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HOMOLOGY PLANES AND ALGEBRAIC CURVES

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1. Introduction and results

This paper is concerned with the construction of acyclic affine surfaces from plane algebraic curves.

Surfaces will always be connected, non-singular, quasi-projective, algebraic surfaces over the complex numbers. Let R be a subring of the rational numbers. A surface V is called *R-homology plane* if $H_i(V; R)=0$ for $i>0$. In the case $R=\boldsymbol{Z}$ we simply refer to this as a *homology plane*.

We investigate homology planes via their compactifications. The compactifications give rise to an algorithmic construction of surfaces from curves in the projective plane $\boldsymbol{P}^2$. If X is a projective surface and $C\subset X$ a curve we call (X, C) or X a *compactification* of any surface V which is isomorphic to the complement $X\backslash C$.

Ramanujam produced the first example of a homology plane. (See RAMANUJAM [1971]). His homology plane was in fact contractible and produced a counterexample to the conjecture that a smooth contractible affine surface over the complex numbers was the standard plane. Any smooth affine variety (over the reals) is the interior of a smooth manifold with boundary. In the case of a homology plane, it follows that the boundary is a homology sphere. When the homology plane is contractible, this observation gives homology planes bounding contractible manifolds—a point of interest in topology. It wasn't until 1987 that other homology planes were found. Gurjar-Miyanishi (see GURJAR—MIYANISHI [1987]) produced all homology planes of logarithmic Kodaira dimension 1. In this paper they ask whether there are an infinite number of contractible homology planes of Kodaira dimension 2. We announced the first affirmative solution to this problem in TOM DIECK—PETRIE [1989]. Among the results of this paper are the details of the announcement. Infinite families of homology planes of Kodaira dimension 2 result from the main theorems A and B here. See (3.20) and (3.21).

Another application of the main results here is the production of homology planes which have non trivial finite order automorphisms. These planes and finite order automorphisms produce counterexamples to the conjecture that non trivial homology planes have no finite order automorphisms. This conjecture appears in PETRIE [1989] where it is also shown that the contractible homology planes of Kodaira dimension 1 have no finite order automorphisms. See TOM DIECK [1990a] and MIYANISHI-SUGIE [1991], [1991a] for the construction of homology planes with automorphisms.

There is recent interest in the higher dimensional analogs of homology planes, i.e. acyclic affine varieties of arbitrary dimension. Some are obtained as products of homology planes with C^k, see ZAIDENBERG [1991]. Others are produced as hypersurfaces in C^n in TOM DIECK—PETRIE [1990]. See also PETRIE [1992], TOM DIECK [1992a], [1992b], DIMCA [1990], KALIMAN [1991]. These analogs are related to the Abhyankar-Moh Problem: Is any hypersurface in C^n which is isomorphic to C^{n-1} actually ambiently isomorphic to C^{n-1}? The point is that it is possible to produce acyclic hypersurfaces in C^n but the question remains whether they are isomorphic to C^{n-1} and therefore potential negative responses to the problem. One related question is: For which homology planes X can $X \times C^h$ be a hypersurface in C^n? See TOM DIECK—PETRIE [1990] for the case $n=3$. Perhaps the explicit homology planes produced from Theorems A and B will yield answers to these problems.

In this paper we give a construction of all homology planes. This is the content of Theorem A. The construction produces other affine surfaces as well. Theorem B and Theorem 3.13 tell which among these are homology planes and the Determinant Algorithm 4.27 gives an effective method of applying Theorems B and 3.13. The results of the first 4 sections deal with the existence of homology planes. The constructions begin with a curve in a minimal projective surface (which can be the projective plane). If there is one homology plane associated to such a curve, there are infinitely many distinct homology planes associated with the same curve. It can even happen that the same homology plane can be associated with different curves. Any classification of homology planes must take this into account. In subsequent notes we illustrate this point by using e.g. the group of birational automorphisms of the projective plane to produce isomorphisms between some of the homology planes constructed here and elsewhere.

Our first theorem explains the construction of homology planes from curves. In order to state it we need some notation.

By an *expansion* $p: X \to Y$ of Y we mean a composition

$$p: X = X_{k+1} \xrightarrow{p_k} X_k \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{p_0} X_0 = Y,$$

where each morphism p_j blows up a single point. Blowing up a single point is also called a *σ-process*. An expansion $p: X \to Y$ of Y is called a *contraction* of X. The *exceptional set* of any expansion $p: X \to Y$ is

$$\Sigma_p = \Sigma(p) = \{y \in Y \mid p^{-1}(y) \text{ is not a point}\} .$$

The *exceptional divisors* of p are the components of $p^{-1}(x)$ for $x \in \Sigma_p$. Let $\mathcal{E} = \mathcal{E}(p)$ denote the set of exceptional divisors.

We now explain the basic setting for the whole paper. Let $D = D_1 \cup \cdots \cup D_n \subset Y$ be a curve in a minimal rational projective surface Y such that the D_i are rational and the (irreducible) components of D. There is a unique minimal expansion $\pi: X \to Y$ such that $\pi^{-1}(D) = C$ has normal crossings (embedded resolution of singularities). Minimal means that any other expansion with this property factors over π.

Choose a partition $\mathcal{E} = \mathcal{E}_0 \amalg \mathcal{E}_1$ and define $d: \mathcal{E} \to \{0, 1\}$ by $d^{-1}(i) = \mathcal{E}_i$. Let $D' = p^{\#} D$ denote the proper transform of D. Consider the curve (=reduced effective divisor, additive notation) $D(d)$ in X

$$D(d) := D' + \sum_{E \in \mathcal{E}_1} E \subset p^{-1}(D) = D' + \sum_{E \in \mathcal{E}} E . \tag{1.1}$$

Then $D(d)$ is a normal crossing curve and has a dual graph $\Gamma D(d)$ with vertex set $\Gamma_0 D(d)$ and edge set $\Gamma_1 D(d)$. If Γ is a graph we set $s(\Gamma) = 1 - \chi(\Gamma)$, where χ denotes the Euler characteristic. If Γ is connected, then $s(\Gamma)$ is the number of its (independent) cycles. We refer to the beginning of section 2 for our terminology relating to graphs. In the case of $\Gamma = \Gamma D(d)$ we set

$$s(d) = 1 - \chi(\Gamma D(d)), \quad r(d) = |\mathcal{E}_0| . \tag{1.2}$$

The function $d: \mathcal{E} \to \{0,1\}$ is called a *selection function* for D, if the following holds:

$$\begin{aligned} &(1) \quad \Gamma D(d) \text{ is connected} \\ &(2) \quad r(d) + s(d) = n - b_2(Y) \end{aligned} \qquad (b_2 = \text{second Betti number}) . \tag{1.3}$$

Affine varieties have a connected compactification divisor (1.3.1). The equality (1.3.2) is basic for varieties of general type. This fact will be explained in the proof of Proposition (2.3). The term 'selection function' simply refers to the fact that one has in general a choice for such a function.

We want to cut the cycles of $\Gamma D(d)$. We call $\Phi \subset \Gamma_1 D(d)$ a *cutting set* for (D, d) if the subgraph with vertex set $\Gamma_0 D(d)$ and edge set $\Gamma_1 D(d) \setminus \Phi$ is a tree. (Thus $|\Phi| = s(d)$, provided (1.3.1) holds. The case $\Phi = \emptyset$ is not excluded.)

Given (D, d, Φ) we let $\mathrm{B1}(D, d, \Phi)$ denote the set (of isomorphism classes) of expansions $p: Z(p) \to X$ with the following properties (1.4). According to our terminology in section 2, the set Φ is a set of points in X, so that the statement (1.4.1) makes sense.

(1.4)
(1) $\Sigma_p = \Phi$.
(2) For each $x \in \Phi$ the graph of $A'(x) \cup B'(x) \cup p^{-1}(x)$ is a linear tree. Here $A(x)$ and $B(x)$ are the curves which intersect in x.

(3) Each curve $p^{-1}(x)$, $x\in\Phi$, contains a unique component $E_0(x)$ with self-intersection -1 (called (-1)-curve).

Expansions with these properties are referred to as *standard*. Later in this paper these expansions are called subdivisional. This terminology is taken from FUJITA [1982], (3.4). Let

$$B(p)=p^{-1}(D(d))-\sum_{x\in\Phi}E_0(x)\subset Z(p)\,. \tag{1.5}$$

The family of surfaces

$$V(p)=Z(p)\backslash B(p),\quad p\in \mathrm{B1}(D,d,\Phi) \tag{1.6}$$

is called the *tower* of surfaces belonging to (D, d, Φ). If Φ is empty, the tower consists of the single element $X\backslash D(d)$. Open surfaces V have a logarithmic Kodaira dimension $\bar{\kappa}(V)\in\{-\infty, 0, 1, 2\}$. It is known that homology planes $V\neq \boldsymbol{C}^2$ with $\bar{\kappa}(V)\in\{-\infty, 0\}$ don't exist.

Theorem A. *Let V be a homology plane of general type. Then there exists a rational curve D in some minimal rational surface Y, a selection function d for D and a cutting set Φ for (D, d) such that V is isomorphic to $V(p)$ for some $p\in \mathrm{B1}(D, d, \Phi)$.*

In fact, Y can always be chosen to be $\boldsymbol{P}^2$. But for some considerations other minimal surfaces Y are more appropriate.

The homology planes with $\bar{\kappa}=1$ have been described by GURJAR—MIYANISHI [1987]. Therefore we concentrate on surfaces with $\bar{\kappa}=2$, called of *general type*.

General considerations suggest that an algorithm like Theorem A exists for construction of homology planes. By compactification, a homology plane V is $Z\backslash B$ where (Z, B) is a compactification of V. The exceptional divisors lying on B can be contracted giving a new surface X and a curve Y such that $V=X\backslash Y$. Further considerations show in fact X can be a minimal rational surface and Y rational. Still given a rational minimal surface X and a rational curve $Y\subset X$, it is not at all obvious if one can reverse this process and produce a homology plane. Theorem A and its companions Theorems B and 4.27 give an effective algorithm for deciding this.

