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THE MALLIAVIN CALCULUS FOR SDE WITH JUMPS AND THE PARTIALLY HYPOELLIPTIC PROBLEM

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1. Introduction

It has been studied by many authors to give natural conditions under which the law of the solution to a stochastic differential equation admits a smooth density. There are many approaches to this problem in the theory of partial differential equations, and it is well known that the smoothness of the fundamental solution for a parabolic differential operator holds under the Hörmander condition which is a condition on the dimension of the Lie algebra generated by the vector fields associated with the coefficients of the differential operator (cf. [8]). In [17] and [18], Malliavin introduced differential calculus on the Wiener space, and applied it to a probabilistic proof of the Hörmander theorem. It is well known that the Kusuoka-Stroock-Norris lemma plays an important role (cf. [3], [9], [14], [19] and [20]).

On the other hand, Bismut discussed the case of jump processes in [4]. He used the Girsanov transformation, and proved formulas of integration by parts. A generalization to stochastic differential equations with jumps has been done since Bismut (cf. [2], [7], [10], [15] and [16]). Picard introduced new calculus on the Poisson space in [21]. He considered, what is called, the duality formula without using the Girsanov formula, instead of formulas of integration by parts, and applied his calculus to the existence of smooth densities. In [12], Komatsu and the present author gave a new approach to the existence of smooth densities. There the Girsanov transformation is used, and the formula of integration by parts is considered not on the Poisson space but on the cád-lág space. It is essential to discuss the integrability of the inverse of the Malliavin covariance matrix via the Sobolev inequalities. For this problem, certain fundamental inequlalities about semimartingales are considered, and the exponential decay of the Laplace transform for the law of the Malliavin covariance is proved. This inequality can be proved by an elementary stochastic calculus. In particular, in the case of diffusion processes, the fundamental inequality can be showed so easily that a simple proof of the Hörmander theorem is obtained ([11]). Moreover using the fundamental inequality, a generalization of the Hörmander condition is obtained for SDE with jumps ([12]). In [13], Kunita discussed the case of canonical stochastic differential equations with jumps by using Picard's method and certain fundamental inequalities on semimartingales obtained in [12].

In this paper, we shall consider the smoothness of the density for the image of the solution to a stochastic differential equation with jumps through a smooth mapping on a Euclidean space. The smoothness of the density for image processes is often called the partial hypoellipticity. In the case of diffusion processes, Bismut-Michel ([5]), Stroock ([24]), and Taniguchi ([26]) have already studied it by using the Malliavin calculus. Their results are a natural generalization of well-known Hörmander's theorem. Taniguchi considered the partially hypoelliptic problem on Riemannian manifolds. Our main purpose is an attempt to generalize their results for diffusion processes to jump processes on Euclidean spaces. We shall discuss the partially hypoelliptic problem by using the Malliavin calculus on the cád-lág space, for which we have to investigate the exponential decay of the Laplace transform for the law of the Malliavin covariance stated above. Hence the crucial point is to use certain fundamental inequalities about some semimartingales considered in [11] and [12]. When we apply the fundamental inequalities, we are faced with a difficulty how to treat random variables at a terminal time of processes included in the Malliavin covariance. In order to overcome the difficulty, we use time-reversed stochastic differential equations. After all, we need to consider an estimate similar to the one considered in [11] and [12]. It might be expected that the partial hypoellipticity for stochastic differential equations with jumps holds under a condition similar to the one obtained by Bismut-Michel, Stroock, and Taniguchi. However, we have to give some additional condition to the coefficient of the jump term in stochastic differential equations. The condition may seem to be technical, but it remains open to remove the condition. Adding the condition, we obtain main results on the partially hypoelliptic problem for stochastic differential equations with jumps. The non-degenerate condition in our main theorem seems to be complicated, but it is essentially a generalization of the one for diffusion processes.

The organization of this paper is as follows: in Section 2, we shall give some preparation and state main results. In order to understand the result for general processes with jumps, we divide into two cases. In Section 3, we shall study time-reversed stochastic differential equations. In Section 4, we shall give a proof of the result for diffusion processes. This yields a new proof of Taniguchi's result ([26]). In Section 5, we shall finish the proof for general processes with jumps.

2. Preliminaries and main results

Let Ω be the cád-lág space $D(\boldsymbol{R}_+ \to \boldsymbol{R}^d)$. For $t \geq 0$, let X_t be the projection from Ω to $\boldsymbol{R}^d$ such that $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Put σ-fields $\mathcal{W}_t = \bigcap_{\varepsilon>0} \sigma[X_s\,;\ s \leq t+\varepsilon]$ and $\mathcal{W} = \bigvee_{t\geq 0} \mathcal{W}_t$ as usual. Set $\Delta X_s = X_s - X_{s-}$ and $\nu(d\theta) = |\theta|^{-d-\alpha}\,d\theta$ for $0<\alpha<2$. We shall consider a probability measure P on the measurable space $(\Omega, \mathcal{W})$ such that

$$J(dt, d\theta) = \sharp\{s \in dt\ ;\ 0 \neq \Delta X_s \in d\theta\}$$

is a Poisson random measure on $\boldsymbol{R}_+ \times \boldsymbol{R}^d$ with the intensity $\nu(d\theta)\,dt$, and that process $\{w_t\}_t$ defined by

$$w_t = X_t - X_0 - \int_0^t \int_{|\theta|\leq 1} \theta\, \widetilde{J}(ds, d\theta) - \int_0^t \int_{|\theta|>1} \theta\, J(ds, d\theta)$$

is a d-dimensional $(\mathcal{W}_t, P)$-Brownian motion starting at $0 \in \boldsymbol{R}^d$, where

$$\widetilde{J}(dt, d\theta) = J(dt, d\theta) - \nu(d\theta)\,dt.$$

We remark that the measure $\widetilde{J}(dt, d\theta)$ generates $(\mathcal{W}_t, P)$-martingales.

Throughout this paper, $c.$'s denote certain positive absolute constants, which may change every lines. Let $\{A_i = a_i(x) \cdot \partial_x\}_{i=0}^d$ be a family of smooth vector fields on $\boldsymbol{R}^m$ such that derivatives of all orders of a_i' are bounded, where the symbol $'$ denotes the derivative with respect to $x \in \boldsymbol{R}^m$, that is,

$$\phi'(x) = \Big(\frac{\partial \phi^j}{\partial x_k}(x)\Big)_{1\leq j,\, k\leq m}$$

for any $\boldsymbol{R}^m$-valued C^1-mapping $\phi(x)$ defined on $\boldsymbol{R}^m$. Let $b(x,\theta)$ be an $\boldsymbol{R}^m$-valued C^1-mapping on $\boldsymbol{R}^m \times \boldsymbol{R}^d$ such that $b(x,\theta)$ and $\partial_\theta b(x,\theta)$ are smooth in $x \in \boldsymbol{R}^m$. Moreover, we assume that it satisfies the following conditions:

$$\begin{aligned}
&b(x,0) = 0, \quad |b(0,\theta)| \leq c., \quad x \cdot \int b(x,\theta)\nu(d\theta) \leq c.\ (1 + |x|^2),\\
&\int \{|b(x,\theta)|^2 \wedge (1 + |x|^2)\}\ \nu(d\theta) \leq c.\ (1 + |x|^2),\\
&\|b'(x,\theta)\| \leq c. < 1, \quad \Big\|\int b'(x,\theta)\ \nu(d\theta)\Big\| + \int \big(\|b'(x,\theta)\|^2 \wedge 1\big)\ \nu(d\theta) \leq c.,\\
&\boldsymbol{R}^m \ni x \longmapsto \int b'(x,\theta)\ \nu(d\theta) \in \boldsymbol{R}^m \otimes \boldsymbol{R}^m \text{ is continuous},\\
&\|\partial_\theta b(0,\theta)\| + \|\partial_\theta\big((\theta\cdot\partial_\theta)b\big)(x,\theta)\| \leq c.,\\
&\partial_\theta b'(x,\theta) \text{ is bounded and continuous},\\
&\boldsymbol{R}^m \ni x \longmapsto x + b(x,\theta) \in \boldsymbol{R}^m \text{ is a homeomorphism},
\end{aligned}$$

where integrals by the measure $\nu(d\theta)$ are defined in the sense of the principal value:

$$\int \psi(x,\theta)\ \nu(d\theta) = \lim_{\varepsilon\downarrow 0} \int_{|\theta|>\varepsilon} \psi(x,\theta)\ \nu(d\theta)$$

for any $\boldsymbol{R}^m$-valued mapping $\psi(x,\theta)$ defined on $\boldsymbol{R}^m \times \boldsymbol{R}^d$. Let $f(x,\theta)$ be the $\boldsymbol{R}^m$-valued mapping on $\boldsymbol{R}^m \times \boldsymbol{R}^d$ satisfying

$$y = x + b(x,\theta) \Longleftrightarrow x = y + f(y,\theta).$$

Let $\Theta_2 = \boldsymbol{R}^d \backslash \{0\}$ and $\Theta = \Theta_2 \bigcup \{0, 1, \ldots, d\}$. Define an operator $\wp_\theta$ $(\theta \in \Theta)$ by

$$\wp_\theta \Phi \ \left(\equiv \left(\wp_\theta \phi\right)(x) \cdot \partial_x\right)$$
$$= \begin{cases} [A_\theta, \Phi] + \dfrac{1}{2} \displaystyle\sum_{i=1}^d [A_i, [A_i, \Phi]] & \text{if } \theta = 0, \\ [A_\theta, \Phi] & \text{if } \theta \in \{1, \ldots, d\}, \\ \left\{ \left(I + f'(x, \theta)\right)^{-1} \phi\left(x + f(x, \theta)\right) - \phi(x) \right\} \cdot \partial_x & \text{if } \theta \in \Theta_2 \end{cases}$$

for any vector field $\Phi = \phi(x) \cdot \partial_x$, where $[A_\theta, \Phi] = A_\theta \Phi - \Phi A_\theta$.

For $x_0 \in \boldsymbol{R}^m$, we shall consider the following stochastic differential equation

$$(1) \qquad x_t = x_0 + \int_0^t a_0(x_s)\, ds + \int_0^t \sum_{i=1}^d a_i(x_s) \circ dw_s^i + \int_0^t \int b(x_{s-}, \theta)\ J(ds, d\theta),$$

where $\circ dw_s^i$ denotes the stochastic integral in the Stratonovich sense. Here we give a remark. The precise meaning of the last term of (1) is the following sum:

$$\int_0^t \int_{|\theta| \le 1} b(x_{s-}, \theta)\ \widetilde{J}(ds, d\theta) + \int_0^t \int_{|\theta| > 1} b(x_{s-}, \theta)\ J(ds, d\theta)$$
$$+ \int_0^t \int_{|\theta| \le 1} b(x_s, \theta)\ \nu(d\theta)\, ds.$$

From now on, we often make use of such simple representations without any comments. Using the stochastic integral in the Ito sense, we can rewrite (1) as follows:

$$x_t = x_0 + \int_0^t \widetilde{a}_0(x_s)\, ds + \int_0^t \sum_{i=1}^d a_i(x_s)\ dw_s^i + \int_0^t \int b(x_{s-}, \theta)\ J(ds, d\theta),$$

where

$$\widetilde{a}_0(x) = a_0(x) + \frac{1}{2} \sum_{i=1}^d a_i'(x)\ a_i(x).$$

From the assumption on the coefficients of (1), there exists a pathwise unique solution $x_t(x_0) = x_t(x_0, X)$. For $s \le t$, define

$$x_{s,t}(x, X) = x_{t-s}(x, \theta_s X),$$

where θ_s is the shift operator on Ω such that $(\theta_s X)_u = X_{s+u}$. By a discussion as in [6], we see that the mapping $x_{s,t}$ defines a stochastic flow of diffeomorphisms on $\boldsymbol{R}^m$ (cf. [25]).

Let $\pi = \pi(x)$ be an $\boldsymbol{R}^n$-valued smooth mapping on $\boldsymbol{R}^m$ such that all derivatives of any orders are bounded, where $m \geq n$. We shall call $\{\pi(x_t)\}_t$ *the image process* of $\{x_t\}_t$ by the mapping π. Our main purpose is to study the partially hypoelliptic problem for jump processes. The main theorem on the problem may seem strange at first sight, and its proof also seems to be complicated. However, considering the theorem only for diffusion processes, it will be understood that the theorem is essentially a generalization of well-known Hörmander's theorem, and the proof is simple and natural. So we shall give a proof of the result restricted to the case for diffusion processes before doing it of the general result. This would make our method of the proof clear and our main theorem easy to understand.

First we shall mention the result about diffusion processes, which yields a new proof of Taniguchi's results ([26]) on Euclidean spaces. Let $\mathcal{A}_k$ $(k \geq 0)$ be families of vector fields on $\boldsymbol{R}^m$ defined by

$$\mathcal{A}_0 = \{A_1, \dots, A_d\},$$
$$\mathcal{A}_k = \left\{\wp_i \Phi \ ; \ \Phi \in \mathcal{A}_{k-1}, \quad i = 0,\ 1, \dots, d\right\} \quad (k \geq 1).$$

Theorem 1. *Assume that there exist non-negative integer l such that*

$$\inf_{z \in S^{n-1}} \sum_{k=0}^{l} \sum_{\phi(x)\cdot\partial_x \in \mathcal{A}_k} \left(z \cdot \pi'(x)\phi(x)\right)^2 > 0 \tag{2}$$

for any $x \in \boldsymbol{R}^m$. Then the law of random variable $\pi(x_T)$ admits a smooth density with respect to the Lebesgue measure on $\boldsymbol{R}^n$.

The proof of Theorem 1 will be given in Section 4.

REMARK 1. Condition (2) in Theorem 1 is called the Hörmander condition for the partially hypoelliptic problem. It is equivalent to the following condition:

$$\dim (d\pi)_x \, \mathcal{L}\left[\bigcup_{k=0}^{l} \mathcal{A}_k\right]\Bigg|_x = n \tag{3}$$

for any $x \in \boldsymbol{R}^m$, where $(d\pi)_x$ denotes the differential of mapping π at x and $\mathcal{L}[\mathcal{A}]|_x$ denotes the linear space generated by a family $\mathcal{A}$ of vector fields at x.

EXAMPLE 1. Let $m = 3$, $n = 2$, $d = 1$ and $\pi((x_1, x_2, x_3)^*) = (x_1, x_2)^*$. We shall consider the following stochastic differential equation:

$$\begin{cases} x_1(t) = t + \int_0^t x_2(s) \circ dw_s, \\ x_2(t) = \int_0^t x_3(s) \circ dw_s, \\ x_3(t) = t + \int_0^t \big(x_2(s) + x_3(s)\big) \circ dw_s. \end{cases}$$

Since $x_3(t) = x_1(t) + x_2(t)$, it is clear that the law of $x(T)$ doesn't admit a smooth density. In this case

$$\begin{aligned} A_0 &= (1, 0, 1)^* \cdot \partial_x, & A_1 &= (x_2,\ x_3,\ x_2 + x_3)^* \cdot \partial_x, \\ \wp_0 A_1 &= (0, 1, 1)^* \cdot \partial_x, & \wp_1 \wp_0 A_1 &= (-1, -1, -2)^* \cdot \partial_x. \end{aligned}$$

Since the dimension of the linear space spanned by the vectors

$$(0, 1)^* = \pi\big((0, 1, 1)^*\big), \quad (-1, -1)^* = \pi\big((-1, -1, -2)^*\big)$$

is equal to 2, the law of random variable $\pi(x(T))$ admits a smooth density from Theorem 1 (cf. Remark 1).

