Harada, K.
Osaka J. Math.
15 (1978), 633–635

THE AUTOMORPHISM GROUP AND THE SCHUR MULTIPLER OF THE SIMPLE GROUP OF ORDER $2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$

KOICHIRO HARADA[1)]

(Received August 8, 1977)

As in [1], F denotes the simple group of order $2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$. F is popularly called F_5 as it appears in the centralizer of an element of order 5 of the so called "Monster."

The simple group F has been constructed by S. Norton [2] and the automorphism group of it has also been determined by him. From his construction of F it can be seen that F has an outer automorphism of order 2.

In this note, we shall give an alternate proof of the fact that $|\mathrm{Aut}(F): F|\leq 2$. We also show that the Schur multiplier of F is trivial.

Theorem A. $|\mathrm{Aut}(F): F|=2$ and $H^2(F, C^*)=0$.

By [1, Proposition 2.13], F contains a subgroup F_0 isomorphic to the alternating group A_{12} of degree 12. It is easy to see the following:

Lemma 1. *F_0 is maximal in F. Every subgroup of F isomorphic to F_0 is conjugate in F to F_0.*

Proof of the first part of Theorem A. Suppose that $|\mathrm{Aut}(F): F|>2$. Then there exists an element $\alpha\in\mathrm{Aut}(F)$ of order p, p a prime, such that $C_F(\alpha)\supseteq F_0$. Let x be an element of $F_0\cong A_{12}$ of type (12345). Then by [1, Lemma 2.17], $C_F(x)\cong Z_5\times U_3(5)$. Since no element of $\mathrm{Aut}(U_3(5))^\#$ centralizes a subgroup of $U_3(5)$ isomorphic to A_7, $\langle C_F(x), \alpha\rangle\cong\langle\alpha\rangle\times Z_5\times U_3(5)$. Hence by the maximality of F_0, $[F, \alpha]=1$. This contradiction shows that $|\mathrm{Aut}(F): F|\leq 2$.

Proof of the second part of Theorem A. Let $m(F)$ be the order of the Schur multiplier of F. We denote by $m_p(F)$ the p-part of $m(F)$. $\tilde{F}$ will denote a central extension of F. For a subgroup A of F, $\tilde{A}$ will denote the inverse image of A in $\tilde{F}$.

Lemma 2. $m_2(F)=1$.

Proof. Let $\tilde{F}$ be a group such that $\tilde{F}/Z(\tilde{F})\cong F$ and $Z(\tilde{F})\cong Z_2$. F contains

1) This research was supported in part by NSF Grant MCS 76–07253.

an involution 2_A such that $2_A \in C_F(2_A)'$ is the double cover of Higman-Sims group. As the Schur multiplier of Higman-Sims group is of order 2, 2_A lifts to an involution of $\tilde{F}$. As 2_A is conjugate to (12) (34) of $F_0 \cong A_{12}$, $\tilde{F}_0 \cong Z_2 \times A_{12}$. This implies that the involution $2_B \sim (12)(34)(56)(78)$ also lifts to an involution in $\tilde{F}$. If $\tilde{i}$ is an involution of $Z(\tilde{F})$, we have shown that $\tilde{i}$ is not a square in $\tilde{F}$. Let $M = C_F(2_B)$ and $R = O_2(M)$. Then $M/R \cong A_5 \wr Z_2$, R is an extra special group of order 2^9 and all elements of R of order 4 are conjugate in M [1, Lemma 2.9]. Hence $\Phi(\tilde{R})$ does not contain $\tilde{i}$. Hence $\tilde{R} = \langle \tilde{i} \rangle \times \tilde{R}_1$ where $\tilde{R}_1 \cong R$. Let 3_B be an element of order 3 in M which acts fixed-point-free on $R/Z(R)$ [1, Lemma 2.8, $3_B = \sigma_1$]. Then $C_M(3_B) \cong Z_3 \times SL(2, 5)$ [1, Lemma 2.15]. We may take $\tilde{R}_1 = [\tilde{R}, \tilde{3}_B]$. If $3'_B$ is an element of order 3 in $C_M(3_B)'$, then $3_B \sim 3'_B$ in M. Hence $\tilde{R}_1 = [\tilde{R}, \tilde{3}'_B]$ and so $\tilde{R}_1 \lhd \tilde{M}$. As $C_F(3_B)$ is an extension of an extra special group of order 3^5 by $SL(2, 5)$ [1, Lemma 2.16], we conclude that $C_{\tilde{F}}(\tilde{3}_B)/O(C_{\tilde{F}}(\tilde{3}_B))Z(\tilde{R}_1) \cong Z_2 \times A_5$. A similar isomorphism holds for $C_{\tilde{F}}(\tilde{3}'_B)$. Hence $\tilde{M}'\langle \tilde{i} \rangle / \tilde{R}_1 \cong Z_2 \times A_5 \times A_5$. As $|M : M'| = 2$, $\tilde{i} \notin \tilde{M}'$. Hence $m_2(F) = 1$.