By Poincaré duality the irreducible components of D or $D(d)$ define elements in $H^2(Y)$ or $H^2(X)$. This is explained in the beginning of section 3. Using this fact we state:

Addendum to Theorem A. *Under the hypothesis of Theorem A the irreducible components of D (resp. of $D(d)$) generate $H^2(Y)$ (resp. $H^2(X)$).*

The next theorem tells us which surfaces $V(p)$ in a tower are homology planes. Suppose the edge $x \in \Phi$ is an intersection of the components $A(x), B(x)$ of $D(d)$, denoted $\partial(x)=\{A(x), B(x)\}$. Let $m_p(A(x))$ denote the multiplicity of $E_0(x)$ in the total transform $p^*A(x)$ (pullback of $A(x)$ as Cartier divisor, compare Definition (3.6)). Then $m_p(A(x))$ and $m_p(B(x))$ are coprime positive integers. Let $\boldsymbol{Z}(\partial x)$ denote the free abelian group with basis $\{A(x), B(x)\}$. Let

$$m_p \in \bigoplus_{x \in \Phi} \boldsymbol{Z}(\partial x)$$

be the element with x-component $m_p(A(x))\,A(x)+m_p(B(x))\,B(x)$. More generally a *multiplicity function* m for (D, d, Φ) assings to each $x \in \Phi$ a pair of coprime integers $(m(A(x)), m(B(x)))$. We also denote by m the corresponding element in $\bigoplus_{x \in \Phi} \boldsymbol{Z}(\partial x)$.

Theorem B. *Let D be a curve with rational components in a minimal rational projective surface Y. Let d be a selection function for D such that the components of $D(d)$ generate $H^2(X)$. Let $\Phi \neq \emptyset$ be a cutting set for (D, d). Then there exists a multilinear function*

$$\Delta = \Delta(D, d, \Phi)\colon \bigoplus_{x \in \Phi} \boldsymbol{Z}(\partial x) \to \boldsymbol{Z}$$

such that:

(1) *$V(p)$, $p \in \mathrm{B1}(D, d, \Phi)$, is a $\boldsymbol{Q}$-homology plane if and only if $\Delta(m_p) \neq 0$.*

(2) *$V(p)$ is a $\boldsymbol{Z}$-homology plane if and only if $\Delta(m_p)=\pm 1$.*

(3) *If $\Delta(m_p) \neq 0$, then $|H^2(V(p); \boldsymbol{Z})| = |H_1(V(p); \boldsymbol{Z})| = |\Delta(m_p)|$.*

The function $\Delta(D, d, \Phi)$ will be called the *discriminant* of (D, d, Φ). The discriminant is defined up to sign.

The multiplicity functions can be used to parametrize elements in $\mathrm{B1}(D, d, \Phi)$. Using the notation set up so far we can state:

(1.7) Proposition. *The assignment $p \mapsto m_p$ sets up a bijection between* $\mathrm{B1}(D, d, \Phi)$ *and the set* $\mathrm{Mul}(D, d, \Phi)$ *of multiplicity functions for (D, d, Φ).*

The following elementary algebraic remarks use (1.7). By linearity, Theorem B determines $|\Delta(m)|$ for each $m \in \bigoplus \boldsymbol{Z}(\partial x)$. Two bilinear maps Δ_1 and Δ_2 which have the same absolute value $|\Delta_1(m)| = |\Delta_2(m)|$ for all arguments m are equal up to sign. Therefore there exists up to sign at most one bilinear map Δ which has the properties stated in Theorem B.

By elementary algebra we see that if Δ assumes the value 1, then it does so for infinitely many arguments. Therefore if a tower contains homology planes at all, then it contains an infinite family.

The proof of Theorem B will show that the discriminants is effectively computable. We do not claim that members of a tower are always surfaces of general type. There are usually some degenerate cases in a tower. We shall see examples later. Different towers may contain isomorphic varieties.

Theorem A is proved in section 2; Theorem B is proved in section 3. In section 4 we prove the Addendum to Theorem A and Proposition (1.7). For an annoucement and further information see TOM DIECK—PETRIE [1989].

2. Towers of surfaces

This section is devoted to the proof of Theorem A.

We begin by introducing some terminology. A graph $\Gamma=(\Gamma_0, \Gamma_1)$ consists of two sets Γ_0, the set of vertices, and Γ_1, the set of edges, and a map ∂ which assigns to each $x\in\Gamma_1$ a non empty set $\partial x\subset\Gamma_0$ of at most two elements. The geometric realization of Γ is a one-dimensional CW-complex with 0-skeleton Γ_0 and with a 1-simplex corresponding to each $x\in\Gamma_1$ attached to ∂x in the obvious manner. We do not distinguish notationally between Γ and its geometric realization and apply a geometric terminology to Γ, like: cycle, subdivision, Euler characteristic, the edge x connects the points in ∂x etc.

A curve $C\subset X$ is said to have *normal crossings* if its irreducible components are non-singular, two components intersect in at most one point and such a point is a transverse intersection, and there are no triple points. A normal crossing curve C has a *weighted dual graph* $(\Gamma C, w)$. The graph $\Gamma C=(\Gamma_0 C, \Gamma_1 C)$ has as vertex set $\Gamma_0 C$ the set of irreducible components of C and as edge set $\Gamma_1 C$ the set of double points. An edge connects the vertices which intersect in it, i.e. if $A\cap B=\{x\}$, then $\partial(x)=\{A, B\}$. Therefore $\Gamma_1 C$ is also a set of points in X, but the Γ-notation always refers to the graph. The *weight function* w: $\Gamma_0 C\to\boldsymbol{Z}$ associates to $A\in\Gamma_0 C$ its self-intersection $A\cdot A$ in X. The intersection matrix is $\Gamma_0 C\times\Gamma_0 C\to\boldsymbol{Z}$, $(A, B)\mapsto A\cdot B$ and its determinant is denoted $\det(\Gamma C, w)$.

(2.1) Proposition. *Suppose C is a normal crossing curve in the rational projective surface X. Then the following are equivalent:*

(1) *$V=X\setminus C$ is a $\boldsymbol{Q}$-homology plane.*

(2) *The components of C are rational curves and ΓC is a tree. The inclusion $C\subset X$ induces an isomorphism $H_2(C;\boldsymbol{Q})\to H_2(X;\boldsymbol{Q})$.*

Proof. We use co-homology with coefficients in $\boldsymbol{Q}$. From the homological structure of topological surfaces one deduces easily: $H_1(C)=0$ if and only if the components of C are 2-spheres and ΓC is a tree.

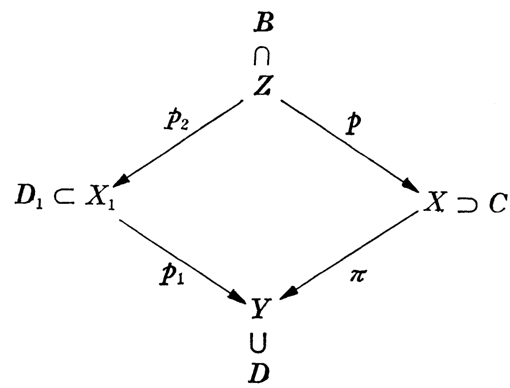
We combine the exact homology sequence of (X, C) with Poincaré duality $H^{4-i}(X, C)\cong H_i(V)$ and $H^*(A)\cong H_*(A)$ for the aprpopriate spaces A.

(1)$\Rightarrow$(2) By hypothesis $H_i(X, C)\cong H^{4-i}(V)\cong 0$ for $0\leq i\leq 3$. The exact sequence of (X, C) and $H_1(X)\cong 0$ implies $H_1(C)\cong 0$ and the isomorphism $H_2(C)$

$\cong H_2(X)$.

(2)$\Rightarrow$(1) From the hypothesis and the remark at the beginning $H_1(C)\cong 0$ and $H_3(X)\cong 0$. The isomorphism $H_2(C)\cong H_2(X)$ and the exact sequence of (X, C) imply $H_2(X, C)\cong H_3(X, C)\cong 0$ and hence $H_1(V)\cong H_2(V)\cong 0$. Since C is connected we obtain $H_3(V)\cong H_4(V)\cong 0$ from the exact sequence. □

Let V be a homology plane of general type. By the rationality theorem of GURJAR—SHASTRI [1989], there exists a rational projective surface X_1 and a curve $D_1\subset X_1$ such that V is isomorphic to $X_1\backslash D_1$. By embedded resolution of singularities (BARTH—PETERS—VAN DE VEN [1984], II(7.2)) we can assume that D_1 has normal crossings. Let p_1: $X_1\to Y$ be a contraction to a minimal rational surface and set $D=p_1(D_1)$. We can expand further by p_2: $Z\to X_1$ such that $(p_1 p_2)^{-1} D$ has normal crossings. Let $\pi=\pi_D$: $X=X_D\to Y$ denote the minimal expansion such that $\pi^{-1}(D)=C$ has normal crossings. By minimality of π we have a factorization $p_1 p_2=\pi p$ with an expansion p: $Z\to X$. Let $B=p_2^{-1}(D_1)$. We have $V\cong Z\backslash B$ by construction. We collect some properties of (p, B). For the convenience of the reader we display the data above in the following diagram.



(2.2) Proposition. *Let V be a $\boldsymbol{Q}$-homology plane. The pair (p, B) has the following properties*:

(1) $\pi p\,(B) = D$.

(2) $\Sigma(\pi p)\subset D$.

(3) ΓB *is a tree.*

(4) $|\Gamma_0\,(\pi p)^{-1} D| - |\Gamma_0 B| = |\Gamma_0 D| - b_2(Y)$.

Proof. The equality (1) holds by construction.

(2) If $x\in\Sigma(\pi p)\backslash D$, then $p^{-1}(x)\subset Z\backslash B=V$ and at least one component of $p^{-1}(x)$ is a (-1)-curve E. Contract E to obtain V'. Then $H^2(V)\cong \boldsymbol{Z}\oplus H^2(V')$ contradicting $H^2(V;\boldsymbol{Q})=0$.

(3) is given by (2.1).

(4) We have $|\Gamma_0(\pi p)^{-1} D| - |\Gamma_0 D| = b_2(Z)-b_2(Y)$ as a general property

of expansions and $|\Gamma_0 B| = b_2(Z)$ by (2.1). This yields the result. $\square$

The curve $p(B)$ is contained in $\pi^{-1}(D)$ but may be different from it. There is a function $d\colon \mathcal{E}(\pi) \to \{0, 1\}$ such that $p(B) = D(d)$, see (1.1).

(2.3) Proposition. *d is a selection function for D.*

Proof. We have to verify (1.3). Since V is a $\boldsymbol{Q}$-homology plane, ΓB is a tree, so the curve B is connected; hence so is $p(B) = D(d)$ and therefore its weighted dual graph $\Gamma D(d)$.

In order to verify (1.3.2) we have to analyze in detail the passage from X to $\boldsymbol{Z}$.

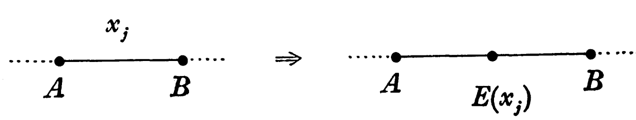
We write p as a composition of σ-processes

$$p\colon \boldsymbol{Z} = \boldsymbol{Z}_{k+1} \xrightarrow{p_k} \boldsymbol{Z}_k \to \cdots \to \boldsymbol{Z}_1 \xrightarrow{p_0} \boldsymbol{Z}_0 = X .$$

The ρ-process p_j blows up a point x_j to $E(x_j)$.

