

$$\chi(\eta)=\begin{cases}0 & \text{if } \eta\in U\\ 1 & \text{if } \eta\in S^{n-1}\backslash V\,.\end{cases}$$

Then

$$\begin{aligned}\hat{u}_\varepsilon^\pm(y) &= \int_{k_\varepsilon^y}(\alpha(D_x)u_\pm(x))\Phi_\varepsilon^\pm(y-x)\,dx\\ &= \int_{k_\varepsilon^y}(\alpha(D_x)u_\pm(x))\Phi_\varepsilon^\pm(y-x)\chi(x/|x|)dx\\ &\qquad +\int_{k_\varepsilon^y}(\alpha(D_x)u_\pm(x))\Phi_\varepsilon^\pm(y-x)(1-\chi(x/|x|))dx\\ &=:I_\varepsilon^1+I_\varepsilon^2\,.\end{aligned}$$

From (3.7)

$$|I_\varepsilon^2| \leqq \int_V C_{\alpha u} C_{\Phi,y}(\eta)(1-\chi(\eta))d\eta \int_{\tilde{k}_\varepsilon} |x|^{-(n-1)/2}\varepsilon^{(n+1)/2} d|x|$$
$$= \int_V C_{\alpha u} C_{\Phi,y}(\eta)(1-\chi(\eta))d\eta \int_{(2/3)\varepsilon^{-1}}^{4\varepsilon^{-1}} \varepsilon^{-(n-1)/2}\varepsilon^{(n+1)/2}\varepsilon^{-n+1} d|x|$$
$$\leqq \text{Const. } \delta .$$

Let

$$J_\varepsilon = \int_{k_\varepsilon^y} \chi(\eta)\{\alpha u_\pm(x) - \tilde{R}_\pm(\eta, D_x)\alpha u_\pm(x)\}\Phi_\varepsilon^\pm(y-x)dx .$$

By Lemma 3.5

$$\lim_{\varepsilon \searrow 0}(I_\varepsilon^1 - J_\varepsilon) = 0 . \tag{3.8}$$

Now

$$J_\varepsilon = \int_{k_\varepsilon^y} \chi(\eta)\{\alpha u_\pm(x) - (\sum_{\nu,\mu} a_\nu(\eta)b_\mu(\eta)D_x^{\nu+\mu})(\alpha u_\pm(x))\}\Phi_\varepsilon^\pm(y-x)dx .$$

Since $P(D_x)\Phi_\varepsilon^\pm(y-x)=0$ on ∂k_ε^y (for any polynomial P), we have by integration by parts

$$J_\varepsilon = \int_{k_\varepsilon^y} (\alpha u_\pm(x))\{\Phi_\varepsilon^\pm(y-x)\chi(\eta) - \sum_{\nu,\mu} D_x^{\nu+\mu}(\chi(\eta)a_\nu(\eta)b_\mu(\eta)\Phi_\varepsilon^\pm(y-x))dx .$$

Now $|D_x^\sigma(\chi(\eta)a_\nu(\eta)b_\mu(\eta)| \leqq C_{\sigma,\chi}|x|^{-|\sigma|}$. Since χ does not have its support near $\bar{Z}_W$, the following integration on k_ε^y is integrable with respect to η. Then we have

$$\left|\int_{k_\varepsilon^y} (\alpha u_\pm(x))(D_j\chi(\eta)a_\nu(\eta)b_\mu(\eta))\Phi_\varepsilon^\pm(y-x)dx\right|$$
$$\leqq C\int_{(2/3)\varepsilon^{-1}}^{4\varepsilon^{-2}} \varepsilon^{(n-1)/2}\cdot\varepsilon\cdot\varepsilon^{(n+1)/2}\cdot\varepsilon^{-n+1} d|x| \leqq C'_\chi \varepsilon .$$

Thus

$$J_\varepsilon = \int_{k_\varepsilon^y} (\alpha u_\pm(x))\chi(\eta)\{\Phi_\varepsilon^\pm(y-x)$$
$$-[\sum_{\nu,\mu} a_\nu(\eta)b_\mu(\eta)(-1)^{|\nu+\mu|}D_x^{\nu+\mu}]\Phi_\varepsilon^\pm(y-x)\}dx + Q_\varepsilon ,$$

where $|Q_\varepsilon| \leqq C_\chi \varepsilon$.

Now the transformation of variables x of the right hand side of J_ε to $-x$ and the fact that $D_x^\alpha\{\Phi_\varepsilon^\pm(y-x)\} = (-1)^{|\alpha|}(D_x^\alpha \Phi_\varepsilon^\pm)(y-x)$ imply

$$J_\varepsilon=(-1)^n\int_{k_\varepsilon^{-y}}(\alpha u_\pm(-x))\chi(-\eta)\{\Phi_\varepsilon^\pm(x+y)$$
$$-[\sum_{\nu,\mu}a_\nu(-\eta)b_\mu(-\eta)D_x^{\nu+\mu}]\Phi_\varepsilon^\pm(x+y)\}\,dx+Q_\varepsilon$$
$$=(-1)^n\int_{k_\varepsilon^{-y}}(\alpha u_\pm(-x))\chi(-\eta)\{1-\tilde{R}_\pm(-\eta,D_x)\}\,\Phi_\varepsilon^\pm(x+y)dx+Q_\varepsilon$$
$$=(-1)^n\int_{k_\varepsilon^{-y}}\chi(-\eta)(\alpha u_\pm(-x))\Psi_\varepsilon^{\pm,y}(x)dx+Q_\varepsilon\,.$$

Hence

$$|J_\varepsilon|\leqq\text{Const.}\int_{(2/3)\varepsilon^{-1}}^{4\varepsilon^{-1}}\varepsilon^{(n-1)/2}\varepsilon^{(n+1)/2+\sigma}\varepsilon^{-n+1}d\,|x|+Q_\varepsilon \tag{3.9}$$
$$\leqq C'_\chi(\varepsilon^\sigma+\varepsilon)=2C'_\chi\varepsilon^\sigma\,.$$

Thus (3.8) and (3.9) give

$$\lim_{\varepsilon\searrow 0}I_\varepsilon^1=\lim_{\varepsilon\searrow 0}(I_\varepsilon^1-J_\varepsilon)+\lim_{\varepsilon\searrow 0}J_\varepsilon=0\,.$$

Hence

$$\overline{\lim_{\varepsilon\searrow 0}}\,|\hat{u}_\varepsilon^\pm(y)|\leqq\lim_{\varepsilon\searrow 0}|I_\varepsilon^2|\leqq C\delta\,.$$

Since δ can be taken arbitrally, we have

$$\lim_{\varepsilon\searrow 0}\hat{u}_\varepsilon^\pm(y)=0\,. \qquad \text{Q.E.D.}$$

Lemma 3.7. *Let $u_\pm$ be a solution of $(\Lambda^0-\lambda)u=0$ satisfying $\pm$ radiation condition. Then we have*

$$\alpha(D_x)u_\pm(x)=0$$

for any $\alpha(\xi)$ with supp $\alpha(\xi)\subset\boldsymbol{R}^n\backslash\bar{Z}_S$.

Proof. Operate $\alpha(D_x)$ to the both sides of $(\Lambda^0-\lambda)u=0$. Then

$$(\Lambda^0-\lambda)\alpha(D_x)u_\pm=0\,.$$

From Theorem 2.1

$$\alpha u_\pm=(\alpha u_\pm)*\Phi_\varepsilon^\pm(x)\qquad\text{for any}\quad\varepsilon>0\,.$$

On the other hand Lemma 3.6 implies

$$\alpha u_\pm(x)=\lim_{\varepsilon\searrow 0}(\alpha u_\pm)*\Phi_\varepsilon^\pm(x)=0$$

for any $x\in\boldsymbol{R}^n$. This gives the conclusion. Q.E.D.

Finally we prove the following fact which is related to the Sobolev spaces.

Lemma 3.8. *If a distribution u satisfies*

i) $u \in H^{-s}$ for $s<1$

ii) supp $u \subset M$, *which is an at most $n-2$ dimensional compact submanifold,*

then $u \equiv 0$.

Proof. If $s \leqq 0$, the lemma is clear. So we consider the case of $s>0$. Since u has compact support, the conclusion of lemma is equivalent to that $(u, \varphi)=0$ for any $\varphi \in C^\infty(\boldsymbol{R}^n)$.

First we show $(u, 1)=0$. Put

$$\varphi_\varepsilon(x) = \rho(\operatorname{dist}(x, M)/\varepsilon),$$

where $\rho \in C_0^\infty(\boldsymbol{R}^1)$ with

$$\rho(t) = \begin{cases} 1 & \text{if } |t|<1/2 \\ 0 & \text{if } |t|>1 . \end{cases}$$

Then

$$(u, 1) = (u, \varphi_\varepsilon)+(u, 1-\varphi_\varepsilon) .$$

Since supp $u \subset M$, $(u, 1-\varphi_\varepsilon)=0$. Thus

$$(u, 1) = (u, \varphi_\varepsilon) . \tag{3.10}$$

The assumption i) gives

$$|(u, \varphi_\varepsilon)| \leqq \|u\|_{-s} \cdot \|\varphi_\varepsilon\|_s . \tag{3.11}$$

By the fact that $0<s<1$ the interpolation inequality implies for any δ with $0<\delta<1$

$$\|\varphi_\varepsilon\|_s \leqq \delta \|\varphi_\varepsilon\|_1 + C\delta^{-s/(1-s)} \|\varphi_\varepsilon\|_0 . \tag{3.12}$$

From the definition of φ_ε and the fact that dim $M \leqq n-2$ we have

$$\|\varphi_\varepsilon\|_1 \leqq C_1 \quad \text{and} \quad \|\varphi_\varepsilon\|_0 \leqq C_0 \varepsilon^2 . \tag{3.13}$$

The estimates (3.12) and (3.13) imply

$$\|\varphi_\varepsilon\|_s \leqq C_1 \delta + C \cdot C_0 \varepsilon^2 \delta^{-s/(1-s)} .$$

Then, by putting $\delta = \varepsilon^{(1-s)/s}$ we obtain

$$\|\varphi_\varepsilon\|_s \leqq C_1 \varepsilon^{(1-s)/s} + C \cdot C_0 \varepsilon .$$

Hence $\|\varphi_\varepsilon\|_s \to 0$ as $\varepsilon \to 0$. (3.10), (3.11) and the above fact give

$$|(u, 1)| \to 0 .$$

Thus $(u, 1)=0$ follows.

Next we show $(u, \varphi)=0$ for any $\varphi\in C^\infty(\boldsymbol{R}^n)$. Note that φu also satisfies the conditions i) and ii). Thus the above arguments gives

$$(\varphi u, 1) = 0 .$$

Then we have

$$(u, \varphi) = (\varphi u, 1) = 0 . \qquad \text{Q.E.D.}$$

We denote the set $\{g\in L^2(\boldsymbol{R}^n);\ \text{supp}\ g \text{ is compact}\}$ by $L^2_{\text{vox}}(\boldsymbol{R}^n)$.

Now we prove the uniqueness theorem of $(\Lambda^0-\lambda)u=g$ for $g\in L^2_{\text{vox}}(\boldsymbol{R}^n)$.

Theorem 3.9. *Let g be a function of $L^2_{\text{vox}}(\boldsymbol{R}^n)$. For $\lambda\in\boldsymbol{R}^1\backslash\{0\}$ the solution of*

$$\begin{cases} (\Lambda^0-\lambda)u_\pm = g \\ u_\pm \textit{ satisfies } \pm \textit{ radiation condition} \end{cases} \tag{3.14}$$

is unique if it exists.

Proof. Obviously it suffices to show that a solution $w_\pm$ of

$$\begin{cases} (\Lambda^0-\lambda)w_\pm = 0 \\ w_\pm \text{ satisfies } \pm \text{ radiation condition} . \end{cases}$$

must be identically zero. We shall see that the Fourier transformation $\hat{w}_\pm$ satisfies the condition i) and ii) of Lemma 3.8. Then $\hat{w}_\pm\equiv 0$, that is $w_\pm\equiv 0$.

From Lemma 3.7

$$\alpha(D_x)w_\pm(x) = 0$$

for any $\alpha=\alpha(\xi)$ with supp $\alpha\subset\boldsymbol{R}^n\backslash\bar{Z}_S$. The Fourier transformation gives

$$\alpha(\xi)\hat{w}_\pm(\xi) = 0 .$$

Here $\hat{w}_\pm(\xi)$ is a distribution of $\mathcal{S}'_\xi$. Since α is any function which does not have its support in $\bar{Z}_S$,

$$\text{supp}\ \hat{w}_\pm\subset\bar{Z}_S \tag{3.15}$$

follows. On the other hand it holds that

$$\text{supp}\ w\subset\lambda S . \tag{3.16}$$

This fact follows from

$$(\hat{w}_\pm(\cdot), \psi(\cdot)) = 0 \tag{3.17}$$

for any $\psi\in\mathcal{S}$ with supp $\psi\subset\boldsymbol{R}^n\backslash\lambda S$. In order to prove (3.17) put

$$\varphi(\xi)=\begin{cases}(\Lambda^0(\xi)-\lambda I)^{-1}\psi(\xi) & \text{if } \xi\in\operatorname{supp}\psi\\ 0 & \text{if } \xi\notin\operatorname{supp}\psi .\end{cases}$$

Since $\Lambda^0(\xi)-\lambda I$ is non-singular on support of ψ, φ is a function of $\mathcal{S}$. Then

$$\begin{aligned}(\hat{w}_\pm(\xi), \psi(\xi)) &= (\hat{w}_\pm(\xi), (\Lambda^0(\xi)-\lambda I)\varphi(\xi))\\ &= ((\Lambda^0(\xi)-\lambda I)\hat{w}_\pm(\xi), \varphi(\xi)) .\end{aligned}$$

Here $w_\pm$ satisfies $(\Lambda^0-\lambda)w_\pm=0$. Then by the Fourier transformation

$$(\Lambda^0(\xi)-\lambda I)\hat{w}_\pm(\xi)=0 .$$

This implies (3.17).

From (3.15) and (3.16)

$$\operatorname{supp}\hat{w}_\pm(\xi)\subset\lambda S\cap\bar{Z}_S \tag{3.18}$$

follows. Here $\lambda S\cap\bar{Z}_S=\lambda Z_S$, and from Si) and Sii) λZ_S is an at most $n-2$ dimensional submanifold. This shows the condition ii) of Lemma 3.8.

