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ON A GENERALIZATION OF LUKEŠ' THEOREM

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0. In the classical potential theory, M.V. Keldych [4] proved the existence of positive harmonic function h in a bounded domain D possessing the property that h can be extended continuously onto $\bar{D}$ and takes the value zero only at the prescribed regular boundary point. In virtue of this result it is possible to obtain interesting results such as characterization of generalized Dirichlet solutions H_f by its functional aspect and identity of the set of regular boundary points with the Choquet boundary. In the axiomatic framework, the former problem was studied by J. Lukeš [5] thoroughly and it was cleared up that for a relatively compact open set the classical result holds as well only when the axiom of polarity is assured. As for the latter one, Bliedtner-Hansen [1] proved completely in their deep result that it remains valid under some condition even in an open set not necessarily relatively compact. In view of this, it seems to be adequate to investigate the former problem for an arbitrary open set. The purpose of this note is to extend Lukeš' theorem to an open set and normalized solutions of Dirichlet problem.

1. Let X be a $\mathcal{P}$-harmonic space with countable base in the sense of [3], and let P be the set of all continuous potentials on X. We consider an open subset U of X with non-empty boundary ∂U. We define

$\boldsymbol{C}_P(E)$: $=\{f;\ f$ is continuous on E, $\exists p\in P$ such that $|f|\leq p\}$ for $E\subset X$,

$S(U)$: $=\{s\in \boldsymbol{C}_P(\bar{U});\ s$ is superharmonic on $U\}$.

For an extended real valued function f on ∂U, we consider

$$\bar{H}^0_f(a):=\inf\left\{v(a);\ \begin{array}{l}\text{hyperharmonic on } U,\text{ bounded below,}\\ \liminf v\geq f \text{ on } \partial U,\ v\geq 0 \text{ outside a compact}\\ \text{subset of } X\end{array}\right\}$$

and

$$\underline{H}^0_f:=-\bar{H}^0_{(-f)}$$

If $\bar{H}^0_f=\underline{H}^0_f$ and is harmonic, f is called resolutive and $\bar{H}^0_f=\underline{H}^0_f=H^0_f$ is called a normalized solution. It is known that all functions of $\boldsymbol{C}_P(\partial U)$ are resolutive, thus, for each $a\in U$ there exists a Borel measure λ_a such that

$$\lambda_a(f)=H^0_f(a)\qquad \text{for every } f\in \boldsymbol{C}_P(\partial U)\,.$$

λ_a is nothing but the balayaged measure ε_a^{CU}. A point $x \in \partial U$ is called regular if

$$\lim_{a \to x} H_f^0(a) = f(x) \qquad \text{for every } f \in \boldsymbol{C}_P(\partial U).$$

The set of all regular points is denoted by U_{reg}.

From the deep result of [1], we know that

(1) $S(U)$ is simplicial;

(2) the following (i) and (ii) are equivalent:

(i) $\mu_x = \varepsilon_x^{CU}$ for every $x \in U$,

(ii) $\varepsilon_x^{CU}(\partial U \setminus U_{\text{reg}}) = 0$ for every $x \in U$;

(3) (ii) implies $Ch_{S(U)}\bar{U} = U_{\text{reg}}$,

where μ_x is the unique minimal measure of $\mathcal{H}_x := \{\mu;$ positive measure on $\bar{U}$, $\mu(s) \leq s(x) \forall s \in S(U)\}$ with respect to the order defined by $S(U)$.

2. Resorting to the above result of Bliedtner-Hansen, J. Lukeš [5] obtained the following theorem in the case where U is relatively compact:

Theorem (Lukeš). *The following statements are equivalent*:

(i) $\partial U \setminus Ch_{S(U)}\bar{U}$ *is negligible*;

(ii) *if* $\mathcal{L}$ *is a Keldych operator then* $\mathcal{L}_f = H_f$.

(iii) $\partial U \setminus U_{\text{reg}}$ *is negligible*.

Here, a Keldych operator $\mathcal{L}$ *is a mapping of* $\boldsymbol{C}(\partial U)$ *into the space of all harmonic function on* U *such that*

1) $\mathcal{L}$ *is linear and positive,*

2) $\mathcal{L}_s \leq s$ *for every* $s \in S(U)$.

Our purpose is to extend this theorem for arbitrary U. To this end, we define a Keldych operator $\mathcal{L}$ is to be a mapping of $\boldsymbol{C}_P(\partial U)$ into the space of harmonic functions on U satisfying above 1), 2).

3. We remark that $\bar{H}_f^0 = \bar{H}_f^U$, where

$$\bar{H}_f^U(a) := \inf\left\{v(a);\ \begin{matrix}\text{hyperharmonic on } U, \liminf v \geq f \text{ on } \partial U, \\ \exists p \ni P \text{ such that } v \geq -p\end{matrix}\right\}.$$

In fact, let v be hyperharmonic on U, lower bounded, $\liminf v \geq f$ on ∂U and $v \geq 0$ on $U \setminus K$ for a compact subset K of X. Then $\inf\{v;\ \overline{U \cap K}\} \geq -\alpha$ for a positive number α. We have a potential p on X such that $p \geq \alpha$ on $\overline{U \cap K}$, thus $\bar{H}_f^0 \geq \bar{H}_f^U$. Next, let v be hyperharmonic on U, $\liminf v \geq f$ on ∂U and $v \geq -p$ for some potential p, and let $\{K_n\}$ be a compact exhaustion of X such that $K_n \subset \mathring{K}_{n+1}$. $p_n = \hat{R}_p^{CK_n}$ decreases to zero. It is readily seen that $v + p_n$ is lower bounded and $v + p_n \geq 0$ on $U \setminus K_{n+1}$, thus $v + p_n \geq \bar{H}_f^0$ and finally $\bar{H}_f^U \geq \bar{H}_f^0$.