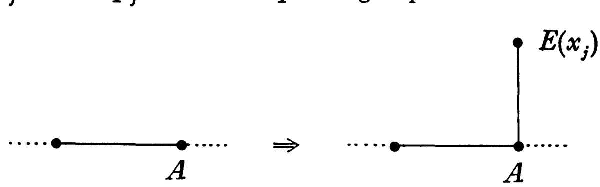
Let $D_j \subset \boldsymbol{Z}_j$ denote the total transform of $D(d) =: D_0$ and $B_j \subset \boldsymbol{Z}_j$ the image of B. Then $B_j \subset D_j$. The graphs ΓD_j and ΓD_{j+1} are related in either one of the following ways:

(2.4) The process p_j blows up a double point x_j of D_j to $E(x_j)$. Then ΓD_{j+1} is obtained from ΓD_j by subdividing the edge $x_j \in \Gamma_1 D_j$. This p_j is called *subdivisional* expansion.



$\Gamma D_j \quad \Rightarrow \quad \Gamma D_{j+1}$

(2.5) The process p_j blows up a regular point x_j of D_j in the component A. Then ΓD_{j+1} is obtained from ΓD_j by attaching a new edge to ΓD_j at the vertex $A \in \Gamma_0 D_j$. This p_j is called a *sprouting* expansion.



$\Gamma D_j \quad \Rightarrow \quad \Gamma D_{j+1}$

The graphs ΓB_j and ΓB_{j+1} are related in either one of the following ways:

(2.6) B_{j+1} is the total transform of B_j. In this case ΓB_{j+1} arises from ΓB_j

either by subdividing as in (2.4) or by sprouting as in (2.5).

(2.7) B_{j+1} is the proper transform of B_j. If p_j is subdivisional, then ΓB_{j+1} is obtained from ΓB_j by removing the edge $x_j \in \Gamma_1 B_j$. If p_j is sprouting, then ΓB_j and ΓB_{j+1} are combinatorially isomorphic.

Of course, passage from B_j to B_{j+1} changes the weights.

The geometric effect in (2.7) is the following. If p_j in (2.7) is subdivisional, then the edge $x_j \in \Gamma_1 B_j$ must be contained in a cycle, for otherwise ΓB_{j+1} and hence ΓB would not be connected. Therefore in this case $s(\Gamma B_{j+1}) = s(\Gamma B_j) - 1$, i.e. the number of cycles decreases by one (see the notation introduced before (1.2)). If p_j in (2.7) is sprouting, then ΓB_{j+1} and ΓB_j are homeomorphic. This shows:

Since $\Gamma D(d)$ has $s(d)$ cycles and ΓB is a tree there are exactly $s(d)$ subdivisional expansions (2.7).

The total number $w = |\Gamma_0 p^{-1} D(d)| - |\Gamma_0 B|$ of expansions (2.7) is counted as follows.

$$\begin{gathered} |\Gamma_0(\pi p)^{-1} D| - |\Gamma_0 B| = \\ |\Gamma_0(\pi p)^{-1} D| - |\Gamma_0 p^{-1} D(d)| + |\Gamma_0 p^{-1} D(d)| - |\Gamma_0 B| = \\ |\Gamma_0 \pi^{-1} D| - |\Gamma_0 D(d)| + |\Gamma_0 p^{-1} D(d)| - |\Gamma_0 B| = \\ r(d) + w\,. \end{gathered}$$

This uses the definition of $r(d)$ in (1.2). From this string of equalities, (2.2.4) and the previous paragraph we obtain

$$|\Gamma_0 D| - b_2(Y) = r(d) + w \geq r(d) + s(d)\,.$$

This is half of (1.3.2)

In order to prove equality we have to use the assumption that V is a surface of general type. For if we would not have equality, then a sprouting expansion p_j would occur in (2.7). In that case $E(x_j)$ intersects B_{j+1} in a single point and $E_j = E(x_j) \setminus E(x_j) \cap B_{j+1}$ is a curve in $Z_{j+1} \setminus B_{j+1}$ which is isomorphic to the affine line $\boldsymbol{C}$. The proper transform of this curve in $V = Z \setminus B$ is also isomorphic to $\boldsymbol{C}$. This contradicts a result of ZAIDENBERG [1988] and of MIYANISHI—TSUNODA [1990], which asserts that a homology plane of general type does not contain such a curve. □

We use the discussion in the proof of (2.3) to state another fact. We call the expansion $p: Z \to X$ minimal if the following holds. Let $p_1: Z_1 \to X$ be an expansion and $B_1 \subset Z_1$ a curve such that (2.2) holds for (p_1, B_1) in place of (p, B). Suppose moreover that $\omega: Z \to Z_1$ is an expansion such that $\omega^{-1}(B_1) = B$ and $p_1 \omega = p$. If in any such situation ω is an isomorphism, then (p, B) is called *minimal*.

(2.8) Proposition. *If $p: Z \to X$ is minimal there are no sprouting expansions* (2.6) *and* (2.7).

Proof. In the proof of (2.3), by verifying (1.3.2), we have already seen that there are no sprouting expansions (2.7). Suppose p_j is a sprouting expansion (2.6). Then $E(x_j) \subset B_{j+1}$ is a (-1)-curve. It represents a vertex in $\Gamma_0 B_{j+1}$ which is not a branch point, i.e. is contained in at most two edges, and which is not contained in a cycle. One now shows by induction that ΓB_r for $r \geq j+1$ contains a vertex with weight -1 which is not a branch point and which is not contained in a cycle. The fact that $B_{j+1}=B$ contains such a curve shows that $p: Z \to X$ is not minimal. □

Using (2.8) it is now easy to verify by induction on j:

(2.9) Proposition. *If $p: Z \to X$ is minimal, then ΓD_j is a subdivision of ΓD_i, $i<j$, and ΓB_j is a subcomplex of ΓD_j.*

Proof of Thoerem A. Let V be a homology plane of general type. In the preceding discussion we have already obtained the following data: A minimal rational surface Y, a rational curve D in Y, a selection function d for D, an expansion $p: Z \to X$ and a curve $B \subset Z$ such that $V \cong Z \setminus B$.

If (p, B) were not minimal we could pass to a minimal situation by suitable contractions. So let us assume that (p, B) is minimal. This is then perhaps no longer the expansion which appears in the diagram before (2.2), but (2.2) and (2.3) still hold and therefore also (2.8) and (2.9).

We claim that Σ_p is a cutting set for (D, d) and satisfies (1.4).

Property (1.4.2): This follows from (2.8).

Property (1.4.3): In geometric terms $\Sigma_p \subset \Gamma_1 D(d)$ is precisely the set of those edges which are subdivided when we pass from $\Gamma D(d)$ to ΓD_{k+1}. By minimality of (p, B), the curve B does not contain the (-1)-curves in $p^{-1}(x)$, $x \in \Sigma_p$. If $p^{-1}(x)$ contains more than one (-1)-curve, then ΓB would not be connected; the proof is similar to the proof of (2.8). Therefore (1.4.3) holds.

Property (1.4.1): The edge $x_j \in \Gamma_1 B_j$ of a subdivisional expansion (2.7) is contained in an edge of $\Gamma D(d)$ and this edge is contained in a cycle of $\Gamma D(d)$. Conversely, each edge in $\Sigma_p \subset \Gamma_1 D(d)$ contains exactly one edge belonging to a sudivisional expansion (2.7). Since ΓB is a tree, Σ_p must be a cutting set. □

A detailed description of expansions together with their weighted dual graphs will be given in section 4.

We add the following remark. Let $P=\{p_j \mid j \in J(x)\}$ be the set of expansions whose exceptional divisor is mapped to $x \in \Sigma_p$. There is a unique expansion of type (2.7) in P and this is p_j with maximal $j \in J(x)$.

3. The discriminant

This section is devoted to the proof of Theorem B. We need a computation of $H^2(V)$ in terms of multiplicities. Cohomology with integral coefficients will be used. We think of V as being one of the surfaces in a tower.

We begin by recalling some general facts. A curve A in a projective surface X defines an element $\langle A\rangle \in H^2(X)$. There are two ways of explaining this fact. Firstly, if $A \subset X$ is an irreducible curve, then A is the continuous image under $f\colon \tilde{A} \to X$ of an oriented closed 2-manifold $\tilde{A}$. The fundamental class of $\tilde{A}$ has an image $z_A \in H_2(X)$ and $\langle A\rangle \in H^2(X)$ corresponds to z_A under Poincaré duality. Secondly, the divisor A defines an element in the Picard group $\mathrm{Pic}(X)$ and $\langle A\rangle$ is its image under the first Chern class homomorphism $\mathrm{Pic}(X) \to H^2(X)$.

If $\mathcal{E}$ is a set of irreducible curves in X we let $\boldsymbol{Z}(\mathcal{E})$ denote the free abelian group on the elements of $\mathcal{E}$. We thus have a homomorphism

(3.1) $$\alpha(\mathcal{E})\colon \boldsymbol{Z}(\mathcal{E}) \to H^2(X)$$

which maps $A \in \mathcal{E}$ to $\langle A\rangle$.

The following Proposition collects some general facts about expansions.

(3.2) Proposition. *Let $p\colon X \to Y$ be an expansion with set of exceptional divisors $\mathcal{E}_p$. Then the following holds:*

(1) *If $B \subset Y$ is any curve and $B' = p^{\#} B$ denotes its proper transform, then $p^*\langle B\rangle - \langle B'\rangle = \alpha(\mathcal{E}_p)(y)$ for a unique element $m(B) := y \in \boldsymbol{Z}(\mathcal{E}_p)$.*

(2) *The element $m(B)$, written as a linear combination of the basis elements $E \in \mathcal{E}$, has non-negative coefficients.*

(3) *The homomorphism*

$$\langle p^*, \alpha(\mathcal{E}_p)\rangle\colon H^2(Y) \oplus \boldsymbol{Z}(\mathcal{E}_p) \to H^2(X), \quad (x, y) \mapsto p^*(x) + \alpha(y)$$

is an isomorphism.

Proof. Suppose p is a σ-process. Then (3) follows from the Mayer-Vietoris sequence in cohomology by using the fact that a σ-process is the connected sum with a projective plane. Using this, (1) and (2) follow from HARTSHORNE [1977], V(3.6). The general case is now a straightforward induction on the number of σ-processes involved in the expansion. □

By linear extension the elements $m(B)$ in (3.2.1) yield a homomorphism

(3.3) $$m\colon \boldsymbol{Z}(\Gamma_0 D) \to \boldsymbol{Z}(\mathcal{E}_p)$$

for any curve $D \subset Y$ and any expansion $p\colon X \to Y$. We also use the elements $m(A)$ in the next definition.