From Rii) and Riii) of the radiation condition it follows that for any $\sigma>0$ and for any θ with $0<\theta<1$

$$|w_\pm(x)|^2\leqq C_\sigma^{1-\theta}C(\eta)^{1+\theta}\langle x\rangle^{-(1+\theta)(n-1)/2+\sigma(1-\theta)} .$$

The case of $n>3$. Since $C(\eta)\in L^p(S^{n-1})$ for any p with $1\leqq p<2$, $C(\eta)^{1+\theta}$ is integrable on S^{n-1} for any θ with $0<\theta<1$. If θ is sufficiently close to 1 and σ sufficiently small, it follows

$$(1+\theta)(n-1)/2-\sigma(1-\theta)>n-2 .$$

So

$$(1+\theta)(n-1)/2-\sigma(1-\theta)+2s>n$$

for some s with $0<s<1$. Then

$$\begin{aligned}&\int_{R^n}\langle x\rangle^{-2s}|w_\pm(x)|^2dx\\ &\leqq C_\sigma^{1-\theta}\int_{S^{n-1}}C(\eta)^{1+\theta}d\eta\int_1^\infty\langle x\rangle^{-n-\nu}|x|^{n-1}d|x|\\ &\qquad+\int_{|x|\leqq 1}\langle x\rangle^{-2s}|w_\pm(x)|^2dx\\ &<+\infty.\end{aligned} \tag{3.19}$$

This implies that $\hat{w}_\pm(\xi)\in H^{-S}$, that is, $\hat{w}_\pm$ satisfies the condition i) of Lemma 3.8.

The case of $n=3$. Since $C(\eta)\in L^p(S^{n-1})$ for some $p>1$, $C(\eta)^{1+\theta}$ is integrable on S^{n-1} for sufficiently small θ.

$$(1+\theta)(n-1)/2-\sigma(1-\theta) = (1+\theta)-\sigma(1-\theta)>(1+\theta)-\sigma$$

for $n=3$. Then

$$(1+\theta)-\sigma(1-\theta)>1$$

for sufficiently small σ. So

$$(1+\theta)-\sigma(1-\theta)+2s>3 \quad (=n)$$

for some s with $0<s<1$. Then in the same way as (3.19)

$$\int_{R^n} \langle x\rangle^{-2s} |w_\pm(x)|^2 dx < +\infty .$$

This implies that $\hat{w}_\pm(\zeta)\in H^{-s}$, the condition i) of Lemma 3.8. Q.E.D.

From Theorems 3.1 and 3.9 we have

Theorem 3.10. *Let* $f\in C_0^\infty(\boldsymbol{R}^n)$ *and* $\lambda\in \boldsymbol{R}^n\setminus\{0\}$. *Then there exists a unique solution of the problem*

$$\begin{cases} (\Lambda^0-\lambda)u_\pm = f \\ u_\pm \text{ satisfies } \pm \text{ radiation condition}. \end{cases}$$

4. The Rellich uniqueness theorem for the perturbed system

This section is devoted to prove Theorem 1.1. Since the proof is very long, we divide the proof into some steps.

4.1. Outline of the proof.

In [8] J.R. Schulenberger and C.H. Wilcox proved the Rellich uniqueness theorem for the steady-state wave propagation problem for inhomogeneous anisotropic media which is uniformly propagative out of a compact set. We shall extend their method and prove Theorem 1.1.

Let $v_\pm$ be a function of $L^2_{\mathrm{loc}}(\bar{\Omega})$ satisfying (1.3). We take a cut off function

$$\beta(x) = \begin{cases} 0 & |x|\leqq R_0 \\ 1 & |x|\geqq R_0+1 , \end{cases} \tag{4.1}$$

where R_0 is a number larger than R of Aiii), and put $w_\pm=\beta(x)v_\pm$. Then

$$(\Lambda-\lambda)w_\pm = \beta(x)(\Lambda-\lambda)v_\pm+(\Lambda^0\beta)v_\pm=\Lambda^0(\nabla\beta)v_\pm . \tag{4.2}$$

Since supp $\nabla\beta\subset\{R_0\leqq|x|\leqq R_0+1\}$, $\Lambda^0(\nabla\beta)v_\pm$ has a compact support. We put

$$g_\pm = \Lambda^0(\nabla\beta)v_\pm . \tag{4.3}$$

Then we have a steady-state wave propagation problem in homogeneous media

$$(4.4)\qquad (\Lambda^0-\lambda)w_\pm = g_\pm.$$

From Ri) of the radiation condition we may assume that $v_\pm(x)$ is smooth in $|x|<R_0$. Then, $g_\pm\in C_0^\infty(\boldsymbol{R}^n)$ from (4.3) because the support of $\nabla\beta$ is compact in Ω. From Theorem 3.10 $w_\pm$ is the unique solution of (4.4) and it is represented as

$$w_\pm = w_\pm(x, \lambda) = \lim_{\varepsilon\searrow 0} \mathcal{F}^{-1}[(\Lambda^0(\xi)-(\lambda\pm i\varepsilon)I)^{-1}\hat{g}_\pm(\xi)]\,.$$

Moreover Theorem 2.3 implies

$$(4.5)\qquad w_\pm(x, \lambda) = \sum_{\gamma=1}^{p(\pm\eta)} e^{i\lambda|x||T(s)|^{-1}}|x|^{-(n-1)/2}|\lambda|^{(n-1)/2}$$
$$\cdot|K(s)|^{-1/2}|T(s)|^{-1}\hat{P}(s)\hat{g}_\pm(\lambda s)\psi_{\mathrm{sign}(\pm\lambda)}(s)|_{s=s^{(\gamma)}(\pm\eta)}+q_\pm(x, \lambda)\,.$$

We call the summation part of (4.5) the leading term and $q_\pm(x, \lambda)$ the remainder term. As in the argument in [8] the main part of ours is also to show that the leading term is equal to zero. When the system is uniformly propagative, it is known that the remainder term has the estimate which assures the square integrability. But in our case the decreasing order of the remainder term is not so good that the square integrability cannot be obtained automatically. Thus even after showing that the leading term vanishes, we need more delicate considerations in order to show that $w_\pm$ belongs to $L^2(\Omega)$. This is given in section 4.6.

In order to show that the leading term vanishes, first remark that

$$(4.6)\qquad \int_{tS^{n-1}} v_\pm^*\Lambda^0(\eta)v_\pm dS = 0$$

holds. Indeed by a fundamental calculus we can show for any u which satisfies $u(x)\in N(x)$

$$(4.7)\qquad ((\Lambda-\lambda)u, u)_{B_t}-(u, (\Lambda-\lambda)u)_{B_t} = (1/i)\int_{S^{n-1}} u^*\Lambda^0(\eta)udS\,,$$

where $B_t=\{|x|<t\}\cap\Omega$ and

$$(u, v)_D = \int_D u^*E(x)vdx \qquad \text{for} \quad D\subset\Omega\,.$$

By substituting $u=v_\pm$ into (4.7) we get (4.6). Note that we cannot substitute the expansion (4.5) into (4.6) because $q_\pm$ has singularities, we shall use a cut off function $\alpha(\xi)$ which does not have its support near $\bar{Z}_S$, and operate $\alpha(D_x)$ (pseudodifferential operator with symbol α) to the both sides of (4.4). Then we have

$$(4.8) \qquad (\Lambda^0-\lambda)\alpha(D_x)w_\pm = \alpha(D_x)g_\pm .$$

From Theorem 3.10 $\alpha(D_x)w_\pm$ is a unique solution of (4.6) satisfying the radiation condition. In section 4.3 we calculate the leading term of $\alpha(D_x)w_\pm$ by the use of the expansion formula of $\alpha(D_x)w_\pm$ which corresponds to (4.5), and in section 4.4 we show a suitable remainder estimate for large t. The conclusion here is

$$(4.9) \qquad \begin{aligned} &\int_{tS^{n-1}}[\alpha(D_x)w_\pm]^*\Lambda^0(\eta)[\alpha(D_x)w_\pm]dS \\ &= \sum_{\beta=1}^{\tilde{p}}\sum_{\gamma=1}^{\beta}\int_{S^{\beta\gamma}}\alpha(\lambda s)^2(|K(s)|^{-1/2}|T(s)|^{-1}|\hat{P}(s)\hat{g}_\pm(\lambda s)|)^2\tilde{J}(s)dS+o(1) . \end{aligned}$$

($\tilde{J}$ is a positive function defined on S. It is given later). Since $w_\pm=v_\pm$ for $|x|>R_0$, we have

$$\int_{tS^{n-1}}[\alpha(D_x)w_\pm]^*\Lambda^0(\eta)[\alpha(D_x)w_\pm]dS \to \int_{tS^{n-1}}v_\pm^*\Lambda^0(\eta)v_\pm dS = 0$$

as $\alpha\to 1$. Therefore, by taking the limit of $t\to\infty$ and $\alpha\to 1$, we can obtain that the left hand side of (4.9) tends to zero, and then we can obtain that the leading term of (4.5) is equal to zero. This part of the proof is given in section 4.5.

As stated above we prove that the remainder term belongs to L^2 in section 4.6. Here we use Lemma 3.8 and some conditions of the slowness surface.

4.2. Preparations.

Hereafter we consider only the case of $+$ radiation condition, and write $g_+=g$ for symplicity. (The same arguments are applicable for the case of $-$ radiation condition).

Let U,V be two open sets of S^{n-1} satisfying

$$\bar{Z}_S\cap S^{n-1}\subset V \quad \text{and} \quad U\subset\subset \text{Interior of } V^c$$

and let $\alpha(\xi)$ be a function of $C^\infty(\boldsymbol{R}^n)$ satisfying

$$\alpha(\xi)=\alpha_1(\xi)\alpha_2(\xi), \qquad 0\leqq\alpha_j(\xi)\leqq 1 \qquad (j=1,2),$$

$$(4.10) \qquad \alpha_1(\xi)=\begin{cases}1 & \text{if } \xi/|\xi|\in U \\ 0 & \text{if } \xi/|\xi|\in V\end{cases}$$

and

$$(4.11) \qquad \alpha_2(\xi)=\begin{cases}1 & \text{if } |\xi|>(2/3)C_S \\ 0 & \text{if } |\xi|<(1/2)C_S,\end{cases}$$

where C_S is a constant which satisfies

$$S\subset\{C_S^{-1}<|\xi|<C_S\}.$$

Such a C_S exists since the system is strongly propagative. We take this function α and consider (4.6). Since $g\in C_0^\infty(\boldsymbol{R}^n)$, $\alpha(D_x)g\in\mathcal{S}$. Thus (2.3) implies

$$\begin{aligned}\alpha(D_x)w(x,\zeta) &= u^0(x,\zeta)+\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\\ &\cdot\Big(\int_S e^{irx\cdot s}\hat{P}(s)|T(s)|^{-1}\alpha(rs)\hat{g}(rs)\phi_1(rs)dS\Big)dr,\end{aligned}\tag{4.12}$$

where $w(x,\zeta)$ is a solution of $(\Lambda^0-\zeta)w=g(\zeta\in\boldsymbol{C}\backslash\boldsymbol{R})$. Clearly $\alpha(D_x)w(x,\zeta)$ satisfies $(\Lambda^0-\zeta)\alpha w=\alpha g$. Since $w(x,\lambda+i\varepsilon)$ converges to $w_+(x,\lambda)$ in $\mathcal{S}'$, the weak continuity of pseudodifferential operators implies that $\alpha(D_x)w(x,\zeta)$ converges to $\alpha(D_x)w_+(x,\lambda)$ as $\varepsilon\to 0$ weakly in $\mathcal{S}'$. Now put $u(x,\zeta)=\alpha(D_x)w(x,\zeta)$. Then

$$u_+(x,\lambda):=\lim_{\varepsilon\searrow 0}u(x,\lambda+i\varepsilon)=\alpha(D_x)w_+(x,\lambda).$$

Next decompose $u(x,\zeta)$ as

$$u(x,\zeta)=u^0(x,\zeta)+\sum_{\beta=1}^{\tilde{\rho}}\sum_{\gamma=1}^{\beta}u^{\beta\gamma}(x,\zeta),$$

where u^0 is that of (4.12) and

$$u^{\beta\gamma}(x,\zeta)=\int_{-\infty}^{\infty}\frac{|r|^{n-1}}{r-\zeta}\Big(\int_{S^{\beta\gamma}}e^{irx\cdot s}\hat{P}(s)|T(s)|^{-1}\alpha(rs)\hat{g}_i(rs)\phi_1(rs)dS\Big)dr.\tag{4.13}$$

(For $S^{\beta\gamma}$ see [4, page 607]). Put

$$u_+^{\beta\gamma}(x,\lambda)=\lim_{\varepsilon\searrow 0}u^{\beta\gamma}(x,\lambda+i\varepsilon).$$

Then

$$u_+(x,\lambda)=\sum_{\beta=1}^{\tilde{\rho}}\sum_{\gamma=1}^{\beta}u_+^{\beta\gamma}(x,\lambda)+u_+^0(x,\lambda).$$

By (4.10) $\alpha(\xi)$ does not have its support in the neighborhood of $\bar{Z}_S$. Thus the integrand of (4.13) has its support in the interior of a smooth surface. Then we can use Proposition 3.2 of [4] to let $\varepsilon\to 0$, and we have

$$u_+^{\beta\gamma}(x,\lambda)=u_+^{\beta\gamma,\infty}(x,\lambda)+q_+^{\beta\gamma}(x,\lambda),\tag{4.14}$$

where

$$u_+^{\beta\gamma,\infty}(x,\lambda)=\begin{cases}e^{i\lambda|x||T(s)|^{-1}}|x|^{-(n-1)/2}|\lambda|^{(n-1)/2}|K(s)|^{-1/2}|T(s)|^{-1}\\ \quad\cdot\hat{P}(s)\,\alpha(\lambda s)\hat{g}(\lambda s)\psi_{\operatorname{sign}\lambda}(s)|_{s=s^{\beta\gamma}(\eta)}\qquad(\eta\in\Omega^\beta)\\ 0\qquad(\eta\notin\Omega^\beta)\end{cases}\tag{4.15}$$

and

$$|q_+^{\beta\gamma}(x, \lambda)| \leqq C|x|^{-n/2},$$

where C is independent of $\eta \in S^{n-1}$ and depends on λ and α. For simplicity we set

$$(4.16) \quad \psi_+^{\beta\gamma}(\eta, \lambda) = \begin{cases} |K(s)|^{-1/2}|T(s)|^{-1}\hat{P}(s)\hat{g}(\lambda s)\psi_{\operatorname{sign}\lambda}(s)|_{s=s^{\beta\gamma}(\eta)} & (\eta \in \Omega^\beta) \\ 0 & (\eta \notin \Omega^\beta) \end{cases}$$