It may be expected that adding a certain regularity condition on $b(x, \theta)$, the partial hypoellipticity for SDE with jumps holds under a condition corresponding to Hörmander's one. But this conjecture remains open. Here we shall consider the case: the coefficient $b(x, \theta)$ of the jump term of (1) has the following expression

$$b(x_1, x_2, \theta) = \begin{pmatrix} b_1(x_1, 0\ , \theta) \\ b_2(x_1, x_2, \theta) \end{pmatrix},$$

where $x = (x_1, x_2) \in \boldsymbol{R}^n \times \boldsymbol{R}^{m-n}$, $b_1(x, \theta)$ is an $\boldsymbol{R}^n$-valued mapping defined on $\boldsymbol{R}^m \times \boldsymbol{R}^d$, and $b_2(x, \theta)$ is an $\boldsymbol{R}^{m-n}$-valued mapping defined on $\boldsymbol{R}^m \times \boldsymbol{R}^d$.

We say that an $\boldsymbol{R}^m$-valued mapping $h(x, \theta)$ defined on $\boldsymbol{R}^m \times \boldsymbol{R}^d$ belongs to the class $C_\nu^{\infty,b}$ if the mapping

$$h_{\mu\gamma}(x, \theta) = (\partial_x)^\mu\ (\theta \cdot \partial_\theta)^\gamma h(x, \theta)$$

is bounded and continuous for any $\mu \in \boldsymbol{Z}_+^m$ and any $\gamma \in \boldsymbol{Z}_+$, and satisfies that

$$\sup_x \left\{ \left| \int h_{\mu\gamma}(x, \theta)\ \nu(d\theta) \right|^2 + \int |h_{\mu\gamma}(x, \theta)|^2\ \nu(d\theta) \right\} < \infty,$$

where $(\partial_x)^\mu = (\partial_{x_1})^{\mu_1} \cdots (\partial_{x_m})^{\mu_m}$ for $\mu = (\mu_1, \dots, \mu_m) \in \boldsymbol{Z}_+^m$. From now on, we always assume the following condition.

ASSUMPTION ([R]). Each component of $b'(x, \theta)$ belongs to the class $C_\nu^{\infty, b}$.

We shall introduce some notations. Choose ρ such that $0 < \rho < ((2\alpha) \wedge 1)/4$. Set $\tau = 2\alpha/(\alpha + 2)$,

$$\sigma(\theta) = \begin{cases} (1 - 4\rho)/4 & \text{if } \theta \in \{0, 1, \dots, d\} \ , \\ (\alpha - 2\rho)/(\alpha + 2) & \text{if } \theta \in \Theta_2, \end{cases}$$

and $\tau(\theta_1, \dots, \theta_k) = \sigma(\theta_1) \cdots \sigma(\theta_k)\tau$ for $\theta_1, \dots, \theta_k \in \Theta$. Define a probability measure $\overline{\boldsymbol{\nu}}(d\theta)$ on Θ as follows:

$$\overline{\boldsymbol{\nu}}(d\theta) = c. \left\{ \sum_{i=0}^{d} I_{\{i \in d\theta\}} + \widetilde{\nu}(d\theta) I_{\{\theta \in \Theta_2\}} \right\},$$

where $\widetilde{\nu}(d\theta) = \left(|\theta|^2 \wedge 1\right) \nu(d\theta)$. Let $\wp_\theta(\lambda)$ be an operator acting on vector fields on $\boldsymbol{R}^m$ such that

$$\begin{aligned} &\wp_\theta(\lambda)\Phi \ \left(\equiv \left(\wp_\theta(\lambda)\phi\right)(x) \cdot \partial_x\right) \\ &= \begin{cases} \wp_\theta \Phi & \text{if } \theta \in \{0, 1, \dots d\}, \\ \lambda^{1-\sigma(\theta)} \ \wp_{\theta/\lambda^{1-\sigma(\theta)}} \Phi & \text{if } \theta \in \Theta_2. \end{cases} \end{aligned}$$

For $\theta_0 \in \Theta \backslash \{0\}$, and $\theta_1, \dots, \theta_k \in \Theta$, let $\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k} = \phi^\lambda_{\theta_0\theta_1\cdots\theta_k}(x) \cdot \partial_x$ be the vector fields on $\boldsymbol{R}^m$ defined by

$$\begin{aligned} \Phi^\lambda_{\theta_0} &= \begin{cases} A_{\theta_0} & \text{if } \theta_0 \in \{1, \dots, d\}, \\ \widetilde{A}^\lambda_{\theta_0} & \text{if } \theta_0 \in \Theta_2, \end{cases} \\ \Phi^\lambda_{\theta_0\theta_1\cdots\theta_k} &= \wp_{\theta_k}\left(\lambda^{\tau(\theta_1, \dots, \theta_{k-1})}\right) \cdots \wp_{\theta_1}\left(\lambda^\tau\right) \Phi^\lambda_{\theta_0}, \end{aligned}$$

where $\widetilde{f}_\theta(x) = \partial_\theta f(x, \theta)\theta$ and $\widetilde{A}^\lambda_\theta = \lambda^{2-\tau} \widetilde{f}_{\theta/\lambda^{2-\tau}}(x) \cdot \partial_x$.

Theorem 2. *Assume that there exist non-negative integer l and three positive constants κ, ε and σ with $\sigma + 5\varepsilon < 2\varepsilon_l$ such that*

$$\begin{aligned} (4) \qquad &\inf_{|x| \le \lambda^\varepsilon} \inf_{z \in S^{n-1}} \sum_{k=0}^{l} \int_{\theta_0 \neq 0} \overline{\boldsymbol{\nu}}(d\theta_0) \int \overline{\boldsymbol{\nu}}(d\theta_1) \cdots \overline{\boldsymbol{\nu}}(d\theta_k) \\ &\times \lambda^{2\sigma} \left\{ \left(z \cdot \pi'(x) \phi^\lambda_{\theta_0\theta_1\cdots\theta_k}(x)\right)^2 \wedge 1 \right\} \ge c. \ \lambda^{2\kappa} \end{aligned}$$

for sufficiently large λ, *where*

$$\varepsilon_l = \frac{\tau}{2}\left\{\left(\frac{1-4\rho}{4}\right)\wedge\left(\frac{\alpha-2\rho}{\alpha+2}\right)\right\}^l.$$

Then the law of random variable $\pi(x_T)$ *admits a smooth density with respect to the Lebesgue measure on* $\boldsymbol{R}^n$.

Let $\{A_{d+i}\}_{i=1}^d$ be the vector fields on $\boldsymbol{R}^m$ defined by

$$A_{d+i}\big(\equiv a_{d+i}(x)\cdot\partial_x\big) = \frac{\partial f}{\partial\theta^i}(x,0)\cdot\partial_x.$$

Set $\wp_{d+i}\Phi = [A_{d+i},\Phi]$ $(i=1,\dots,d)$ for any vector field Φ, and let $\tilde{\mathcal{A}}_k$ $(k\geq 0)$ be a family of vector fields on $\boldsymbol{R}^m$ defined by

$$\begin{aligned}\tilde{\mathcal{A}}_0 &= \{A_1,\dots,A_d,A_{d+1},\dots,A_{2d}\},\\ \tilde{\mathcal{A}}_k &= \{\wp_i\Phi\,;\,i=0,\ 1,\dots,d,\ d+1,\dots,2d,\ \Phi\in\tilde{\mathcal{A}}_{k-1}\}\quad(k\geq 1).\end{aligned}$$

Corollary 1. *Assume that there exists a non-negative integer* l *such that*

$$\inf_x\inf_{z\in S^{n-1}}\sum_{k=0}^{l}\sum_{\phi(x)\cdot\partial_x\in\tilde{\mathcal{A}}_k}\big(z\cdot\pi'(x)\phi(x)\big)^2>0. \tag{5}$$

Then the assumption of Theorem 2 *is satisfied, and the law of random variable* $\pi(x_T)$ *admits a smooth density with respect to the Lebesgue measure on* $\boldsymbol{R}^n$.

In Section 5, we will prove Theorem 2 and Corollary 1.

REMARK 2. We remark that condition (5) is a generalization of condition (2) in Theorem 1 and the conditions obtained by Léandre in [15] and [16].

EXAMPLE 2. Let $m=4$, $n=3$, $d=1$ and $\pi\big((x_1,x_2,x_3,x_4)^*\big)=(x_1,x_2,x_3)^*$. Let $\delta>0$ be a sufficiently small constant, and set $\delta_1=1+\delta$. We shall consider the following stochastic differential equation:

$$\begin{cases} x_1(t) = -\int_0^t \int b_1(\theta)\ J(ds, d\theta), \\ x_2(t) = \int_0^t x_4(s)\, ds + \int_0^t x_1(s) \circ dw_s, \\ x_3(t) = -\int_0^t \int x_2(s-) b_3(\theta)\ J(ds, d\theta), \\ x_4(t) = \int_0^t \int \big(-x_1(s-) b_1(\theta) + 2^{-1} b_1(\theta)^2 - x_2(s-) b_3(\theta)\big)\ J(ds, d\theta), \end{cases}$$

where $b_1(\theta)$ and $b_3(\theta)$ are certain bounded functions on $\boldsymbol{R}$ such that $b_1(\theta) = 2^{-1}|\theta|^\delta \theta$ and $b_3(\theta) = 2^{-1}\theta$ for $|\theta| \leq 1$.

Since $x_4(t) = 2^{-1} x_1(t)^2 + x_3(t)$, it is clear that the law of $x(T)$ doesn't admit a smooth density. In this case

$$\begin{aligned} & A_1 = (0, x_1, 0, 0)^* \cdot \partial_x, \\ & f(x, \theta) = 2^{-1}(|\theta|^\delta \theta,\ 0,\ \theta x_2,\ |\theta|^\delta \theta x_1 + \theta x_2 + 4^{-1}|\theta|^{2\delta+2})^*, \\ & \widetilde{A}_\theta = \widetilde{f}_\theta(x) \cdot \partial_x = 2^{-1}(\delta_1 |\theta|^\delta \theta,\ 0,\ \theta x_2,\ \delta_1 |\theta|^\delta \theta x_1 + \theta x_2 + 2^{-1}\delta_1 |\theta|^{2\delta+2})^* \cdot \partial_x, \\ & A_2 = \partial_\theta f(x, 0) \cdot \partial_x = 2^{-1}(0, 0, x_2, x_2)^* \cdot \partial_x, \\ & \wp_1 \widetilde{A}_\theta = 2^{-1}(0,\ -\delta_1 |\theta|^\delta \theta,\ \theta x_1,\ \theta x_1)^* \cdot \partial_x, \\ & \wp_\eta \wp_1 \widetilde{A}_\theta = 4^{-1}(\delta_1 |\theta|^\delta + |\eta|^\delta)\theta\eta \quad (0, 0, 1, 1)^* \cdot \partial_x \end{aligned}$$

for $|\theta|$, $|\eta| \leq 1$. This example doesn't satisfy condition (5) in Corollary 1, but satisfies the assumption in Theorem 2. In fact,

$$\begin{aligned} I_x^z \ =\ & \int_{\theta \in \Theta_2} \left\{ \big(\widetilde{z} \cdot \phi_\theta^\lambda(x)\big)^2 \wedge 1 \right\} \overline{\boldsymbol{\nu}}(d\theta) \\ & + \int_{\theta \in \Theta_2} \left\{ \big(\widetilde{z} \cdot \phi_{\theta 1}^\lambda(x)\big)^2 \wedge 1 \right\} \overline{\boldsymbol{\nu}}(d\theta) \\ & + \int_{\theta \in \Theta_2} \int_{\eta \in \Theta_2} \left\{ \big(\widetilde{z} \cdot \phi_{\theta 1 \eta}^\lambda(x)\big)^2 \wedge 1 \right\} \overline{\boldsymbol{\nu}}(d\eta)\, \overline{\boldsymbol{\nu}}(d\theta) \\ \geq\ & c. \int_{|\theta| \leq 1} \left\{ (\lambda^{-(2-\tau)\delta} \delta_1 |\theta|^\delta z_1 + x_2 z_3)^2 \wedge 1 \right\} |\theta|^4\ \nu(d\theta) \\ & + c. \int_{|\theta| \leq 1} \left\{ (-\lambda^{-(2-\tau)\delta} \delta_1 |\theta|^\delta z_2 + x_1 z_3)^2 \wedge 1 \right\} |\theta|^4\ \nu(d\theta) \\ & + c. \int_{|\theta| \leq 1} \int_{|\eta| \leq 1} \left\{ \big(\lambda^{-\widetilde{\sigma}\delta}(\delta_1 |\theta|^\delta + |\eta|^\delta) z_3\big)^2 \wedge 1 \right\} |\theta|^4 |\eta|^4\ \nu(d\eta)\ \nu(d\theta) \\ \geq\ & c.\lambda^{-2(2-\tau)\delta}({z_1}^2 + {z_2}^2) \int_{|\theta| \leq 1} (\delta_1 |\theta|^{2\delta} \wedge 1) |\theta|^4\ \nu(d\theta) I_{\{z_3 = 0\}} \end{aligned}$$

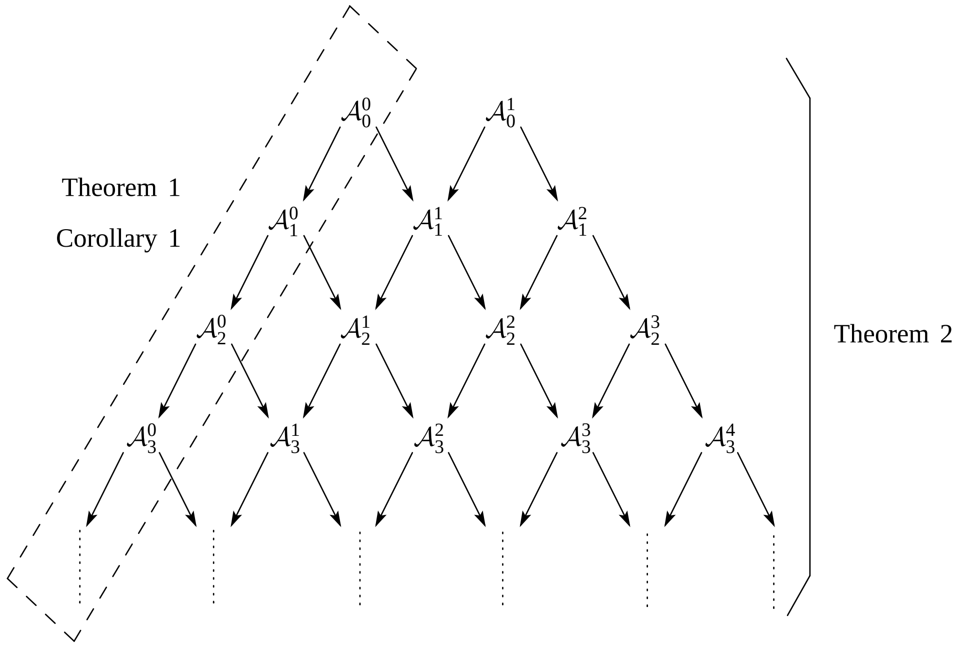
$$+ c.\lambda^{-2\widetilde{\sigma}\delta} {z_3}^2 \int_{|\theta|\le 1}\int_{|\eta|\le 1} \Big\{(\delta_1|\theta|^{\delta} + |\eta|^{\delta})^2 \wedge 1\Big\} |\theta|^4 |\eta|^4 \, \nu(d\eta)\, \nu(d\theta) I_{\{z_3\neq 0\}}$$
$$\ge c.\big(I_{\{z_3=0\}} + {z_3}^2 I_{\{z_3\neq 0\}}\big)\lambda^{-2\widetilde{\sigma}\delta},$$

where $\widetilde{\sigma} = (2-\tau)\vee\{\tau(2\mu+2)/(\alpha+2)\}$. If $0<\delta<\sigma/\widetilde{\sigma}$, condition (4) in Theorem 2 is satisfied. Hence the law of random variable $\pi(x(T))$ admits a smooth density.

