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Author(s)	Matsumoto, Kazuo
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Matsumoto, Kazuo
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Decision Procedure for Modal Sentential Calculus S3

By Kazuo MATSUMOTO

Some trials to solve the decision problem for modal sentential calculus *S3*[1] have been tried by W. T. Parry, S. Halldén, A. R. Anderson and some others. That is, in 1932, W. T. Parry [8] showed that γ^* is provable in *S3* if and only if γ^* is provable in *S5* where γ^* is of degree at most 1[2),3)].

In 1950, S. Halldén [4] showed that the decision problem for *S3* can be reduced to that for a new system *S7*[4], which enlarges *S3* by adjoining $\Diamond\Diamond p$ as an axiom to *S3*.

It is reported[5] that A. R. Anderson [1] solved the decision problem for *S3* in 1953 using the method of von Wright [10].

The object of this paper is to give a Gentzen type decision procedure for modal sentential calculus *S3*.

The author wishes to express his cordial thanks to Mr. Masao Ohnishi for his suggestions and instructions in connection with this paper.

§1. Definitions of $Q3$ and Q^* and their equivalence.

Our formulations of *Q3* and *Q** are based upon "Sequenzenkalkül *LK*", which was constructed by G. Gentzen [3]. Namely:

- logical symbols:
  - $\cdot$ (and), $\sim$ (not), $\vee$ (or), $\supset$ (if $\cdots$, then)
- rules of inference (*LK*-rules):
  - structural rules
    - weakening, contraction, exchange and cut.
  - logical rules
    - $(\rightarrow\cdot)$, $(\rightarrow\vee)$, $(\rightarrow\sim)$, $(\rightarrow\supset)$,
    - $(\cdot\rightarrow)$, $(\vee\rightarrow)$, $(\sim\rightarrow)$, $(\supset\rightarrow)$.

Numbers in brackets refer to the bibliography at the end of this paper.

1) C. I. Lewis and C. H. Langford [5].

2) For the definition of "a formula of degree n", see A. R. Anderson [2], p. 203.

3) S. Halldén [4] remarked that γ^* is provable in *S2* if and only if γ^* is provable in *S5* where γ^* is of degree at most 1. See M. Ohnishi and K. Matsumoto [7], p. 119.

4) The decision problem for *S7* has not been solved.

5) Recently Prof. Anderson wrote me the essential part of his solution for the decision problem of *S3*, but it seems to me that his solution is incorrect. (Added in proof.)

Next, we add to LK a new logical symbol $\Box$ (necessary), and we define as follows: if α is a formula, then $\Box\alpha$ is also a formula.

Giving rules for the modality symbol $\Box$, we define Q^* and $Q3$ as follows:

Definition of Q^*

$$\begin{cases} \text{Rules}: & \begin{cases} LK\text{-rules} \\ \dfrac{\alpha,\ \Gamma \to \Theta}{\Box\alpha,\ \Gamma \to \Theta}\ (\Box \to) \\ \dfrac{\Sigma \to \alpha}{\Box\Sigma \to \Box\alpha}\ (\to \Box), \quad \text{where } \Sigma \text{ is non-empty.} \end{cases} \\ \text{Axiom}: & \alpha \prec\!\!\!\!\!\!\rightarrow \beta \to \Box\alpha \prec\!\!\!\!\!\!\rightarrow \Box\beta \end{cases}$$

[6),7)].

Definition of $Q3$

$$\text{Rules}: \begin{cases} LK\text{-rules} \\ \dfrac{\alpha,\ \Gamma \to \Theta}{\Box\alpha,\ \Gamma \to \Theta}\ (\Box \to) \\ \dfrac{\Sigma \to p \prec\!\!\!\!\!\!\rightarrow p,\ \alpha \qquad \Box\Sigma \to \alpha}{\Box\Sigma \to \Box\alpha}\ (\to \Box), \end{cases}$$

where Σ is non-empty and p is a predetermined sentence-variable.

By Σ, Γ, $\Theta \cdots$ we mean a series of formulas as in LK. $\Box\Sigma$ means the series of formulas which is formed by prefixing $\Box$ in front of each formula of Σ.

Now we shall prove the equivalence of Q^* and $Q3$.