(3.4) Definition. If $A \in \Gamma_0 D$ and $m(A) = \sum_{E \in \mathcal{E}_p} m(A, E)\, E$, then $m(A, E) \in \boldsymbol{Z}$

is called the *multiplicity* of E in the total transform $p^*\langle A\rangle$ of A.

(3.5) Proposition. *Let $p\colon X\to Y$ be an expansion with set of exceptional divisors $\mathcal{E}_p$. Let $A\subset Y$ be an irreducible curve. The multiplicity $m(A, E), E\in\mathcal{E}_p$, is non-zero if and only if $p(E)\subset A$.*

Proof. This follows by induction on the number of σ-processes from HARTSHORNE [1977], V(3.5.2), (3.6). □

After this recollection we turn to the cohomological study of complements of curves.

(3.6) Assumptions. Let C be a curve in the rational projective surface X with the following properties:

(1) The components of C are rational.

(2) $H_1(C)=0$.

(3) $H_2(C)$ is free abelian. A basis is given by the fundamental classes of the components of C.

(4) $|\Gamma_0 C|=b_2(X)$. ($\Gamma_0 C$=set of components).

We do not assume that C has normal crossings.

(3.7) Proposition. *Suppose (X, C) satisfies (3.6). Let $j\colon V=X\backslash C\to X$ denote the inclusion. Then the following holds:*

(1) *The sequence*

$$\boldsymbol{Z}(\Gamma_0 C)\xrightarrow{\alpha(\Gamma_0 C)} H^2(X)\xrightarrow{j^*} H^2(V)\longrightarrow 0$$

is exact.

(2) *V is a $\boldsymbol{Z}$- (resp. $\boldsymbol{Q}$-) homology plane if and only if $H^2(V)$ is zero (resp. finite).*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & H^2(V) & \xleftarrow{j^*} & H^2(X) & & \\ & & \downarrow P & & \downarrow P & & \\ H_1(C) & \longleftarrow & H_2(X, C) & \longleftarrow & H_2(X) & \xleftarrow{i^*} & H_2(C)\,. \end{array}$$

The maps P are Poincaré duality isomorphisms. The bottom row shows part of the homology sequence of (X, C). By (3.6.3) we can identify $H_2(C)=\boldsymbol{Z}(\Gamma_0 C)$ and then $P^{-1}\circ i_*$ becomes $\alpha(\Gamma_0 C)$. Thus (3.7.1) follows.

The exact homology sequence of the pair (X, C), the assumptions (3.6) and the fact that $H_1(X)\cong H_3(X)\cong 0$ for a rational projective surface imply: The groups $H_i(X, C)$ are zero for $0\le i\le 3$ if and only if $H_2(C)\to H_2(X)$ is an isomorphism. By Poincaré duality this implies: $\tilde{H}^*(V)=0$ if and only if $H_2(C)\to$

$H_2(X)$ is an isomorphism. This holds for integral or rational coefficients.

Since $H_2(C)$ and $H_2(X)$ have the same rank (3.6.4), the assertions of the last paragraph are easily translated into the statment (3.7.2). □

We set $L(C)=\text{image } \alpha(\Gamma_0 C)$ and obtain under the hypothesis of (3.7) an isomorphism $H^2(V)\cong H^2(X)/L(C)$.

We consider a pair (X, C) that arises from (Y, D) in a particular way:

(3.8) Data and assumptions on $(p\colon X\to Y, D, C)$.

(1) $p\colon X\to Y$ is an expansion.
(2) $D\subset Y$ is a curve with rational components.
(3) $\Sigma_p\subset D$.
(4) $\mathcal{E}:=\mathcal{E}(p)=\mathcal{E}_0\cup\mathcal{E}_1$ disjoint union.
(5) $C=D'+\sum_{E\in\mathcal{E}_1} E$. $(D'=p^{\#}D)$
(6) C satisfies (3.6).
(7) $\alpha(\Gamma_0 D)$ is surjective and has kernel R.
(8) $m_0\colon \boldsymbol{Z}(\Gamma_0 D)\to\boldsymbol{Z}(\mathcal{E}_0)$ is the composition of m in (3.3) with the canonical projection $\boldsymbol{Z}(\mathcal{E})\to\boldsymbol{Z}(\mathcal{E}_0)$.

(3.9) Proposition. *Suppose data and assumptions* (3.8) *are given. Then*

$$H^2(V)\cong H^2(X)/L(C)\cong \boldsymbol{Z}(\mathcal{E}_0)/m_0(R)\,.$$

Proof. Diagram chasing in the following commutative diagram.

$$\begin{array}{ccc}
H^2(X) & \overset{\supset}{\longleftarrow} & L(C) \\
\uparrow \langle p^*, \alpha(\mathcal{E})\rangle & & \\
H^2(Y)\oplus\boldsymbol{Z}(\mathcal{E}) & & \uparrow \alpha(\Gamma_0 D'\cup\mathcal{E}_1) \\
\uparrow \alpha(\Gamma_0(D))\oplus id & & \\
\boldsymbol{Z}(\Gamma_0 D)\oplus\boldsymbol{Z}(\mathcal{E}) & \overset{\delta}{\longleftarrow} & \boldsymbol{Z}(\Gamma_0 D')\oplus\boldsymbol{Z}(\mathcal{E}_1) \\
\uparrow & & \\
R & &
\end{array}$$

The map δ is defined as $\delta(x', y)=(x, y-m(x))$.

We express (3.9) more explicitly for the cases of interest. Let $R_1, \cdots, R_k$ be a basis for $R=\text{kernel } \alpha(\Gamma_0 D)$, called *relation basis*. Then

$$k=|\Gamma_0 D|-b_2(Y),\quad b_2(Y)+|\mathcal{E}|=b_2(X)=|\Gamma_0 C|=|\Gamma_0 D|+|\mathcal{E}_1|,$$

whence $k=|\mathcal{E}_0|$. Consider the elements of $\Gamma_0 D$ as a basis for $\boldsymbol{Z}(\Gamma_0 D)$. Then R_j is an integral linear combination

$$R_j=\sum_{A\in\Gamma_0 D} c_j(A)\,A,\quad c_j(A)\in\boldsymbol{Z}\,. \tag{3.10}$$

For $A\in\Gamma_0 D$ the element $m_0(A)$, see (3.8.8), is an integral linear combination

$$m_0(A) = \sum_{E\in\mathcal{E}_0} m(A, E)\, E\,. \tag{3.11}$$

From (3.10) and (3.11) we obtain an equality of the following type

$$m_0(R_j) = \sum_{E\in\mathcal{E}_0} \lambda(j, E)\, E, \quad \lambda(j, E)\in \boldsymbol{Z}\,. \tag{3.12}$$

The coefficients $a(j, E)$ constitute a (k, k)-matrix $\Lambda=(\lambda(j, E))$, called *multiplicity relation matrix,* or MR-matrix for short.

Now (3.9) can be reformulated as follows.

(3.13) Theorem. *Suppose data and assumptions* (3.8) *are given. Then* $V=X\backslash C$ *is a* $\boldsymbol{Q}$*-homology plane if and only if the MR-matrix* $\Lambda=(\lambda(j, E))$ *has non-zero determinant. If* $\det\Lambda\neq 0$, *then* $|H^2(V)|=|\det\Lambda|$. *The surface* V *is a homology plane if and only if* $|\det\Lambda|=1$. □

Proof of Theorem B. We apply (3.13) to $(p\colon Z(p)\to X, D(d), B(p))$ for $p\in \mathrm{Bl}(D, d, \Phi)$ in place of $(p\colon X\to Y, D, C)$. We have to check (3.8). We have assumed that $\alpha(\Gamma_0 D(d))$ is surjective. It only remains to check that $B(p)$ satisfies (3.6). Part (3.6.1) holds by construction; (3.6.2) and (3.6.3) follow from (3.6.1) and because $\Gamma B(p)$ is a tree.

In order to verify $|\Gamma_0 B(p)|=b_2(Z)$ we note:

$$\begin{aligned} b_2(X)-b_2(Y) &= |\Gamma_0 \pi^{-1} D|-|\Gamma_0 D| \\ &= |\Gamma_0 D(d)|+|d^{-1}(0)|-|\Gamma_0 D| \end{aligned}$$

and the definition of $s(d)$ imply

$$b_2(X) = |\Gamma_0 D(d)|-s(d)\,.$$

Moreover we have

$$\begin{aligned} b_2(Z)-b_2(X) &= |\Gamma_0 p^{-1} D(d)|-|\Gamma_0 D(d)| \\ &= |\Gamma_0 B(p)|+|\Phi|-|\Gamma_0 D(d)| \\ &= |\Gamma_0 B(p)|+s(d)-|\Gamma_0 D(d)|\,. \end{aligned}$$

The two relations imply the desired equality.

The MR-matrix is again computed from (3.10) and (3.11) which reads in this case

$$R_i = \sum_{A\in\Gamma_0 D(d)} c_j(A)\, A$$

and

$$m_0(A) = \sum_{x\in\Phi} m(A, E_0(x))\, E_0(x)\,.$$

The MR-matrix therefore has the entries

$$\lambda(j, x) = \sum_{A \in \Gamma_0 D(d)} c_j(A)\, m(A, E_0(x))\,. \tag{3.14}$$

By (3.5) and (1.4.1) there are exactly two $A \in \Gamma_0 D(d)$ with $m(A, E_0(x)) \neq 0$, namely the elements called $A(x)$ and $B(x)$ before the statement of Theorem B. Thus $m_p(A(x)) = m(A(x), E_0(x))$ and similarly for $B(x)$. From (3.14) we see now that the x-column of the MR-matrix Λ is a vector of the form

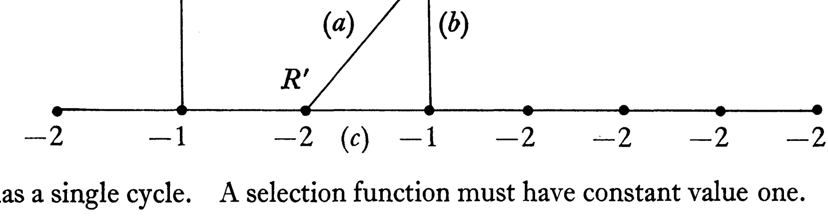
$$(c_j(A(x))\, m_p A(x) + c_j(B(x))\, m_p B(x))\,.$$

Since the $c_j(A)$ only depend on the choice of the relation basis they are independent of p. The discriminant $\Delta(D, p, \Phi)$ is now defined to be the determinant of Λ, viewed formally as a function in the variables $(m_p A(x), m_p B(x))$, $x \in \Phi$. Statements (1)—(3) in Theorem B are finally a restatement of (3.13). □

Theorem (3.13) is more general than Theorem B and allows calculation of the discriminant without passing to a normal crossing curve. An example of this type is given in (3.15).