REMARK 3. Let $\mathcal{A}_k^j$ $(k\ge 0,\ 0\le j\le k+1)$ be a family of vector fields on $\boldsymbol{R}^m$ defined by

$$\mathcal{A}_0^0 = \tilde{A}_0,\ \ \mathcal{A}_0^1 = \{\tilde{A}_\theta^\lambda\}$$
$$\mathcal{A}_k^j = \{\wp_i(\lambda^{\tau(\theta_1,\dots,\theta_{k-1})})\Psi\ ;\ i=0,\ 1,\dots,2d,\ \sharp(k)=j,\ \Psi\in\mathcal{A}_{k-1}^j\}$$
$$\bigcup\{\wp_\theta(\lambda^{\tau(\theta_1,\dots,\theta_{k-1})})\Phi\ ;\ \theta\in\Theta_2,\ \sharp(k)=j-1,\ \Phi\in\mathcal{A}_{k-1}^{j-1}\}\ (k\ge 1),$$

where $\sharp(k)=\sharp\{h\in\{1,\dots,k-1\}\,;\,\theta_h\in\Theta_2\}$. Then vector field $\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}$ is an element in $\bigcup_{j=0}^{k+1}\mathcal{A}_k^j$. Theorem 1 and Corollary 1 corresponds to the case of $\bigcup_{k\ge 0}\mathcal{A}_k^0$. Theorem 2 says that it is possible to extend Theorem 1 and Corollary 1 to the case of $\bigcup_{k\ge 0}\bigcup_{j=0}^{k+1}\mathcal{A}_k^j$.



We shall prove Theorem 1, Theorem 2 and Corollary 1 by using the Malliavin calculus. The Jacobi matrix $Z_t = ((\partial/\partial x_0^k)x_{0,t}^j(x_0,X))_{1\le j,k\le m}$ of diffeomorphism

$x_{0,t}(x_0, X)$ satisfies the following linear SDE:

(6)
$$Z_t = I + \int_0^t a_0'(x_s) Z_s \, ds + \int_0^t \sum_{i=1}^d a_i'(x_s) Z_s \circ dw_s^i + \int_0^t \int b'(x_{s-}, \theta) Z_{s-} \; J(ds, d\theta),$$

where $I = (\delta_k^j)_{1 \le j, k \le m}$. Let U_t be the solution to the linear SDE

(7)
$$U_t = I - \int_0^t U_s a_0'(x_s) \, ds - \int_0^t \sum_{i=1}^d U_s a_i'(x_s) \circ dw_s^i + \int_0^t \int U_{s-} \{ \big(I + b'(x_{s-}, \theta)\big)^{-1} - I \} \; J(ds, d\theta).$$

From the Ito formula, we see that

$$Z_t U_t = U_t Z_t = I.$$

Since a_0', $\{a_i', a_i''\}_{i=1}^d$ and b' are bounded, it is a routine work to prove that

(8)
$$E_P \left[\sup_{0 \le s \le t} \big(|x_s|^p + \|Z_s\|^p + \|U_s\|^p \big) \right] < \infty$$

for all $p > 1$ and $t \ge 0$ (cf. [25]). For a matrix B, we will denote its transpose by B^*. Define an $(m \times m)$-matrices valued process $V_t = V_t(X)$ as follows:

$$V_t = Z_t \int_0^t \frac{1}{A_{s-}} \left\{ \sum_{i=1}^d \big(U_s a_i(x_s)\big) \big(U_s a_i(x_s)\big)^* \, ds + \int \big(U_{s-} \widetilde{b}_\theta(x_{s-})\big) \big(U_{s-} \widetilde{b}_\theta(x_{s-})\big)^* J(ds, d\theta) \right\},$$

where $A_s = (1 + |x_s|^2)(1 + \|U_s\|^2)$ and $\widetilde{b}_\theta(x) = (I + b'(x, \theta))^{-1} \partial_\theta b(x, \theta) \theta$. Let $K_t = K_t(X)$ be a non-negative definite, $(n \times n)$-matrices valued process defined by

$$K_t = \pi'(x_t) \; V_t \; [\pi'(x_t) Z_t]^*.$$

We call K_t *the Malliavin covariance matrix* for the functional $\pi(x_t(X))$ on Ω.

Finally we shall consider a variation of the process $X = \{X_t\}_t$. First we shall introduce some notations. Set

$$l_i(t, X) = A_t^{-1} U_t a_i(x_t), \qquad l(t, X) = \big(l_1(t, X), \ldots, l_d(t, X)\big),$$

$$h(t,X,\theta) = A_t^{-1}U_t\widetilde{b}_\theta(x_t), \qquad H^\xi_{t,X}(\theta) = \exp\{\xi\cdot h(t,X,\theta)\}\theta$$

for $\xi \in \boldsymbol{R}^m$ such that $|\xi| < 1$. Define a perturbed process $X^\xi = \{X^\xi_t\}_t$ by

$$X^\xi_t = X_0 + w_t + \int_0^t l(s,X)^*\xi\,ds + \int_0^t\!\!\int H^\xi_{s,X}(\theta)\,J(ds,d\theta).$$

Then P and the law $P^\xi = P\circ(X^\xi)^{-1}$ of the process X^ξ are mutually absolutely continuous ([23]). We shall denote the conditional expectation of the Radon-Nikodym derivative $E_P[dP^\xi/dP \mid \mathcal{W}_t]$ by M^ξ_t. Set

$$\begin{aligned} x^\xi_t &= x_t(x_0, X^\xi),\ \ Z^\xi_t = Z_t(X^\xi),\ \ U^\xi_t = U_t(X^\xi),\\ V^\xi_t &= V_t(X^\xi),\quad K^\xi_t = K_t(X^\xi). \end{aligned}$$

Lemma 2.1. *For $T > 0$, if $(\det K_T)^{-1} \in \bigcap_{p>1} L^p(\Omega, P)$, then the law of $\pi(x_T)$ admits a smooth density with respect to the Lebesgue measure on $\boldsymbol{R}^n$.*

Proof. We shall give an outline of the proof. From the absolute continuity between P and P^ξ, the equality

$$\begin{aligned} (9)\qquad & E_P\Big[\, f\big(\pi(x_T)\big)\ \big[\big(\pi'(x_T)Z_T\big)^* K_T^{-1}\big]^j_k\,\Big]\\ &= E_P\Big[\, f\big(\pi(x^\xi_T)\big)\ \big[\big(\pi'(x^\xi_T)Z^\xi_T\big)^*(K^\xi_T)^{-1}\big]^j_k\, M^\xi_T\,\Big] \end{aligned}$$

holds for any test function f on $\boldsymbol{R}^n$, $1 \le j \le m$ and $1 \le k \le n$. By using the same argument as stated in [12], we see that random variables

$$\frac{\partial}{\partial\xi_j}\Big\{ f\big(\pi(x^\xi_T)\big)\big[\big(\pi'(x^\xi_T)Z^\xi_T\big)^*(K^\xi_T)^{-1}\big]^j_k M^\xi_T\Big\}$$

are integrable by P uniformly in ξ. Taking the differential of both sides of (9) at $\xi = 0$, we see that

$$\begin{aligned} (10)\quad & E_P\left[\Big(\frac{\partial f}{\partial x_k}\Big)\big(\pi(x_T)\big)\right]\\ &= -\,E_P\left[f\big(\pi(x_T)\big)\sum_{j=1}^m \frac{\partial}{\partial\xi_j}\Big\{\big[\big(\pi'(x^\xi_T)Z^\xi_T\big)^*(K^\xi_T)^{-1}\big]^j_k\, M^\xi_T\Big\}\Big|_{\xi=0}\right]. \end{aligned}$$

Define operators $\mathcal{D}(t) = (\mathcal{D}_1(t),\dots,\mathcal{D}_n(t))$ acting for smooth functionals $F(X)$ on Ω by

$$\big(\mathcal{D}_k(t)F\big)(X) = \sum_{j=1}^m \frac{\partial}{\partial\xi_j}\Big\{\big[\big(\pi'(x^\xi_T)Z^\xi_T\big)^*(K^\xi_T)^{-1}\big]^j_k M^\xi_T F(X^\xi)\Big\}\Big|_{\xi=0}.$$

Then equality (10) is expressed as follows:

$$E_P\left[\left(\frac{\partial f}{\partial x_k}\right)\left(\pi(x_T)\right)\right] = E_P\left[f\left(\pi(x_T)\right)\left((-\mathcal{D}_k(T))1\right)(X)\right].$$

Set $F_t(X) = (-\mathcal{D}_l(t)1)(X)$ and $g(x) = (\partial/\partial x_k)f(x)$. By an argument as in [12], we see that random variables

$$\frac{\partial}{\partial \xi_j}\left\{f\left(\pi(x_T^\xi)\right)\ \left[\left(\pi'(x_T^\xi)Z_T^\xi\right)^*(K_T^\xi)^{-1}\right]_k^j\ F_T(X^\xi)\ M_T^\xi\right\}$$

are integrable by P uniformly in ξ. Hence we have

$$\begin{aligned} E_P\left[\left(\frac{\partial^2 f}{\partial x_l \partial x_k}\right)\left(\pi(x_T)\right)\right] &= E_P\left[\left(\frac{\partial g}{\partial x_l}\right)\left(\pi(x_T)\right)\right] \\ &= E_P\left[g\left(\pi(x_T)\right)\left((-\mathcal{D}_l(T))1\right)(X)\right] \\ &= E_P\left[\left(\frac{\partial f}{\partial x_k}\right)\left(\pi(x_T)\right)\left((-\mathcal{D}_l(T))1\right)(X)\right] \\ &= E_P\left[f(\pi(x_T))((-\mathcal{D}_k(T))((-\mathcal{D}_l(T))1))(X)\right]. \end{aligned}$$

We remark $((-\mathcal{D}_k(T))((-\mathcal{D}_l(T))1))(X) \in \bigcap_{p>1} L^p(\Omega, P)$ by the same method as stated in [12]. Repeating such arguments, we obtain

$$\begin{aligned} &E_P\left[\left(\partial_x{}^\mu f\right)\left(\pi(x_T)\right)\right] = E_P\left[f\left(\pi(x_T)\right)\left((-\mathcal{D}(t))^\mu 1\right)(X)\right], \\ &\left((-\mathcal{D}(t))^\mu 1\right)(X) \in \bigcap_{p>1} L^p(\Omega, P) \end{aligned}$$

for any multi-index $\mu \in \boldsymbol{Z}_+^n$. The assertion of the lemma follows from the Sobolev lemma. □

3. Time-reversed processes

In this section, we shall give some remarks on time-reversed stochastic differential equations. Fix $0 \le t \le T$. Let $\widehat{X}_t = X_{T-} - X_{(T-t)-}$. Then $\{\widehat{X}_t\}_{t\in[0,T]}$ is also a Lévy process, and the law of $\{\widehat{X}_t\}_{t\in[0,T]}$ coincides with that of $\{X_t\}_{t\in[0,T]}$. In particular, $\{\widehat{w}_t\}_{t\in[0,T]}$ defined by $\widehat{w}_t = w_T - w_{T-t}$ is also a d-dimensional Brownian motion starting at $0 \in \boldsymbol{R}^d$, and $N(dt, d\theta)$ defined by

$$N\left([0,t); A\right) = J\left([T-t, T); A\right) \quad \left(t \in [0,T],\ \ A \in \mathcal{B}(\boldsymbol{R}^d)\right)$$

is also a Poisson random measure with the intensity $\nu(d\theta)\,dt$ (cf. [1], [22]). Put $\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)\,dt$.

For $x \in \boldsymbol{R}^m$, let $\widehat{x}_t$ be the solution to the following SDE:

$$\widehat{x}_t = x - \int_0^t a_0(\widehat{x}_s)\,ds - \int_0^t \sum_{i=1}^d a_i(\widehat{x}_s) \circ dw_s^i + \int_0^t \int f(\widehat{x}_{s-}, \theta)\; J(ds, d\theta). \tag{11}$$

From the assumption on the coefficients of (11), there exists a pathwise unique solution $\widehat{x}_t(x, X)$. For any $s \le t$ $(\le T)$, let

$$\widehat{x}_{s,t}(x, X) = \widehat{x}_{t-s}(x, \theta_s X).$$

Then the mapping $\widehat{x}_{s,t}$ defines a stochastic flow of diffeomorphisms on $\boldsymbol{R}^m$ as in [6]. The Jacobi matrix $\widehat{Z}_t = ((\partial/\partial x^k)\ \widehat{x}^j_{0,t}(x))_{1\le j,\,k\le m}$ of the mapping of diffeomorphism $\widehat{x}_t(x)$ satisfies the linear SDE:

$$\begin{aligned}\widehat{Z}_t \;=\; I &- \int_0^t a_0'(\widehat{x}_s)\widehat{Z}_s\,ds - \int_0^t \sum_{i=1}^d a_i'(\widehat{x}_s)\widehat{Z}_s \circ dw_s^i \\ &+ \int_0^t \int f'(\widehat{x}_{s-}, \theta)\widehat{Z}_{s-}\; J(ds, d\theta).\end{aligned} \tag{12}$$

Let $\widehat{U}_t$ be a solution to the linear SDE:

$$\begin{aligned}\widehat{U}_t \;=\; I &+ \int_0^t \widehat{U}_s a_0'(\widehat{x}_s)\,ds + \int_0^t \sum_{i=1}^d \widehat{U}_s a_i'(\widehat{x}_s) \circ dw_s^i \\ &+ \int_0^t \int \widehat{U}_{s-}\{(I + f'(\widehat{x}_{s-}, \theta))^{-1} - I\}\; J(ds, d\theta).\end{aligned} \tag{13}$$

Since $(I + f'(x, \theta))^{-1} - I = b'(x + f(x, \theta), \theta)$, equations (12) and (13) are parallel with equations (6) and (7). By using the Ito formula, we can easily check that

$$\widehat{Z}_t\widehat{U}_t = \widehat{U}_t\widehat{Z}_t = I.$$

Moreover, the following proposition holds.

Proposition 3.1.

$$\sup_{|x|\le\zeta} E_P\left[\sup_{\tau\le t}\{\,|(\partial_x)^\mu \widehat{x}_\tau(x)|^p + \|(\partial_x)^\mu \widehat{U}_\tau(x)\|^p\,\}\right] \le c_p\zeta^p \tag{14}$$

for any $p > 1$, $t \in [0, T]$, $\mu \in \boldsymbol{Z}_+^m$ *and* $\zeta > 1$.