Our purpose here is to prove

$$(4.17) \qquad \hat{P}(s)\hat{g}(\lambda s) = 0 \qquad \text{for} \quad s \in S \backslash \bar{Z}_S .$$

It is equivalent to

$$\psi_+^{\beta\gamma}(\eta, \lambda) = 0 \qquad \text{for} \quad \beta = 1, 2, \cdots, \tilde{\rho}, \ \gamma = 1, 2, \cdots, \beta, \ \eta \in S^{n-1} \backslash \bar{Z}_W .$$

Note the following equalities:

$$\begin{aligned}
&(v_+, (\Lambda-\lambda)v_+)_{B_t} - ((\Lambda-\lambda)v_+, v_+)_{B_t} \\
&= \int_{B_t} \{v_+^* E(x) \cdot E(x)^{-1}(\sum_{j=1}^n A_j(x)D_j v_+ + B(x)v_+) \\
&\qquad -(\sum_{j=1}^n A_j(x)D_j v_+ + B(x)v_+)^* E(x)^{-1}E(x)v_+\} dx \\
&= \int_{B_t} \{\sum_{j=1}^n [(A_j(x)v_+)^* D_j v_+ - (D_j v_+)^* A_j(x)v_+] \\
&\qquad + v_+^*(B(x)-B(x)^*)v_+\} dx \\
&= (1/i)\int_{B_t} \{\sum_{j=1}^n [(A_j(x)v_+)^* \partial_j v_+ + (\partial_j v_+)^* A_j(x)v_+] \\
&\qquad + v_+^* \cdot i(B(x)-B(x)^*)v_+\} dx \\
&= (1/i)\int_{B_t} \{\sum_{j=1}^n [v_+^* \cdot \partial_j(A_j(x)v_+) + (\partial_j v_+)^* A_j(x)v_+] \\
&\qquad - \sum_{j=1}^n v_+^*(\partial_j A_j(x))v_+ + v_+^* \cdot i(B(x)-B(x)^*)v_+\} dx \\
&= (1/i)\int_{B_t} \sum_{j=1}^n \partial_j(v_+^* A_j(x)v_+) dx \\
&= (1/i)\int_{\partial B_t} \sum_{j=1}^n v_+^* A_j(x)v_+ \cdot n_j(x) dS \\
&= (1/i)\int_{tS^{n-1}} v_*^+(\sum_{j=1}^n A_j \cdot (x_j/|x|))v_+ dS \\
&= (1/i)\int_{tS^{n-1}} v_+^* \Lambda^0(\eta)v_+ dS
\end{aligned}$$

for $t \geqq R_0+1$. Since $(\Lambda-\lambda)v_+=0$ in Ω and $v_+=w_+$ for $t \geqq R_0+1$, we have

$$\int_{tS^{n-1}} w_+^* \Lambda^0(\eta) w_+ dS = 0\,. \tag{4.18}$$

In order to prove (4.17) we prove (4.9). The equality

$$u_+(x, \lambda) = \alpha(D_x) w_+(x, \lambda) = \sum_{\beta=1}^{\tilde{\rho}} \sum_{\gamma=1}^{\beta} u_+^{\beta\gamma}(x, \lambda) + u_+^0(x, \lambda)$$

implies

$$\begin{aligned}
&\int_{tS^{n-1}} u_+^* \Lambda^0(\eta) u_+ dS \\
&= \sum_{\beta=1}^{\tilde{\rho}} \sum_{\gamma=1}^{\beta} \int_{tS^{n-1}} (u_+^{\beta\gamma})^* \Lambda^0(\eta) u_+^{\beta\gamma} dS \\
&\quad + \Big\{ \sum_{(\beta,\gamma)\neq(\tilde{\beta},\tilde{\gamma})} \int_{tS^{n-1}} (u_+^{\beta\gamma})^* \Lambda^0(\eta) u_+^{\tilde{\beta}\tilde{\gamma}} dS \\
&\quad + \sum_{\beta,\gamma} \int_{tS^{n-1}} (u_+^{\beta\gamma})^* \Lambda^0(\eta) u_+^0 dS + \sum_{\beta,\gamma} \int_{tS^{n-1}} (u_+^0)^* \Lambda^0(\eta) u_+^{\beta\gamma} dS \Big\} \\
&\quad + \int_{tS^{n-1}} (u_+^0)^* \Lambda^0(\eta) u_+^0 dS \\
&=: I_1 + I_2 + I_3\,.
\end{aligned} \tag{4.19}$$

4.3. The leading term of $\alpha(D_x)w_+$.

In this subsection we calculate I_1. Put

$$W_t^{\beta\gamma} = \{tx;\ x \in W^{\beta\gamma}\},$$
$$\Omega_t^{\beta\gamma} = \{x;\ x = \tau w,\ w \in W^{\beta\gamma},\ 0 \leqq \tau \leqq t\} \cap \Omega$$

and

$$\Gamma_t^{\beta\gamma} = \partial(\Omega_t^{\beta\gamma} \Delta B_t) \cap \bar{Z}_W\,.$$

(Δ denotes the symmetric difference). Then, since $u_+^{\beta\gamma}$ is smooth in a neighborhood of $\Omega_t^{\beta\gamma}\Delta B_t$ and $\partial(\Omega_t^{\beta\gamma}\Delta B_t)$ is piecewise smooth, it follows from the same way as that of (4.18) that

$$\begin{aligned}
I_1^{\beta\gamma} :&= \int_{tS^{n-1}} (u_+^{\beta\gamma})^* \Lambda^0(\eta) u_+^{\beta\gamma} dS \\
&= \int_{W_t^{\beta\gamma}} (u_+^{\beta\gamma})^* \Lambda^0(\bar{N}(x)) u_+^{\beta\gamma} dW \\
&\quad + \int_{\Gamma_t^{\beta\gamma} \cup t(S^{n-1}\setminus\Omega^\beta)} (u_+^{\beta\gamma})^* \Lambda^0(n(x)) u_+^{\beta\gamma} d\Gamma \\
&\quad + i\,\{(u_+^{\beta\gamma}, (\Lambda^0-\lambda)u_+^{\beta\gamma})_{B_t\setminus\Omega_t^{\beta\gamma}} - ((\Lambda^0-\lambda)u_+^{\beta\gamma}, u_+^{\beta\gamma})_{B_t\setminus\Omega_t^{\beta\gamma}} \\
&\quad - (u_+^{\beta\gamma}, (\Lambda^0-\lambda)u_+^{\beta\gamma})_{\Omega_t^{\beta\gamma}\setminus B_t} + ((\Lambda^0-\lambda)u_+^{\beta\gamma}, u_+^{\beta\gamma})_{\Omega_t^{\beta\gamma}\setminus B_t}\} \\
&=: I_{11} + I_{12} + I_{13}
\end{aligned} \tag{4.20}$$

(Ω^β is defined in [4, page 607]).

Lemma 4.1. $\lim_{t\to\infty} I_{12}=0$. *The convergence may not be uniform for* α.

Proof. Since $\Gamma_t^{\beta\gamma}\subset \bar{Z}_W$, $\alpha(s^{\beta\gamma}(\eta))$ does not have its support in a neighborhood of $\Gamma_t^{\beta\gamma}$. In fact $s^{\beta\gamma}$ maps a neighborhood of $\Gamma_t^{\beta\gamma}$ into a neighborhood of $\bar{Z}_S$, and α does not have its support there. Thus in a neighborhood of $\Gamma_t^{\beta\gamma}$

$$u_+^{\beta\gamma}(x, \lambda) = 0\,,$$

that is,

$$u_+^{\beta\gamma}(x, \lambda) = q_+^{\beta\gamma}(x, \lambda)\,.$$

On the other hand, since $\Lambda^0(\eta)$ is analytic, there exists a constant C_{Λ^0} such that $|\Lambda^0(\eta)|\leqq C_{\Lambda^0}$. Then it follows from (4.15) that

$$\begin{aligned}|I_{12}| &\leqq \text{Const.} \int_{\Gamma_t^{\beta\gamma}} |x|^{-n} d\Gamma \leqq \text{Const.} \int_{c_0 t}^{c_1 t} |x|^{-n}|x|^{n-2} d|x| \\ &= \text{Const.}\,(c_0^{-1}-c_1^{-1})t^{-1}\,.\end{aligned}$$

Hence $I_{12}\to 0$ as $t\to\infty$. Q.E.D.

Next I_{13} is treated. We decompose $u_+^{\beta\gamma}$ in the following way. Put

$$S_k^{\beta\gamma} = S_k \cap S^{\beta\gamma} \subset S$$

and

$$\chi_k^{\beta\gamma}(\xi) = \begin{cases} 1 & \text{if} \quad \xi\in C_k^{\beta\gamma} := \{\xi = rs;\ r>0,\ s\in S_k^{\beta\gamma}\} \\ 0 & \text{otherwise}\,. \end{cases}$$

Let

$$u_k^{\beta\gamma}(x, \zeta) = \mathcal{F}^{-1}[\alpha(\xi)\chi_k^{\beta\gamma}(\xi)\hat{P}_k(\xi)\hat{g}(\xi)/(\lambda_k(\xi)-\zeta)]\,.$$

Note that α does not have its support in a neighborhood of $\bar{Z}_S$. Then the following can be verified in the same way as the case of the Green function:

$$\begin{aligned}u_k^{\beta\gamma}(x, \zeta) = u_{k0}^{\beta\gamma}(x, \zeta) &+ \int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} \\ &\cdot\Big(\int_{S_k} e^{irx\cdot s}\hat{P}_k(s)|T(s)|^{-1}\alpha(rs)\chi_k^{\beta\gamma}(rs)\hat{g}(rs)dS\Big)dr\,,\end{aligned}$$

where

$$|u_{k0}^{\beta\gamma}(x, \zeta)| \leqq C_l |x|^{-l}$$

with C_l independent of $\eta=x/|x|$ and $\zeta\in\Delta$. Here $\chi_k^{\beta\gamma}(rs)=0$ if $s\notin S_k^{\beta\gamma}$ and $\chi_k^{\beta\gamma}(rs)=1$ if $s\in S_k^{\beta\gamma}$. Then

$$\begin{aligned}u_k^{\beta\gamma}(x, \zeta) = u_{k0}^{\beta\gamma}(x, \zeta) &+ \int_{-\infty}^{\infty} \frac{|r|^{n-1}}{r-\zeta} \\ &\cdot\Big(\int_{S_k^{\beta\gamma}} e^{irx\cdot s}\hat{P}_k(s)|T(s)|^{-1}\alpha(rs)\hat{g}(rs)dS\Big)dr\,.\end{aligned} \tag{4.21}$$

Thus the fact $S^{\beta\gamma}=\bigcup_{k=1}^{p} S_k^{\beta\gamma}$ and (4.10) imply

$$\sum_{k=1}^{p} u_k^{\beta\gamma}(x, \zeta) = \sum_{k=1}^{p} u_{k0}^{\beta\gamma}(x, \zeta)+u^{\beta\gamma}(x, \zeta)\,.$$

Put $u_0^{\beta\gamma}(x, \zeta)=-\sum_{k=1}^{p} u_{k0}^{\beta\gamma}(x, \zeta)$. Then

$$u^{\beta\gamma}(x, \zeta) = \sum_{k=0}^{p} u_k^{\beta\gamma}(x, \zeta)\,.$$

Moreover, since α does not have its support in the neighborhood of the boundary of $S_k^{\beta\gamma}$, (4.21) implies the existence of

$$u_{k,+}^{\beta\gamma}(x, \lambda) := \lim_{\varepsilon\searrow 0} u_k^{\beta\gamma}(x, \lambda+i\varepsilon) \qquad \text{in} \quad \mathcal{S}'$$

and the equality

$$\sum_{k=0}^{p} u_{k,+}^{\beta\gamma}(x, \lambda) = u_+^{\beta\gamma}(x, \lambda)$$

in the similar way as before. Then

$$I_{13} = \sum_{k,l=0}^{p} i\{(u_{k,+}^{\beta\gamma}, (\Lambda^0-\lambda)u_{l,+}^{\beta\gamma})_{B_t\backslash\Omega_t^{\beta\gamma}}-(\cdots)_{B_t\backslash\Omega_t^{\beta\gamma}}-(\cdots)_{\Omega_t^{\beta\gamma}\backslash B_t}+(\cdots)_{\Omega_t^{\beta\gamma}\backslash B_t}\}\,.$$

Lemma 4.2. *Let Ω_t be an open domain of Ω depending on $t>0$ which satisfies*

$$\operatorname{dist}(\Omega_t, 0)\geqq ct$$

for some constant c. Then for any k, $l\neq 0$ it holds that

$$(u_{k,+}^{\beta\gamma}, (\Lambda^0-\lambda)u_{l,+}^{\beta\gamma})_{\Omega_t} = o(1) \qquad \textit{as} \quad t\to\infty\,.$$

Proof. From (4.21)

$$|u_{k,+}^{\beta\gamma}|\leqq C|x|^{-(n-1)/2}$$

follows. Moreover (4.20) implies

$$(\Lambda^0-\zeta)u_k^{\beta\gamma}(x, \zeta) = \alpha(D_x)\chi_k^{\beta\gamma}(D_x)\hat{P}_k(D_x)g$$

($\zeta=\lambda+i\varepsilon$). Since $u_k^{\beta\gamma}(x, \zeta)\to u_{k,+}^{\beta\gamma}(x, \lambda)$ in $\mathcal{S}'$, the limit as $\varepsilon\to 0$ of the above equality is

$$(\Lambda^0-\lambda)u_{k,+}^{\beta\gamma}(x, \lambda) = \alpha(D_x)\chi_k^{\beta\gamma}(D_x)\hat{P}_k(D_x)g\,.$$

Thus

$$(u_{k,+}^{\beta\gamma}, (\Lambda^0-\lambda)u_{l,+}^{\beta\gamma}) = (u_{k,+}^{\beta\gamma}, \alpha(D_x)\chi_l^{\beta\gamma}(D_x)\hat{P}_l(D_x)g)\,.$$

Here it holds that

(4.22) $$\alpha(D_x)\chi_l^{\beta\gamma}(D_x)\hat{P}_l(D_x)g=O(|x|^{-n})$$

uniformly for η. In fact from (4.10) and (4.11) it follows that α does not have its support on the discontinuous points of $\chi_l^{\beta\gamma}$ and $\hat{P}_l$. So $\alpha(D_x)\chi_l^{\beta\gamma}(D_x)\hat{P}_l(D_x)g$ is smooth in $\boldsymbol{R}^n$. Then (4.22) follows from the integration by parts. Hence

$$|(u_{k,+}^{\beta\gamma},(\Lambda^0-\lambda)u_{l,+}^{\beta\gamma})_{\Omega_t}|\leqq\int_{ct}^{\infty}\text{Const.}|x|^{-(n-1)/2}|x|^{-n}|x|^{n-1}d|x|$$
$$=\text{Const. }t^{-(n-1)/2}. \qquad \text{Q.E.D.}$$

Lemma 4.3. $\lim_{t\to\infty}I_{13}=0$. *The limit may not be uniform for* α.