$\underline{Q^* \Longrightarrow Q3}$

We have only to show the Q^*-admissibility of the rule $(\to \Box)$ of $Q3$. Therefore it is sufficient to show the Q^*-admissibility of

$$\frac{\beta,\ \gamma \to p \prec\!\!\!\!\!\!\rightarrow p,\ \alpha \qquad \Box\beta,\ \Box\gamma \to \alpha}{\Box\beta,\ \Box\gamma \to \Box\alpha},$$

which is the special case that Σ consists of two formulas β and γ.[8)]

First, we shall prove the following $1^\circ \sim 3^\circ$:

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6) Without this axiom Q^* becomes $Q2$. See M. Ohnishi and K. Matsumoto [6].

7) $\alpha \prec\!\!\!\!\!\!\rightarrow \beta$ is the abbreviation of $\Box(\alpha \supset \beta)$.

8) In case that Σ consists of n formulas $\alpha_1, \alpha_2, \cdots, \alpha_n$, we can prove this using the Q^*-provability of $\Box\alpha_1 \cdot \cdots \cdot \Box\alpha_n \rightleftarrows \Box(\alpha_1 \cdot \cdots \cdot \alpha_n)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \to \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$.

1°

$$\dfrac{\dfrac{\dfrac{\dfrac{\beta\to\beta}{\beta\cdot\gamma\to\beta}}{\Box(\beta\cdot\gamma)\to\Box\beta}\qquad\dfrac{\dfrac{\gamma\to\gamma}{\beta\cdot\gamma\to\gamma}}{\Box(\beta\cdot\gamma)\to\Box\gamma}}{\Box(\beta\cdot\gamma)\to\Box\beta\cdot\Box\gamma}\qquad\dfrac{\dfrac{\text{(hypothesis)}}{\Box\beta,\ \Box\gamma\to\alpha}}{\Box\beta\cdot\Box\gamma\to\alpha}}{\Box(\beta\cdot\gamma)\to\alpha}$$

2°

$$\dfrac{\dfrac{\dfrac{\dfrac{\beta\cdot\gamma\to\beta\cdot\gamma}{p\supset p,\ \beta\cdot\gamma\to\beta\cdot\gamma}}{\beta\cdot\gamma\to p\supset p.\supset.\beta\cdot\gamma}}{\Box(\beta\cdot\gamma)\to p\supset p.\strictif.\beta\cdot\gamma}\qquad\dfrac{\text{(axiom of } Q^*\text{)}}{p\supset p.\strictif.\beta\cdot\gamma\to p\strictif p.\strictif.\Box(\beta\cdot\gamma)}}{\Box(\beta\cdot\gamma)\to p\strictif p.\strictif.\Box(\beta\cdot\gamma)}$$

3°

$$\dfrac{\dfrac{\dfrac{\beta\to\beta}{\beta,\ \gamma\to\beta}\qquad\dfrac{\gamma\to\gamma}{\beta,\ \gamma\to\gamma}}{\beta,\ \gamma\to\beta\cdot\gamma}}{\Box\beta,\ \Box\gamma\to\Box(\beta\cdot\gamma)}$$

Then we have the following proof-figure which was to be desired:

$$\dfrac{\dfrac{3^\circ}{\Box\beta,\ \Box\gamma\to\Box(\beta\cdot\gamma)}\qquad\dfrac{\dfrac{\dfrac{2^\circ}{\Box(\beta\cdot\gamma)\to p\strictif p.\strictif.\Box(\beta\cdot\gamma)}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\text{(hypothesis)}}{\beta,\ \gamma\to p\strictif p,\ \alpha}}{\beta\cdot\gamma\to p\strictif p,\ \alpha}\qquad\dfrac{\dfrac{1^\circ}{\Box(\beta\cdot\gamma)\to\alpha}}{\beta\cdot\gamma,\ \Box(\beta\cdot\gamma)\to\alpha}}{\beta\cdot\gamma,\ p\strictif p.\supset.\Box(\beta\cdot\gamma)\to\alpha}}{\Box(\beta\cdot\gamma),\ p\strictif p.\strictif.\Box(\beta\cdot\gamma)\to\Box\alpha}}{\Box(\beta\cdot\gamma),\ \Box(\beta\cdot\gamma)\to\Box\alpha}}{\Box(\beta\cdot\gamma)\to\Box\alpha}}{\Box\beta,\ \Box\gamma\to\Box\alpha}$$

<u>$Q3\Longrightarrow Q^*$</u>

We have only to show the $Q3$-admissibility of the rule $(\to\Box)$ of Q^* and the $Q3$-provability of the axiom for Q^*.