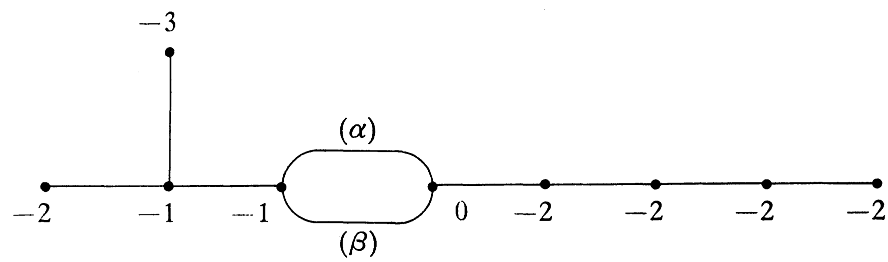
(3.15) The Ramanujam tower. The first non-trivial example of a contractible homology plane was published by RAMANUJAM [1971]. The starting point was a curve $D = R \cup S$ in $\boldsymbol{P}^2$. It consists of a cuspidal cubic $R \subset \boldsymbol{P}^2$ and a regular quadric $S \subset \boldsymbol{P}^2$ with intersection pattern $R \cdot S = P + 5Q$. Let $p\colon Z(p) \to \boldsymbol{P}^2$ be a standard expansion (1.4) with $\Sigma_p = \{P\}$. We consider the tower consisting of surfaces $V(p) = Z(p) \setminus B(p)$. Then $(Z(p), B(p))$ satisfies (3.6) and $(p\colon Z(p) \to \boldsymbol{P}^2, D, B(p))$ satisfies (3.8) with $\mathcal{E}_0$ the (-1)-curve E in $p^{-1}(P)$. We have $H^2(\boldsymbol{P}^2) = \boldsymbol{Z}$ and $\alpha(\Gamma_0 D)\colon aR + bS \mapsto 3a - 2b$ is surjective with relation basis given by $R_1 = 2R - 3S$. The expansion p is determined by the multiplicities $m(R, E)$, $m(S, E)$ and $m_0(R_1) = (2m(R, E) - 3m(S, E))\,E$. Hence $2m(R, E) - 3m(S, E)$ is the value of the discriminant. The case $m(R, E) = m(S, E) = 1$ was the example given by Ramanujam. The surface with $m(R, E) = 2$, $m(S, E) = 1$ can be shown to be isomorphic to an example in GURJAR—MIYANISHI [1987].

In the systematic treatment (i.e. in the tower algorithm of Theorem B) we would first pass to the minimal resolution $\pi\colon X \to \boldsymbol{P}^2$ of D. The weighted dual graph of $\pi^{-1}(D) = C$ is

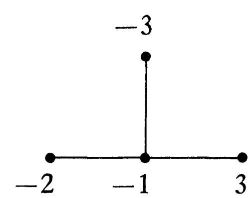


It has a single cycle. A selection function must have constant value one. There are three cutting sets, consisting of the edges (a), (b), or (c).

The (-1)-curve S' can be contracted giving $\pi'\colon X'\to \boldsymbol{P}^2$ and $(\pi')^{-1}(D)$ has weighted dual graph

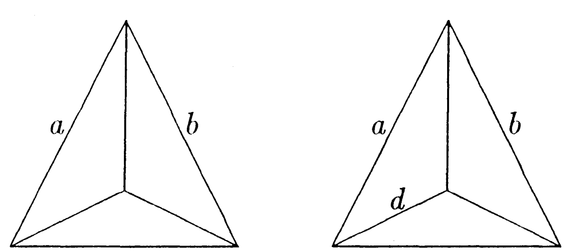


(Dual graphs can obviously be defined, more generally, when all singularities are ordinary double points.) Because of symmetry of the graph one expects that the cutting sets (α) and (β) lead to isomorphic towers. This is indeed the case because one can show that the situation carries an involution which interchanges (α) and (β). The tower of the last graph contains surfaces which are not contained in the Ramanujam tower, e.g. cut the cycle at α or β with multiplicities $(1, 1)$. The resulting tree is not minimal and can be contracted to



It belongs to a surface with $\bar{\kappa}=-\infty$. By contraction of the part with negative weights we see that the surface is isomorphic to the complement of a cuspidal cubic in $\boldsymbol{P}^2$.

(3.16) Four Lines in general position. Let $L_1, L_2, L_3, L_4 \subset \boldsymbol{P}^2$ be four lines in general position. Then $D=L_1\cup L_2\cup L_3\cup L_4$ is already a normal crossing curve. The dual graph $\Gamma(D)$ is the 1-skeleton of a 3-simplex. Up to symmetry there are two cutting sets Φ and Ψ, as indicated in the next figure.



$\Phi = \{a, b, c\}$ $\Psi = \{a, b, d\}$

The discriminant in case Ψ is computed in TOM DIECK—PETRIE [1989], (2.10).

We illustrate the computation in terms of the present algorthm. Since $\langle L_i \rangle$ is a generator of $H^2(\boldsymbol{P}^2)$ a relation basis is

$$R_1 = L_1 - L_2,\ R_2 = L_1 - L_3,\ R_3 = L_1 - L_4\,. \tag{3.17}$$

The indexing of the lines is chosen so that with respect to the figures above

$$\begin{aligned} \partial a &= \{L_1, L_4\} & \partial b &= \{L_2, L_4\} \\ \partial c &= \{L_1, L_2\} & \partial d &= \{L_1, L_3\}\,. \end{aligned} \tag{3.18}$$

We use the following notation for the multiplicities: $m(L_j, E_0(x)) = x_j$. The values (3.14) are then by (3.17)

$$\begin{aligned} \lambda(1, x) &= x_1 - x_2 \\ \lambda(2, x) &= x_1 - x_3 \\ \lambda(3, x) &= x_1 - x_4\,. \end{aligned}$$

In case Φ only $a_1, a_4, b_2, b_4, c_1, c_2$ are non-zero. Therefore in this case Φ

$$\begin{aligned} \Lambda &= \begin{vmatrix} a_1 & -b_2 & c_1 - c_2 \\ a_1 & 0 & c_1 \\ a_1 - a_4 & -b_4 & c_1 \end{vmatrix} \\ &= b_2\, c_1\, a_4 + b_4\, a_1\, c_2\,. \end{aligned} \tag{3.19}$$

Since the multiplicities are positive integers we see that the discriminant never has the value one. The tower does not contain homology planes. The simplest element in this tower arises when all multiplicities are 1. This amounts to blowing up the points a, b, c once. The resulting tree has 4 vertices, a central one with weight 1 and three edges attached to it with terminal weights -1. By contracting the three (-1)-curves one arrives at a single curve with weight 4 in $\boldsymbol{P}^2$. This must be a quadric.

A similar computation in case Ψ yields the discriminant

$$\Lambda = (b_2 - b_4)\, d_3\, a_1 + (d_1 - d_3)\, b_2\, a_4\,. \tag{3.20}$$

This tower contains infinitely many homology planes. They are actually contractible: The next Proposition shows that they are simply-connected, and a simply-connected acyclic manifold is contractible by the Whitehead-Theorem of algebraic topology.

(3.21) Proposition. *Surfaces in the towers of the* Four Lines *arrangement have abelian fundamental group.*

Proof. The fundamental group of any surface V in the tower is a quotient of the fundamental group of $\boldsymbol{P}^2\setminus\bigcup L_j=W$, since W is obtained from V by removing a submanifold of real codimension two. It is well known that the fundamental group of W is abelian. □

In many interesting cases the selection function must have constant value one. In this case we can state:

(3.22) Proposition. *If $\alpha(\Gamma_0 D)$ is surjective, then $\alpha(\Gamma_0 D(d))$ is surjective.*

Proof. The proof will also show how to obtain a relation basis for $\alpha(\Gamma_0 D(d))$ from a relation basis for $\alpha(\Gamma_0 D)$. We have

$$D(d) = C = \pi^{-1}(D) = D' \cup \{E \mid E \in \mathcal{E}_\pi\} .$$

We construct a commutative diagram with horizontal isomorphisms

$$\begin{array}{ccc} H^2(Y)\oplus \boldsymbol{Z}(\mathcal{E}_\pi) & \xrightarrow{\langle \pi^*, \alpha(\mathcal{E}_\pi)\rangle} & H^2(X) \\ \uparrow \alpha(\Gamma_0 D)\oplus \mathrm{id} & & \uparrow \alpha(\Gamma_0 C) \\ \boldsymbol{Z}(\Gamma_0 D)\oplus \boldsymbol{Z}(\mathcal{E}_\pi) & \xrightarrow{\sigma} & \boldsymbol{Z}(\Gamma_0 C) . \end{array}$$

From (3.4) we have $\pi^*\langle D_j\rangle=\langle D_j'\rangle+\alpha(\mathcal{E})(y_j)$ for suitable $y_j\in\boldsymbol{Z}(\mathcal{E}_\pi)$. In order to make the diagram commutative we have to define σ by $\sigma(D_j)=(D_j', y_j)$ and $\sigma(E)=E$ for $E\in\mathcal{E}_\pi$. The map σ sends a relation basis to a relation basis. □

4. The weighted graph

A basic invariant of a homology plane or any other variety in a tower is the weighted dual graph. This section collects some facts about graphs and trees which are related to the tower algorithm:

1. The computation of the weighted dual tree from multiplicities and vice versa.
2. The computation of the discriminant from the weighted trees.

Weighted graphs and trees are convenient for practical computations. They can be used to illustrate the theory. Finally they are useful as a heuristic tool.

Many of the results of this section are known, but there seems to be no convenient reference for our purposes.