Proof. In the case of k or $l=0$ we follow the calculus of (4.18) conversely, and we have

$$i\{\cdots\}=\int_{tS^{n-1}\cup W_t^{\beta\gamma}\cup\Gamma_t^{\beta\gamma}}(u_{0,+}^{\beta\gamma})^*\Lambda^0(n(x))u_{l,+}^{\beta\gamma}dS.$$

Since (4.21) imlies $|u_{0,+}^{\beta\gamma}|\leqq C|x|^{-n}$ and $|u_{l,+}^{\beta\gamma}|\leqq C|x|^{-(n-1)/2}$, it follows that

$$|i\{\cdots\}|\leqq\text{Const.}\int_{tS^{n-1}\cup W_t^{\beta\gamma}\cup\Gamma_t^{\beta\gamma}}|x|^{-(3n-1)/2}dS$$
$$\leqq\text{Const. }t^{-(3n-1)/2}t^{n-1}=\text{Const. }t^{-(n+1)/2}.$$

Hence

$$i\{\cdots\}=o(1) \qquad \text{as} \quad t\to\infty$$

for k or $l=0$.

In the case of $k\neq0$ and $l\neq0$ we use Lemma 4.2 for $\Omega_t=\Omega_t^{\beta\gamma}\backslash B_t$. It is clear that $B_t\backslash\Omega_t^{\beta\gamma}$ and $\Omega_t^{\beta\gamma}\backslash B_t$ satisfy the condition of Lemma 4.2. Then

$$I_{13}=o(1) \qquad \text{as} \quad t\to\infty$$

follows. Q.E.D.

Lemma 4.4. *It holds that*

$$I_{11}=\int_{S^{\beta\gamma}}\alpha(\lambda s)^2|\psi_+^{\beta\gamma}(N(s),\lambda)|^2\tilde{J}(s)dS+o(1) \qquad as \quad t\to\infty,$$

where $\tilde{J}(s)$ *satisfies* $0<c_\alpha\leqq\tilde{J}(s)\leqq C_\alpha$ *for some constants* c_α *and* C_α.

Proof. From (4.14)

$$I_{11}=\int_{W_t^{\beta\gamma}}(u_+^{\beta\gamma,\infty}+q_+^{\beta\gamma})^*\Lambda^0(\bar{N}(x))(u_+^{\beta\gamma,\infty}+q_+^{\beta\gamma})dW$$
$$=\int_{W_t^{\beta\gamma}}(u_+^{\beta\gamma,\infty})^*\Lambda^0(\bar{N}(x))u_+^{\beta\gamma,\infty}dW+\int_{W_t^{\beta\gamma}}[(u_+^{\beta\gamma,\infty})^*\Lambda^0(\bar{N}(x))q_+^{\beta\gamma}$$
$$+(q_+^{\beta\gamma})^*\Lambda^0(\bar{N}(x))u_+^{\beta\gamma,\infty}+(q_+^{\beta\gamma})^*\Lambda^0(\bar{N}(x))q_+^{\beta\gamma}]dW$$
$$=:J_1+J_2.$$

Here (4.15) implies $|u_+^{\beta\gamma,\infty}|\leqq C|x|^{-(n-1)/2}$ and $|q_+^{\beta\gamma}|\leqq C|x|^{-n/2}$, where constants are independent of η. Then

$$(4.23)\qquad |J_2|\leqq \text{Const.}\int_{W_t^{\beta\gamma}}|x|^{-(n-1)/2}|x|^{-n/2}dW=\text{Const.}\ t^{-n/2}.$$

Next we consider J_1. From (4.15) and (4.16)

$$J_1=\int_{W_t^{\beta\gamma}}|x|^{-(n-1)}\alpha(\lambda s^{\beta\gamma}(\eta))^2\psi_+^{\beta\gamma}(\eta,\lambda)^*\Lambda^0(\bar N(x))\psi_+^{\beta\gamma}(\eta,\lambda)dW.$$

Here a change of variables is introduced as

$$W_t^{\beta\gamma}\ni x\mapsto s\in S^{\beta\gamma},\qquad x=tT(s).$$

The transformation is well-defined because T is a diffeomorphism from the interior of $S^{\beta\gamma}$ to the interior of $W^{\beta\gamma}$. Then the fact $\bar N(x)=\bar N(x/t)=s/|s|$ and (4.16) imply

$$\begin{aligned}\Lambda^0(\bar N(x))\psi_+^{\beta\gamma}(\eta,\lambda)&=\Lambda^0(s/|s|)\psi_+^{\beta\gamma}((tT(s))/|tT(s)|,\lambda)\\&=\Lambda^0(s/|s|)\psi_+^{\beta\gamma}(N(s),\lambda)\\&=\Lambda^0(s/|s|)|K(s)|^{-1/2}|T(s)|^{-1}\hat P(s)g(\lambda s)\psi_{\operatorname{sign}\lambda}(s)\\&=\Lambda^0(s)\hat P(s)\cdot|s|^{-1}|K(s)|^{-1/2}|T(s)|^{-1}\hat g(\lambda s)\psi_{\operatorname{sign}\lambda}(s).\end{aligned}$$

Since

$$\Lambda^0(s)\hat P(s)=\Lambda^0(s)\hat P_k(s)=\lambda_k(s)\hat P_k(s)=\hat P_k(s)$$

for $s\in S_k^{\beta\gamma}$, it follows that

$$(4.24)\qquad \Lambda^0(\bar N(x))\psi_+^{\beta\gamma}(\eta,\lambda)=|s|^{-1}\psi_+^{\beta\gamma}(N(s),\lambda).$$

Let $J^{\beta\gamma}(s)$ be the Jacobian of T^{-1}: $W^{\beta\gamma}\to S^{\beta\gamma}$. Then

$$dW_t^{\beta\gamma}=t^{n-1}J^{\beta\gamma}(s)dS^{\beta\gamma}\qquad (J^{\beta\gamma}(s)>0).$$

Thus (4.24) implies

$$(4.25)\qquad \begin{aligned}J_1=\int_{S^{\beta\gamma}}&|tT(s)|^{-(n-1)}\psi_+^{\beta\gamma}(N(s),\lambda)^*\psi_+^{\beta\gamma}(N(s),\lambda)\\&\cdot\alpha(\lambda s)^2t^{n-1}J^{\beta\gamma}(s)'dS,\end{aligned}$$

where $J^{\beta\gamma}(s)'=|s|^{-1}J^{\beta\gamma}(s)>0$,

$$=\int_{S^{\beta\gamma}}\alpha(\lambda s)^2|\psi_+^{\beta\gamma}(N(s),\lambda)|^2|T(s)|^{-(n-1)}J^{\beta\gamma}(s)'dS.$$

Hence (4.23) and (4.25) imply the conclusion of the lemma with $\tilde J(s)=|T(s)|^{-(n-1)}J^{\beta\gamma}(s)'$. Q.E.D.

Now it follows from the results of Lemmas 4.1, 4.3 and 4.4, (4.19) and (4.20) that

$$(4.26)\qquad I_1=\sum_{\beta,\gamma}\int_{S^{\beta\gamma}}\alpha(\lambda s)^2|\psi_+^{\beta\gamma}(N(s),\lambda)|^2 J(s)dS+o(1)\qquad\text{as}\quad t\to\infty\,.$$

4.4. The remainder estimates.

In this subsection we prove the following lemma.

Lemma 4.5. $\lim_{t\to\infty}(I_2+I_3)=0$.

Proof. Put

$$I_{(\beta,\gamma),(\tilde\beta,\tilde\gamma)}=\int_{tS^{n-1}}(u_+^{\beta\gamma})^*\Lambda^0(\eta)u_+^{\tilde\beta\tilde\gamma}dS\,.$$

u_+^0 is regarded as the case of $(\beta,\gamma)=0$. If one of (β,γ) and $(\tilde\beta,\tilde\gamma)=0$, then $|I_{(\beta,\gamma),(\tilde\beta,\tilde\gamma)}|\leqq ct^{-1/2}$. If $\beta\neq\tilde\beta$, it follows that

$$\text{supp}\,u_+^{\beta\gamma,\infty}(x,\lambda)\cap\text{supp}\,u_+^{\tilde\beta\tilde\gamma,\infty}(x,\lambda)=\phi\,,$$

and this implies

$$|\int_{tS^{n-1}}(u_+^{\beta\gamma})^*\Lambda^0(\eta)u_+^{\tilde\beta\tilde\gamma}dS|\leqq C\int_{tS^{n-1}}|x|^{-(n-1)/2}|x|^{-n/2}dS$$
$$=Ct^{-1/2}\,.$$

This shows

$$(4.27)\qquad I_{(\beta,\gamma),(\tilde\beta,\tilde\gamma)}=o(1)$$

for such cases. In the case of $\beta=\tilde\beta$, it follows from (4.15) and (4.16) that

$$\begin{aligned}I_{(\beta,\gamma),(\beta,\tilde\gamma)}&=\int_{tS^{n-1}}e^{i\lambda|x|(|T(s^{\beta\tilde\gamma}(\eta))|^{-1}-|T(s^{\beta\gamma}(\eta))|^{-1})}|x|^{-(n-1)}\\&\quad\cdot\alpha(\lambda s^{\beta\gamma}(\eta))\alpha(\lambda s^{\beta\tilde\gamma}(\eta))\psi_+^{\beta\gamma}(\eta,\lambda)\Lambda^0(\eta)\psi_+^{\beta\tilde\gamma}(\eta,\lambda)dS+o(1)\\&=\int_{t\Omega^\beta}e^{i\lambda|x|(\cdots)}t^{-(n-1)}\phi_+^{\beta\gamma}(\eta,\lambda)^*\hat P(s^{\beta\gamma}(\eta))\Lambda^0(\eta)\hat P(s^{\beta\tilde\gamma}(\eta))\\&\quad\cdot\phi_+^{\beta\tilde\gamma}(\eta,\lambda)dS+o(1)\,,\end{aligned}$$

where $\phi_+^{\beta\gamma}(\eta,\lambda)=|K(s)|^{-1/2}|T(s)|^{-1}\hat g(\lambda s)\psi_{\text{sign}\,\lambda}(s)\alpha(\lambda s)|_{s=s^{\beta\gamma}(\eta)}$,

$$=\int_{\Omega^\beta}e^{it\lambda(\cdots)}\{\cdots\}dS+o(1)\,.$$

Here put $\Phi(\eta)=\lambda(|T(s^{\beta\gamma}(\eta))|^{-1}-|T(s^{\beta\tilde\gamma}(\eta))|^{-1})$. Since $\Phi(\eta)$ is an analytic function in Ω^β, the set of points where grad $\Phi(\eta)=0$ is either a closed null set of

Ω^β or Ω^β itself. (Note that the consideration of the situation of the neighborhood of $\partial\Omega^\beta$ is not needed because $\alpha(\lambda s^{\beta\gamma}(\eta))=0$ near $\partial\Omega^\beta$).

Case 1) grad $\Phi(\eta)=0$ if and only if $\eta\in N$, a closed null set of Ω^β.

Let $\rho_\varepsilon(\eta)$ be a function of $C_0^\infty(\Omega^\beta)$ which satisfies

$$\rho_\varepsilon=\begin{cases}1 & \text{on } N\\ 0 & \text{near } \partial\Omega^\beta\end{cases} \quad \text{and} \quad |\operatorname{supp}\rho_\varepsilon|<\varepsilon$$

for any $\varepsilon>0$. Since the measure of N is zero, such a function surely exists. If we write

$$\begin{aligned}I_{(\beta,\gamma),(\beta,\tilde{\gamma})}&=\int_{\Omega^\beta}e^{it\lambda\Phi(\eta)}\{\cdots\}\rho_\varepsilon dS+\int_{\Omega^\beta}e^{it\lambda\Phi(\eta)}\{\cdots\}(1-\rho_\varepsilon)dS\\&=:K_1+K_2\,,\end{aligned}$$

then

$$|K_1|\leqq\int_{\operatorname{supp}\rho_\varepsilon}|\{\cdots\}|\rho_\varepsilon dS\leqq C_{\beta,\gamma,\tilde{\gamma},\alpha}\varepsilon\,.$$

For K_2, since $\operatorname{supp}_\eta\{(1-\rho_\varepsilon)\alpha(\lambda s^{\beta\gamma}(\eta))\}$ is compact and $\operatorname{grad}_\eta\Phi(\eta)\neq 0$ on this set, we have by the stationary phase method

$$|K_2|\leqq C_l(\varepsilon)t^{-l} \qquad \text{for any integer } l\,.$$

Thus

$$|I_{(\beta,\gamma),(\beta,\tilde{\gamma})}|\leqq C_l(\varepsilon)t^{-l}+C_{\beta,\gamma,\tilde{\gamma},\alpha}\varepsilon\,.$$

This implies

$$\overline{\lim_{t\to\infty}}\,|I_{(\beta,\gamma),(\beta,\tilde{\gamma})}|\leqq C_{\beta,\gamma\,\tilde{\gamma},\alpha}\varepsilon\,.$$

Since ε is any positive number, it follows

$$\overline{\lim_{t\to\infty}}\,|I_{(\beta,\gamma),(\beta,\tilde{\gamma})}|=0\,.$$

Case 2) $\operatorname{grad}_\eta\Phi(\eta)=0$ for any $\eta\in\Omega^\beta$.