Proof. The continuity and the differentiability of $\widehat{U}_t(x)$ with respect to x can be proved by a method as in [6]. From the assumptions for the coefficients of equations, the assertion of the proposition follows. □

Lemma 3.1. *For almost all ω, the equality*

$$x_{t-}(x, X) = \widehat{x}_{T-t}(x_T(x, X), \widehat{X}) \tag{15}$$

holds for any $t \in [0, T]$ and $x \in \boldsymbol{R}^m$.

Proof. For $n = 1, 2, \ldots$, define processes $w_n(t)$ and $\widehat{w}_n(t)$ by

$$\begin{aligned} w_n(t) &= \frac{n}{T}\left\{\left(\frac{k+1}{n}T - t\right) w\left(\frac{k}{n}T\right) + \left(t - \frac{k}{n}T\right) w\left(\frac{k+1}{n}T\right)\right\}, \\ \widehat{w}_n(t) &= \frac{n}{T}\left\{\left(\frac{k+1}{n}T - t\right) \widehat{w}\left(\frac{k}{n}T\right) + \left(t - \frac{k}{n}T\right) \widehat{w}\left(\frac{k+1}{n}T\right)\right\} \end{aligned}$$

for $t \in [(kT)/n, \{(k+1)T\}/n)$ $(k = 0, 1, \ldots, n-1)$. It is easy to check that

$$\widehat{w}_n(t) = w(T) - w_n(T-t)$$

for $t \in [0, T]$. We shall consider the following ordnary differential equations:

$$\begin{aligned} y_n(t) &= x + \int_0^t \left\{a_0\big(y_n(s)\big) + \sum_{i=1}^d a_i\big(y_n(s)\big)\frac{dw_n^i}{ds}(s)\right\} ds, \\ \widehat{y}_n(t) &= x - \int_0^t \left\{a_0\big(\widehat{y}_n(s)\big) + \sum_{i=1}^d a_i\big(\widehat{y}_n(s)\big)\frac{dw_n^i}{ds}(s)\right\} ds. \end{aligned}$$

Denote solutions of the above equations by $y_n(t, x, X)$ and $\widehat{y}(t, x, X)$, respectively. From the uniqueness of solutions, it is easy to see that for almost all $\omega \in \Omega$,

$$y_n(t, x, X) = \widehat{y}_n\big(T - t, y_n(T, x, X), \widehat{X}\big) \tag{16}$$

for any $t \in [0, T]$ and $x \in \boldsymbol{R}^m$. Note that when process $X = (X_t)$ is replaced by process $\widehat{X} = (\widehat{X}_t)$, system $\{X, (w_t), J(ds, d\theta) \text{ etc.}\}$ shall be replaced by system $\{\widehat{X}, (\widehat{w}_t), N(ds, d\theta) \text{ etc.}\}$.

Let $\{c_l\}_l$ be a decreasing sequence of positive numbers such that $c_l \downarrow 0$ as $l \to \infty$. Secondly we shall consider the following stochastic differential equations:

$$\begin{aligned} \eta_{l,n}(t) &= x + \int_0^t \left\{a_0\big(\eta_{l,n}(s)\big) + \sum_{i=1}^d a_i(\eta_{l,n}(s))\frac{dw_n^i}{ds}(s)\right\} ds \\ &\quad + \int_0^t \int_{|\theta|>c_l} b\big(\eta_{l,n}(s-), \theta\big)\, J(ds, d\theta), \\ \widehat{\eta}_{l,n}(t) &= x - \int_0^t \left\{a_0\big(\widehat{\eta}_{l,n}(s)\big) + \sum_{i=1}^d a_i\big(\widehat{\eta}_{l,n}(s)\big)\frac{dw_n^i}{ds}(s)\right\} ds \end{aligned}$$

$$+\int_0^t\int_{|\theta|>c_l} f\left(\widehat{\eta}_{l,n}(s-),\theta\right)\ J(ds,d\theta).$$

Solutions will be denoted by $\eta_{l,n}(t,x,X)$ and $\widehat{\eta}_{l,n}(t,x,X)$, respectively. Then it is easy to check that for almost all $\omega\in\Omega$,

$$\eta_{l,n}(t-,x,X)=\widehat{\eta}_{l,n}\left(T-t,\eta_{l,n}(T,x,X),\widehat{X}\right) \tag{17}$$

for all $t\in[0,T]$ and $x\in\boldsymbol{R}^m$.

Thirdly we shall consider the following stochastic differential equations:

$$\begin{aligned} x_l(t) &= x+\int_0^t a_0\big(x_l(s)\big)\,ds+\int_0^t\sum_{i=1}^d a_i\left(x_l(s)\right)\circ dw_s^i \\ &\quad +\int_0^t\int_{|\theta|>c_l} b\big(x_l(s-),\theta\big)\ J(ds,d\theta), \\ \widehat{x}_l(t) &= x-\int_0^t a_0\big(\widehat{x}_l(s)\big)\,ds-\int_0^t\sum_{i=1}^d a_i\left(\widehat{x}_l(s)\right)\circ dw_s^i \\ &\quad +\int_0^t\int_{|\theta|>c_l} f\left(\widehat{x}_l(s-),\theta\right)\ J(ds,d\theta). \end{aligned}$$

From the assumptions for the coefficients, there exist unique solutions $x_l(t,x,X)$ and $\widehat{x}_l(t,x,X)$ in the pathwise sense. For a positive integer M, set

$$\zeta(M)=\inf\left\{t>0\ ;\ \int_0^t\int_{|\theta|>M} J(ds,d\theta)\neq 0\right\}.$$

By an arguments as in Chapter VI, Section 7 of [9], we can show that for $N>0$,

$$\begin{aligned} &\sup_{|x|\le N}\ \sup_l\ E_P\left[\sup_{\tau\le t\wedge\zeta(M)}|(\partial_x)^\mu\eta_{l,n}(\tau,x,X)-(\partial_x)^\mu x_l(\tau,x,X)|^p\right]\longrightarrow 0, \\ &\sup_{|x|\le N}\ \sup_l\ E_P\left[\sup_{\tau\le t\wedge\zeta(M)}|(\partial_x)^\mu\widehat{\eta}_{l,n}(\tau,x,X)-(\partial_x)^\mu \widehat{x}_l(\tau,x,X)|^p\right]\longrightarrow 0 \end{aligned}$$

as $n\to\infty$ for any $p>1$, $t\in[0,T]$, and $\mu\in\boldsymbol{Z}_+^m$. It follows immediately by the Sobolev inequality that

$$\begin{aligned} &\sup_l\ E_P\left[\sup_{|x|\le N}\ \sup_{\tau\le t\wedge\zeta(M)}|(\partial_x)^\mu\eta_{l,n}(\tau,x,X)-(\partial_x)^\mu x_l(\tau,x,X)|^p\right]\longrightarrow 0, \\ &\sup_l\ E_P\left[\sup_{|x|\le N}\ \sup_{\tau\le t\wedge\zeta(M)}|(\partial_x)^\mu\widehat{\eta}_{l,n}(\tau,x,X)-(\partial_x)^\mu \widehat{x}_l(\tau,x,X)|^p\right]\longrightarrow 0 \end{aligned}$$

as $n \to \infty$ for any $p > 1$, $t \in [0, T]$, and $\mu \in \boldsymbol{Z}_+^m$. By taking the limits in (17), we can obtain that for almost all $\omega \in \{\omega\,;\ \zeta(M) > T\}$,

$$x_l(t-, x, X) = \widehat{x}_l\big(T - t, x_l(T, x, X), \widehat{X}\big) \tag{18}$$

for all $t \in [0, T]$ and $x \in \boldsymbol{R}^m$.

Finally by the same argument as stated above, we can prove that

$$E_P\left[\sup_{|x|\le N}\ \sup_{\tau\le t\wedge\zeta(M)} |(\partial_x)^\mu x_l(\tau, x, X) - (\partial_x)^\mu x(\tau, x, X)|^p\right] \longrightarrow 0,$$
$$E_P\left[\sup_{|x|\le N}\ \sup_{\tau\le t\wedge\zeta(M)} |(\partial_x)^\mu \widehat{x}_l(\tau, x, X) - (\partial_x)^\mu \widehat{x}(\tau, x, X)|^p\right] \longrightarrow 0$$

as $l \to \infty$ for any $p > 1$, $t \in [0, T]$, and $\mu \in \boldsymbol{Z}_+^m$. By taking the limits in (18), we can obtain that for almost all $\omega \in \{\omega\,;\ \zeta(M) > T\}$,

$$x(t-, x, X) = \widehat{x}(T - t, x(T, x, X), \widehat{X}) \tag{19}$$

for all $t \in [0, T]$ and $x \in \boldsymbol{R}^m$. Since $P[\zeta(M) \le T] \to 0$ as $M \to \infty$, the assertion of the lemma holds. □

By Lemma 2.1, our goal is to find sufficient conditions under which

$$(\det K_T)^{-1} \in \bigcap_{p>1} L^p(\Omega, P)$$

is satisfied. Since

$$E_P\left[(\det K_T)^{-p}\right] \le E_P\left[\Big(\inf_{z\in S^{n-1}}(z\cdot K_T z)\Big)^{-pn}\right]$$
$$\le c.\ \sup_{z\in S^{n-1}} E_P\left[(z\cdot K_T z)^{-(pn+4n-4)}\right] + c.$$

and

$$E_P\left[(z\cdot K_T z)^{-p}\right] = \Gamma(p)^{-1}\int_0^\infty \lambda^{p-1} E_P\left[\exp\{-\lambda(z\cdot K_T z)\}\right]\, d\lambda,$$

it suffices to find conditions under which

$$\sup_{z\in S^{n-1}} E_P\left[\exp\{-\lambda(z\cdot K_T z)\}\right] = o(\lambda^{-p}) \quad (\lambda \to \infty) \tag{20}$$

is satisfied for all $p > 1$.

First we shall give some notations and remarks. For $\lambda > 1$, $z \in S^{n-1}$, $x \in \boldsymbol{R}^m$, and a vector field Φ, we shall introduce the following notations:

$$\mathcal{Q}_T^z(x, \lambda, \Phi) = \lambda^4 \int_0^T \left(z \cdot \pi'(x)\widehat{U}_s(x)\phi(\widehat{x}_s(x))\right)^2 ds,$$
$$\widehat{\mathcal{Q}}_T^z(x, \lambda, \Phi) = \lambda^4 \int_0^T \left\{\left(z \cdot \pi'(x)\widehat{U}_s(x)\phi(\widehat{x}_s(x))\right)^2 \wedge \frac{1}{\lambda^2}\right\} ds,$$
$$\mathcal{N}_T^z(x, \lambda, \widetilde{A}.) = \lambda^4 \int_0^T \int \left(z \cdot \pi'(x)\widehat{U}_{s-}(x)\widetilde{f}_\theta(\widehat{x}_{s-}(x))\right)^2 N(ds, d\theta),$$

where $\widetilde{A}_\theta = \widetilde{f}_\theta(x) \cdot \partial_x$. Define an $(n \times n)$-matrices valued process $\widetilde{K}_T$ by

$$\widetilde{K}_T = \left(\pi'(x_T)Z_T\right) \int_0^T \left\{\sum_{i=1}^d \left(U_s a_i(x_s)\right)\left(U_s a_i(x_s)\right)^* ds \right.$$
$$\left. + \int \left(U_{s-}\widetilde{b}_\theta(x_{s-})\right)\left(U_{s-}\widetilde{b}_\theta(x_{s-})\right)^* J(ds, d\theta)\right\} \left(\pi'(x_T)Z_T\right)^*.$$

Since we see that

$$Z_T U_{t-} = \widehat{U}_{T-t}$$

from Lemma 3.1, we can express as follows:

$$z \cdot \widetilde{K}_T z = \sum_{i=1}^d \mathcal{Q}_T^z(x_T, 1, A_i) + \mathcal{N}_T^z(x_T, 1, \widetilde{A}.).$$

In order to give a lower estimate of $z \cdot \widetilde{K}_T z$, we shall prove the following lemma. Though this lemma is elementary, it plays an important role in our argument. For $\zeta > 1$, define random variables L_T and $\widetilde{L}_T$ by

$$L_T = \sup_{|x| \le \zeta} \int_0^T \sum_{i=1}^d \left|\partial_x \left(z \cdot \pi'(x)\widehat{U}_s(x)a_i(\widehat{x}_s(x))\right)^2\right| ds,$$
$$\widetilde{L}_T = \sup_{|x| \le \zeta} \int_0^T \int \left|\partial_x \left(z \cdot \pi'(x)\widehat{U}_{s-}(x)\widetilde{f}_\theta(\widehat{x}_{s-}(x))\right)^2\right| N(ds, d\theta).$$

Put $B(0, \zeta) = \{v \in \boldsymbol{R}^m \ ; \ |v| \le \zeta\}$.

Lemma 3.2. *Let ρ be a positive constant. There exist a positive integer N and points $\{x_j\}_{j=1}^N$ in $B(0,\zeta)$ such that*

$$\inf_{|x|\le\zeta}\sum_{i=1}^d \mathcal{Q}_T^z(x,1,A_i)\ge\sum_{i=1}^d \mathcal{Q}_T^z(x_j,1,A_i)-\rho\,L_T,$$
$$\inf_{|x|\le\zeta}\mathcal{N}_T^z(x,1,\widetilde{A}.)\ge\mathcal{N}_T^z(x_j,1,\widetilde{A}.)-\rho\,\widetilde{L}_T.$$

for some $1\le j\le N$, and furthermore that

$$E_P[L_T^p]+E_P[\tilde{L}_T^p]\le c_p\,\zeta^{2p+m}.$$

Proof. Put $x^\sigma(x')=x+\sigma(x'-x)$ for $\sigma\in[0,1]$, and $x,\ x'\in B(0,\zeta)$. It is easy to see that

$$\begin{aligned}&\Big|\sum_{i=1}^d\{\mathcal{Q}_T^z(x,1,A_i)-\mathcal{Q}_T^z(x',1,A_i)\}\Big|\\&\le\int_0^T\int_0^1\sum_{i=1}^d\Big|\frac{d}{d\sigma}\big(z\cdot\pi'(x^\sigma(x'))\widehat{U}_s(x^\sigma(x'))a_i(\widehat{x}_s(x^\sigma(x')))\big)^2\Big|\,d\sigma\,ds\\&\le L_T|x-x'|\end{aligned}$$

and

$$\begin{aligned}&\Big|\mathcal{N}_T^z(x,1,\widetilde{A}.)-\mathcal{N}_T^z(x',1,\widetilde{A}.)\Big|\\&\le\int_0^1\int_0^T\int\Big|\frac{d}{d\sigma}\big(z\cdot\pi'(x^\sigma(x'))\widehat{U}_{s-}(x^\sigma(x'))\widetilde{f}_\theta(\widehat{x}_{s-}(x^\sigma(x')))\big)^2\Big|\,N(ds,d\theta)\,d\sigma\\&\le\widetilde{L}_T|x-x'|.\end{aligned}$$