The weighted graphs of standard expansions are obtained from the multiplicities by a continued fraction algorithm. Therefore we recall some facts about continued fractions. The symbol $[c_1, \cdots, c_r]$ for a continued fraction is defined inductivley by $[c_1]=c_1$ and

$$[c_1, \cdots, c_r] = c_1-[c_2, \cdots, c_r]^{-1} . \tag{4.1}$$

If $1\leq q<p$ are coprime integers, there is a unique ontinued fraction expansion

(4.2) $$\frac{p}{q} = [c_1, \cdots, c_r]$$

with integers $c_i \geq 2$. The determinant of the (r, r)-matrix (a_{ij}) with $a_{ii} = c_i$ and $a_{ij} = -1$ if $|i-j| = 1$ is denoted $D(c_1, \cdots, c_r)$. The intersection matrix of the weighted tree

is the negative of the matrix just considered. Expanding the determinant with respect to the first row yields for $r \geq 3$

(4.3) $$D(c_1, \cdots, c_r) = c_1 D(c_2, \cdots, c_r) - D(c_3, \cdots, c_r) .$$

We compare this with (4.1) and obtain by induction on $r \geq 2$

(4.4) $$[c_1, \cdots, c_r] = \frac{D(c_1, \cdots, c_r)}{D(c_2, \cdots, c_r)} .$$

The relation (4.3) can be used to prove by induction on $r \geq 3$ the following identity

(4.5) $$D(c_1, \cdots, c_{r-1}) D(c_2, \cdots, c_r) - D(c_1, \cdots, c_r) D(c_2, \cdots, c_{r-1}) = 1 .$$

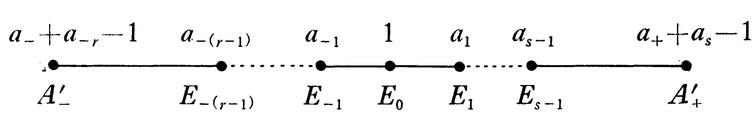
This relation shows that the right hand side of (4.4) is a quotient of coprime integers. It also shows what happens, if continued fractions are turned around. Namely:

(4.6) If $p/q = [c_1, \cdots, c_r]$, $q \cdot q^* \equiv 1 \bmod p$, and $1 \leq q^* < p$, then $p/q^* = [c_r, \cdots, c_1]$.

Our aim is to characterize standard expansions by weights and multiplicities. We use the following notation. Let $M_0(A_-)$ and $M_0(A_+)$ be a pair of coprime integers. We set

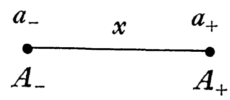
(4.7) $$\frac{M_0(A_-)+M_0(A_+)}{M_0(A_+)} = [a_{-r}, \cdots, a_{-1}]$$
$$\frac{M_0(A_-)+M_0(A_+)}{M_0(A_-)} = [a_s, \cdots, a_1] .$$

Associated to these data and to two further integers a_-, a_+ is the following weighted linear tree.

(4.8)


Suppose A_- and A_+ are irreducible curves in a projective surface X and $x \in A \cap B$

is a regular point of transverse intersection of A and B (=an ordinary double point of $A \cup B$). Although it is not necessary to assume at this point that A_- and A_+ belong to a normal crossing curve, we use the graph notation and illustrate this situation by (4.9).



Here $-a_\pm = A_\pm \cdot A_\pm$ are the weights. We may think of (4.9) as being an edge in some larger weighted graph. An expansion $p: Z \to X$ with exceptional set $\{x\}$ is called *standard* if it has the following properties (compare (1.4)).

(4.10)

(1) The dual graph of $p^{-1}(x)$ is a linear tree.

(2) $p^{-1}(x)$ contains a unique (-1)-curve E_0.

(3) If $p^{-1}(x) \neq E_0$, then $p^{-1}(x)$ has two terminal components. One of them intersects A_- transversely and is disjoint from A_+ and the other one intersects A_+ transversely and is disjoint from A_-.

We denote, in accordince with (4.8), the exceptional divisors of a standard expansion p by E_j, $-r<j<s$, and assume that $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Finally let $A'_\pm$ denote the proper transform of $A_\pm$ and assume that $A'_- \cap E_{-(r-1)} \neq \emptyset$ and $A'_+ \cap E_{s-1} \neq \emptyset$. Forgetting about points in $A \cap B$ which differ from x, the intersection pattern of $A'_- \cup p^{-1}(x) \cup A'_+$ is codified by a weighted graph of the shape (4.8), called the ngraph of $A'_- \cup p^{-1}(x) \cup A'_+$. More generally the ngraph of a normal crossing curve is the weighted graph obtained from the weighted dual graph by changing the signs of the weights. That the weights are actually given by a continued fraction algorithm (4.7) is the partial content of the next Theorem.

(4.11) Theorem. *Let $p: Z \to X$ be a standard expansion with exceptioanl set $\{x\} \subset A \cap B$ and use the notation introduced above. Let $M_0(A_\pm)$ denote the multiplicity of E_0 in $p^*(A_\pm)$. Then the following holds:*

(1) $M_0(A_+) = D(a_{-(r-1)}, \cdots, a_{-1})$
$M_0(A_-) = D(a_{s-1}, \cdots, a_1)$.

If $r=1$ or $s=1$, then the corresponding multiplicities are 1.

(2) *The multiplicities $M_0(A_-)$ and $M_0(A_+)$ are coprime integers and the ngraph of $A'_- \cup p^{-1}(x) \cup A'_+$ is* (4.8) *with weights determined by* (4.7).

Given the curves $A_\pm$ and x, the standard expansion with exceptional set x is uniquely determined by the pair $(M_0(A_-), M_0(A_+))$ and any pair of coprime positive integers arises from some such standard expansion.

Proof. We already know from (3.2) that there exists a relation of the tpye

$$p^*(A_\pm) = A'_\pm + \sum_{j=-(r-1)}^{s-1} M_j(A_\pm)\, E_j\,. \tag{4.12}$$

(We let E_j denote the cohomology class determined by this curve. One can also read (4.12) as an equality of Cartier divisors.) By induction over the number of σ-processes involved in p one shows:

$$M_{s-1}(A_-) = 1, \quad M_{-(r-1)}(A_+) = 1\,. \tag{4.13}$$

In order to simplify the exposition we assume $r \geq 2$, $s \geq 2$ and leave the cases $r=1$ or $s=1$ to the reader.

Denote the intersection pairing by $\langle -, - \rangle$. For any expansion $p\colon Z \to X$ the relation $\langle p^*C, D\rangle = \langle C, p_* D\rangle$ holds for divisoıs C in X and D in Z (see HARTSHORNE [1977], V(3.2)). In cohomological terms, p_*, is the Hopf Umkehr homomorphism associated to p. This relation implies in our case

$$\langle p^*(A_+), E_j \rangle = 0\,. \tag{4.14}$$

Let

$$\alpha_+ = \sum_{j=-(r-1)}^{-1} M_j(A_+)\, E_j\,.$$

From (4.14) we conclude

$$\langle \alpha_+, E_{-1} \rangle = -M_0(A_+), \quad \langle \alpha_+, E_j \rangle = 0 \quad \text{for} \quad -r<j<-2\,. \tag{4.15}$$

This can be viewed as a linear system for the unknown $M_j(A_+)$, $-r<j<0$. With the help of Cramer's rule we compute

$$M_0(A_+) = M_{-(r-1)}(A_+)\, D(a_{-(q-1)}, \cdots, a_{-1})\,. \tag{4.16}$$

Thus (4.13) and (4.16) yield the first assertion of (4.11.1).

We intend to show (4.11.2) by induction on the number of σ-processes in p. We leave it to the reader to check that the induction starts correctly. Given the expansion p, a further σ-process has to blow up either $E_0 \cap E_1$ or $E_0 \cap E_{-1}$. This follows from the fact that we only allow standard expansions. Let us treat the case that $E_0 \cap E_1$ is blown up. Denote the new multiplicities by the letter N. Then we have

$$N_0(A_\pm) = M_0(A_\pm) + M_1(A_\pm)\,. \tag{4.17}$$

For the induction step we have to show

$$\begin{aligned} \frac{N_0(A_+)+N_0(A_-)}{N_0(A_-)} &= [a_s, \cdots, a_2, a_1+1] \\ \frac{N_0(A_+)+N_0(A_-)}{N_0(A_+)} &= [a_{-r}, \cdots, a_{-1}, 2]\,. \end{aligned} \tag{4.18}$$

We use the identity

$$M_0(A_+)\,M_1(A_-)-M_0(A_-)\,M_1(A_+) = -1\,. \tag{4.19}$$

This identity is shown by induction on the number of σ-processes by using (4.17). Instead of verifying the first identity in (4.18) we use (4.6) and verify (with $N_\pm=N_0(A_\pm)$ etc.)

$$\frac{N_+ + N_-}{N_-^*} = 1+\frac{M_+ + M_-}{M_-^*}\,. \tag{4.20}$$

From (4.17) and (4.19) we calculate

$$(M_+ + M_-)\,N_- -(N_+ + N_-)\,M_- = -1$$

and this implies easily using (4.6) and (4.17)

$$M_-^* = M_{1+}+M_{1-} = N_-^*\,. \tag{4.21}$$

Then (4.17) and (4.21) yield (4.20). The second identity in (4.18) requires the verification of

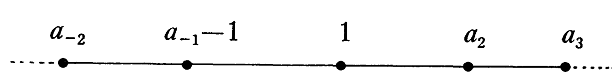
$$\frac{N_+ + N_-}{N_+^*} = 2-\frac{M_+^*}{M_+ + M_1}\,;$$

it uses $(M_+ + M_-)\,N_+ -(N_+ + N_-)\,M_+ = 1$ and $N_+^* = M_+ + M_-$, $M_+^* = M_+ - M_{1+} + M_- - M_{1-}$. This finishes the inductive proof of (4.11.2).

Once we know the multiplicities $M_0(A_\pm)$ of the expansion p we know by (4.11.2) the weigths. The weights, on the other hand, determine uniquely the inverse contraction, because p, being a standard expansion, has a unique (-1)-curve at every step in the iteration of σ-processes (proof by induction). Hence p is uniquely determined by the weights.

Finally, we show by induction that any pair of coprime positive integers arises as the pair of multiplicities for some standard expansion. Given such integers $M_0(A_-)$ and $M_0(A_+)$ we can formally write donw the ngraph (4.8). By properties of continued fractions, either $a_{-1}=2$, $a_1>2$ or $a_{-1}>2$, $a_1=2$. Suppose $a_1=2$. Then we can formally contract the (-1)-curve and arrive at a tree with weights

(4.22)



These weights arise again from a continued fraction expansion. Namely, if

$$\frac{U+V}{U} = [2, b_1, \cdots, b_r],\quad \frac{U+V}{V} = [c_1, \cdots, c_s]\,,$$

then

$$\frac{U}{U-V}=[b_1,\cdots,b_r],\quad \frac{U}{V}=[c_1-1,c_2,\cdots,c_s]\,.$$

Now turn the continued fractions around (4.6). By induction we know that (4.22) is realizable. This finishes the proof of (4.11). □

For completeness, we determine all multiplicities in terms of weights.