1°

$$\dfrac{\dfrac{\Sigma\to\alpha}{\Sigma\to p\strictif p,\ \alpha}\qquad\dfrac{\Sigma\to\alpha}{\Box\Sigma\to\alpha}\ (\Box\to)}{\Box\Sigma\to\Box\alpha}\ (\to\Box)$$

2°

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p\to p}{\to p\supset p}}{\alpha\to p\supset p}}{\Box\alpha\to p\strictif p}\ (1^\circ)}{\Box\alpha\to p\strictif p,\ \Box\beta}}{\to p\strictif p,\ \Box\alpha\supset\Box\beta}}{\alpha\supset\beta\to p\strictif p,\ \Box\alpha\supset\Box\beta}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\alpha\to\alpha}{\alpha\to\alpha,\ \beta}\qquad\dfrac{\beta\to\beta}{\alpha,\ \beta\to\beta}}{\alpha,\ \alpha\supset\beta\to\beta}}{\Box\alpha,\ \alpha\strictif\beta\to\Box\beta}\ (1^\circ)}{\alpha\strictif\beta\to\Box\alpha\supset\Box\beta}}{\alpha\strictif\beta\to\Box\alpha\strictif\Box\beta}$$

§ 2. Hauptsatz for *Q3*.

We shall prove in this § the following Hauptsatz (cut-elimination theorem) for *Q3*.

Hauptsatz (CUT-ELIMINATION THEOREM). *Any Q3-proof-figure can be transformed into a Q3-proof-figure with the same endsequent and without any cut as a rule of inference.*

The proof is treated along the line of G. Gentzen [3].

We replace cut-rule by mix (Mischung)-rule as in Gentzen. Then, we have only to prove the following

Lemma: *Any proof-figure which has a mix-rule only as its lowest rule and does not include this rule elsewhere, can be transformed into a proof-figure which has the same endsequent and has no mix at all.*

Grade (Grad) and rank being the same as in *LK*, the proof of our lemma can be treated by the induction on rank and grade.

The cases which are to be added to the proof for *LK* are the following:

(1) When $\rho=2$, and the outermost symbol of the mix-formula is $\square$, the mix has the following form:

$$\dfrac{\dfrac{\Sigma \to p \dashv\!\!\!\!\!3\, p,\ \alpha \qquad \square\Sigma \to \alpha}{\square\Sigma \to \square\alpha}\ (\to \square) \qquad \dfrac{\alpha,\ \Gamma \to \Theta}{\square\alpha,\ \Gamma \to \Theta}\ (\square \to)}{\square\Sigma,\ \Gamma \to \Theta}\ (\text{mix of } \square\alpha)$$

(Γ does not contain $\square\alpha$).

We transform this into:

$$\dfrac{\dfrac{\square\Sigma \to \alpha \qquad \alpha,\ \Gamma \to \Theta}{\square\Sigma,\ \Gamma^* \to \Theta}\ (\text{mix of } \alpha)}{\square\Sigma,\ \Gamma \to \Theta}$$

This shows that we can omit the mix from the assumption of the induction, as the grade of the mix formula is decreased by 1.

(2) When $\rho>2$, and the left rank $\rho_l=1$ and the upper sequent on the left side of mix is the lower sequent of the rules of $\square$, we have to treat the following four cases:

(2-1)
$$\dfrac{\dfrac{\Sigma \to p \dashv\!\!\!\!\!3\, p,\ \alpha \qquad \square\Sigma \to \alpha}{\square\Sigma \to \square\alpha}\ (\to \square) \qquad \dfrac{\alpha,\ \square\alpha,\ \Gamma \to \Theta}{\square\alpha,\ \square\alpha,\ \Gamma \to \Theta}\ (\square \to)}{\square\Sigma,\ \Gamma^* \to \Theta}\ (\text{mix of } \square\alpha)$$

We transform this into:

$$\dfrac{\dfrac{\dfrac{\Box\Sigma \to \alpha \qquad \dfrac{\Box\Sigma \to \Box\alpha \qquad \alpha, \Box\alpha, \Gamma \to \Theta}{\Box\Sigma, \alpha, \Gamma^* \to \Theta}\;(\text{mix of }\Box\alpha)}{\Box\Sigma, \Box\Sigma^\dagger, \Gamma^{*\dagger} \to \Theta}\;(\text{mix of }\alpha)}{\Box\Sigma, \Box\Sigma, \Gamma^* \to \Theta}}{\Box\Sigma, \Gamma^* \to \Theta}$$

This shows that we can omit two mixes from the assumption of the induction, as the rank and the grade of the mix formulas are decreased by 1 respectively.