Since $B(0,\zeta)$ is compact in $\boldsymbol{R}^m$, there exist a positive integer N with $N\le c.(\zeta/\rho)^{m-1}$ and points $\{x_j\}_{j=1}^N$ in $B(0,\zeta)$ such that

$$B(0,\zeta)=\bigcup_{j=1}^N\{v\in B(0,\zeta)\ ;\ |v-x_j|\le\rho\}.$$

Hence we obtain

$$\begin{aligned}\sum_{i=1}^d\mathcal{Q}_T^z(x,1,A_i)&\ge\sum_{i=1}^d\mathcal{Q}_T^z(x_j,1,A_i)-L_T|x-x_j|\\&\ge\sum_{i=1}^d\mathcal{Q}_T^z(x_j,1,A_i)-\rho\,L_T\end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_T^z(x, 1, \widetilde{A}.) &\geq \mathcal{N}_T^z(x_j, 1, \widetilde{A}.) - \widetilde{L}_T|x - x_j| \\ &\geq \mathcal{N}_T^z(x_j, 1, \widetilde{A}.) - \rho\, \widetilde{L}_T \end{aligned}$$

for some $1 \leq j \leq N$. Moreover, from Proposition 3.1 and the Sobolev inequality, we see that for any positive integer l

$$\begin{aligned} &E_P[L_T^p] \\ &\leq E_P\left[\left(\sum_{|\mu|\leq l} \sup_{|x|\leq\zeta} \int_0^T \sum_{i=1}^d \left|(\partial_x)^{\mu+1}\big(z \cdot \pi'(x)\widehat{U}_s(x)a_i(\widehat{x}_s(x))\big)^2\right| ds\right)^p\right] \\ &\leq c_p E_P\left[\left\{\sum_{|\mu|\leq l}\left(\int_{|x|\leq\zeta}\left\{\sup_{s\leq T}\sum_{i=1}^d\left|(\partial_x)^{\mu+1}\left(z\cdot\pi'(x)\widehat{U}_s(x)a_i\left(\widehat{x}_s(x)\right)\right)^2\right|\right\}^p dx\right)^{1/p}\right\}^p\right] \\ &\leq c_p \sum_{|\mu|\leq l} E_P\left[\int_{|x|\leq\zeta} \sup_{s\leq T}\left(\sum_{i=1}^d\left|(\partial_x)^{\mu+1}\left(z\cdot\pi'(x)\widehat{U}_s(x)a_i\left(\widehat{x}_s(x)\right)\right)^2\right|\right)^p dx\right] \\ &\leq c_p \operatorname{vol}\{x\,;\ |x| \leq \zeta\} \\ &\qquad \times \sum_{|\mu|\leq l} \sup_{|x|\leq\zeta} E_P\left[\sup_{s\leq T}\left(\sum_{i=1}^d\left|(\partial_x)^{\mu+1}\left(z\cdot\pi'(x)\widehat{U}_s(x)a_i\left(\widehat{x}_s(x)\right)\right)^2\right|\right)^p\right] \\ &\leq c_p\, \zeta^{2p+m}. \end{aligned}$$

Similarly we can prove that

$$E_P[\tilde{L}_T^p] \leq c_p\, \zeta^{2p+m}.$$

The proof is complete. □

We shall consider estimate (20). Let $0 < \varepsilon < (2\varepsilon_l)/5$, where we consider $\varepsilon_l = 2^{-4l}$ in the case of diffusion processes. At first we see that

$$\begin{aligned} I &:= E_P\left[\exp\{-\lambda^{4+4\varepsilon}(z \cdot K_T z)\}\right] \\ &\leq E_P^1\left[\exp\{-\lambda^4(z \cdot \widetilde{K}_T z)\}\right] + P[\Omega_1] + P[\Omega_2] + P[\Omega_3] \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $\gamma > 4 + 3\varepsilon$, $3\varepsilon < \beta < \gamma - 4$, and

$$\begin{aligned} \Omega_1 &= \left\{\omega\,;\ \sup_{0\leq s\leq T}(1 + |x_s|^2 + \|U_s\|^2) > \lambda^{2\varepsilon}\right\}, \\ \Omega_2 &= \left\{\omega\,;\ L_T > \lambda^\beta\right\}, \quad \Omega_3 = \left\{\omega\,;\ \widetilde{L}_T > \lambda^\beta\right\}, \end{aligned}$$

$$E_P^1[\ \cdot\ ] = E_P\left[\ \ \cdot\ \ ;\ \Omega_1^c \cap \Omega_2^c \cap \Omega_3^c\ \right].$$

From the Chebyshev inequality, (8) and Lemma 3.2 with $\zeta = \lambda^{\varepsilon}$, we see that

$$\begin{aligned} I_2 &\le \lambda^{-2\varepsilon p} E_P\left[\sup_{0\le s\le T}(1+|x_s|^2+\|U_s\|^2)^p\right] = O(\lambda^{-2\varepsilon p}),\\ I_3 &\le \lambda^{-\beta p} E_P[L_T^p] = O(\lambda^{-p(\beta-3\varepsilon)}),\\ I_4 &\le \lambda^{-\beta p} E_P[\widetilde{L}_T^p] = O(\lambda^{-p(\beta-3\varepsilon)}) \end{aligned}$$

for any $p > 1$. Moreover, from the Schwarz inequality, inequality (14), and Lemma 3.2 with $\zeta = \lambda^{\varepsilon}$, $\rho = \lambda^{-\gamma}$, we obtain the estimate

$$\begin{aligned} I_1 &\le E_P^1\left[\exp\left\{-\inf_{|x|\le\lambda^{\varepsilon}}\left(\sum_{i=1}^d \mathcal{Q}_T^z(x,\lambda,A_i)+\mathcal{N}_T^z(x,\lambda,\widetilde{A}.)\right)\right\}\right]\\ &\le E_P^1\left[\exp\left\{-\min_{1\le j\le N_\lambda}\left(\sum_{i=1}^d \mathcal{Q}_T^z(x_j,\lambda,A_i)+\mathcal{N}_T^z(x_j,\lambda,\widetilde{A}.)\right)+2\lambda^{4-\gamma+\beta}\right\}\right]\\ &\le c.\sum_{j=1}^{N_\lambda} E_P^1\left[\exp\left\{-\left(\sum_{i=1}^d \mathcal{Q}_T^z(x_j,\lambda,A_i)+\mathcal{N}_T^z(x_j,\lambda,\widetilde{A}.)\right)\right\}\right]\\ &\le c.\sum_{j=1}^{N_\lambda} E_P^1\left[\exp\left\{-c.\left(\sum_{i=1}^d \mathcal{Q}_T^z(x_j,\lambda,A_i)+\int \widehat{\mathcal{Q}}_T^z(x_j,\lambda^{\tau},\widetilde{A}_\theta^\lambda)\,\nu(d\theta)\right)\right\}\right]^{1/2}\\ &\le c.\sum_{j=1}^{N_\lambda} E_P^1\left[\exp\left\{-c.\int_{\theta_0\ne 0}\widehat{\mathcal{Q}}_T^z(x_j,\lambda^{\tau},\Phi_{\theta_0}^\lambda)\,\overline{\boldsymbol{\nu}}(d\theta_0)\right\}\right]^{1/2}\\ &\le c.\sum_{j=1}^{N_\lambda}\left\{\widetilde{E_P^1}\left[\exp\left\{-c.\int_{\theta_0\ne 0}\widehat{\mathcal{Q}}_T^z(x_j,\lambda^{\tau},\Phi_{\theta_0}^\lambda)\,\overline{\boldsymbol{\nu}}(d\theta_0)\right\}\right]\right.\\ &\qquad\left.+P\left[\sup_{s\le T}\left(1+|\widehat{x}_s|^2+\|\widehat{U}_s\|^2\right)>\lambda^{4\varepsilon}\right]\right\}^{1/2}\\ &\le c.\sum_{j=1}^{N_\lambda}\left\{\widetilde{E_P^1}\left[\exp\left\{-c.\int_{\theta_0\ne 0}\widehat{\mathcal{Q}}_T^z(x_j,\lambda^{\tau},\Phi_{\theta_0}^\lambda)\,\overline{\boldsymbol{\nu}}(d\theta_0)\right\}\right]^{1/2}+o(\lambda^{-\varepsilon p})\right\}, \end{aligned}$$

where N_λ is a certain positive constant satisfying $N_\lambda \le c.\lambda^{(\varepsilon+\gamma)(m-1)}$, j is a certain positive integer less than N_λ, and

$$\widetilde{E_P^1}[\ \cdot\ ] = E_P^1\left[\ \cdot\ ;\ \sup_{s\le T}\left(1+|\widehat{x}_s|^2+\|\widehat{U}_s\|^2\right)\le\lambda^{4\varepsilon}\right].$$

Hence we have the following lemma.

Lemma 3.3. *If there exists a positive integer j less than N_λ such that*

$$\sup_{z\in S^{n-1}} \sum_{j=1}^{N_\lambda} \widetilde{E_P^1}\left[\exp\left\{-c.\int_{\theta_0\neq 0} \widehat{\mathcal{Q}}_T^z(x_j, \lambda^\tau, \Phi_{\theta_0}^\lambda)\, \bar{\boldsymbol{\nu}}(d\theta_0)\right\}\right] = o(\lambda^{-p}) \tag{21}$$

for any $p > 1$, then estimate (20) *holds for any $p > 1$.*

4. Proof of Theorem 1

In this section, we shall give the proof of Theorem 1. For $\lambda > 1$, define a subset $\Gamma_k(\lambda)$ $(k = 0, 1, \dots, l)$ of Ω by

$$\Gamma_k(\lambda) = \begin{cases} \emptyset & (k = 0), \\ \displaystyle\bigcup_{\Phi\in\mathcal{A}_{k-1}} \left\{\sum_{i=0}^{d}(\|\wp_i\Phi\|^2 + \|\wp_i\wp_0\Phi\|^2) > \lambda^{2\varepsilon_k}\right\} & (1 \le k \le l), \end{cases}$$

where $\|\Psi\|^2 = \sup_{0\le s\le T} |\pi'(y)\widehat{U}_s(y)\psi(\widehat{x}_s(y))|^2$ for any vector field $\Psi = \psi(x)\cdot\partial_x$. We shall introduce the fundamental estimate on a certain continuous semimartingale considered in [11]. The following lemma plays an important role in our argument.

Lemma 4.1. *Let $F(t) \equiv F(t, \omega)$ be a continuous semimartingale such that*

$$dF(s) = f_0(s)\,ds + \sum_{i=1}^{d} f_i(s)\,dw_s^i\ ,\ \ df_0(s) = f_{00}(s)\,ds + \sum_{i=1}^{d} f_{0i}(s)\,dw_s^i.$$

Then there exist a positive random variable $M_T^{(\lambda)}$ with $E[M_T^{(\lambda)}] \le 1$, and positive constants C_0, C_1, C_2 independent of λ and $F(\cdot)$ such that the inequality

$$C_0\int_0^T \lambda^4 F(s)^2\,ds + \lambda^{-1/8}\log M_T^{(\lambda)} \ge C_1\int_0^T \lambda^{1/4}\sum_{i=0}^{d} f_i(s)^2\,ds - C_2.$$

holds on the complement of the set

$$\left\{\omega\ ;\ \sum_{i=0}^{d}\|f_i{}^2 + f_{0i}{}^2\| > \lambda^{1/4}\right\}$$

for sufficiently large λ, where $\|h\| = \sup_{0\le s\le T}|h(s)|$ for any function $h = h(s)$.

We shall consider estimate (21) in Lemma 3.3. For the sake of simplicity of notations, put

$$\mathcal{Q}_T^z(x_j, \lambda, \mathcal{A}_0) = \sum_{\Phi\in\mathcal{A}_0} \mathcal{Q}_T^z(x_j, \lambda, \Phi), \quad \zeta = 2\varepsilon_l - \sigma,$$

where $0 < \sigma < 2\varepsilon_l - 3\varepsilon$. Then we have

$$\begin{aligned}
&\widetilde{E^1_P}\left[\exp\{-\mathcal{Q}^z_T(x_j, \lambda, \mathcal{A}_0)\}\right] \\
&\le E^2_P\left[\exp\{-\mathcal{Q}^z_T(x_j, \lambda, \mathcal{A}_0)\}\right] + P\left[\bigcup_{k=0}^{l} \Gamma_k(\lambda)\right] + P[\widetilde{\Omega}] \\
&= \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3,
\end{aligned}$$

where $2\varepsilon < \delta < \zeta - \varepsilon$ and

$$\begin{aligned}
\widetilde{\Omega} &= \left\{\omega\ ;\ \sup_{0\le s\le \lambda^{-2\zeta}} \left(|\widehat{x}_s(x_j) - x_j| + \|\widehat{U}_s - I\|\right) > \lambda^{-\delta}\right\}, \\
E^2_P[\ \cdot\ ] &= \widetilde{E^1_P}\left[\ \cdot\ ;\ \widetilde{\Omega}^c \cap \bigcap_{k=0}^{l} \Gamma_k(\lambda)^c\right].
\end{aligned}$$

From the Chebyshev inequality, the Burkholder inequality and Proposition 3.1, we obtain estimates

$$\widetilde{I}_2 \le \sum_{k=0}^{l} \sum_{\Phi\in\mathcal{A}_k} \lambda^{-2p\varepsilon_k}\ E_P\left[\left\{\sum_{i=0}^{d}\left(\|\wp_i\Phi\|^2 + \|\wp_i\wp_0\Phi\|^2\right)\right\}^p\right] = O(\lambda^{-2p(\varepsilon_l - 2\varepsilon)})$$

and

$$\widetilde{I}_3 \le \lambda^{p\delta}\ E_P\left[\sup_{0\le s\le\lambda^{-2\zeta}}\left(|\widehat{x}_s(x_j) - x_j|^p + \|\widehat{U}_s - I\|^p\right)\right] = O(\lambda^{-(\zeta-\delta-\varepsilon)p})$$

for all $p > 1$. Finally we shall consider the estimate of $\widetilde{I}_1$. From the Ito formula, we see that

$$d\left(z\cdot\pi'(x_j)\widehat{U}_s\Phi_{\widehat{x}_s}\right) = \left(z\cdot\pi'(x_j)\widehat{U}_s(\wp_0\Phi)_{\widehat{x}_s}\right) ds + \sum_{i=1}^{d}\left(z\cdot\pi'(x_j)\widehat{U}_s(\wp_i\Phi)_{\widehat{x}_s}\right) dw^i_s$$

for $\Phi = \Phi_x \in \mathcal{A}_k$. From Lemma 4.1, there exists a positive random variable $M^{(k,\lambda)}_{T,z}$ with $E_P[M^{(k,\lambda)}_{T,z}] \le 1$ such that the inequality

$$c.\mathcal{Q}^z_T(x_j, \lambda^{\varepsilon_{k-1}}, \mathcal{A}_{k-1}) \ge -\lambda^{-2\varepsilon_k}\log M^{(k,\lambda)}_{T,z} + c.\mathcal{Q}^z_T(x_j, \lambda^{\varepsilon_k}, \mathcal{A}_k) - c. \tag{22}$$

holds on $\Gamma_k(\lambda)^c$. Put $\mathcal{A}'_l = \bigcup_{k=0}^{l}\mathcal{A}_k$. By the iterative application of (22), we have

$$\begin{aligned}
c.\mathcal{Q}^z_T(x_j, \lambda^{\varepsilon_0}, \mathcal{A}_0) &\ge -\lambda^{-2\varepsilon_1}\log M^{(1,\lambda)}_{T,z} + c.\mathcal{Q}^z_T(x_j, \lambda^{\varepsilon_1}, \mathcal{A}'_1) - c. \\
&\ge \cdots\cdots
\end{aligned}$$

$$\geq -\sum_{k=1}^{l} \lambda^{-2\varepsilon_k} \log M_{T,z}^{(k,\lambda)} + c.\mathcal{Q}_T^z(x_j, \lambda^{\varepsilon_l}, \mathcal{A}_l') - c.$$