(4.23) Proposition. Under the hypothesis of (4.11) the multiplicities $M_j(A_\pm)$ in (4.12) are given as follows:

(1) $M_j(A_+)=D(a_s-1,a_{s-1},\cdots,a_{(j+1)}),\qquad 0<j<s\,.$

(2) $M_j(A_+)=D(a_{-(r-1)},\cdots,a_{-(j+1)}),\qquad -r<j\le 0\,.$

(3) $M_j(A_-)=D(a_{-r}-1,a_{-(r-1)},\cdots,a_{-j}),\qquad -r<j<0\,.$

(4) $M_j(A_-)=D(a_{s-1},\cdots,a_{j+1}),\qquad 0\le j\le s\,.$

Proof. The proofs for (2) and (4) are similar to the proof in the case $j=0$. Let us prove (1) by downward induction on j. We have

$$\begin{aligned}-a_+&=\langle A_+,A_+\rangle=\langle p^*A_+,A'_+\rangle\\&=\langle A'_+,A'_+\rangle+M_{s-1}(A_+)\langle A_+,E_{s-1}\rangle\\&=-a_+-a_s+1+M_{s-1}(A_+)\,.\end{aligned}$$

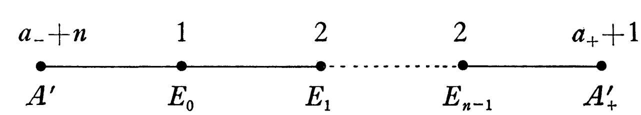
This starts the induction. The induction step follows from

$$0=\langle p^*A_+,E_j\rangle=M_{j+1}(A_+)-M_j(A_+)\,a_j+M_{j-1}(A_+)$$

by using (4.3), with $M_{j+1}(A_+)$ replaced by 1 in case $j=s-1$.

We mention the special case that one of the multiplicities $M_0(A_\pm)$ equals 1. This case was left to the reader in (4.11) and (4.23). If $M_0(A_+)=1$ and $M_0(A_-)=n$, then (4.8) has the form (4.24).

(4.24)



Our next task is the computation of the discriminant in terms of weights.

(4.25) Theorem. *Let $p\colon Z(p)\to X$, $B(p)$ be an object in a tower* (1.6), $p\in Bl(D,d,\Phi)$. *Denote by $D(p)$ the determinant* $\det(\Gamma_0 B(p),w)$ *of the weighted dual graph $B(p)$ (see section* 2) *and by Δ the discriminant of the tower (see Theorem **B**). Then $|D(p)|=|\Delta(m_p)|^2$.*

The proof of this Theorem is based on some general topological facts which we recall first.

(4.26) Lemma. *Let B be a compact, connected, oriented 4-manifold with* *boundary* $S=\partial B$. *Suppose* $\tilde{H}_*(B;\boldsymbol{Q})=0$. *Then* S *is a* $\boldsymbol{Q}$*-homology* 3*-sphere,* $H_2(S;\boldsymbol{Z})=0$ *and*

$$|H_1(S;\boldsymbol{Z})|=|\text{kernel } i\colon H_1(B;\boldsymbol{Z})\to H_1(B,S;\boldsymbol{Z})|^2.$$

Proof. By Poincaré duality $H_2(B,S;\boldsymbol{Q})\cong H^2(B;\boldsymbol{Q})=0$. The exact homology sequence of (B,S) yields $H_1(S;\boldsymbol{Q})=0$. Since $H_1(S,\boldsymbol{Z})$ is a torsion group we obtain from duality and universal coefficients

$$H_2(S)\cong H^1(S)\cong \text{Hom}(H_1(S),\boldsymbol{Z})=0.$$

For similar reasons we have a commutative diagram

$$\begin{array}{ccccc} H_2(B) & \xrightarrow{\cong} & H^2(B,S) & \xrightarrow{\cong} & \text{Ext}(H_1(B,S),\boldsymbol{Z}) \\ \downarrow & & \downarrow & & \downarrow i^* \\ H_2(B,S) & \xrightarrow{\cong} & H^2(B) & \xrightarrow{\cong} & \text{Ext}(H_1(B),\boldsymbol{Z}). \end{array}$$

By algebra, cokernel $i^*\cong$kernel i. The exact sequence

$$0\to \text{cok}\, i^*\to H_1(S;\boldsymbol{Z})\to \ker i\to 0$$

yields the last assertion of (4.26).

Proof of (4.25). We apply the preceding Lemma in the following situation. Let D be a normal crossing curve in the projective surface $\boldsymbol{Z}$. Let $U\subset \boldsymbol{Z}$ be a closed tubular neighbourhood of D in Z, diffeomorphic to the manifold which is obtained by plumbing the normal disk bundles of the components of D. Then the interior of $B=X\backslash U^\circ$ is diffeomorphic to V and B is a manifold with boundary $\partial B=S$. If $V=Z\backslash D$ is a $\boldsymbol{Q}$-homology plane in a rational projective surface Z, then the exact homology sequence of (Z,D) implies $H_1(Z,D)=0$. because $H_1(Z)=0$ and D is connected. By excision and the homotopy equivalence $D\simeq U$ we obtain $H_1(B,S)\cong H_1(Z,U)\cong H_1(Z,D)=0$. Therefore (4.26) yields in this case

$$|H_1(S,\boldsymbol{Z})|=|H_1(V;\boldsymbol{Z})|^2.$$

We know already that $|H_1(V,\boldsymbol{Z})|=|\Delta(m_p)|$ for $V=Z(p)\backslash B(p)$. It remains to be shown that $|H_1(S,\boldsymbol{Z})|=|D(p)|$. This uses the fact that $S=\partial U$. Since $H_2(S)=0$, $H_1(U)=0$ we have an exact sequence

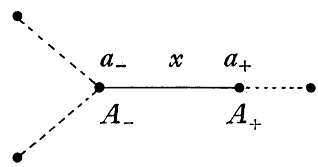
$$0\longrightarrow H_2(U)\xrightarrow{\ \rho\ } H_2(U,S)\longrightarrow H_1(S)\longrightarrow 0$$

and by duality and universal coefficients, $H_2(U,S)\cong H^2(U)\cong\text{Hom}(H_2(U),\boldsymbol{Z})$, the map ρ is transformed into the adjoint $H_2(U)\to\text{Hom}(H_2(U),\boldsymbol{Z})$ of the intersection pairing $H_2(U)\times H_2(U)\to\boldsymbol{Z}$. By the very construction of U this intersection pairing is specified by the intersection matrix of $\Gamma_0 B$. Hence $|H_1(S)| = |\mathrm{cok}\ \rho|$ is the absolute value of the determinant of the intersection matrix. □

(4.27) The Determinant Algorithm.

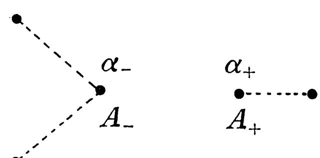
We use (4.25) to derive an algorithm which yields a computation of the discriminant as a multilinear function from the data (D, d, Φ) which specify a tower. Suppose x is an edge in the weighted dual graph of $D(d)$ which belongs to $x \in \Phi$. Then the ngraph of $D(d)$ looks like:

(4.28)



(The dots illustrate the fact that x is part of a larger graph.) We have to "cut the cycle at x". The first step in the algorithm removes the edge x and replaces the adjacent weights in (4.28) formally by

(4.29)



The weights $\alpha_\pm$ are now formal expressions

$$\alpha_\pm = a_\pm + 1 - \frac{M_0(A_+) + M_0(A_-)}{M_0(A_\mp)}$$

in indeterminates $M_0(A_\pm)$, called *multiplicity variables*. Having applied this process successively to all edges $x \in \Phi$ we obtain a tree where some of the weights are no longer integers but rational functions in the multiplicity variables. Nevertheless, we still have the determinant of the weighted tree. Multiply the resulting determinant by the product of all multiplicity variables.

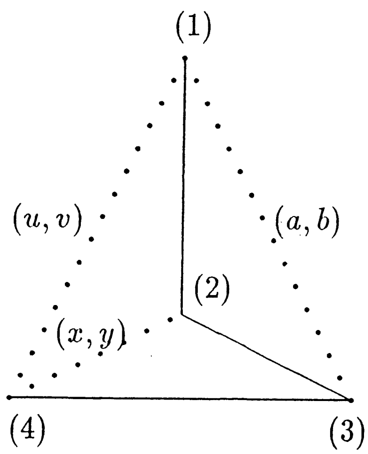
(4.30) Proposition. *The rational function, resulting from the preceding algorithm, is (up to sign) the square of the discriminant* $\Delta(D, d, \Phi)$.

Proof. The proof is a generalization of the continued fraction algorithm (4.4) for the computation of the determinant of a linear tree. Namely, consider the determinants of the weighted trees T and T' which are related in the following manner: There is an edge in T with weights b and c at its boundary vertices. The vertex with weight b is a terminal vertex of T. Remove this edge and replace the weight c by $c - b^{-1}$ and call the resulting weighted tree T'. Then

(4.31) $$\det(T) = b\det(T').$$

The proof is the same as for (4.3). The ngraph of $B(p)$ is obtained by replacing an edge (4.9) by (4.8) and then removing the vertex E_0 and the adjacent edges. The wighted dual graph is obtained by reversing signs of weights in the ngraph. Now use (4.31) and induction on the length r, s of the continued fractions. □

(4.32) Example. We apply (4.27), (4.30) to find the weighted dual graph $D(d)$ for the D and Ψ in (3.16). We obtain the following weighted dual graph:



$$(1) = 3 - \frac{a+b}{a} - \frac{u+v}{v}$$

$$(2) = 2 - \frac{x+y}{x}$$

$$(3) = 2 - \frac{a+b}{b}$$

$$(4) = 3 - \frac{u+v}{u} + \frac{x+y}{y}$$

((1)-(4) show the actual weights and not their negatives. The labeled dotted lines indicate the "cuts" and their multiplicity variables.) The reader may wish to compute the determinant and check the result against (3.20).

(4.33) Remark. The algorithm (4.27) can be applied to each connected weighted graph and leads to a rather curious combinatorial problem. Cutting cycles always yields a quadratic form in the multiplicity variables. In the geometric case this form is the square of a linear form. Simple examples show that this is not always the case. Which graphs have the property, that all possible cycle cuttings yields squares of linear forms?

Proof of the Addendum to Theorem A. If $V(p) = Z(p) \backslash B(p)$ is a homology plane, then by (3.7.1) the map $\alpha(\Gamma_0 B(p))$ is surjective. There is a commutative diagram

$$\begin{array}{ccc} \boldsymbol{Z}(\Gamma_0 B(p)) & \xrightarrow{\alpha(\Gamma_0(B(p))} & H^2(Z(p)) \\ \downarrow q & & \downarrow p_* \\ \boldsymbol{Z}(\Gamma_0 D(d)) & \xrightarrow{\alpha(\Gamma_0 D(d))} & H^2(X) \end{array}$$

where q maps a component L of $B(p)$ to $p_*(L)$; this is either a component of $D(d)$ or zero. Since p_* is surjective too, so is $\alpha(\Gamma_0 D(d))$. □

5. Birational maps

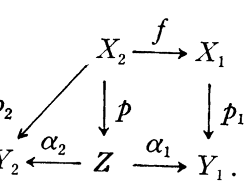
A fixed homology plane V can be contained in towers arising from different pairs (Y, D), D a curve in the minimal rational surface Y. We want to study this ambiguity more closely. This section only indicates some examples. A systematic investigation will be published elsewhere.