(2-2) $$\dfrac{\dfrac{\Sigma \to p \mathrel{-\!\!\!3} p, \alpha \qquad \Box\Sigma \to \alpha}{\Box\Sigma \to \Box\alpha}\;(\to\Box) \qquad \dfrac{\beta, \Box\alpha, \Gamma \to \Theta}{\Box\beta, \Box\alpha, \Gamma \to \Theta}\;(\Box\to)}{\Box\Sigma, \Box\beta, \Gamma^* \to \Theta}\;(\text{mix of }\Box\alpha)$$

where $\alpha \neq \beta$.

We transform this into:

when $\beta \neq \Box\alpha$

$$\dfrac{\dfrac{\Box\Sigma \to \Box\alpha \qquad \beta, \Box\alpha, \Gamma \to \Theta}{\Box\Sigma, \beta, \Gamma^* \to \Theta}\;(\text{mix of }\Box\alpha)}{\Box\Sigma, \Box\beta, \Gamma^* \to \Theta}\;(\Box\to)$$

when $\beta = \Box\alpha$

$$\dfrac{\dfrac{\Box\Sigma \to \Box\alpha \qquad \beta, \Box\alpha, \Gamma \to \Theta}{\Box\Sigma, \Gamma^* \to \Theta}\;(\text{mix of }\Box\alpha)}{\Box\beta, \Box\Sigma, \Gamma^* \to \Theta}$$

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula is decreased by 1.

(2-3)

$$\dfrac{\dfrac{\Gamma \to p \mathrel{-\!\!\!3} p, \alpha \qquad \Box\Gamma \to \alpha}{\Box\Gamma \to \Box\alpha}\;(\to\Box) \qquad \dfrac{\alpha, \Sigma \to p \mathrel{-\!\!\!3} p, \beta \qquad \Box\alpha, \Box\Sigma \to \beta}{\Box\alpha, \Box\Sigma \to \Box\beta}\;(\to\Box)}{\Box\Gamma, (\Box\Sigma)^* \to \Box\beta}\;(\text{mix of }\Box\alpha)$$

We transform this into:

$$\dfrac{\dfrac{\dfrac{\Gamma \to p \mathrel{-\!\!\!3} p, \alpha \qquad \alpha, \Sigma \to p \mathrel{-\!\!\!3} p, \beta}{\Gamma, \Sigma^\dagger \to p \mathrel{-\!\!\!3} p, p \mathrel{-\!\!\!3} p, \beta}\;(\text{mix of }\alpha)}{\Gamma, \Sigma^\dagger \to p \mathrel{-\!\!\!3} p, \beta} \qquad \dfrac{\Box\Gamma \to \Box\alpha \qquad \Box\alpha, \Box\Sigma \to \beta}{\Box\Gamma, (\Box\Sigma)^* \to \beta}\;(\text{mix of }\Box\alpha)}{\Box\Gamma, (\Box\Sigma)^* \to \Box\beta}\;(\to\Box)$$

because of $\Box(\Sigma^\dagger) = (\Box\Sigma)^*$.

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula of the right mix is decreased by 1 and the grade of the mix formula of the left mix is decreased by 1.

When the upper sequent on the left side of mix is the lower sequent of the rules of *LK*, we have only to treat the following:

(2-4)
$$\dfrac{\dfrac{\Sigma \to p \mathrel{-\!\!3} p,\ \alpha \qquad \Box\Sigma \to \alpha}{\Box\Sigma \to \Box\alpha}\,(\to \Box) \qquad \dfrac{\Box\alpha,\ \Delta' \to \Lambda'}{\Box\alpha,\ \Delta \to \Lambda}\,P_f}{\Box\Sigma,\ \Delta^* \to \Lambda}\ (\text{mix of } \Box\alpha)$$

(P_f is any one of *LK*-rules).