From the above observation it is obvious that the normalized solutions H_f^0 form a Keldych operator.

4. A Keldych operator $\mathcal{L}_f(a)$ defines a measure ν_a on ∂U, *i.e.*,

$$\nu_a(f)=\mathcal{L}_f(a) \qquad \text{for every } f\in \boldsymbol{C}_P(\partial U)\,.$$

We define $U_{\text{reg}}^{\mathcal{L}}:=\{x\in\partial U;\ \lim_{a\to x}\mathcal{L}_f(a)=f(x)\ \forall f\in \boldsymbol{C}_P(\partial U)\}$. It is clear that $x\in U_{\text{reg}}^{\mathcal{L}}$ if and only if $\lim_{a\to x}\mathcal{L}_s(a)=s(x)$ for every $s\in S(U)$. We prove the following:

Proposition (Lukeš).

$$Ch_{S(U)}\bar{U}\subset U_{\text{reg}}^{\mathcal{L}}\subset U_{\text{reg}}\,.$$

Proof. We show first

$$(*) \qquad \mu_a<\nu_a<\lambda_a \qquad \text{for every } a\in U\,,$$

where μ_a is the unique minimal measure of $\mathcal{H}_a$ and λ_a is the harmonic measure and the order $<$ is introduced by $S(U)$. If $s\in S(U)$ and $|s|\le p$ with $p\in P$, then $\limsup_x \mathcal{L}_s\le s(x)$ and $\mathcal{L}_s\le\mathcal{L}_p\le p$ implies that $\mathcal{L}_s\le H_s^0\le s$, thus we conclude $(*)$.

Since $\mu_x(s)$ is lower semi-continuous on $\bar{U}$,

$$\mu_x(s)\le\liminf_{\substack{a\to x\\ a\in U}}\mu_a(s)\le\liminf_{\substack{a\to x\\ a\in U}}\nu_a(s)\le\limsup_{\substack{a\to x\\ a\in U}}\nu_a(s)\le s(x)\,.$$

From this we have $Ch_{S(U)}\bar{U}\subset U_{\text{reg}}^{\mathcal{L}}$, similarly we have $U_{\text{reg}}^{\mathcal{L}}\subset U_{\text{reg}}$.

5. Now we shall prove Lukeš' theorem ($n^0 2$) for an arbitrary open set U: the equivalence of (i) and (iii) is due to the result of Blidetner-Hansen ($n^0 1$).

(i)$\Rightarrow$(ii): let $f\in\boldsymbol{C}_P(\partial U)$ and $|f|\le p$ with $p\in P$. There exists $p_1\in P$ such that for every $\varepsilon>0$ there is a compact subset K_ε of X with $p\le\varepsilon p_1$ on $X\backslash K_\varepsilon$. By hypothesis, we may find a non-negative hyperharmonic function w on U such that w is finite at a prescribed point of U and $\lim_x w=+\infty$ for every $x\in\partial U\backslash Ch_{S(U)}\bar{U}$. The function $v=2\varepsilon p_1+\varepsilon w+[H_f^0-\mathcal{L}_f]$ is hyperharmonic on U, $v\ge 0$ on $U\backslash K_\varepsilon$ and $\liminf v\ge 0$ on ∂U. Thus, by boundary minimum principle, $v\ge 0$ on U and letting $\varepsilon\to 0$ we have $H_f^0\ge\mathcal{L}_f$, similarly $\mathcal{L}_f\ge H_f^0$.

(ii)$\Rightarrow$(i): suppose that $\partial U\backslash Ch_{S(U)}\bar{U}$ is not negligible. Since there is a sequence $\{s_n\}$ of $S(U)$ such that

$$Ch_{S(U)}\bar{U}=\bigcap_{n=1}^{\infty}\{x\in\partial U;\ \mu_x(s_n)=s_n(x)\}\ ,$$

we may find $s\in S(U)$ and $a\in U$ such that

$$\lambda_a(\{x \in \partial U;\ \mu_x(s) \neq s(x)\}) > 0 .$$

We note that $f^*(x) = \mu_x(f)$ is a Borel function on ∂U and thus resolutive [2] for every $f \in \boldsymbol{C}_P(\partial U)$. Thus if we define

$$\mathcal{L}_f(a) = H^0_{f^*}(a) ,$$

then $\mathcal{L}_f$ is a Keldych operator. However, $\mathcal{L}_f$ is not H^0_f, since

$$\mathcal{L}_s(a) = H^0_{s^*}(a) = \lambda_a(s^*) \neq \lambda_a(s) = H^0_s(a) .$$

6. It seems to be an interesting problem to extend Lukeš' theorem to a resolutive compactification of X.

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