on $\bigcap_{k=0}^{l} \Gamma_k(\lambda)^c$. We remark that if

$$\sup_{0\leq s\leq T} \left(1 + |\widehat{x}_s|^2 + \|\widehat{U}_s\|^2\right) \leq \lambda^{4\varepsilon}, \quad \sup_{0\leq s\leq \lambda^{-2\zeta}} (|\widehat{x}_s(x_j) - x_j| + \|\widehat{U}_s - I\|) \leq \lambda^{-\delta},$$

then the inequality

$$\begin{aligned} &\left(z \cdot \pi'(x_j)\widehat{U}_s(x_j)\phi(\widehat{x}_s(x_j))\right)^2 \\ &\quad \geq \frac{1}{2}\left(z \cdot \pi'(x_j)\phi(x_j)\right)^2 - 2\|\pi'\|^2 \lambda^{4\varepsilon - 2\delta}(C[\phi] + L[\phi]) \end{aligned} \tag{23}$$

holds for any $s \in [0, \lambda^{-2\zeta}]$, where for any $\boldsymbol{R}^m$-valued mapping ψ on $\boldsymbol{R}^m$, $C[\psi]$ and $L[\psi]$ denote the best constants satisfying the inequality

$$|\psi(x)|^2 \leq C[\psi]\ (1 + |x|^2), \quad |\psi(x) - \psi(x')|^2 \leq L[\psi]\ |x - x'|^2$$

for $x,\ x' \in \boldsymbol{R}^m$. From the Jensen inequality, we see

$$E_P\left[\prod_{k=1}^{l}\left(M_{T,z}^{(k,\lambda)}\right)^{2\lambda^{-2\varepsilon_k}}\right] \leq \prod_{k=1}^{l} E_P\left[\left(M_{T,z}^{(k,\lambda)}\right)^{2l\lambda^{-2\varepsilon_k}}\right]^{1/l} \leq 1$$

for sufficiently large λ. Hence from (2) and (23), we have

$$\begin{aligned} \widetilde{I}_1 &\leq c.\ E_P^2\left[\prod_{k=1}^{l}\left(M_{T,z}^{(k,\lambda)}\right)^{\lambda^{-2\varepsilon_k}} \exp\{-c.\mathcal{Q}_T^z(x_j, \lambda^{\varepsilon_l}, \mathcal{A}_l')\}\right] \\ &\leq c.\ E_P^2\left[\exp\{-c.\mathcal{Q}_T^z(x_j, \lambda^{\varepsilon_l}, \mathcal{A}_l')\}\right]^{1/2} \\ &\leq c.\ E_P^2\left[\exp\left\{-c.\mathcal{Q}_{\lambda^{-2\zeta}}^z(x_j, \lambda^{\varepsilon_l}, \mathcal{A}_l')\right\}\right]^{1/2} \\ &\leq c.\ \exp\left\{-c.\lambda^{2\sigma}\left(\inf_{z\in S^{n-1}} \sum_{\Phi\in\mathcal{A}_l'} (z\cdot\pi'(x_j)\phi(x_j))^2 - c.\lambda^{4\varepsilon-2\delta}\right)\right\} \\ &= O(\exp\{-c.\lambda^{2\kappa}\}). \end{aligned}$$

Therefore we obtain the estimate

$$\sum_{j=1}^{N_\lambda} \sup_{z\in S^{n-1}} \widetilde{E_P^1}\left[\exp\{-\mathcal{Q}_T^z(x_j, \lambda, \mathcal{A}_0)\}\right] = o(\lambda^{-p})$$

for any $p > 1$. From Lemma 3.3, the law of random variable $\pi(x_T)$ admits a smooth density. □

5. Proof of Theorem 2 and Corollary 1

In this section, we shall discuss the case of general processes with jumps. At first, we prepare the following lemma on a certain semimartingale with jumps considered in [12], which plays an important role in the proof of Theorem 2.

Lemma 5.1. *Let $F(t) \equiv F(t,\omega)$ be a semimartingale such that*

$$
\begin{aligned}
dF(s) &= f_0(s)\,ds + \sum_{i=1}^{d} f_i(s)\,dw_s^i + \int_{|\theta|\le 1} g_\theta(s)\,d\widetilde{J} + \int_{|\theta|>1} g_\theta(s)\,dJ,\\
df_0(s) &= f_{00}(s)\,ds + \sum_{i=1}^{d} f_{0i}(s)\,dw_s^i + \int_{|\theta|\le 1} g_{0\theta}(s)\,d\widetilde{J} + \int_{|\theta|>1} g_{0\theta}(s)\,dJ.
\end{aligned}
$$

For $0 < \rho < ((2\alpha)\wedge 1)/4$, there exist positive constants C, $\{C_i\}_{i=0}^3$ independent of λ and $F(\cdot)$, and a positive random variable $M_T^{(\lambda)}$ with $E_P[M_T^{(\lambda)}] \le 1$ such that

$$
\begin{aligned}
&\lambda^2 \int_0^T (\lambda F(s))^2 \wedge 1\,ds + C_0\lambda^{-\rho}\log M_T^{(\lambda)} + C\\
&\quad\ge C_1\lambda^{1-4\rho}\int_0^T f_0(s)^2\,ds + C_2\lambda^{2-2\rho}\int_0^T \sum_{i=1}^{d} f_i(s)^2\,ds\\
&\qquad + C_3\lambda^{-2\rho}\int_0^T\!\!\int (\lambda g_\theta(s))^2\wedge 1\ \nu(d\theta)\,ds
\end{aligned}
$$

on the complement of the set

$$
\left\{\omega\ ;\ \sum_{i=0}^{d} \|{f_i}^2 + {f_{0i}}^2\| + \int_{|\theta|\le 1} \|{g_\theta}^2 + {g_{0\theta}}^2\|\ \nu(d\theta) + \sup_{|\theta|>1}\|{g_{0\theta}}^2\| > \lambda^{2\rho}\right\},
$$

for sufficiently large λ, where $\|h\| = \sup_{0\le s\le T}|h(s)|$ for any function $h = h(s)$.

Define a subset $\Gamma(\lambda,\Psi)$ as follows:

$$
\begin{aligned}
\Gamma(\lambda^\rho,\Psi) = \Bigg\{\omega\ ;\ &\sum_{i=0}^{d}\left(\|\wp_i\Psi\|^2 + \|\wp_i\wp_0\Psi\|^2\right) + \sup_{|\eta|>1}\|\wp_\eta\wp_0\Psi\|^2\\
&+ \int_{|\eta|\le 1}\left(\|\wp_\eta\Psi\|^2 + \|\wp_\eta\wp_0\Psi\|^2\right)\nu(d\eta) > \lambda^{2\rho}\Bigg\},
\end{aligned}
$$

where $\|\Phi\|^2 = \sup_{0\le s\le T}|\pi'(x_j)\widehat{U}_s(x_j)\phi(\widehat{x}_s(x_j))|^2$ for any vector field $\Phi = \phi(x)\cdot\partial_x$. By using the Ito formula, we see that for $\Psi = \Psi_x$,

$$
d\left(z\cdot\pi'(x_j)\widehat{U}_s\Psi_{\widehat{x}_s}\right) = \left(z\cdot\pi'(x_j)\widehat{U}_s(\wp_0\Psi)_{\widehat{x}_s}\right)ds
$$

$$+\sum_{i=1}^{d}\left(z\cdot\pi'(x_j)\widehat{U}_s(\wp_i\Psi)_{\widehat{x}_s}\right)dw_s^i$$
$$+\int\left(z\cdot\pi'(x_j)\widehat{U}_{s-}(\wp_\eta\Psi)_{\widehat{x}_{s-}}\right)J(ds,d\eta).$$

From Lemma 5.1, there exists a positive random variable $M_{T,z}(\lambda,\Psi)$ satisfying $E_P[M_{T,z}(\lambda,\Psi)]\leq 1$ such that

$$c.\widehat{\mathcal{Q}}_T^z(x_j,\lambda,\Psi)+c.\lambda^{-\rho}\log M_{T,z}(\lambda,\Psi)+c. \geq c.\int\widehat{\mathcal{Q}}_T^z(x_j,\lambda^{1(\eta)},\wp_\eta(\lambda)\Psi)\,\bar{\boldsymbol{\nu}}(d\eta) \tag{24}$$

holds on the complement of $\Gamma(\lambda^\rho,\Psi)$. Set $\overline{\Gamma}_0(\rho,\lambda,\Psi)=\emptyset$ and

$$\overline{\Gamma}_l(\rho,\lambda,\Psi) = \Gamma(\lambda^\rho,\Psi)\cup\bigcup_{k=1}^{l-1}\bigcup_{\theta_1\cdots\theta_k}\Gamma\left((\lambda^{1(\theta_1,\ldots,\theta_k)})^\rho,\wp_{\theta_k}(\lambda^{1(\theta_1,\ldots,\theta_{k-1})})\cdots\wp_{\theta_1}(\lambda)\Psi\right)$$

for $l\geq 1$ and any vector field Ψ. For $l\geq 1$, define $M_{l,z}(\rho,\lambda,\Psi)$ by

$$\begin{aligned}\log M_{l,z}(\rho,\lambda,\Psi) &= \lambda^{-\rho}\log M_{T,z}(\lambda,\Psi)\\ &\quad+c.\sum_{k=1}^{l-1}\int\bar{\boldsymbol{\nu}}(d\theta_1)\cdots\bar{\boldsymbol{\nu}}(d\theta_k)\\ &\qquad\times(\lambda^{1(\theta_1,\ldots,\theta_k)})^{-\rho}\log M_{T,z}\left(\lambda^{1(\theta_1,\ldots,\theta_k)},\wp_{\theta_k}(\lambda^{1(\theta_1,\ldots,\theta_{k-1})})\cdots\wp_{\theta_1}(\lambda)\Psi\right).\end{aligned}$$

From the Jensen inequality, it is easy to see that for any $1<p<\lambda^{\rho\varepsilon_l}$,

$$E_P[M_{l,z}(\rho,\lambda,\Psi)^p]\leq 1$$

for sufficiently large λ. Hence by iterating (24), the inequality

$$\begin{aligned}&\widehat{\mathcal{Q}}_T^z(x_j,\lambda,\Psi)+\log M_{l,z}(\rho,\lambda,\Psi)+C_l\\ &\geq c.\widehat{\mathcal{Q}}_T^z(x_j,\lambda,\Psi)+c.\sum_{k=1}^{l}\int\bar{\boldsymbol{\nu}}(d\theta_1)\cdots\bar{\boldsymbol{\nu}}(d\theta_k)\\ &\qquad\times\widehat{\mathcal{Q}}_T^z\left(x_j,\lambda^{1(\theta_1,\ldots,\theta_k)},\wp_{\theta_k}(\lambda^{1(\theta_1,\ldots,\theta_{k-1})})\cdots\wp_{\theta_1}(\lambda)\Psi\right)\end{aligned} \tag{25}$$

holds on the complement of the set $\overline{\Gamma}_l(\rho,\lambda,\Psi)$ for sufficiently large λ.

We shall consider estimate (21) in Lemma 3.3.

$$\begin{aligned}
&\widetilde{E^1_P}\Big[\exp\Big\{-c.\int_{\theta_0\neq 0}\widehat{\mathcal{Q}}^z_T(x_j,\lambda^\tau,\Phi^\lambda_{\theta_0})\,\overline{\boldsymbol{\nu}}(d\theta_0)\Big\}\Big] \\
&\quad\le E^2_P\Big[\exp\Big\{-c.\int_{\theta_0\neq 0}\widehat{\mathcal{Q}}^z_T(x_j,\lambda^\tau,\Phi^\lambda_{\theta_0})\,\overline{\boldsymbol{\nu}}(d\theta_0)\Big\}\Big] + P\left[\bigcup_{\theta_0\neq 0}\overline{\Gamma}_l(\rho,\lambda^\tau,\Phi^\lambda_{\theta_0})\right] \\
&\quad= \widehat{I}_1+\widehat{I}_2,
\end{aligned}$$

where

$$E^2_P[\ \cdot\ ]=\widetilde{E^1_P}\left[\ \cdot\ ;\ \bigcap_{\theta_0\neq 0}\overline{\Gamma}_l(\rho,\lambda^\tau,\Phi^\lambda_{\theta_0})^c\ \right].$$

Since $a_i'(x)$ $(i=0,1,\ldots,d)$ are bounded smooth mappings and $b(x,\theta)$ satisfies assimption $[R]$, it is a routine work to show the estimate

$$\begin{aligned}
(26)\qquad &\sum_{k=0}^{l}\sup_{\theta_0\neq 0}\sup_{\theta_1,\ldots,\theta_k\in\Theta}\Bigg\{\sum_{i=0}^{d}\big(\|\wp_i\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}\|^2+\|\wp_i\wp_0\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}\|^2\big) \\
&\qquad+\int_{|\eta|\le 1}\big(\|\wp_\eta\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}\|^2+\|\wp_\eta\wp_0\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}\|^2\big)\,\nu(d\theta) \\
&\qquad+\sup_{|\eta|>1}\|\wp_\eta\wp_0\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k}\|^2\Bigg\} \\
&\le \overline{C}(1+|x_j|^2),
\end{aligned}$$

where $\overline{C}$ is a constant independent of λ. From the Chebyshev inequality, Proposition 3.1 and (26), we can estimate $\widehat{I}_2$ as follows:

$$\widehat{I}_2=P\left[\bigcup_{\theta_0\neq 0}\overline{\Gamma}_l(\rho,\lambda^\tau,\Phi^\lambda_{\theta_0})\right]=O(\lambda^{-2p(\varepsilon_l-2\varepsilon)}).$$

By using the Jensen inequality and (25), we have

$$\begin{aligned}
(27)\qquad \widehat{I}_1 &= E^2_P\Big[\exp\Big\{-c.\int_{\theta_0\neq 0}\widehat{\mathcal{Q}}^z_T(x_j,\lambda^\tau,\Phi^\lambda_{\theta_0})\,\overline{\boldsymbol{\nu}}(d\theta_0)\Big\}\Big] \\
&\le c.\,E^2_P\Bigg[\exp\left\{c.\int_{\theta_0\neq 0}\log M_{l,z}(\rho,\lambda^\tau,\Phi^\lambda_{\theta_0})\,\overline{\boldsymbol{\nu}}(d\theta_0)\right\} \\
&\qquad\times\exp\left\{-c.\sum_{k=0}^{l}\int_{\theta_0\neq 0}\overline{\boldsymbol{\nu}}(d\theta_0)\int\overline{\boldsymbol{\nu}}(d\theta_1)\cdots\overline{\boldsymbol{\nu}}(d\theta_k)\right.
\end{aligned}$$

$$\left.\left.\times \widehat{\mathcal{Q}}^z_T(x_j, \lambda^{\tau(\theta_1,\dots,\theta_k)}, \Phi^\lambda_{\theta_0\theta_1\cdots\theta_k})\right\}\right]$$

$$\le c.\, E^2_P\left[\exp\left\{-c.\sum_{k=0}^{l}\int_{\theta_0\neq 0}\bar{\boldsymbol{\nu}}(d\theta_0)\int\bar{\boldsymbol{\nu}}(d\theta_1)\cdots\bar{\boldsymbol{\nu}}(d\theta_k)\right.\right.$$
$$\left.\left.\times\, \widehat{\mathcal{Q}}^z_T(x_j, \lambda^{\tau(\theta_1,\dots,\theta_k)}, \Phi^\lambda_{\theta_0\theta_1\cdots\theta_k})\right\}\right]^{c.},$$

where $\tau(\theta_1,\dots,\theta_k) = \tau$ for $k = 0$. Therefore it suffices to estimate

$$\sup_{z\in S^{n-1}}\sum_{j=1}^{N_\lambda} E^2_P\left[\exp\{-\widehat{\mathcal{Q}}^z_T(x_j, \lambda^\gamma, \Psi)\}\right] \tag{28}$$

for $\gamma > 2\varepsilon_l - \sigma + \bar{\delta}$ and a vector field Ψ.