For this case $\Phi(\eta)=c_0'$ (a constant on Ω^β), that is, $|T^{\beta\gamma}(s(\eta))|^{-1}-|T(s^{\beta\gamma}(\eta))|^{-1}=\lambda^{-1}c_0'=:c_0$. Put $C^\beta=\{r\eta\,;\,r>0,\eta\in\Omega^\beta\}$. Then similar to the case of the slowness surface there exists two analytic functions μ^γ and $\mu^{\tilde{\gamma}}$ with positively homogeneity of degree 1 defined on C^β such that

$$\begin{cases}W^{\beta\gamma}=\{x\in C^\beta;\ \mu^\gamma(x)=1\}\\ W^{\beta\tilde{\gamma}}=\{x\in C^\beta;\ \mu^{\tilde{\gamma}}(x)=1\}\,.\end{cases}$$

Note that $x=|x|\eta\in W^{\beta\gamma}$ is equivalent to $\mu^\gamma(|x|\eta)=1$, and this is equivalent to

$|x|=1/\mu^{\gamma}(\eta)$. Similar facts hold for the case of $\tilde{\gamma}$. Then, $T(s^{\beta\gamma}(\eta))\in W^{\beta\gamma}$ implies $|T(s^{\beta\gamma}(\eta))|^{-1}=\mu^{\gamma}(\eta)$. Similarly $|T(s^{\beta\tilde{\gamma}}(\eta))|^{-1}=\mu^{\tilde{\gamma}}(\eta)$. Hence

$$\mu^{\tilde{\gamma}}(\eta)-\mu^{\gamma}(\eta)=c_0\,.$$

The homogeneity of $\mu^{\tilde{\gamma}}$ and μ^{γ} gives

$$\mu^{\tilde{\gamma}}(x)=\mu^{\gamma}(x)+c_0|x|\,. \tag{4.28}$$

Note the fact that $x\in W^{\beta\gamma}$ is equivalent to $\nabla\mu^{\gamma}(x)\in S^{\beta\gamma}$. Then for $\eta\in\Omega^{\beta}$

$$s^{\beta\gamma}(\eta)=\nabla\mu^{\gamma}(r^{\gamma}(\eta)\cdot\eta)=\nabla\mu^{\gamma}(\eta)\,,$$

where $r^{\gamma}(\eta)$ denotes the length of a point vector x of $W^{\beta\gamma}$ which is parallel to η. Here the fact that $\nabla\mu^{\gamma}$ is positively homogeneous of degree 0 is used. Similarly

$$s^{\beta\tilde{\gamma}}(\eta)=\nabla\mu^{\tilde{\gamma}}(\eta)\,.$$

From (4.28)

$$\nabla\mu^{\tilde{\gamma}}(x)=\nabla\mu^{\gamma}(x)+c_0x/|x|\,.$$

Thus

$$s^{\beta\tilde{\gamma}}(\eta)=\nabla\mu^{\gamma}(\eta)+c_0\eta=s^{\beta\gamma}(\eta)+c_0\eta\,.$$

Hence

$$s^{\beta\tilde{\gamma}}(\eta)-s^{\beta\gamma}(\eta)=c_0\eta\,.$$

Then

$$\begin{aligned} I_{(\beta,\gamma),(\beta,\tilde{\gamma})}&=\int_{\Omega^{\beta}}e^{it\lambda c_0}\phi_{+}^{\beta\gamma}(\eta,\lambda)^{*}\hat{P}(s^{\beta\gamma}(\eta))\\ &\qquad\cdot\Lambda^0((s^{\beta\tilde{\gamma}}(\eta)-s^{\beta\gamma}(\eta))/c_0)\hat{P}(s^{\beta\tilde{\gamma}}(\eta))\phi_{+}^{\beta\tilde{\gamma}}(\eta,\lambda)dS+o(1)\\ &=\int_{\Omega^{\beta}}e^{it\lambda c_0}c_0^{-1}\phi_{+}^{\beta\gamma}(\eta,\lambda)^{*}\{\hat{P}(s^{\beta\gamma}(\eta))[\Lambda^0(s^{\beta\tilde{\gamma}}(\eta))\\ &\qquad-\Lambda^0(s^{\beta\gamma}(\eta))]\hat{P}(s^{\beta\gamma}(\eta))\}\,\phi_{+}^{\beta\tilde{\gamma}}(\eta,\lambda)dS+o(1)\,. \end{aligned}$$

Here it holds that

$$\begin{aligned} &\hat{P}(s^{\beta\gamma}(\eta))[\Lambda^0(s^{\beta\gamma}(\eta))-\Lambda^0(s^{\beta\tilde{\gamma}}(\eta))]\hat{P}(s^{\beta\tilde{\gamma}}(\eta))\\ &=\hat{P}(s^{\beta\gamma}(\eta))\sum_{|\nu|=1}^{p}\lambda_{\nu}(s^{\beta\gamma}(\eta))\hat{P}_{\nu}(s^{\beta\gamma}(\eta))\hat{P}(s^{\beta\tilde{\gamma}}(\eta))\\ &\qquad-\hat{P}(s^{\beta\gamma}(\eta))\sum_{|\nu|=1}^{p}\lambda_{\nu}(s^{\beta\tilde{\gamma}}(\eta))\hat{P}_{\nu}(s^{\beta\tilde{\gamma}}(\eta))\hat{P}(s^{\beta\tilde{\gamma}}(\eta))\,. \end{aligned}$$

Suppose $s^{\beta\gamma}(\eta)\in S_k$ and $s^{\beta\tilde{\gamma}}(\eta)\in S_{\tilde{k}}$. Then

$$\begin{aligned} I_{(\beta,\gamma),(\beta,\tilde{\gamma})}&=\hat{P}_k(s^{\beta\gamma}(\eta))\lambda_k(s^{\beta\gamma}(\eta))\hat{P}_{\tilde{k}}(s^{\beta\tilde{\gamma}}(\eta))\\ &\qquad-\hat{P}_k(s^{\beta\gamma}(\eta))\lambda_{\tilde{k}}(s^{\beta\gamma}(\eta))\hat{P}_{\tilde{k}}(s^{\beta\tilde{\gamma}}(\eta)) \end{aligned}$$

$$= \hat{P}_k(s^{\beta\gamma}(\eta))\hat{P}_{\tilde{k}}(s^{\tilde{\beta}\tilde{\gamma}}(\eta)) - \hat{P}_k(s^{\beta\gamma}(\eta))\hat{P}_{\tilde{k}}(s^{\tilde{\beta}\tilde{\gamma}}(\eta))$$
$$(\lambda_k(s^{\beta\gamma}(\eta)) = 1,\ \lambda_{\tilde{k}}(s^{\tilde{\beta}\tilde{\gamma}}(\eta)) = 1)$$
$$= 0 .$$

Thus (4.19), (4.27) and the results of case 1) and case 2) imply the conclusion of the lemma. Q.E.D.

From (4.19), (4.27) and Lemma 4.5 we have

$$\int_{tS^{n-1}} (u_+)^* \Lambda^0(\eta) u_+ dS = \sum_{\beta=1}^{\tilde{\rho}} \sum_{\gamma=1}^{\beta} \int_{S^{\beta\gamma}} \alpha(\lambda s)^2 |\psi_+^{\beta\gamma}(N(s), \lambda)|^2 \tilde{J}(s) dS + o(1) .$$

This shows (4.9). (See (4.16)).

4.5. Proof of (4.17).

In this subsection we complete the proof of (4.17). From (4.18) we have

$$\begin{aligned}
&\int_{tS^{n-1}} (u_+)^* \Lambda^0(\eta) u_+ dS \\
&= \int_{tS^{n-1}} \alpha(D_x) w_+{}^* \Lambda^0(\eta) \alpha(D_x) w_+ dS \\
&= \int_{tS^{n-1}} \alpha(D_x) w_+{}^* \Lambda^0(\eta) \alpha(D_x) w_+ dS - \int_{tS^{n-1}} w_+{}^* \Lambda^0(\eta) w_+ dS \\
&= \int_{tS^{n-1}} \alpha((D_x) w_+ - w_+)^* \Lambda^0(\eta) w_+ dS \\
&\qquad + \int_{tS^{n-1}} (\alpha(D_x) w_+ - w_+)^* \Lambda^0(\eta) \alpha(D_x) w_+ dS \\
&=: I_1' + I_2' .
\end{aligned}$$

First we treat I_2'. The expansion formulas of $(\alpha(D_x)-1)w_+$ and $\alpha(D_x)w_+$ corresponding to (4.5) give

$$\begin{aligned}
&(\alpha(D_x)-1) w_+{}^* \Lambda^0(\eta) \alpha(D_x) w_+ \\
&= \sum_{(\beta,\gamma),(\tilde{\beta},\tilde{\gamma})} e^{i\lambda(|T(s^{\tilde{\beta}\tilde{\gamma}}(\eta))|^{-1} - |T(s^{\beta\gamma}(\eta))|^{-1})|x|} \\
&\qquad \cdot (\alpha(\lambda s^{\beta\gamma}(\eta)) - 1) \psi_+^{\beta\gamma}(\eta, \lambda)^* \Lambda^0(\eta) \psi_+^{\tilde{\beta}\tilde{\gamma}}(\eta, \lambda) \alpha(\lambda s^{\tilde{\beta}\tilde{\gamma}}(\eta)) |x|^{-(n-1)} \\
&+ \sum_{\beta,\gamma} e^{-i\lambda|T(s^{\beta\gamma}(\eta))|^{-1}|x|} (\alpha(\lambda s^{\beta\gamma}(\eta)) - 1) \psi_+^{\beta\gamma}(\eta, \lambda)^* \Lambda^0(\eta) \tilde{q}_+^{\alpha}(x, \lambda) |x|^{-(n-1)/2} \\
&+ \sum_{\beta,\gamma} e^{i\lambda|T(s^{\tilde{\beta}\tilde{\gamma}}(\eta))|^{-1}|x|} q_+^{\alpha}(x, \lambda)^* \Lambda^0(\eta)\, \alpha(\lambda s^{\tilde{\beta}\tilde{\gamma}}(\eta)) \psi_+^{\tilde{\beta}\tilde{\gamma}}(\eta, \lambda) |x|^{-(n-1)/2} \\
&+ q_+^{\alpha}(x, \lambda)^* \Lambda^0(\eta) \tilde{q}_+^{\alpha}(x, \lambda) ,
\end{aligned}$$

where q_+^{α} and $\tilde{q}_+^{\alpha}$ are the remainder terms of $(\alpha(D_x)-1)w_+$ and $\alpha(D_x)w_+$, respectively, and they have the estimates

$$|q_+^\alpha(x, \lambda)| \leqq C_{\alpha,p}(\eta)|x|^{-(n-1)-\nu_p} \qquad (\nu_p>0)$$

for any p with $1\leqq p<2$ if $n>3$, some $p>1$ if $n=3$ and $C_{\alpha,p}(\eta)\in L^p(S^{n-1})$, and

$$|\tilde{q}_+^\alpha(x, \lambda)| \leqq C_\alpha |x|^{-n/2}$$

for some constant C_α independent of η. Then

$$\begin{aligned}|I_2'| \leqq C_{\Lambda^0} \sum \int_{tS^{n-1}} &(\alpha(\lambda s^{\beta\gamma}(\eta))-1)\alpha(\lambda s^{\tilde{\beta}\tilde{\gamma}}(\eta))|(\psi_+^{\beta\gamma})^*\psi_+^{\tilde{\beta}\tilde{\gamma}}|\,|x|^{-(n-1)}dS \\ &+C_{\Lambda^0}\sum\int_{tS^{n-1}}(\alpha(\lambda s^{\beta\gamma}(\eta))-1)|(\psi_+^{\beta\gamma})^*|C_\alpha|x|^{-(n-1)-1/2}dS \\ &+C_{\Lambda^0}\sum\int_{tS^{n-1}}\alpha(\lambda s^{\beta\gamma}(\eta))|\psi_+^{\beta\gamma}|\cdot C_{\alpha,p}(\eta)|x|^{-(n-1)-\nu_p}dS \\ &+C_{\Lambda^0}\int_{tS^{n-1}}C_\alpha C_{\alpha,p}(\eta)|x|^{-(n-1)-1/2-\nu_p}dS\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\overline{\lim_{t\to\infty}}\,|I_2'| \\ (4.29)\qquad &\leqq C_{\Lambda^0}\sum\int_{S^{n-1}}(\alpha(\lambda s^{\beta\gamma}(\eta))-1)\alpha(\lambda s^{\tilde{\beta}\tilde{\gamma}}(\eta))|(\psi_+^{\beta\gamma})^*\psi_+^{\tilde{\beta}\tilde{\gamma}}|\,dS \\ &=C_{\Lambda^0}\sum_{\beta=1}^{\tilde{p}}\sum_{\gamma=1}^{\beta}\int_{\Omega^\beta}(\alpha(\lambda s^{\beta\gamma}(\eta))-1)\alpha(\lambda s^{\beta\gamma}(\eta))|\psi_+^{\beta\gamma}|^2dS\end{aligned}$$

since $|x|=t$.