We transform this into:

$$\dfrac{\dfrac{\Box\Sigma \to \Box\alpha \qquad \Box\alpha,\ \Delta' \to \Lambda'}{\Box\Sigma,\ \Delta'^* \to \Lambda'}\ (\text{mix of } \Box\alpha)}{\Box\Sigma,\ \Delta^* \to \Lambda}\,P_f$$

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula is decreased by 1.

Remark: In case that P_f is a weakening rule and $\Box\alpha$ is a weakening formula and is not included in Δ, we can easily derive the desired sequent from the upper sequent of P_f.

(3) When $\rho > 2$, and the left rank $\rho_l > 1$, and the upper sequent on the left side of mix is the lower sequent of the rules of $\Box$, we have to treat the following two cases:

(3-1)
$$\dfrac{\dfrac{\alpha,\ \Gamma \to \Theta}{\Box\alpha,\ \Gamma \to \Theta}\,(\Box \to) \qquad \Sigma \to \Pi}{\Box\alpha,\ \Gamma,\ \Sigma^* \to \Theta^*,\ \Pi}\ (\text{mix of } \mu)$$

We transform this into:

$$\dfrac{\dfrac{\alpha,\ \Gamma \to \Theta \qquad \Sigma \to \Pi}{\alpha,\ \Gamma,\ \Sigma^* \to \Theta^*,\ \Pi}\ (\text{mix of } \mu)}{\Box\alpha,\ \Gamma,\ \Sigma^* \to \Theta^*,\ \Pi}\,(\Box \to)$$

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula is decreased by 1.

(3-2) When the left upper sequent of the mix is the lower sequent of $(\to \Box)$ and a principal formula of this $(\to \Box)$ is $p \mathrel{-\!\!3} p$, a part of the proof-figure is as follows:

$$\dfrac{\dfrac{\Sigma \to p \mathrel{-\!\!3} p,\ p \supset p \qquad \Box\Sigma \to p \supset p}{\Box\Sigma \to p \mathrel{-\!\!3} p}\,(\to \Box) \qquad \dfrac{}{p \mathrel{-\!\!3} p,\ \Delta \to \Lambda}\,P_f}{\Box\Sigma,\ \Delta^* \to \Lambda}\ (\text{mix of } p \mathrel{-\!\!3} p)$$

When P_f represents any one of LK-rules, $(\Box\rightarrow)$ or $(\rightarrow\Box)$, a part of the proof-figure and its transformation is what we can obtain by replacing α in (2-4), (2-1), (2-2), (1) or (2-3) by $p\supset p$.

Thus we can also omit the mix from the assumption of the induction in the case of (3-2).

§ 3. Reduction of *S3* to Q^*.

In this §, we shall treat the formulation[9] of *S3* by L. Simons [9]. That is,

Axioms:

$H1$: $\alpha \prec \alpha\cdot\alpha$,
$H2$: $\alpha\cdot\beta \prec \beta$,
$H3$: $((\gamma\cdot\alpha)\cdot\sim(\beta\cdot\gamma)) \prec (\alpha\cdot\sim\beta)$,
$H4$: $\Box\alpha\supset\alpha$,
$H5$: $\sim\alpha \prec \sim\Box\alpha$,
$H6$: $\sim\alpha \prec \sim\beta : \prec : \Box\beta \prec \Box\alpha$.

Rule: Detachment for material implication.

Now we shall prove the following

Theorem. *γ is provable in S3 if and only if $p\prec p\rightarrow\gamma$ is provable in Q^*.*

Proof.

(Necessity) Suppose that γ is provable in *S3*.

1° Let γ be an axiom. If γ is any one of the axioms except $H4$, then, as the outermost symbol of γ is $\prec$, we write simply γ' the formula which we get by replacing the outermost symbol $\prec$ of γ by $\supset$. As $p\supset p\rightarrow\gamma'$ is provable in Q^*, $p\prec p\rightarrow\gamma$ is also provable in Q^*. If γ is an axiom of $H4$, Q^*-provability of $p\prec p\rightarrow\Box\alpha\supset\alpha$ is clear.

2° Let γ be the result of detachment for material implication.