To simplify our argument, we shall consider $\pi = \pi(x)$ as a canonical projection from $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$. In fact, it suffices to consider a diffeomorphism $\widetilde{\pi}$ on $\boldsymbol{R}^m$ such that n elements of $\widetilde{\pi}(x)$ is equal to $\pi(x)$, that is,

$$\widetilde{\pi}(x) = \begin{pmatrix}\pi(x)\\ *\end{pmatrix}.$$

Let $\bar{Z}_t$, $\bar{U}_t$ be the solutions to following linear stochastic differential equations:

$$\bar{Z}_t = I - \int_0^t a_0'(\widehat{x}_s)\bar{Z}_s\,ds - \int_0^t\sum_{i=1}^d a_i'(\widehat{x}_s)\bar{Z}_s\circ dw^i_s,$$
$$\bar{U}_t = I + \int_0^t \bar{U}_s a_0'(\widehat{x}_s)\,ds + \int_0^t\sum_{i=1}^d \bar{U}_s a_i'(\widehat{x}_s)\circ dw^i_s.$$

Then by using the Ito formula, we get

$$\begin{aligned} d(\widehat{U}_t\bar{Z}_t) = {}& -\widehat{U}_t a_0'(\widehat{x}_t)\bar{Z}_t\,dt - \sum_{i=1}^d \widehat{U}_t a_i'(\widehat{x}_t)\bar{Z}_t\circ dw^i_t \\ & + \widehat{U}_t a_0'(\widehat{x}_t)\bar{Z}_t\,dt + \sum_{i=1}^d \widehat{U}_t a_i'(\widehat{x}_t)\bar{Z}_t\circ dw^i_t \\ & + \int \widehat{U}_{t-}\{\big(I + f'(\widehat{x}_{t-},\theta)\big)^{-1} - I\}\bar{Z}_{t-}\, J(dt, d\theta) \\ = {}& \int \widehat{U}_{t-}\{\big(I + f'(\widehat{x}_{t-},\theta)\big)^{-1} - I\}\bar{Z}_{t-}\, J(dt, d\theta), \end{aligned}$$

that is,

$$\widehat{U}_t = \overline{U}_t + \int_0^t \int \widehat{U}_{s-} \{ \big(I + f'(\widehat{x}_{s-}, \theta) \big)^{-1} - I \} \overline{Z}_{s-} J(ds, d\theta) \overline{U}_t.$$

Furthermore, since a_0' and $\{a_i', a_i''\}_{i=1}^d$ are bounded, we see that

$$E_P \left[\sup_{0 \le s \le t} \big(\|\overline{Z}_s\|^p + \|\overline{U}_s\|^p \big) \right] < \infty \tag{29}$$

for any $t \in [0, T]$ and $p > 1$.

For any $(m \times m)$-matrix A, we shall decompose A to four elements. Let A^1 be an $(n \times n)$-matrix, A^2 an $(n \times (m-n))$-matrix, A^3 an $((m-n) \times n)$-matrix, and A^4 an $((m-n) \times (m-n))$-matrix. Then we shall express A as follows:

$$A = \begin{pmatrix} A^1 & A^2 \\ A^3 & A^4 \end{pmatrix}.$$

Denote

$$\widehat{U}_t = \begin{pmatrix} \widehat{U}_t^1 & \widehat{U}_t^2 \\ \widehat{U}_t^3 & \widehat{U}_t^4 \end{pmatrix}, \quad \overline{Z}_t = \begin{pmatrix} \overline{Z}_t^1 & \overline{Z}_t^2 \\ \overline{Z}_t^3 & \overline{Z}_t^4 \end{pmatrix}, \qquad \overline{U}_t = \begin{pmatrix} \overline{U}_t^1 & \overline{U}_t^2 \\ \overline{U}_t^3 & \overline{U}_t^4 \end{pmatrix},$$

$$\big(I + f'(\widehat{x}_t, \theta) \big)^{-1} - I = \begin{pmatrix} A_t^1(\theta) & 0 \\ A_t^3(\theta) & A_t^4(\theta) \end{pmatrix}, \quad \widetilde{z} = \pi'(x_j)^* z = \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

In particular, we can express $\widehat{U}_t^2$ as follows:

$$\begin{aligned} \widehat{U}_t^2 = \overline{U}_t^2 + \int_0^t \int \Big[& \widehat{U}_{s-}^1 A_{s-}^1(\theta) \big(\overline{Z}_{s-}^1 \overline{U}_t^2 + \overline{Z}_{s-}^2 \overline{U}_t^4 \big) \\ & + \widehat{U}_{s-}^2 A_{s-}^3(\theta) \big(\overline{Z}_{s-}^1 \overline{U}_t^2 + \overline{Z}_{s-}^2 \overline{U}_t^4 \big) \\ & + \widehat{U}_{s-}^2 A_{s-}^4(\theta) \big(\overline{Z}_{s-}^3 \overline{U}_t^2 + \overline{Z}_{s-}^4 \overline{U}_t^4 \big) \Big] J(ds, d\theta). \end{aligned}$$

Before giving an upper estimate of the higher order moment of $\widehat{U}_t^2$, we shall introduce the following lemma. It can be easily proved by applying the Burkholder inequality, so we omit the proof (cf. [25]).

Lemma 5.2. *Let k be a positive integer, and $h(s, \theta)$ a predictable process with*

$$E_P \left[\int_0^t \left\| \int h(s, \theta) \, \nu(d\theta) \right\|^{2^k} ds \right] < \infty,$$

$$E_P \left[\int_0^t \int \|h(s, \theta)\|^{2^j} \, \nu(d\theta) \, ds \right] < \infty \quad (j = 1, \dots, k-1).$$

Then the following estimate holds

$$
\begin{aligned}
&E_P\left[\sup_{0\le u\le t}\left\|\int_0^u\int h(s,\theta)\,N(ds,d\theta)\right\|^{2^k}\right]\\
&\le c_k\left\{E_P\left[\int_0^t\left\|\int h(s,\theta)\,\nu(d\theta)\right\|^{2^k}ds\right]\right.\\
&\qquad\left.+\sum_{j=1}^k E_P\left[\left(\int_0^t\int\|h(s,\theta)\|^{2^j}\,\nu(d\theta)\,ds\right)^{2^{k-j}}\right]\right\},
\end{aligned}
$$

where c_k is a positive constant depending on k.

By using the above lemma, we obtain the following lemma on an upper estimate of the higher order moment of $\widehat{U}_t^2$.

Lemma 5.3. *For $p>1$, there exists a positive constant c_p such that*

$$
\begin{aligned}
(30)\quad &E_P^3\left[\sup_{0\le u\le t}\|\widehat{U}_u^2\|^p\right]\\
&\le c_p\left\{E_P^2\left[\sup_{0\le u\le t}\|\overline{U}_u^2\|^p\right]+E_P^3\left[\sup_{0\le u\le t}\|\overline{Z}_u^1\overline{U}_t^2+\overline{Z}_u^2\overline{U}_t^4\|^{2p}\right]^{1/2}\right.\\
&\qquad\left.+\eta t\,E_P^2\left[\sup_{0\le u\le t}\|\overline{Z}_u^1\overline{U}_t^2+\overline{Z}_u^2\overline{U}_t^4\|^{2p}\sup_{0\le u\le t}\|\widehat{U}_u^1\|^{2p}\right]^{1/2}\right\}e^{c_p\eta t},
\end{aligned}
$$

where η is a positive constant and

$$
E_P^3[\ \cdot\ ]=E_P^2\left[\ \cdot\ ;\ \sup_{0\le u\le t}\|\overline{Z}_u^3\overline{U}_t^2+\overline{Z}_u^4\overline{U}_t^4\|\le\eta\right].
$$

Proof. In this lemma, we may take $p=2^k$ for a positive integer k. Under

$$
\sup_{0\le u\le t}\|\overline{Z}_u^3\overline{U}_t^2+\overline{Z}_u^4\overline{U}_t^4\|\le\eta,
$$

we see from the Hölder inequality that

$$
\begin{aligned}
&E_P^3\left[\sup_{0\le u\le t}\|\widehat{U}_u^2\|^p\right]\\
&\le c_p\left\{E_P^3\left[\sup_{0\le u\le t}\|\overline{U}_u^2\|^p\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + E_P^3\left[\sup_{0\le u\le t}\left\|\int_0^u \widehat{U}^1_{s-}A^1_{s-}(\theta)\left(\overline{Z}^1_{s-}\overline{U}^2_t+\overline{Z}^2_{s-}\overline{U}^4_t\right)J(ds,d\theta)\right\|^p\right]\\
&\quad + E_P^3\left[\sup_{0\le u\le t}\left\|\int_0^u\int \widehat{U}^2_{s-}A^3_{s-}(\theta)\left(\overline{Z}^1_{s-}\overline{U}^2_t+\overline{Z}^2_{s-}\overline{U}^4_t\right)J(ds,d\theta)\right\|^p\right]\\
&\quad + E_P^3\left[\sup_{0\le u\le t}\left\|\int_0^u\int \widehat{U}^2_{s-}A^4_{s-}(\theta)\left(\overline{Z}^3_{s-}\overline{U}^2_t+\overline{Z}^4_{s-}\overline{U}^4_t\right)J(ds,d\theta)\right\|^p\right]\Bigg\}\\
&\le c_p\Bigg\{E_P^3\left[\sup_{0\le u\le t}\|\overline{U}^2_u\|^p\right]\\
&\quad + E_P^3\left[\int_0^t\left\|\int \widehat{U}^1_s A^1_s(\theta)\left(\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\right)\nu(d\theta)\right\|^p ds\right]\\
&\quad + E_P^3\left[\int_0^t\left\|\int \widehat{U}^2_s A^3_s(\theta)\left(\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\right)\nu(d\theta)\right\|^p ds\right]\\
&\quad + E_P^3\left[\int_0^t\left\|\int \widehat{U}^2_s A^4_s(\theta)\left(\overline{Z}^3_s\overline{U}^2_t+\overline{Z}^4_s\overline{U}^4_t\right)\nu(d\theta)\right\|^p ds\right]\\
&\quad + \sum_{j=1}^k E_P^3\left[\left(\int_0^t\int \|\widehat{U}^1_s A^1_s(\theta)\left(\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\right)\|^{2^j}\nu(d\theta)\,ds\right)^{2^{k-j}}\right]\\
&\quad + \sum_{j=1}^k E_P^3\left[\left(\int_0^t\int \|\widehat{U}^2_s A^3_s(\theta)\left(\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\right)\|^{2^j}\nu(d\theta)\,ds\right)^{2^{k-j}}\right]\\
&\quad + \sum_{j=1}^k E_P^3\left[\left(\int_0^t\int \|\widehat{U}^2_s A^4_s(\theta)\left(\overline{Z}^3_s\overline{U}^2_t+\overline{Z}^4_s\overline{U}^4_t\right)\|^{2^j}\nu(d\theta)\,ds\right)^{2^{k-j}}\right]\Bigg\}\\
&\le c_p\Bigg\{E_P^3\left[\sup_{0\le u\le t}\|\overline{U}^2_u\|^p\right]\\
&\quad + E_P^3\left[\sup_{0\le s\le t}\|\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\|^p\int_0^t\left\|\int \widehat{U}^1_s A^1_s(\theta)\nu(d\theta)\right\|^p ds\right]\\
&\quad + E_P^3\left[\sup_{0\le s\le t}\|\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\|^p\int_0^t\left\|\widehat{U}^2_s A^3_s(\theta)\nu(d\theta)\right\|^p ds\right]\\
&\quad + E_P^3\left[\sup_{0\le s\le t}\|\overline{Z}^3_s\overline{U}^2_t+\overline{Z}^4_s\overline{U}^4_t\|^p\int_0^t\left\|\int \widehat{U}^2_s A^4_s(\theta)\nu(d\theta)\right\|^p ds\right]\\
&\quad + \sum_{j=1}^k E_P^3\left[\sup_{0\le s\le t}\|\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\|^p\left(\int_0^t\int \|\widehat{U}^1_s A^1_s(\theta)\|^{2^j}\nu(d\theta)\,ds\right)^{2^{k-j}}\right]\\
&\quad + \sum_{j=1}^k E_P^3\left[\sup_{0\le s\le t}\|\overline{Z}^1_s\overline{U}^2_t+\overline{Z}^2_s\overline{U}^4_t\|^p\left(\int_0^t\int \|\widehat{U}^2_s A^3_s(\theta)\|^{2^j}\nu(d\theta)\,ds\right)^{2^{k-j}}\right]
\end{aligned}
$$

$$+ \sum_{j=1}^{k} E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^3 \bar{U}_t^2 + \bar{Z}_s^4 \bar{U}_t^4\|^p \left(\int_0^t \int \|\widehat{U}_s^2 A_s^4(\theta)\|^{2^j} \nu(d\theta)\, ds \right)^{2^{k-j}} \right] \Bigg\}$$

$$\le c_p \Bigg\{ E_P^3 \left[\sup_{0\le u\le t} \|\bar{U}_u^2\|^p \right]$$

$$+ E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^p \int_0^t \|\widehat{U}_s^1\|^p\, ds \sup_{0\le s\le t} \left\| \int A_s^1(\theta)\nu(d\theta) \right\|^p \right]$$

$$+ \sum_{j=1}^{k} E_P^3 \Bigg[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^p$$

$$\times \int_0^t \|\widehat{U}_s^1\|^p\, ds \left\{ \int_0^t \left(\int \|A_s^1(\theta)\|^{2^j} \nu(d\theta) \right)^{2^k/(2^k-2^j)} ds \right\}^{(2^k-2^j)/2^j} \Bigg]$$

$$+ E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^p \int_0^t \|\widehat{U}_s^2\|^p\, ds \sup_{0\le s\le t} \left\| \int A_s^3(\theta)\nu(d\theta) \right\|^p \right]$$

$$+ \sum_{j=1}^{k} E_P^3 \Bigg[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^p$$

$$\times \int_0^t \|\widehat{U}_s^2\|^p\, ds \left\{ \int_0^t \left(\int \|A_s^3(\theta)\|^{2^j} \nu(d\theta) \right)^{2^k/(2^k-2^j)} ds \right\}^{(2^k-2^j)/2^j} \Bigg]$$

$$+ E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^3 \bar{U}_t^2 + \bar{Z}_s^4 \bar{U}_t^4\|^p \int_0^t \|\widehat{U}_s^2\|^p\, ds \sup_{0\le s\le t} \left\| \int A_s^4(\theta)\nu(d\theta) \right\|^p \right]$$

$$+ \sum_{j=1}^{k} E_P^3 \Bigg[\sup_{0\le s\le t} \|\bar{Z}_s^3 \bar{U}_t^2 + \bar{Z}_s^4 \bar{U}_t^4\|^p$$

$$\times \int_0^t \|\widehat{U}_s^2\|^p\, ds \left\{ \int_0^t \left(\int \|A_s^4(\theta)\|^{2^j} \nu(d\theta) \right)^{2^k/(2^k-2^j)} ds \right\}^{(2^k-2^j)/2^j} \Bigg] \Bigg\}$$

$$\le c_p \Bigg\{ E_P^3 \left[\sup_{0\le u\le t} \|\bar{U}_u^2\|^p \right] + E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^{2p} \right]^{1/2}$$

$$+ \eta t E_P^3 \left[\sup_{0\le s\le t} \|\bar{Z}_s^1 \bar{U}_t^2 + \bar{Z}_s^2 \bar{U}_t^4\|^{2p} \sup_{0\le s\le t} \|\widehat{U}_s^1\|^{2p} \right]^{1/2} + \eta E_P^3 \left[\int_0^t \|\widehat{U}_s^2\|^p\, ds \right] \Bigg\}$$