Lemma 3. $m_3(F) = 1$.

Proof. Let $\tilde{F}$ be a group such that $\tilde{F}/Z(\tilde{F}) \cong F$ and $|Z(\tilde{F})| = 3$. Let A be a subgroup of $F_0 \cong A_{12}$ which corresponds to $\langle (123), (456), (789), (10, 11, 12) \rangle$. Using $C_F((123)) \cong Z_3 \times A_9 \subseteq A_{12}$ and the fusion $(123) \sim (123)(456) \sim (123)(456)(789)(10, 11, 12)$, we can compute that $N_F(A)/A$ is a group of order $2^7 \cdot 3^2$. In particular, $N_F(A)$ contains a Sylow 3-subgroup of F. As $\tilde{F}_0 \cong Z_3 \times A_{12}$, $\tilde{A}$ is elementary of order 3^5. Let $\tilde{z}$ be an involution of $\tilde{F}_0$ which maps onto (12) (45) (78) (10, 11). Then z inverts A. Further $z \sim 2_B$ in F. We have that $C_{\tilde{A}}(\tilde{z}) = Z(\tilde{F})$ and $\widetilde{N_F(A)} = [\tilde{A}, \tilde{z}]\, \widetilde{(C_F(z) \cap N_F(A))}$. By the structure of $C_F(z) \cong C_F(2_B)$ we obtain that Sylow 3-subgroups of $\widetilde{C_F(z) \cap N_F(A)}$ are elementary of order 3^3. Hence $Z(\tilde{F}) \not\subseteq \tilde{F}'$. Thus $m_3(F) = 1$.

Lemma 4. $m_5(F) = 1$.

Proof. Let $\tilde{F}$ be a group with $\tilde{F}/Z(\tilde{F}) \cong F$ and $|Z(\tilde{F})| = 5$. A Sylow 5-subgroup S of F is described as follows:

$$S = \langle z, \alpha, \beta, \gamma, \partial, \chi \rangle$$
$$z^5 = \alpha^5 = \beta^5 = \gamma^5 = \partial^5 = \chi^5 = 1\,,$$
$$[\alpha, \beta] = [\alpha, \gamma] = [\alpha, \partial] = [\gamma, \beta] = z\,,$$
$$[\alpha, \chi] = \beta, \quad [\beta, \chi] = \gamma, \quad [\gamma, \chi] = \partial\,,$$

with all the other commutators of pairs of generators being trivial. We have that $\langle z, \alpha, \beta, \gamma, \partial \rangle$ is an extra special group of order 5^5. We can also check that all elements of $V=\langle z, \partial\rangle^{\#}$ are conjugate in F and $N_F(V)/C_F(V)\cong SL(2, 5)*Z_4$, $C_F(V)=\langle z, \beta, \gamma, \partial, \chi\rangle$. We have that $V=Z(C_F(V))$ and $C_F(V)/V$ is a nonabelian group of order 5^3. The $SL(2, 5)$ acts faithfully on $C_F(V)/V$. If $\tilde{V}$ is nonabelian, then $\widetilde{C_F(V)}=\tilde{V}*C_{\tilde{F}}(\tilde{V})$. Clearly then $Z(C_F(V))\supset V$. Hence $\tilde{V}$ is elementary and $\tilde{V}=Z(\widetilde{C_F(V)})$. Let $\tilde{z}$ be an involution of $\widetilde{N_F(V)}'$. Then $C_{\tilde{V}}(\tilde{z})=Z(\widetilde{F})$ and $[\tilde{z}, \tilde{V}]\lhd \widetilde{N_F(V)}$. As $\widetilde{C_F(V)}/[\tilde{z}, \tilde{V}]$ is of class 2, $Z(\widetilde{F})\not\subseteq \widetilde{C_F(V)}'$. Hence $Z(\widetilde{F})\not\subseteq \widetilde{N_F(V)}'$. This implies that $m_5(F)=1$.

As $m_7(F)=m_{11}(F)=m_{19}(F)=1$, this completes the proof of the theorem.

THE OHIO STATE UNIVERSITY

References

[1] K. Harada: *On the simple group F of order* $2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$, Proceedings of the conference on finite group, Academic Press Inc., 1976, 119–276.

[2] S.P. Norton: *Construction and properties of a new simple group*, Doctoral Dissertation, University of Cambridge, 1975.