We have only to show the Q^*-provability of $p\prec p\rightarrow\gamma$ assuming the Q^*-provabilities of $p\prec p\rightarrow\alpha$ and $p\prec p\rightarrow\alpha\supset\gamma$. The proof is as follows:

$$\dfrac{\dfrac{p\prec p\rightarrow\alpha \qquad p\prec p\rightarrow\alpha\supset\gamma}{p\prec p\rightarrow\alpha,\ \alpha\supset\gamma} \qquad \dfrac{\alpha\rightarrow\alpha \qquad \gamma\rightarrow\gamma}{\alpha,\ \alpha\supset\gamma\rightarrow\gamma}}{p\prec p\rightarrow\gamma}$$

(Sufficiency) We can prove the following

Lemma: *If $\rightarrow\alpha$ is provable in Q^*, then $\Box\alpha$ is provable in S3.*

9) L. Simons [9] adopts the symbol $\Diamond$ as a primitive modal symbol.

Now in order to prove this lemma, we assume that a Q^*-proof-figure of $\rightarrow\alpha$ without cut is given. Let an $S3$-formula $\alpha\cdot\beta \mathrel{-\!\!3} \gamma\vee\delta$ correspond to a Q^*-sequent $\alpha, \beta\rightarrow\gamma, \delta$, $\Box\gamma$ to $\rightarrow\gamma$, and $\Box\sim\beta$ to $\beta\rightarrow$.

Then, following this correspondence, $\alpha \mathrel{-\!\!3} \beta \rightarrow \Box\alpha \mathrel{-\!\!3} \Box\beta$ is transformed into $\alpha \mathrel{-\!\!3} \beta . \mathrel{-\!\!3} . \Box\alpha \mathrel{-\!\!3} \Box\beta$ which is clearly provable in $S3$.

Now it is sufficient to show the $S3$-provability of the corresponding formula to the lower sequent, assuming the $S3$-provability of corresponding formula to the upper sequent for each rule of inference in Q^*.

But most of these trials can be carried out without difficulty. Therefore we shall treat here only the rule $(\rightarrow\Box)$. Let $\alpha\cdot\beta \mathrel{-\!\!3} \gamma$, which corresponds to the upper sequent $\alpha, \beta\rightarrow\gamma$, be provable in $S3$. Then using the $S3$-provable formulas $\alpha\cdot\beta \mathrel{-\!\!3} \gamma : \mathrel{-\!\!3} : \Box(\alpha\cdot\beta) \mathrel{-\!\!3} \Box\gamma$ and $\Box(\alpha\cdot\beta) . = . \Box\alpha\cdot\Box\beta$, we can obtain that $\Box\alpha\cdot\Box\beta \mathrel{-\!\!3} \Box\gamma$, which corresponds to the lower sequent $\Box\alpha, \Box\beta\rightarrow\Box\gamma$, is provable in $S3$. See M. Ohnishi and K. Matsumoto [6], pp. 121-122. (Proof of Proposition 2°).

Now assuming that $p \mathrel{-\!\!3} p\rightarrow\gamma$ is provable in Q^*, $p \mathrel{-\!\!3} p . \mathrel{-\!\!3} . \gamma$ is provable in $S3$ by this lemma. Therefore γ is provable in $S3$.

§4. Decision procedure for *S3*.

In this §, we shall modify the definition of "subformula of γ" as follows: We define a "quasi-subformula" of γ as an ordinary subformula of γ or p, $p\supset p$ or $p \mathrel{-\!\!3} p$, where p is the sentence-variable which appears in the rule $(\rightarrow\Box)$.[10]

Then $Q3$ has the "quasi-subformula property". This means that the reduced sequent, of which sequent-formulas are all quasi-subformulas of γ, are finite in number. Therefore we can solve the decision problem for $Q3$ in the analogous way to LK decision procedure by G. Gentzen [3].

Now suppose that ξ is an arbitrary $S3$-formula. Then we have only to examine the decidability of $p \mathrel{-\!\!3} p\rightarrow\xi$ in $Q3$.

Thus we can give a decision procedure for modal sentential calculus $S3$.

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10) For example, the quasi-subformulas of $\Box q \mathrel{-\!\!3} q$ are q, $\Box q$, $\Box q\supset q$, $\Box q \mathrel{-\!\!3} q$, p, $p\supset p$ and $p \mathrel{-\!\!3} p$, where q is a sentence-variable.

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