From the Gronwall inequality, we obtain inequality (30). □

Let us consider estimate (28). Put $\zeta = 2\varepsilon_l - \sigma$. Then we have

$$\begin{aligned} &E_P^2\left[\exp\left\{-\widehat{\mathcal{Q}}_T^z(x_j, \lambda^\gamma, \Psi)\right\}\right] \\ &\le E_P^2\left[\exp\left\{-\widehat{\mathcal{Q}}_{\lambda^{-2\zeta}}^z(x_j, \lambda^\gamma, \Psi)\right\}\right] \\ &\le E_P^4\left[\exp\left\{-\widehat{\mathcal{Q}}_{\lambda^{-2\zeta}}^z(x_j, \lambda^\gamma, \Psi)\right\}\right] + P_3[\widetilde{\Omega}_1] + P[\widetilde{\Omega}_2] \\ &= \overline{I}_1 + \overline{I}_2 + \overline{I}_3, \end{aligned}$$

where $0 < \beta < \zeta - 2\varepsilon$, $2\varepsilon < \delta < \zeta - 3\varepsilon$, and

$$\begin{aligned} \widetilde{\Omega}_1 &= \left\{\omega \,\middle|\, \sup_{0\le s\le \lambda^{-2\zeta}} \|\widehat{U}_s^2\| > \lambda^{-\delta}\right\}, \\ \widetilde{\Omega}_2 &= \left\{\omega \,\middle|\, \sup_{0\le u\le \lambda^{-2\zeta}} \|\overline{Z}_u^3\overline{U}_{\lambda^{-2\zeta}}^2 + \overline{Z}_u^4\overline{U}_{\lambda^{-2\zeta}}^4\| > \lambda^{-\beta}\right\}, \\ E_P^3[\,\cdot\,] &= E_P^2\left[\ \cdot\ ;\ \widetilde{\Omega}_2^c\right], \quad E_P^4[\,\cdot\,] = E_P^2\left[\ \cdot\ ;\ \widetilde{\Omega}_1^c \cap \widetilde{\Omega}_2^c\right]. \end{aligned}$$

By using the Chebyshev inequality and Lemma 5.3, we get

$$\begin{aligned} \overline{I}_2 &\le \lambda^{p\delta}\, E_P^3\left[\sup_{0\le s\le\lambda^{-2\zeta}} \|\widehat{U}_s^2\|^p\right] \\ &\le c_p\lambda^{p\delta}\, \Bigg\{E_P^3\left[\sup_{0\le s\le\lambda^{-2\zeta}} \|\overline{U}_s^2\|^p\right] \\ &\qquad + E_P^3\left[\sup_{0\le s\le\lambda^{-2\zeta}} \|\overline{Z}_s^1\overline{U}_{\lambda^{-2\zeta}}^2 + \overline{Z}_s^2\overline{U}_{\lambda^{-2\zeta}}^4\|^{2p}\right]^{1/2} \\ &\qquad + \lambda^{-2\zeta}\, E_P^3\Bigg[\sup_{0\le s\le\lambda^{-2\zeta}} \|\overline{Z}_s^1\overline{U}_{\lambda^{-2\zeta}}^2 + \overline{Z}_s^2\overline{U}_{\lambda^{-2\zeta}}^4\|^{2p} \\ &\qquad\qquad \times \sup_{0\le s\le\lambda^{-2\zeta}} \|\widehat{U}_s^1\|^{2p}\Bigg]^{1/2}\Bigg\} e^{c_p\lambda^{-(2\zeta+\beta)}} \\ &= O(\lambda^{-p(\zeta-3\varepsilon-\delta)}). \end{aligned}$$

By using the Chebyshev inequality, we have

$$\begin{aligned} \overline{I}_3 &\le \lambda^{p\beta}\, E_P\left[\sup_{0\le u\le\lambda^{-2\zeta}} \|\overline{Z}_u^3\overline{U}_{\lambda^{-2\zeta}}^2 + \overline{Z}_u^4\overline{U}_{\lambda^{-2\zeta}}^4\|^p\right] \\ &= O(\lambda^{-p(\zeta-\beta-2\varepsilon)}), \end{aligned}$$

because $\overline{Z}_t$, $\overline{U}_t$ are continuous processes. Finally we shall estimate $\overline{I}_1$. Since

$$\inf_{0\le s\le\lambda^{-2\zeta}} |(\widehat{U}_s^1)^* z|^2 \ge c. - \lambda^{-2\delta}$$

we have

$$\begin{aligned}
\big(\widetilde{z}\cdot\widehat{U}_s\psi(\widehat{x}_s)\big)^2 &\ge \frac{1}{2}\big((z^*\widehat{U}_s^1,0)^*\cdot\psi(\widehat{x}_s)\big)^2 - \big((0,z^*\widehat{U}_s^2)^*\cdot\psi(\widehat{x}_s)\big)^2 \\
&\ge \frac{1}{2}\Big(\Big(\frac{z^*\widehat{U}_s^1}{|(\widehat{U}_s^1)^*z|},0\Big)^*\cdot\psi(\widehat{x}_s)\Big)^2\Big(\inf_{0\le s\le\lambda^{-2\zeta}}|(\widehat{U}_s^1)^*z|^2\Big) \\
&\qquad -|\psi(\widehat{x}_s)|^2\Big(\sup_{0\le s\le\lambda^{-2\zeta}}\|\widehat{U}_s^2\|^2\Big) \\
&\ge c.\Big(\Big(\frac{z^*\widehat{U}_s^1}{|(\widehat{U}_s^1)^*z|},0\Big)^*\cdot\psi(\widehat{x}_s)\Big)^2 - 2C[\psi]\ \lambda^{4\varepsilon-2\delta}
\end{aligned}$$

under

$$\sup_{0\le s\le T}(1+|\widehat{x}_s|^2+\|\widehat{U}_s\|^2)\le\lambda^{4\varepsilon}, \qquad \sup_{0\le s\le\lambda^{-2\zeta}}\|\widehat{U}_s^2\|\le\lambda^{-\delta}.$$

Hence we see that

$$\begin{aligned}
\overline{I}_1 &\le E_P^4\Big[\exp\Big\{-\lambda^{2\gamma}\int_0^{\lambda^{-2\zeta}}(\widetilde{z}\cdot\widehat{U}_s\psi(\widehat{x}_s))^2\wedge 1\,ds\Big\}\Big] \\
&\le E_P^4\Big[\exp\Big\{-c.\lambda^{2\gamma}\int_0^{\lambda^{-2\zeta}}\big(1-\exp\{-\big(\widetilde{z}\cdot\widehat{U}_s\psi(\widehat{x}_s)\big)^2\}\big)\,ds\Big\}\Big] \\
&\le E_P^4\Big[\exp\Big\{-c.\lambda^{2\gamma} \\
&\qquad\times\int_0^{\lambda^{-2\zeta}}\Big(1-c.\exp\{-c.\Big(\Big(\frac{z^*\widehat{U}_s^1}{|(\widehat{U}_s^1)^*z|},0\Big)^*\cdot\psi(\widehat{x}_s)\Big)^2\}\Big)\,ds\Big\}\Big] \\
&\le E_P^4\Big[\exp\Big\{-c.\lambda^{2\gamma}\int_0^{\lambda^{-2\zeta}}\Big(\Big(\frac{z^*\widehat{U}_s^1}{|(\widehat{U}_s^1)^*z|},0\Big)^*\cdot\psi(\widehat{x}_s)\Big)^2\wedge 1\,ds\Big\}\Big] \\
&\le \exp\Big\{-c.\lambda^{2\gamma-2\zeta}\inf_{|y|\le\lambda^{\varepsilon}}\ \inf_{z\in S^{n-1}}\Big((\widetilde{z}\cdot\psi(y))^2\wedge 1\Big)\Big\}.
\end{aligned}$$

Therefore we obtain the estimate

$$\begin{aligned}
(31)\qquad \sup_{z\in S^{n-1}} E_P^2&\Big[\exp\{-\widehat{\mathcal{Q}}_T^z(x_j,\lambda^\gamma,\Psi)\}\Big] \\
&\le \exp\Big\{-c.\lambda^{2\gamma-2\zeta}\inf_{|y|\le\lambda^{\varepsilon}}\ \inf_{z\in S^{n-1}}\big((\widetilde{z}\cdot\psi(y))^2\wedge 1\big)\Big\}+O(\lambda^{-p}).
\end{aligned}$$

We now consider estimate (27). From (4) and (31), we have

$$\begin{aligned}\widehat{I}_1 &\le c.\, E_P^2\Bigg[\exp\Bigg\{-c.\sum_{k=0}^{l}\int_{\theta_0\neq 0}\overline{\boldsymbol{\nu}}(d\theta_0)\int\overline{\boldsymbol{\nu}}(d\theta_1)\cdots\overline{\boldsymbol{\nu}}(d\theta_k)\\ &\qquad\times\widehat{\mathcal{Q}}_T^z(x_j,\lambda^{\tau(\theta_1,\ldots,\theta_k)},\Phi^\lambda_{\theta_0\theta_1\cdots\theta_k})\Bigg\}\Bigg]^{c.}\\ &\le c.\exp\Bigg\{-c.\lambda^{2\sigma}\inf_{|y|\le\lambda^\varepsilon}\inf_{z\in S^{n-1}}\sum_{k=0}^{l}\int_{\theta_0\neq 0}\overline{\boldsymbol{\nu}}(d\theta_0)\int\overline{\boldsymbol{\nu}}(d\theta_1)\cdots\overline{\boldsymbol{\nu}}(d\theta_k)\\ &\qquad\times\big((\widetilde{z}\cdot\phi^\lambda_{\theta_0\theta_1\cdots\theta_k}(y))^2\wedge 1\big)\Bigg\}+O(\lambda^{-p})\\ &= O(\lambda^{-p}).\end{aligned}$$

Hence we get the estimate

$$\sup_{z\in S^{n-1}}\sum_{j=1}^{N_\lambda}\widetilde{E_P^1}\Bigg[\exp\Bigg\{-c.\int_{\theta_0\neq 0}\widehat{\mathcal{Q}}_T^z(x_j,\lambda^\tau,\Phi^\lambda_{\theta_0})\,\overline{\boldsymbol{\nu}}(d\theta_0)\Bigg\}\Bigg]=o(\lambda^{-p})$$

for any $p>1$. From Lemma 3.3, the law of random variable $\pi(x_T)$ admits a smooth density. $\square$

Proof of Corollary 1. It suffices to check that condition (4) in Theorem 2 is satisfied under (5). Let $0<\varepsilon<\big(2\varepsilon_l/5\big)\wedge(1-\sigma_2)$, and ζ, $\overline{\zeta}$ two constants satisfying $\varepsilon<\zeta<(\overline{\zeta}/2)<1-\sigma_2$, where $\sigma_2=\sigma(\theta)=(\alpha-2\rho)/(\alpha+2)$ $(\theta\in\Theta_2)$. Put $\widetilde{z}=\pi'(x)^*z$, and

$$\vartheta[\theta,i]=I_{\{\theta=i\in\{0,1,\ldots,d\}\}}+\theta^{i-d}I_{\{\theta\in\Theta_2\}}I_{\{i\in\{d+1,\ldots,2d\}\}}\quad(i=0,\ 1,\ldots,2d).$$

Let $\psi(x)$ be an $\boldsymbol{R}^m$-valued mapping on $\boldsymbol{R}^m$. Since $b(x,\theta)$ satisfies assumption $[R]$, we see that

$$\widetilde{z}\cdot\big(\wp_\theta(\lambda)\psi\big)(x)=\sum_{i=0}^{2d}\widetilde{z}\cdot\big(\wp_i\psi\big)(x)\,\vartheta[\theta,i]+O(\lambda^{-(1-\sigma_2)})\quad a.e.\ \theta\ (\overline{\boldsymbol{\nu}})$$

for sufficiently large λ. Hence it is a routine work to check that

$$\begin{aligned}&\int\Big\{\big(\widetilde{z}\cdot\big(\wp_\theta(\lambda)\psi\big)(x)\big)^2\wedge 1\Big\}\,\overline{\boldsymbol{\nu}}(d\theta)\\ &\ge c.\sum_{i=0}^{d}\Big\{\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)^2\wedge 1\Big\}+c.\int_{|\theta|\le\lambda^{-\zeta}}\Big(\widetilde{z}\cdot\big(\wp_\theta(\lambda)\big)(x)\Big)^2\,\widetilde{\nu}(d\theta)\\ &\ge c.\sum_{i=0}^{d}\Big\{\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)^2\wedge 1\Big\}\end{aligned}$$

$$
\begin{aligned}
&\quad + c.\int_{|\theta|\le\lambda^{-\zeta}}\left\{\sum_{i=d+1}^{2d}\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)\theta^{i-d}\right\}^2 \widetilde{\nu}(d\theta) + O(\lambda^{-\zeta(2-\alpha)-2(1-\sigma_2)})\\
&= c.\sum_{i=0}^{d}\left\{\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)^2\wedge 1\right\}\\
&\quad + c.\sum_{i=d+1}^{2d}\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)^2\int_{|\theta|\le\lambda^{-\zeta}}|\theta^{i-d}|^2\,\widetilde{\nu}(d\theta) + O(\lambda^{-\zeta(2-\alpha)-2(1-\sigma_2)})\\
&\ge c.\lambda^{-\zeta(4-\alpha)}\sum_{i=0}^{2d}\left\{\big(\widetilde{z}\cdot(\wp_i\psi)(x)\big)^2\wedge 1\right\}.
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
&\int_{\theta_0\neq 0}\overline{\boldsymbol{\nu}}(d\theta_0)\int\overline{\boldsymbol{\nu}}(d\theta_1)\cdots\overline{\boldsymbol{\nu}}(d\theta_k)\left\{\big(\widetilde{z}\cdot\phi^{\lambda}_{\theta_0\theta_1\cdots\theta_k}(x)\big)^2\wedge 1\right\}\\
&\qquad\ge c.\lambda^{-2\varepsilon_0\zeta(4-\alpha)}\sum_{i_0=1}^{2d}\sum_{i_1,\dots,i_k=0}^{2d}\left\{\big(\widetilde{z}\cdot\wp_{i_k}\cdots\wp_{i_1}a_{i_0}(x)\big)^2\wedge 1\right\}.
\end{aligned}
$$

Choosing σ such that $\sigma > \big(\overline{\zeta}(\varepsilon_0+1)(4-\alpha)/2\big)$, we see that condition (4) in Theorem 2 is satisfied under condition (5). The proof is now complete. □

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