We call D a *plane divisor* of V in case $Y=\boldsymbol{P}^2$ and otherwise a *minimal divisor* of V. Plane divisors which are arrangements of lines are classified in TOM DIECK [1990].

If D is a plane or minimal divisor of a homology plane in a tower, it is the plane or minimal divisor of every homology plane in the tower: so it is a function of the tower. It generally happens that some homology planes in a tower with plane or minimal divisor D are isomorphic to homology planes with another plane or minimal divisor.

Suppose the surfaces $X_1\backslash C_1$ and $X_2\backslash C_2$ are isomorphic. An isomorphism is a birational map φ: $X_1 \to X_2$ which can be written as $\alpha_2 \alpha_1^{-1}$ with expansions α_i: $Z \to X_i$ having their exceptional sets $\Sigma(\alpha_i)$ in C_i. Then $C=\alpha_i^{-1}(C_i)$ and α_i induces an isomorphism $Z\backslash C \cong X_i\backslash C_i$.

Let $\varphi=\alpha_2 \alpha_1^{-1}$: $Y_1 \xleftarrow{\alpha_1} Z \xrightarrow{\alpha_2} Y_2$ be a birational isomorphism and let $D_1 \subset Y_1$ be a curve such that $\Sigma(\alpha_1) \subset D_1$. Let p_1: $X_1 \to Y_1$ be a contraction, $C_1 \subset X_1$ a curve and $D_1=p_1(C_1)$. Then there exist expansions f, p, and p_2 such that the following diagram is commutative



Let $C_2=f^{-1}(C_1)$, so that f induces an isomorphism $X_2\backslash C_2 \cong X_1\backslash C_1$ and set $D=p_2(C_2)$. Then $D_2 \subset \alpha_2 \alpha_1^{-1}(D_1)$, but equality does not hold in general. In particular, if Y_1 and Y_2 are minimal rational surfaces and $X_1\backslash C_1$ is a homology plane with minimal divisor D_1, then the same homology plane has minimal divisor D_2. In order to deal with this in practice one has to understand the way in which D_2 differs from $\alpha_2 \alpha_1^{-1}(D_1.)$

We will use the following notation: If p: $Z \to Y$ is an expansion and $P \in \Sigma_p$, then $E(P)$ denotes the exceptional divisor in the first blow up of P occuring in the expansion p.

To see how D_2 differs from $\alpha_2 \alpha_1^{-1}(D)$ in genearl it suffices to see what happens in two simple cases: $\varphi=\alpha_1^{-1}$ or $\varphi=\alpha_2$ and α_i is a single σ-process. The case $\varphi=\alpha_2$ is trivial, because then $D_2=\alpha_2(D_1)$.

Let α_1 blow up the point $x\in D_1$. There are two cases: $x\in\Sigma(p_1)$ or $x\notin\Sigma(p_1)$.

If $x\notin\Sigma(p_1)$ we have a diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{f} & X_1 \\ \downarrow p & & \downarrow p_1 \\ Z & \xrightarrow{\alpha_1} & Y_1 \end{array}$$

and f blows up the "same" point as α_1 does. In this case $pf^{-1}(C_1)=\alpha_1^{-1}(D_1)$.

If $x\in\Sigma(p_1)$ we have a factorization $p_1=\alpha_1 p$

$$\begin{array}{ccc} X_2 & \xrightarrow{=} & X_1 \\ \downarrow p & & \downarrow p_1 \\ Z & \xrightarrow{\alpha_1} & Y_1 \end{array}$$

and the exceptional curve $E(x)=\alpha_1^{-1}(x)$ may or may not be contained in $p(C_1)=p(C_2)$, so that D_2 is either the total or the proper transform of D_1. Therefore this case depends on the properties of (p, C_1).

Now we consider a quadratic transformation

$$\varphi(S)=\alpha_2(S)\,\alpha_1(S)^{-1}\colon \boldsymbol{P}^2\to\boldsymbol{P}^2\,.$$

It is specified by 3 points $S=\{x_1, x_2, x_3\}\subset D_1\subset\boldsymbol{P}^2$ which are not collinear: The points are blown up, $\alpha_1(S)\colon \boldsymbol{Z}\to\boldsymbol{P}^2$, and the proper transforms of their connecting lines are contracted, $\alpha_2(S)\colon \boldsymbol{Z}\to\boldsymbol{P}^2$, see HARTSHORNE [1977], V(4.2.3).

(5.1) Proposition. *If $\varphi(S)$ is a birational map of this type, then the components of D_2 are given as follows*:

(1) *The $\varphi(S)$ images of those components of D_1 which are different from the connecting lines of S (if present in D_1).*

(2) *The $\alpha_2(S)$ images of those exceptional curves $E(x_i)$ of the contraction $\alpha_1(S)$ which are components of $p(C_2)$.*

Thus D_2 may have more or fewer components than D_1.

Proof. This follows immediately from the discussion of the special situations above. □

(5.2) Five Lines. Let $D\subset\boldsymbol{P}^2$ be a configurations of five lines L, L_1, L_2, L_3, L_4, where the L_i are in general position and L passes through two double points P

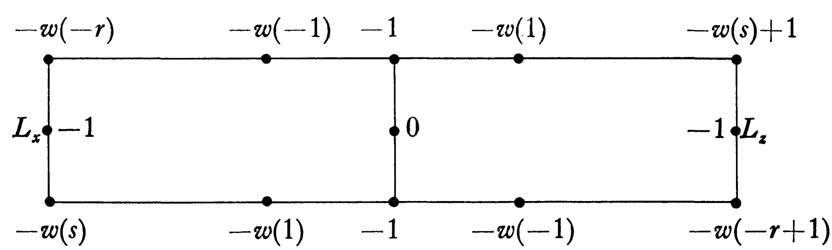
and Q of $\bigcup_{i=1}^{4} L_i$.

Blow up the points P and Q and contract teh proper transform of L. Then one arrives at a configuration of 6 lines in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, up to isomorphism $\boldsymbol{P}^1 \times \{0, 1, \infty\} \cup \{0, 1, \infty\} \times \boldsymbol{P}^1$. For further information about this configuration see MIYANISHI-SUGIE [1991] and ZAIDENBERG [1991]. This configuration has to be used if the proper transform of the exceptional curves $E(P)$ and $E(Q)$ are contained in the compactification divisor of the homology plane having $L \cup L_1 \cup \cdots \cup L_4$ as plane divisor.

If this is not the case, say $E(P)$ is missing, apply the quadratic transformation with $S = \{P, Q, R\}$, where R is a further double point of $\bigcup_{i=1}^{4} L_i$. The resulting curve has the form $M_1 \cup M_2 \cup M_3 \cup M_4$, where $M_i \subset \boldsymbol{P}^2$ are lines, and M_1, M_2, M_3 have a common point. This arrangement has been used (essentially) by GURJAR-MIYANISHI [1987] to construct homology planes with $\bar{\kappa}=1$.

Homology planes which are in towers of Four Lines (3.16) are contained in towers of Five Lines. In order to see this apply again a quadratic transformation with $S = \{P, Q, R\}$. But this time assume that the curve $E(R)$ is not contained in the compactification divisor. Then, by applying (5.1), we see that Four Lines result.

(5.3) Contractible surfaces with $\bar{\kappa}=1$. In TOM DIECK-PETRIE [1990] a simple construction of contractible homology planes with $\bar{\kappa}=1$ was given. We showed indirectly that our surfaces were isomorphic to those of GURJAR-MIYANISHI [1987]. Here is a direct proof by birational maps. Let $C(n, b) \subset \boldsymbol{P}^2$ denote the curve given by $x^n = y^b z^{n-b}$, (n, b) coprime positive integers. Let L_z be the line $z=0$. Blow up a regular point of $C(n, b)$ and let $V(n, b)$ denote the complement of the proper transform $C'(n, b) \cup L'_z$. This was the way we defined the homology planes $V(n, b)$. Now add the lines L_x, $x=0$, and L_y, $y=0$ and consider $D = C(n, b) \cup L_x \cup L_y \cup L_z$. Resolve singularities and get a normal crossing curve with weighted dual graph (see TOM DIECK-PETRIE [1990], (2.7) for the computation)



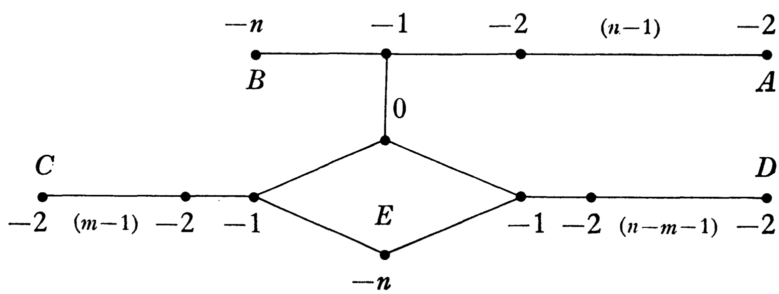
using the continued fraction expansions

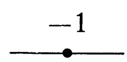
$$n/b = [w(s), \cdots, w(1)], \quad n/a = [w(-r), \cdots, w(-1)]$$

with $a=n-b$, $2a>n$. It is now easy to see that this graph can be contracted to the Gurjar-Miyanishi configuration of four lines: Begin the contraction with the left and right most (-1)-curves.

(5.4) An arrangement of Miyanishi-Sugie [1988]. They have considered the follgwing curve $D_1 \cup D_2 \subset \boldsymbol{P}^2$: The curve D_1 has degree n and a single singularity (cup) of multiplicity $n-1$; the curve D_2 is a line which intersects D_1 in two regular points P, Q with multiplicities $D_1 \cdot D_2 = (n-m)P + mQ$. They show that such arrangements lead to homology planes and, for $m=1$, to contractible surfaces. We demonstrate that their surfaces arise from Five Lines (4.2). This shows, incidentally, that they are contractible.

Resolving the singularities of the Miyanishi-Sugie curve gives the following graph (in brackets: the number of (-2)-curves in a string).



Add (formally) connection of type 

between AE, BC and BD. Then the resulting graph can be (formally) contracted to the graph of the six lines in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ (5.2). Now reverse the procedure to show existence and our claim simultaneously. The three (-1)-curves which are used to cut the cycles correspond after contraction to three lines which have to be added to the Miyanishi-Sugie curve. We leave it as a small exercise for the reader to figure out the position of these lines.

We finally mention without proof:

(5.5) The Ramanujam tower. The tower (3.15) is contained in towers arising from a quadric with three tangents. The Ramanujam surface itself is contained in a tower of Five Lines.

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