Next I_1' is treated. By the same calculus of (4.18) we have

$$\begin{aligned}I_1' &= \int_{tS^{n-1}}(\alpha(D_x)-1)w_+{}^*\Lambda^0(\eta)w_+dS \\ &= i\{((\alpha(D_x)-1)w_+,\,(\Lambda^0-\lambda)w_+)_{B_t}-((\Lambda^0-\lambda)(\alpha(D_x)-1)w_+,\,w_+)_{B_t}\} \\ &= i\{((\alpha(D_x)-1)w_+,\,g)_{B_t}-((\alpha(D_x)-1)g,\,w_+)_{B_t}\}\,.\end{aligned}$$

Here note that $g\in C_0^\infty$, w_+ satisfies Ri) and $\alpha(D_x)=\alpha(D_x)^*$ since α is real valued. Then

$$\begin{aligned}\lim_{t\to\infty} I_1' &= i\{((\alpha(D_x)-1)w_+,\,g)_{R^n}-((\alpha(D_x)-1)g,\,w_+)_{R^n}\} \\ (4.30)\qquad &= i\{(w_+,\,(\alpha(D_x)-1)g)_{R^n}-((\alpha(D_x)-1)g,\,w_+)_{R^n}\} \\ &= 2\,\mathrm{Im}\,((\alpha(D_x)-1)g,\,w_+)_{R^n}\,.\end{aligned}$$

Now by changing U and V of (4.10) and C_S of (4.11), we can take a sequence of functions $\{\alpha^{(m)}(\xi)\}$ with the properties (4.10) and (4.11) which also satisfies

$$(4.31)\qquad \alpha^{(m)}\nearrow 1 \qquad \text{on} \quad \boldsymbol{R}^n\backslash\bar{Z}_S$$

and

$$|\nabla\alpha^{(m)}(\xi)|\leqq \text{Const. } m\,. \tag{4.32}$$

If we take the superior limits of the both side of (4.9) as $t\to\infty$, then it follows from (4.29) and (4.30) that

$$\begin{aligned}&\sum_{\beta=1}^{\rho}\sum_{\gamma=1}^{\beta}\widetilde{\int}_{S^{\beta\gamma}}\alpha^{(m)}(\lambda s)^2|\psi_+^{\beta\gamma}(N(s),\lambda)|^2\tilde{J}(s)dS\\ &\leqq\sum_{\beta=1}^{\tilde{\rho}}\sum_{\gamma=1}^{\beta}C_{\Lambda^0}\int_{\Omega^\beta}(\alpha^{(m)}(\lambda s^{\beta\gamma}(\eta))-1)\alpha^{(m)}(\lambda s^{\beta\gamma}(\eta))|\psi_+^{\beta\gamma}|^2dS\\ &\qquad+2\,\mathrm{Im}\,((\alpha^{(m)}(D_x)-1)g,\,w_+)_{\boldsymbol{R}^n}\,.\end{aligned} \tag{4.33}$$

Then from (4.31) and the Lebesgue convergence theorem we have

$$C_{\Lambda^0}\int_{\Omega^\beta}(\alpha^{(m)}(\lambda s^{\beta\gamma}(\eta))-1)\alpha^{(m)}(\lambda s^{\beta\gamma}(\eta))|\psi_+^{\beta\gamma}|^2dS\to 0 \tag{4.34}$$

as $m\to\infty$ for any β and γ.

On the other hand the radiation condition implies

$$\langle x\rangle^{-s}w_+\in L^2(\boldsymbol{R}^n)\qquad\text{for some}\quad s<1\,.$$

Thus

$$|((\alpha^{(m)}(D_x)-1)g,\,w_+)_{\boldsymbol{R}^n}|\leqq\|\langle x\rangle^s(\alpha^{(m)}(D_x)-1)g\|_{L^2}\|\langle x\rangle^{-s}w_+\|_{L^2}\,.$$

Hence it is sufficient to prove

$$\|\langle x\rangle^s(\alpha^{(m)}(D_x)-1)g\|_{L^2}=\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_s\to 0$$

as $m\to\infty$. It follows from the interval inequality that

$$\begin{aligned}&\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_s\\ &\quad\leqq\delta\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_1+\delta^{-s/(1-s)}\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_0\end{aligned}$$

for any δ with $0<\delta<1$. Since $\hat{g}\in\mathcal{S}$, it holds that

$$|\hat{g}(\xi)|+\sum_{j=1}^{n}|D_j\hat{g}(\xi)|\leqq C_l\langle\xi\rangle^{-l}$$

for any integer l. If we take U, V and C_S suitably for each m, we have

$$\begin{aligned}&\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_0\\ &\quad\leqq C_0(C_S^2+\int_{S^{n-1}}|\alpha^{(m)}(\omega)-1|\,d\omega\int_0^\infty\langle\xi\rangle^{-l}|\xi|^{n-1}d|\xi|)\\ &\quad\leqq C_0(m^{-2}+m^{-1})\leqq C_0m^{-1}\end{aligned}$$

and

$$\begin{aligned}&\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_1\\&\leqq C_1 m(C_S^2+\int_{\mathrm{supp}(\alpha^{(m)}-1)} d\omega\cdot\int_0^\infty \langle\xi\rangle^{-l}|\xi|^{n-1}d|\xi|)\\&\leqq C_1 m(m^{-2}+m^{-1})\leqq C_1\,.\end{aligned}$$

Hence, if we put $\delta=m^{-\sigma}$ $(0<\sigma<(1-s)/s)$, we have

$$\|(\alpha^{(m)}(\cdot)-1)\hat{g}(\cdot)\|_s\leqq C_1 m^{-\sigma}+C_0 m^{\sigma s/(1-s)}m^{-1}\to 0 \quad \text{as} \quad m\to\infty\,.$$

Then

$$((\alpha^{(m)}(D_x)-1)g,\, w_+)_{R^n}\to 0 \qquad \text{as} \quad m\to\infty\,. \tag{4.35}$$

It follows from (4.34) and (4.35) that the limit of (4.33) as $m\to\infty$ gives

$$\sum_{\beta=1}^{\tilde{\rho}}\sum_{\gamma=1}^{\beta}\int_{S^{\beta\gamma}}|\psi_+^{\beta\gamma}(N(s),\lambda)|^2\tilde{J}(s)dS\leqq 0\,.$$

Since $\tilde{J}(s)>0$, we have

$$|\psi_+^{\beta\gamma}(N(s),\lambda)|^2=0 \qquad \text{for any} \quad s\in S^{\beta\gamma}\,.$$

Since N is bijective from $S^{\beta\gamma}$ to Ω^β, we have

$$\psi_+^{\beta\gamma}(\eta,\lambda)=0 \qquad \text{for} \quad \eta\in\Omega^\beta\,,$$

which shows (4.17).

4.6. Proof of $v_+\in L^2(\Omega)$.

The final step is the proof of $v_+\in L^2(\Omega)$.
The Fourier transformation of (4.4) gives

$$(\Lambda^0(\xi)-\lambda I)\hat{w}_+(\xi)=\hat{g}(\xi)\,.$$

If $\xi\neq\lambda s$ for $s\in S$, $\Lambda^0(\xi)-\lambda I$ is non-singular. So if we multiply the both sides by $(\Lambda^0(\xi)-\lambda I)^{-1}$ from the left, we have

$$\hat{w}_+(\xi)=(\Lambda^0(\xi)-\lambda I)^{-1}\hat{g}(\xi) \qquad \text{for} \quad \xi\neq\lambda s\,.$$

Hence $\hat{w}_+(\xi)$ have a decomposition:

$$\hat{w}_+(\xi)=\hat{w}_1(\xi)+\hat{w}_2(\xi)\,,$$

where $\hat{w}_1(\xi)=(\Lambda^0(\xi)-\lambda I)^{-1}\hat{g}(\xi)$ and $\hat{w}_2(\xi)$ is a distribution which satisfies

$$\mathrm{supp}\, w_2\subset\lambda S\,. \tag{4.36}$$

Lemma 4.6. *If $\hat{w}_1(\xi)\in L^2(\boldsymbol{R}^n)$, then $\hat{w}_2(\xi)\equiv 0$.*

Proof. From (4.17) it follows that the principal part of $\alpha(D_x)w_+$ also vanishes. The estimates of the remainder term of $\alpha(D_x)w_+$ gives $\alpha(D_x)w_+\in L^2(\boldsymbol{R}^n)$. By the Fourier transformation we have

$$\alpha(\xi)\hat{w}_+(\xi)=\alpha(\xi)\hat{w}_1(\xi)+\alpha(\xi)\hat{w}_2(\xi)\in L^2(\boldsymbol{R}^n)\,.$$

Thus, if $\hat{w}_1(\xi)$ belongs to $L^2(\boldsymbol{R}^n_\xi)$, $\alpha(\xi)\hat{w}_2(\xi)$ also belongs to $L^2_\xi(\boldsymbol{R}^n)$. But it follows from (4.36) that the measure of supp $\alpha(\xi)\hat{w}_2(\xi)=0$. Then $\alpha(\xi)\hat{w}_2(\xi)=0$. Since α is an arbitrally given function satisfying supp $\alpha\subset\boldsymbol{R}^n\setminus\bar{Z}_S$,

$$\text{supp}\,\hat{w}_2\subset\bar{Z}_S\,. \tag{4.37}$$

From (4.36) and (4.37) supp $\hat{w}_2\subset\lambda S\cap\bar{Z}_S=\lambda Z_S$. On the other hand from (4.17) the principal part of w_+ vanishes. Thus

$$|w_+(x)|\leqq C_p(\eta)|x|^{-(n-1)/2-\nu_p}\qquad(\nu_p>0)$$

for any p with $1\leqq p<2$ if $n>3$, and for some $p<1$ if $n=3$ and $C_p(\eta)\in L^p(S^{n-1})$. From this it is easily proved that $\hat{w}_2\in H^{-s}$ for $s<1$. Then the conclusion follows from Lemma 3.8. Q.E.D.

Lemma 4.7. $\hat{w}_1(\xi)\in L^2(\boldsymbol{R}^n_\xi)$.

Proof. Note that

$$(\Lambda^0(\xi)-\lambda I)^{-1}\hat{g}(\xi)=\sum_{|k|=0}^{\rho}\frac{\hat{P}_k(\xi)\hat{g}(\xi)}{\lambda_k(\xi)-\lambda}\,.$$

If $\xi\in\bar{Z}_S$, ξ has a respresentation: $\xi=rs$ $(r>0,\ s\in S_k)$ for $k=1,2,\cdots,\rho$. So from (4.17)

$$\begin{aligned}\hat{P}_k(\xi)\hat{g}(\xi)&=\hat{P}_k(\lambda s)\hat{g}(\lambda s)+(\lambda_k(rs)-\lambda_k(\lambda s))\int_0^1 s\cdot\vec{\nabla}(\hat{P}_k\hat{g})((r-\lambda)s\theta+\lambda s)d\theta\\&=\hat{P}_k(s)\hat{g}(\lambda s)+(r-\lambda)\int_0^1 s\cdot\vec{\nabla}(\hat{P}_k\hat{g})((r-\lambda)\theta s+\lambda s)d\theta\\&=(r-\lambda)\int_0^1 s\cdot\vec{\nabla}(\hat{P}_k\hat{g})((r-\lambda)\theta s+\lambda s)d\theta\end{aligned}$$

for $k>0$. The assumption Sv) implies

$$|\nabla(\hat{P}_k\hat{g})(\xi)|\leqq C\,\text{dist}\,(\xi,\bar{Z}^{(1)}_S)^{-1}\,,$$

where $\bar{Z}^{(1)}_S=\{rs;\ r\in\boldsymbol{R},\ s\in Z^{(1)}_S\}$. Hence, for $k>0$,

$$\left|\frac{\hat{P}_k(\xi)g(\xi)}{\lambda_k(\xi)-\lambda}\right|\leqq C\,\text{dist}\,(\xi,\bar{Z}^{(1)}_S)^{-1} \tag{4.38}$$

for $\xi\in\boldsymbol{R}^n\backslash\bar{Z}_S$. When $k<0$, the representation $\xi=-rs(r>0, s\in S_{-k})$ is used. When $k=0$, $\lambda_0(\xi)=0$ and $-\lambda^{-1}\hat{P}_0(\xi)\hat{g}(\xi)$is bounded in the neighborhood of λS. So (4.38) holds for any k. On the other hand $\hat{g}\in\mathcal{S}(\boldsymbol{R}^n)$ and $S\subset\{C_S^{-1}\leqq|\xi|\leqq C_S\}$ imply

$$\int_{|\xi|\geqq R}|\hat{w}_1(\xi)|^2d\xi<\infty$$

for sufficiently large R. Then it is left only to prove that $\hat{w}_1$ is square integrable in a neighborhood of $\lambda S\cap\bar{Z}_S^{(1)}=\lambda Z_S^{(1)}$. Let $Q(\lambda, \xi)$ be a minimal polynomial of $\det(\lambda I-\Lambda^0(\xi))$. Then

$$(\Lambda^0(\xi)-\lambda I)^{-1}=\frac{L(\lambda, \xi)}{Q(\lambda, \xi)},$$

where L is an $m\times m$ matrix whose elements are polynomials of λ and ξ (see J.R. Schulenberger and C.H. Wilcox [10]). $Q(\lambda, \xi)$ is given as

$$Q(\lambda, \xi)=(\lambda-\lambda_p(\xi))\cdots(\lambda-\lambda_1(\xi))\lambda^{\pi(r)}(\lambda-\lambda_{-1}(\xi))\cdots(\lambda-\lambda_{-p}(\xi)).$$

(r denotes the deficit.)

Then for $\xi\in\bar{Z}_S^{(1)}$ with $\xi=rs_0$ ($r>0$, $s_0\in Z_S^{(1)}$)

$$|(\Lambda^0(\xi)-\lambda I)^{-1}\hat{g}(\xi)|\leqq C(r-\lambda)^{-l}, \tag{4.39}$$

where l is the multiplicity of S at s_0. Then another coordinate system is introduced in a neighborhood of $\lambda s_0\in\lambda Z_S^{(1)}$, which satisfies

$$Z_S^{(1)}=\{x_1=\cdots=x_d=0,\ x_{d+1}=1\}\ (\dim Z_S^{(1)}=n-1-d),$$
$$\xi=rs_0=(\underbrace{0, \cdots, 0}_{d}, r, x_{d+2}, \cdots, x_n)$$

and

$$\operatorname{dist}(\xi, \bar{Z}_S^{(1)})\sim|x_1|+\cdots+|x_d|.$$

Then (4.38) and (4.39) imply

$$|(\Lambda_0(\xi)-I)^{-1}\hat{g}(\xi)|\leqq C(|x_1|+\cdots+|x_d|+|x_{d+1}-\lambda|^l)^{-1}.$$

From Si) $d\geqq(n+3)/2\geqq3$. Thus from some fundamental calculus we have

$$(|x_1|+\cdots+|x_d|+|x_{d+1}-\lambda|^l)^{-1}\in L^2.$$

Hence

$$\hat{w}_1(\xi)=(\Lambda^0(\xi)-\lambda I)^{-1}\hat{g}(\xi)\in L^2(\boldsymbol{R}_\xi^2). \qquad \text{Q.E.D.}$$

Then Lemma 4.6 implies $\hat{w}_2(\xi)\equiv0$, and then $\hat{w}_+(\xi)=\hat{w}_1(\xi)+\hat{w}_2(\xi)\in L^2(\boldsymbol{R}^n)$. So $w_+=\beta(x)v_+\in L^2(\boldsymbol{R}^n)$. Since $v_+\in L^2_{\text{loc}}(\Omega)$, $v_+\in L^2(\Omega)$ follows.

Thus the proof of Theorem 1.1 is complete.

5. The limiting absorption principle

Our proof of the limiting absorption principle is carried out under the same line of that of J.R. Schulenberger and C.H. Wilcox [9]. However there are some differences because our system is not uniformly propagative and Ω has boundary. Especially the existence of singularities and non-convexity of the slowness surface gives the large difference in the proof. We shall state such differences mainly.

The assumption Ai), Aii), Aiv) and Avi) imply the self-adjointness of Λ in a Hilbert space $\mathcal{H}=L^2(\Omega)$ with the inner product

$$(u, v)_{\mathcal{H}} = \int_\Omega u^* E(x) v dx$$

(P.D. Lax and R.S. Phillips [5]). So all of the spectra of Λ are real, and then there exists for $\zeta \in \boldsymbol{C}\setminus\boldsymbol{R}$ the resolvent $R(\zeta)=(\Lambda-\zeta I)^{-1}$.

Let $\mathcal{H}_{\mathrm{loc}}$ denotes a functional space

$$\mathcal{H}_{\mathrm{loc}} = \{u;\ \text{measurable and } ||u||_K = \int_K u^* E(x) u dx < \infty \quad \text{for any bounded subset } K \text{ of } \Omega\}.$$

A sequence $\{f_n\}$ of $\mathcal{H}_{\mathrm{loc}}$ is said to converge to f when

$$\lim_{n\to\infty} ||f_n - f||_K = 0$$

for any bounded subset K.

The limiting absorption principle is the following theorem.

Theorem 5.1. 1) *Let $\lambda \in \boldsymbol{R}^1\setminus(\sigma_p(\Lambda)\cup\{0\})$ and let $f\in L^2_{\mathrm{vox}}$. Then the limit*

$$\lim_{\sigma\searrow 0} v(\cdot, \lambda\pm i\sigma) = v_\pm(\cdot, \lambda) \tag{5.1}$$

exists in $\mathcal{H}_{\mathrm{loc}}$, where $v(\cdot, \zeta)=R(\zeta)f$. Moreover $v_\pm(\cdot, \lambda)$ is the solution of the steadystate wave propagation problem for the frequency λ:

$$\begin{cases} (\Lambda-\lambda)v_\pm = f & \text{for} \quad x\in\Omega \\ v_\pm(x, \lambda)\in N(x) & \text{for} \quad x\in\partial\Omega \\ v \quad \text{satisfies } \pm \text{ radiation condition}. \end{cases} \tag{5.2}$$

2) *Let $\Delta=[a, b]\subset\boldsymbol{R}^1\setminus(\sigma_p(\Lambda)\cup\{0\})$. Then the convergence of (5.1) is uniform for $\lambda\in\Delta$ in $\mathcal{H}_{\mathrm{loc}}$.*

In order to prove Theorem 5.1 we need some preparations. First we recall the following results.

Theorem 5.2. *Suppose that* λ *is in* $\boldsymbol{R}^1\backslash\{0\}$. *Let* v *be a function in* $L^2(\boldsymbol{R}^n)$ *which satisfies*

$$(\Lambda^0-\lambda)v = 0 \qquad \text{for} \quad |x|>\rho\,.$$

Then v *itself is zero for* $|x|>\rho$.

For the proof we refer to N. Iwasaki [3].

We denote $\{|x|<R\}\cap\Omega$ by Ω_R.

Theorem 5.3. 1) *Let* $\lambda\in\sigma_p(\Lambda)\backslash\{0\}$ *and let* $v\in\mathcal{H}$ *be a corresponding eigenfunction. Then*

$$\operatorname{supp} v\subset\Omega_R \ \ (R \text{ is of } Aiii))\,.$$

2) $\sigma_p(\Lambda)$ *is discrete, that is, there are only a finite number of eigenfunction of* Λ *in any finite interval of* $\boldsymbol{R}^1$. *Moreover each nonzero eigenvalue of* Λ *has finite multiplicity.*

The proof can be carried out in the same line of that of Theorem 2.1 of [9] by using our assumption Avii) instead of the result of their Theorem 1.1.

For the proof of Theorem 5.1 we make use of the Hilbert space $\mathcal{K}_s$ defined by

$$\mathcal{K}_s = \{u\in\mathcal{H}_{\mathrm{loc}};\, (1+|x|)^{-s/2}u\in\mathcal{H}\} \qquad (s>0)$$

and

$$(u,\, v)_s = \int_\Omega u^*E(x)v(1+|x|)^{-s}dx, \ \ |u|_s = (u,\, u)_s^{1/2}\,.$$

We state some lemmas which will be used in the proof later.

Lemmas 5.4. *If* $u_n\to u$ *weakly in* $\mathcal{K}_s$, *then* $u_n\to u$ *weakly in* $\mathcal{H}_{\mathrm{loc}}$.

Here a sequence of functions $u_n\in\mathcal{H}_{\mathrm{loc}}$ is said to converge weakly in $\mathcal{H}_{\mathrm{loc}}$ to a function $u\in\mathcal{H}_{\mathrm{loc}}$ when for each f and each bounded set $K\subset\Omega$

$$\lim_{n\to\infty}(u_n, f)_K = (u, f)_K\,. \tag{5.3}$$

The proof can be carried out in the same line of that of Lemma 3.5 of [9] if we take $B_N=\{|x|<N\}\cap\Omega$, and replace $\mathcal{K}_8$ with the above $\mathcal{K}_s$.

Let $\beta(x)\in C^\infty(\boldsymbol{R}^n)$ be a function of (4.1). Let $f\in\mathcal{H}$ satisfy

$$\operatorname{supp} f\subset\operatorname{supp}(E-I)\subset B_{R_0}\,.$$

Then $v(\cdot,\, \zeta, f):=R(\zeta)f$ satisfies the identity

$$(\Lambda-\zeta)(\beta v) = (\Lambda^0\beta)(x)v(x,\, \zeta)\,.$$

Then it holds that

$$\beta(x)v(x, \zeta) = G(\cdot, \zeta)*(\Lambda^0\beta)(\cdot)v(\cdot, \zeta). \tag{5.4}$$

(Note that $\Lambda=\Lambda^0$ on supp β).

The proof of the next lemma is essentially different from the corresponding lemma of [9]. The difference comes from the existence of the singularity and the non-convexity of the slowness surface. So in the essential part of the proof we use the result of our previous paper [4].

Lemma 5.5. *Let* $\lambda\in \boldsymbol{R}^1\backslash(\sigma_p(\Lambda)\cup\{0\})$. *Put*

$$\Sigma_\pm(\gamma):=\{\zeta\in\boldsymbol{C};\ |\zeta-\lambda|<\gamma,\ \operatorname{Im}\zeta\gtrless 0\}.$$

Choose γ *so that* $0\notin\Sigma_\pm(\gamma)$. *Then for each* $\varepsilon>0$ *there exists a constant* $R_1=R_1(\varepsilon, R_0, n, \lambda, \gamma)$ *such that*

$$\int_{|x|\geqq R} v(x, \zeta, f)^*E(x)v(x, \zeta, f)(1+|x|)^{-s}dx<\varepsilon|v(\cdot, \zeta, f)|_s^2 \tag{5.5}$$

for all $R\geqq R_1$, *all* $\zeta\in\Sigma_\pm(\gamma)$ *and all* $s>(n+1)/2$.

Proof. From (5.4)

$$\begin{aligned} v(x, \zeta, f) &= G(\cdot, \zeta)*(\Lambda^0\beta)(\cdot)v(\cdot, \zeta, f) \\ &= \mathcal{F}^{-1}[(\Lambda^0(\cdot)-\zeta)^{-1}\hat{h}(\cdot)] \end{aligned}$$

if $|x|\geqq R_0+1$, where $h(x)=(\Lambda^0\beta)(x)v(x, \zeta, f)$. Since $\Lambda^0-\zeta$ is hypoelliptic (J.R. Schulenberger and C.H. Wilcox [10]), $v(x, \zeta, f)$ is smooth on $\{|x|\geqq R_0\}$. This implies $h\in C_0^\infty(\boldsymbol{R}^n)$. If we see the proof of Theorem 7.1 of [4] carefully, we can obtain

$$|v(x)|\leqq C(\eta)|x|^{-(n-1)/2}|\hat{h}|_{n+1,\mathcal{B}^\infty}$$

for some $C(\eta)\in L^1(S^{n-1})$ and

$$|v(x)|\leqq C_\nu|x|^\nu|\hat{h}|_{1,\mathcal{B}^\infty} \qquad \text{for any } \nu>0,$$

where

$$|\hat{h}|_{l,\mathcal{B}^\infty}=\sup_{\xi\in\boldsymbol{R}^n}\sum_{|\alpha|\leqq l}|\partial^\alpha\hat{h}(\xi)| \qquad (l\in\boldsymbol{N}\cup\{0\}).$$

Thus if $s-(n+1)/2>\nu$ (that is $(n-1)/2+s-\nu-n>0$),

$$\begin{aligned} &\int_{|x|\geqq R} v(x, \zeta, f)^*E(x)v(x, \zeta, f)(1+|x|)^{-s}dx \\ &\leqq C|\hat{h}|_{n+1,\mathcal{B}^\infty}\int_{|x|\geqq R} C(\eta)|x|^{-(n-1)/2}|x|^\nu(1+|x|)^{-s}dx \end{aligned} \tag{5.6}$$

$$\leqq C'|\hat{h}|_{n+1,\mathcal{B}^\infty}\int_{S^{n-1}}C(\eta)d\eta\int_R^\infty |x|^{-(n-1)/2+\nu}(1+|x|)^{-s}|x|^{n-1}d|x|$$
$$\leqq C''|\hat{h}|_{n+1,\mathcal{B}^\infty}R^{-((n-1)/2+s-\nu-n)}\,.$$

Now $||\hat{u}||_{L^\infty}\leqq||u||_{L^1}$ for any u. This fact implies

$$\begin{aligned}|\hat{h}|_{n+1,\mathcal{B}^\infty}&\leqq||(1+|\cdot|^2)^{(n+1)/2}h(\cdot)||_{L^1}\\ &=\int_{R_0\leqq|x|\leqq R_0+1}|(\Lambda^0\beta)(x)|(1+|x|^2)^{(n+1)/2}|v(x)|dx\\ &\leqq C_{R_0}(\int_{R_0\leqq|x|\leqq R_0+1}|v(x)|^2dx)^{1/2}\\ &\leqq C'_{R_0}|v(\cdot,\zeta,f)|_s\,.\end{aligned}$$

Then (5.5) follows from (5.6) provided R_1 is sufficiently large. Q.E.D.

Lemma 5.6. *Let $\lambda\in\boldsymbol{R}^1\backslash\{0\}$. Let $\zeta_{\pm n}=\lambda_n\pm i\sigma_n(\lambda_n,\ \sigma_n\in\boldsymbol{R})$ be a sequence such that $\sigma_n>0$ and $\zeta_{\pm n}\to\lambda$ when $n\to\infty$. Let $g_n\in\mathcal{H}$ be a sequence such that supp $g_n\subset B_c$ for all n and some $c>0$, and $g_n\to g$ in $\mathcal{H}$ as $n\to\infty$. Put*

$$w_{\pm n}=R(\zeta_{\pm n})g_n$$

and assume that there exists a constant K such that

$$|w_{\pm n}|_s\leqq K \qquad \text{for all}\quad n\,.$$

Then $\{w_{\pm n}\}$ converges weakly in $\mathcal{K}_s$ to a limit $w_\pm\in\mathcal{K}_s$. Moreover $w_\pm$ is the solution of the steady-state wave propagation problem (5.1).

Proof. The same method of (3.22) of [9] implies

$$(\Lambda-\lambda)w=g \qquad \text{in}\quad \mathcal{H}_{\rm loc}\,.$$

Next step is to prove that $w_\pm$ satisfies the $\pm$ radiation condition. Since $(\Lambda-\lambda)w_\pm=g$ and $\Lambda=\Lambda^0$ for $|x|>R$,

$$(\Lambda^0-\lambda)w_\pm=0$$

for $|x|>\max(R,c)=:R_0$ (c of B_c). Thus the hypoellipticity of $\Lambda^0-\lambda$ (J.R. Schulenberger and C.H. Wilcox [10]) implies $w_\pm\in C^\infty(|x|>R_0)$. Then $w_\pm$ satisfies Ri). It follows from (5.4) and (3.19) of [9] that

$$\beta(x)w_{\pm n'}=G(\cdot,\zeta_{\pm n})*(\Lambda^0\beta)w_{\pm n'}\,. \tag{5.7}$$

Now supp $(\Lambda^0\beta)w_{\pm n'}\subset\{R_0\leqq|x|\leqq R_0+1\}$ and hence

$$(\Lambda^0\beta)w_{\pm n'}\to(\Lambda^0\beta)w_\pm \qquad \text{in}\quad \mathcal{E}'\,.$$

Moreover $G(\cdot,\zeta_{\pm n'})\to G_\pm(\cdot,\lambda)$ in $\mathcal{S}'$. It follows from the continuity of the convolution operator on $\mathcal{S}'*\mathcal{E}'$ that making $n\to\infty$ in (5.7) gives

$$\beta(x)w_\pm = G(\cdot, \lambda)*(\Lambda^0\beta)w_\pm ,$$

that is,

$$\begin{aligned}\beta(x)w_\pm &= \lim_{\varepsilon\searrow 0} G(\cdot, \lambda\pm i\varepsilon)*h \qquad (h=(\Lambda^0\beta)w_\pm)\\ &= \lim_{\varepsilon\searrow 0} \mathcal{F}^{-1}[(\Lambda^0(\cdot)-(\lambda\pm i\varepsilon))^{-1}\hat{h}(\cdot)]\,.\end{aligned}$$

As is shown in the proof of Lemma 5.5, $h\in C_0^\infty(\boldsymbol{R}^n)$. Then the same argument as in the proof of Lemma 3.1 show that $w_\pm$ satisfies $\pm$radiation condition.

Finally we shall prove that $w_\pm$ satisfies the boundary condition. Note that $w_{\pm n'}\in\mathcal{D}(\Lambda)$. Especially $w_{\pm n'}$ satisfies the boundary condition. Decompose $w_{\pm n'}$ as

$$w_{\pm n'} = w^1_{\pm n'}+w^2_{\pm n'} ,$$

where $w^1_{\pm n'}\in\mathcal{D}(\Lambda)\ominus\mathcal{N}(\Lambda)$ and $w^2_{\pm n'}\in\mathcal{N}(\Lambda)$. The assumption Avii) implies that for any r' and any r with $R<r'<r$

$$\begin{aligned}||w^1_{\pm n'}||_{H^1(\Omega_{r'})} &\leqq C\,\{||w^1_{\pm n'}||_{\mathcal{H}(\Omega_r)}+||\Lambda w^1_{\pm n'}||_{\mathcal{H}(\Omega_r)}\}\\ &= C\,\{||w^1_{\pm n'}||_{\mathcal{H}(\Omega_r)}+||\Lambda w_{\pm n'}||_{\mathcal{H}(\Omega_r)}\}\\ &\leqq C_r\{||w^1_{\pm n'}||_{\mathcal{H}(\Omega_r)}+||w^2_{\pm n'}||_{\mathcal{H}(\Omega_r)}+|\Lambda w_{\pm n'}|_s\}\\ &\leqq C'_r\{|w_{\pm n'}|_s+|\Lambda w_{\pm n'}|_s\}\\ &= C'_r\{|w_{\pm n'}|_s+|\zeta_{\pm n}w_{\pm n'}+g_{n'}|_s\}\\ &\leqq C''_r<+\infty.\end{aligned}$$

Hence $\{w^1_{\pm n'}\}$ is a bounded subset of $H^1(\Omega_r)$. Then there exists a subsequence $\{w^1_{\pm n''}\}$ which converges weakly to an element $w^1_\pm$ of $H^1(\Omega_r)$. Moreover from the Rellich compact theorem it follows that $w^1_{\pm n''}\to w^1_\pm$ strongly in $\mathcal{H}(\Omega_r)$. Next it is proved that $\{w_{\pm n''}\}$ converges strongly in $\mathcal{H}(\Omega_r)$. It is sufficient to prove that $w^2_{\pm n'}$ converges strongly in $\mathcal{H}(\Omega_r)$ to an ellement $w^2_\pm$ of $\mathcal{K}_s$. Note that $w^2_{\pm n''}\in\mathcal{N}(\Lambda)$. Then

$$0=\Lambda w^2_{\pm n''} = P_0\Lambda w_{\pm n''} = \zeta_{\pm n''}w^2_{\pm n''}+P_0 g_{n''}$$
$$(P_0 \text{ is the projection to } \mathcal{N}(\Lambda) \text{ and } \Lambda P_0=P_0\Lambda)\,.$$

Thus $w^2_{\pm n''}=-P_0 g_{n''}/\zeta_{\pm n''}$, and this converges strongly in $\mathcal{H}(\Omega_r)$. Then $w_{\pm n''}= w^1_{\pm n''}+w^2_{\pm n''}$ converges strongly in $\mathcal{H}(\Omega_r)$, and Lemma 5.5 implies the strong convergence of $w_{\pm n''}$ in $\mathcal{K}_s$. On the other hand $w_{\pm n''}$ converges weakly to $w_\pm$ in $\mathcal{H}_{\text{loc}}$. So $w_{\pm n''}\to w_\pm$ strongly in $\mathcal{K}_s$. The self-adjointness of Λ implies the closedness of Λ. Then for any function $\varphi\in C_0^\infty(\bar{\Omega})$

$$\varphi w_{\pm n''}\to\varphi w_\pm \qquad \text{strongly in } \mathcal{H}$$

and

$$\Lambda(\varphi w_{\pm n''}) \to (\Lambda\varphi)w_{\pm}+\varphi(\lambda w_{\pm}+g) \qquad \text{strongly in } \mathcal{H}\,.$$

Hence $\varphi w_{\pm}\in\mathcal{D}(\Lambda)$ and $\Lambda(\varphi w_{\pm})=(\Lambda\varphi)w_{\pm}+\varphi(\lambda w_{\pm}+g)$. This shows $w_{\pm}$ satisfies the boundary condition. Q.E.D.

Lemma 5.7. *Let $\lambda\in \boldsymbol{R}^1\backslash(\sigma_p(\Lambda)\cup\{0\})$ and $f\in L^2_{\text{vox}}$. Let $\{\zeta_{\pm n}=\lambda_n+i\sigma_n\}$ be a sequence such that $\sigma_n>0$ and $\zeta_{\pm n}\to\lambda$ when $n\to\infty$ and let $\{f_n\}$ be a sequence of functions of $\mathcal{H}$ and supp $f_n\subset K$, a compact set, and $f_n\to f$. Then $v(\cdot,\zeta_{\pm n},f_n)=R(\zeta_{\pm n})f_n$ converges weakly in $\mathcal{K}_s$ for $s>(n+1)/2$ to a limit $v_{\pm}(\cdot,\lambda,f)\in\mathcal{K}_s$. Moreover $v_{\pm}(\cdot,\lambda,f)$ is the solution of the steady-state wave propagation problem* (5.1).

The proof can be carried out in the same line of that of Theorem 3.4 of [9] by replacing their Lemma 3.6 and Lemma 3.7 with our Lemma 5.5 and Lemma 5.6, respectively. For our case the convergence of $w_{\pm n''}$ follows from the same argument of the last step of Lemma 5.6.

Then the proof of 1) of Theorem 5.1 can be carried out in the same line of that of Lemma 3.11 and Theorem 3.1 of [9] by replaicng their Lemma 3.4 with our Lemma 5.7. The proof of 2) of Theorem 5.1 can be carried out in the same line of that of Theorem 3.3 of [9] by replacing their Theorem 2.1 with our Theorem 5.2.

6. The eigenfunction expansion

In this section we shall construct distorted plane waves, and state the theorem of eigenfunction expansions. The proof is almost same as that of J.R. Schulenberger and C.H. Wilcox [11]. So we shall omit it. Here we assume that $E(x)$ is continuously differentiable.

To begin with disrtoted plane waves $\{\Phi_j^{\pm}(x,\xi);\ x\in\Omega,\ \xi\in\boldsymbol{R}^n,\ |j|=1,2,\cdots,\rho\}$ are constructed. Let $\beta(x)$ be a smooth function of (4.1). Consider the following equation in $\mathcal{H}$:

$$(6.2)\quad (\Lambda-\zeta)\Psi_j = -(\Lambda-\lambda_j(\xi))\beta(x)\Phi_j^0+(\lambda_j(\xi)-\zeta)(1-\beta(x))E(x)^{-1/2}\Phi_j^0\,,$$

where

$$(6.3)\qquad \Phi_j^0(x,\xi) = e^{ix\cdot\xi}\hat{P}_j(\xi)\,.$$

Since Λ equals Λ^0 for $|x|>R_0+1$ and $(\Lambda^0-\lambda_j(\xi))\Phi_j^0=0$, it follows that the right hand side of this equation has its support in the ball $\{|x|<R_0+1\}$. Thus, for each $\zeta=\lambda\pm i\sigma(\sigma\neq 0)$, there exists a unique solution $\Psi_j=\Psi_j(\cdot,\xi;\zeta)\in L^2(\Omega)$. Theorem 5.1 implies the existence of the limit

$$\Psi_j(x,\xi;\lambda\pm i0) = \lim_{\sigma\searrow 0}\Psi_j(x,\xi;\lambda\pm i\sigma)$$

if $\lambda \notin \sigma_p(\Lambda) \cup \{0\}$, and it satisfies $\pm$ radiation condition and the boundary condition on $\partial\Omega$.

Put

$$\Phi_j(x, \xi; \zeta) = \beta(x)\Phi_j^0(x, \xi) + \Psi_j(x, \xi; \zeta).$$

Then, since $\beta\Phi_j^0$ satisfies the boundary condition, it follows easily from the equation (6.2) that $\Phi_j(\cdot, \xi; \zeta)$ satisfies

$$(\Lambda - \zeta)\Phi_j = (\lambda_j(\xi) - \zeta)E^{-1/2}\Phi_j^0.$$

Here note that $E(x) = I$ for $|x| > R_0$. Finally the distorted plane waves $\{\Phi_j^\pm(x, \xi)\}$ are defined by

$$\Phi_j^\pm(x, \xi) = \lim_{\sigma \searrow 0} \Phi_j(x, \xi; \lambda_j(\xi) \pm i\sigma)$$

for any $\xi \in \boldsymbol{R}^n \setminus \bar{Z}_S$. Then

$$\begin{cases} (\Lambda - \lambda_j(\xi))\Phi_j^\pm(\xi) = (\lambda_j(\xi) - \lambda)E^{-1/2}\Phi_j^0 & \text{for} \quad x \in \Omega \\ \Phi_j^\pm \text{ satisfies } \pm \text{ radiation condition} \\ \Phi_j^\pm \in N(x) \qquad \text{for} \quad x \in \partial\Omega. \end{cases}$$

In order to give the expansion theorem we have to introduce more notations. For $f \in C_0^\infty(\Omega)$ the transform $\hat{f}_j^\pm(\xi)$ is defined by

$$\hat{f}_j^\pm(\xi) = \int_\Omega \Phi_j^\pm(x, \xi)^* E(x) f(x) dx$$

(the integral in the sense of the limit in mean)

for all $\xi \in \boldsymbol{R}^n \setminus \bar{Z}_S$. Let $\mathcal{F}_j^\pm$ denote

$$(\mathcal{F}_j^\pm f)(\xi) = \hat{f}_j^\pm(\xi)$$

and $\mathcal{F}^\pm = \sum_{|j|=1}^{p} \mathcal{F}_j^\pm$. $\mathcal{H}_p$ denotes a subspace of $\mathcal{H}$ of all eigenfunctions of Λ, and P^p denotes the projection onto it. $\mathcal{H}_c$ denotes the continuous spectral subspaces and P^c the projection onto it.

Theorem 6.1. *The distorted plane waves* $\{\Phi_j^\pm(x, \xi); \xi \in \boldsymbol{R}^n \setminus \bar{Z}_S\}$ *form a complete set of generalized eigenfunctions of* Λ *restricted to* $\mathcal{H}_c$:

1) $\mathcal{F}^\pm$ *is an isometry of* $\mathcal{H}_c$ *onto* $(\mathcal{H}_0)_c$, *the continuous spectral subspace of* $\mathcal{H}_0$. *The adjoint operator* $(\mathcal{F}^\pm)^*$ *is an isometry of* $(\mathcal{H}_0)_c$ *onto* $\mathcal{H}_c$, *and is given by*

$$(\mathcal{F}^\pm)^* = \sum_{|j|=1}^{p} (\mathcal{F}_j^\pm)^*,$$

$$[(\mathcal{F}_j^\pm)^* \hat{f}](x) = \int_{R_n} \Phi_j^\pm(x, \xi) \hat{f}(\xi)\, d\xi$$

(*the integral in the sense of* $\mathcal{H}$, *the limit in mean*).

2) *For any function f of $\mathcal{H}$, it follows that*

$$(P^c f)(x) = (\mathcal{F}^{\pm})^* \mathcal{F}^{\pm} f(x) = \sum_{|j|=1}^{\rho} (\mathcal{F}_j^{\pm})^* \mathcal{F}_j^{\pm} f(x) .$$

3) $\mathcal{F}_j^{\pm} \Lambda \subset \lambda_j(\cdot) \mathcal{F}_j^{\pm}$ $(|j|=1, 2, \cdots, \rho)$.

4) $P_0^c = \mathcal{F}^{\pm} (\mathcal{F}^{\pm})^*$ *(P_0^c denotes the projection onto $(\mathcal{H}_0)_c$).*

References

[1] P. Appel and E. Lacour: Fonctions elliptiques et applications, deuxième édition, Paris, Gauthier-Villars, 1922.

[2] T. Ikebe: *Eigenfunction expansions associated with the Schrödinger operator and their application to scattering theory*, Arch. Rational Mech. Anal. **5** (1960), 1–34.

[3] N. Iwasaki: *Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains*, Publ. RIMS, Kyoto Univ. **5** (1969), 193–218.

[4] K. Kikuchi: *Asymptotic behavior at infinity of the Green function of a class of systems including wave propagation in crystals*, Osaka J. Math. **22** (1985), 575–635.

[5] P.D. Lax and R.S. Phillips: *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math. **13** (1960), 427–455.

[6] K. Mochizuki: *Spectral and scattering theory for symmetric hyperbolic systems in an exterior domain*, Publ. RIMS, Kyoto Univ. **5** (1969), 219–258.

[7] J.R. Schulenberger: *Uniqueness theorem for a class of wave propagation problem*, Arch. Rational Mech. Anal. **34** (1969), 70–84.

[8] J.R. Schulenberger and C.H. Wilcox: *A Rellich uniqueness theorem for steady-state wave propagation in inhomogeneous anisotropic media*, Arch. Rational Mech. Anal. **41** (1971), 18–45.

[9] J.R. Schulenberger and C.H. Wilcox: *The limiting absorption principle and spectral theory for steady-state wave propagation in inhomogeneous anisotropic media*, Arch. Rational Mech. Anal. **41** (1971), 46–65.

[10] J.R. Schulenberger and C.H. Wilcox: *The singularities of the Green's matrix in anisotropic wave motion*, Indiana Univ. Math. J. **20** (1971), 1093–1117.

[11] J.R. Schulenberger and C.H. Wilcox: *Eigenfunction expansions and scattering theory for wave propagation problems of classical physics*, Arch. Rational Mech. Anal. **46** (1972), 280–320.

[12] J.R. Schulenberger and C.H. Wilcox: *A coerciveness inequality for a class of nonelliptic operators of constant deficit*, Ann. Mat. Pura Appl. **92** (1972), 77–84.

[13] H. Tamura: *The principle of limiting absorption for propagative systems in crystal optics with perturbations of long-range class*, Nagoya Math. J. **84** (1981), 169–193.

[14] R. Weder: *Spectral analysis of strongly propagative systems*, J. Reine Angew. Math. **354** (1984), 95–122.

[15] C.H. Wilcox: Scattering theory for the d'Alembert equation in exterior domains, Lecture Notes in Math., No. 442, Springer, New York, 1975.

[16] C.H. Wilcox: *Asymptotic wave functions and energy distributions in strongly propagative anisotropic media,* J. Math. Pures Appl. **57** (1978), 275–321.
[17] C.H. Wilcox: *Electromagnetic signal propagation in crystals,* Applicable Anal. **8** (1978), 83–94.

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan