# THE TRANSFER MAP IN THE $KR_G$-THEORY

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In his work [10] Nishida defined the equivariant transfer maps and studied some properties of the transfer maps in the equivariant $K$-theory. And making use of them, he gave a new proof of the Adams conjecture in complex case. Following his work and introducing the transfer maps in the Real equivariant $K$-theory, we give here a proof of the Adams conjecture in real case.

In §1 we introduce the transfer maps in the $KR_G$-theory, and in §2 we discuss induced representations of Real representations and real representations. Nishida [10] used the monomiality of complex representations [11]. Instead of this fact, we prove in §3 that the identity representation of any odd dimensional orthogonal group is a linear combination of representations which are induced from one or two dimensional representations of appropriate subgroups. Then, by a parallel argument to Nishida [10], the proof of the Adams conjecture in real case is given in §4.

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## 1. The transfer map

Let $G$ be a compact Lie group and $X$ a compact $G$-space. For an admissible $G$-bundle $\xi=(p\colon E\to X)$ [10, 8], Nishida [10] defined a $G$-equivariant trace $t\colon X_+\wedge V^c\to E_+\wedge V^c$ of $\xi$ for a suitable real representation space $V$ of $G$, and proved that it is unique up to stable $G$-homotopies.

Let $G$ be a compact Real Lie group with involution $\tau$ [5]. We denote by $G\times_\tau Z_2$ the semidirect product of $G$ with $Z_2$, the group generated by $\tau$. Atiyah [4] introduced $KR_G$, the Real equivariant $K$-theory, which is a contravariant functor from the category of Real $G$-spaces (that is, $G\times_\tau Z_2$-spaces) to the category of abelian groups. When the involution acts trivially on $G$ and a Real $G$-space $X$, then $KR_G(X)$ is naturally isomorphic to $KO_G(X)$.

Let $V$ be a Real representation space of $G$ and $X$ a Real $G$-space. Atiyah [4] proved the Thom isomorphism $\Phi\colon KR_G(X)\cong KR_G(X\times V)$. Let $\xi=(p\colon E\to X)$ be an admissible $G\times_\tau Z_2$-bundle. We can choose a $G\times_\tau Z_2$-equivariant trace $t\colon X_+\wedge V^c\to E_+\wedge V^c$ of $\xi$ [10] in such a way that $V$ is a Real representation

space of $G$.   Then we define

$$p_! : KR_G(E) \to KR_G(X)$$

the transfer map for $\xi$ in the $KR_G$-theory as the composite of the following sequence

$$KR_G(E) \xrightarrow{\ \Phi\ } KR_G(E \times V) \xrightarrow{\ t^*\ } KR_G(X \times V) \xrightarrow{\ \Phi^{-1}\ } KR_G(X) .$$

This definition is well defined since the trace is unique.   Similarly we define the transfer for an admissible $G$-bundle in the $K_G$-theory.

Let $X$ be a Real G-space.   If we forget the involution on $X$, then we may regard $X$ as a $G$-space, which is denoted by $\psi X$.   We define the forgetful map

$$\psi : KR_G(X) \to K_G(\psi X)$$

by foregtting conjugate linear involutions on vector bundles.   The following lemma is obtained straightforward from the definitions of the Thom elements.

**Lemma 1.**   *The Thom isomorphisms commute with the forgetful maps, i.e., the diagram*

$$
\begin{array}{ccc}
KR_G(X) & \xrightarrow{\ \Phi\ } & KR_G(X \times V) \\
\Big\downarrow p_! & & \Big\downarrow p_! \\
K_G(\psi X) & \xrightarrow{\ \Phi\ } & K_G(\psi X \times \psi V)
\end{array}
$$

*commutes, where $V$ is a Real representation space of $G$.*

Forgetting the involutions, an admissible $G \times_\tau \mathbf{Z}_2$-bundle becomes an admissible $G$-bundle and a $G \times_\tau \mathbf{Z}_2$-equivariant trace becomes a $G$-equivariant trace.   So we have

**Proposition 2.**   *The transfer maps commute with the forgetful maps, i.e., the following diagram commutes*

$$
\begin{array}{ccc}
KR_G(E) & \xrightarrow{\ \psi\ } & K_G(\psi E) \\
\Big\downarrow p_! & & \Big\downarrow p_! \\
KR_G(X) & \xrightarrow{\ \psi\ } & K_G(\psi X) .
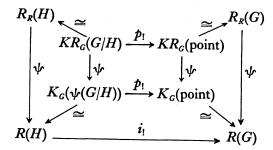\end{array}
$$

## 2.   The induced representation

Let $G$ be a compact Lie group, $H$ a closed subgroup and $i : H \subset G$ the inclusion map.   Segal [11] defined the induction homomorphism $i_! : R(H) \to R(G)$ and Nishida [10] showed that the transfer map for a $G$-bundle ($p : G/H \to point$)

in the $K_G$-theory coincides with the induction homomorphism through the natural isomorphism $K_G(G/H) \cong R(H)$.

Let $G$ be a compact Real Lie group, $H$ a closed Real subgroup and $i: H \subset G$ the inclusion map. $R_R(G)$ denotes the Real representation ring of $G$ [5]. The forgetful map

$$\psi: R_R(G) \to R(G)$$

is defined by forgetting conjugate linear involutions. It is well known that this forgetful map is injective. When the involution acts trivially on $G$, then $R_R(G)$ is naturally isomorphic to $RO(G)$ and the forgetful map coincides with the complexification map $c: RO(G) \to R(G)$. The diagram



commutes by the definition of the natural isomorphism $KR_G(G/H) \cong R_R(H)$, Proposition 2 and [10], Theorem 5.2. We define an induction homomorphism

$$i_!: R_R(H) \to R_R(G)$$

as the composite of the upper horizontal map and two isomorphisms of this diagram. In case the involution is trivial, we have an induction homomorphism

$$i_!: RO(H) \to RO(G).$$

Since the forgetful map and the complexification map preserve the characters, these induction homomorphisms satisfy the character formula [11], p. 119–120.

Let $E$ be a compact Real $G$-space such that $\psi E$ is a free $G$-space. For a Real representation space $M$ of $G$, we define $\alpha(M)$ as a Real $G$-vector bundle $(E \times_G M \to E/G)$. The correspondence $M \to \alpha(M)$ induces a homomorphism

$$\alpha: R_R(G) \to KR(E/G).$$

When the involution acts trivially on $G$ and $E$, we have a homomorphism

$$\alpha: RO(G) \to KO(E/G).$$

**Proposition 3.** *The diagram*

$$R_R(H) \xrightarrow{\ \alpha\ } KR(E/H)$$
$$\downarrow i_! \qquad\qquad \downarrow p_!$$
$$R_R(G) \xrightarrow{\ \alpha\ } KR(E/G)$$

*commutes, where $p_!$ is the transfer for an admissible $Z_2$-bundle $(p: E/H \to E/G)$.*

This proof is parallel to [10] Proposition 5.4, so we omit it. In case the involution is trivial, we have

**Corollary 4.** *The following diagram commutes*

$$RO(H) \xrightarrow{\ \alpha\ } KO(E/H)$$
$$\downarrow i_! \qquad\qquad \downarrow p_!$$
$$RO(G) \xrightarrow{\ \alpha\ } KO(E/G) .$$

## 3. Real representations of the orthogonal group

In this section we put $G=O(2m+1)$ and $H=O(2)\times O(2m-1)$. Let $i: H \subset G$ be the standard inclusion, i.e., $i(B,C)=\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$. Let $\iota$ and $\nu$ be representations of $G$, whose actions are $\iota(A)x=Ax$ and $\nu(A)y=\det A \cdot y$ for $A \in G$, $x \in R^{2m+1}$ and $y \in R$. And let $\mu$ be a representation of $H$, whose action is $\mu(B,C)z=Bz$ for $(B,C) \in H$ and $z \in R^2$.

**Proposition 5.** $\iota=i_!\mu+\nu$

Proof. We take the characters of both representations and we shall see that they are equal as class functions. Since $G$ consists of exactly two connected components, we have two conjugacy classes of Cartan subgroups of $G$ in the sense of Segal [11], and we may choose $T^m$ and $T^m \times Z_2$ as representatives of them, where $T^m$ is the standard maximal torus of $SO(2m+1)$ and $Z_2$ is generated by $-I_{2m+1}$, the diagonal matrix with $-1$ as diagonal entries. Let $g(\theta_1,\theta_2,\cdots,\theta_m;\varepsilon)$ be a matrix

$$\begin{pmatrix} D(\theta_1) & & & \\ & D(\theta_2) & & \\ & & D(\theta_m) & \\ & & & \varepsilon \end{pmatrix}$$

where $D(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $0 \le \theta_k < 2\pi$, $\varepsilon=\pm 1$. Every topological generators of $T^m$ (resp. $T^m \times Z_2$) can be expressed as $g=g(\theta_1,\theta_2,\cdots,\theta_m;1)$ (resp. $g'=g(\theta_1,$

$\theta_1, \cdots, \theta_m; -1))$ such that $\theta_1, \theta_2, \cdots, \theta_m$ and $\pi$ are linearly independent over the rational field. Since the topological generators of Cartan subgroups are dense in $G$, it is sufficient to show that those characters coincide on $g$ and $g'$. It is easy to see that $\chi_\iota(g) = \sum_{k=1}^m 2\cos\theta_k + 1$, $\chi_\iota(g') = \sum_{k=1}^m 2\cos\theta_k - 1$, $\chi_\mu(g) = \chi_\mu(g')$ $= 2\cos\theta_1$, $\chi_\nu(g) = 1$, $\chi_\nu(g') = -1$. By the character formula, the character of $i_! \mu$ is written as

$$\chi_{i_!\mu}(g) = \sum_{x \in F}\chi_\mu(x^{-1}gx)$$
$$\chi_{i_!\mu}(g') = \sum_{y \in F'}\chi_\mu(y^{-1}g'y)$$

where $F$ (resp. $F'$) is the set of representatives of fixed points of the action of $g$ (resp. $g'$) on $G/H$. We shall describe $F'$ explicitly. $y \in F'$ means $y^{-1}gy \in H$, and $y^{-1}gy$ generates a Cartan subgroup $T$ of $H$ which is isomorphic to $T^m \times \mathbf{Z}_2$. Put $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and let $U_0$ be the subgroup of $H$ generated by $\begin{pmatrix} A & \\ & I_{2m-1} \end{pmatrix}$ and $U_1$ the subgroup of $H$ generated by $\begin{pmatrix} A & \\ & -I_{2m-1} \end{pmatrix}$. $T^{m-1}$ denotes the maximal torus of $SO(2m-1)$ which we regard as a subgroup of $H$. $U_0$ and $U_1$ are subgroups of $Z_H(T^{m-1})$, the centralizer of $T^{m-1}$ in $H$. We define $S_0 = U_0 \times T^{m-1}$ and $S_1 = U_1 \times T^{m-1}$. $S_0$ and $S_1$ are isomorphic to $T^{m-1} \times \mathbf{Z}_2$ and they are Cartan subgroups of $H$ which are not conjugate. According to Segal [11], there are just four conjugacy classes of Cartan subgroups of $H$ since $H/H^0$, the group of components is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. And we may take $T^m$, $T^m \times \mathbf{Z}_2$, $S_0$ and $S_1$ as representatives of those conjugacy classes. Thus $T$, the group generated by $y^{-1}g'y$, is conjugate to $T^m \times \mathbf{Z}_2$ in $H$, i.e., there exists an element $h$ of $H$ such that $T^m \times \mathbf{Z}_2 = h^{-1}Th$. Then $yh \in N_G(T^m \times \mathbf{Z}_2)$, the normalizer, and $y$ and $yh$ are in the same coset in $G/H$. So we can take $F'$ as a subset of $N_G(T^m \times \mathbf{Z}_2)$. The natural projection $G \to G/H$ sends $N_G(T^m \times \mathbf{Z}_2)$ to $N_G(T^m \times \mathbf{Z}_2)/(N_G(T^m \times \mathbf{Z}_2) \cap H)$ and evidently $N_G(T^m \times \mathbf{Z}_2) \cap H = N_H(T^m \times \mathbf{Z}_2)$. So we identify $F'$ with $N_G(T^m \times \mathbf{Z}_2)/N_H(T^m \times \mathbf{Z}_2)$. It is easy to see that

$$N_G(T^m \times \mathbf{Z}_2)/(T^m \times \mathbf{Z}_2) \cong \sum_m \int \mathbf{Z}_2$$

$$N_H(T^m \times \mathbf{Z}_2)/(T^m \times \mathbf{Z}_2) \cong \mathbf{Z}_2 \times \sum_{m-1} \int \mathbf{Z}_2$$

So we shall identify them. Let $y = (\sigma; \varepsilon_1, \cdots, \varepsilon_m) \in \sum_m \int \mathbf{Z}_2$ and $y' = (\delta, \rho; \delta_1, \cdots, \delta_{m-1}) \in \mathbf{Z}_2 \times \sum_{m-1} \int \mathbf{Z}_2$. Then

$$y^{-1}g(\theta_1, \cdots, \theta_m; -1)y = g(\varepsilon_1 \theta_{\sigma^{-1}(1)}, \cdots, \varepsilon_m \theta_{\sigma^{-1}(m)}; -1)$$
$$y'^{-1}g(\theta_1, \cdots, \theta_m; -1)y' = g(\delta\theta_1, \delta_1 \theta_{1+\rho^{-1}(1)}, \cdots, \delta_{m-1}\theta_{1+\rho^{-1}(m-1)}; -1).$$

Since $\#F' = m$, $\chi_{i_!\mu}(g') = \sum_{k=1}^m 2\cos\theta_k$. Similarly $\chi_{i_!\mu}(g) = \sum_{k=1}^m 2\cos\theta_k$. This completes the proof.

## 4. The Adams conjecture

We state the Adams conjecture in real case and prove it. Let $F_n$ be the monoid of based homotopy equivalences of $S^n$. Let $BF_n$ be the classifying space of $F_n$ and $BF = \varinjlim BF_n$. The homotopy set $[X_+, BF]$ is isomorphic to the group of stable fibre homotopy equivalence classes of spherical fibre spaces [14]. For a finite CW-complex $X$, an abelian group $Sph(X)$ is defined as $[X_+, BF \times Z]$, and the $J$-homomorphism $J: KO(X) \to Sph(X)$ is defined by $J(\xi) = ([\xi], \dim \xi)$ for a real vector bundle $\xi$ where $[\xi]$ denotes the class of the associated sphere bundle. By Segal [13], $\{O(n)\}$ and $\{F_n\}$ are $\Gamma$-spaces and the map $j = \{j_n: O(n) \to F_n\}$ is a map of $\Gamma$-spaces. So $BO \times Z$ and $BF \times Z$ become infinite loop spaces and $j: BO \times Z \to BF \times Z$ becomes an infinite loop map. Remark that this infinite loop space structure of $BO \times Z$ coincides with the infinite loop space structure induced from the Thom isomorphism [15]. So $Sph(X)$ is a 0-th term of a generalised cohomology theory and $J = j^*$ is a stable natural transformation. By [10], Proposition 4.3, we have

**Lemma 6.** *The transfer commutes with the J-homomorphism.*

Let $q$ be a prime number. For an abelian group $A$, $A \otimes Z\left[\dfrac{1}{q}\right]$ is denoted by $A\left[\dfrac{1}{q}\right]$. Let $\psi^q$ be the $q$-th Adams operation. Since $\alpha: RO(G) \to KO(E/G)$ is a $\lambda$-ring homomorphism, $\alpha$ commutes with $\psi^q$. It is well known that $\psi^q$ is a stable operation on $KO(X)\left[\dfrac{1}{q}\right]$. So we have

**Lemma 7.** *The transfer commutes with $\psi^q$ in the $KO(\ )\left[\dfrac{1}{q}\right]$-theory.*

Now we prove

**Theorem 8** (Adams conjecture).

$$J(\psi^q - 1) = 0; KO(X)\left[\frac{1}{q}\right] \to Sph(X)\left[\frac{1}{q}\right].$$

Proof. Adams [2] proved this theorem for one and two dimensional vector bundles. Since odd dimensional vector bundles generate $KO(X)$ as an abelian group, it is sufficient to prove the theorem for odd dimensional vector bundles. Let $\xi$ be a $(2m+1)$-dimensional real vector bundle over $X$ and $(E \to X)$ the associated principal $O(2m+1)$-bundle. Let $G$ and $H$ be the same groups as in §3. Consider the following commutative diagram

$$
\begin{array}{ccccc}
RO(H)\left[\dfrac{1}{q}\right] & \xrightarrow{\ \alpha\ } & KO(E/H)\left[\dfrac{1}{q}\right] & \xrightarrow{\ J\ } & Sph(E/H)\left[\dfrac{1}{q}\right] \\
\downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_!} \\
RO(G)\left[\dfrac{1}{q}\right] & \xrightarrow{\ \alpha\ } & KO(E/G)\left[\dfrac{1}{q}\right] & \xrightarrow{\ J\ } & Sph(E/G)\left[\dfrac{1}{q}\right]
\end{array}
$$

where $p_!$ is the transfer for the bundle $(p : E/H \rightarrow E/G)$. Clearly $\xi = \alpha(\iota)$. Since $\nu$ is a one dimensional representation and $\mu$ is a two dimensional representation, we have

$$
\begin{aligned}
J(\psi^q - 1)\,(\xi) &= J(\psi^q - 1)\alpha(\iota) \\
&= J(\psi^q - 1)\alpha i_!(\mu) + J(\psi^q - 1)\alpha(\nu) \\
&= J(\psi^q - 1)p_!\alpha(\mu) + J(\psi^q - 1)\alpha(\nu) \\
&= p_! J(\psi^q - 1)\alpha(\mu) + J(\psi^q - 1)\alpha(\nu) \\
&= 0 .
\end{aligned}
$$

This completes the proof.

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## References

[1]   J.F. Adams:   *Vector fields on spheres,* Ann. of Math. **75** (1962), 603–632.

[2]   J.F. Adams:   *On the group J(X)-I.* Topology **2** (1963), 181–195.

[3]   M.F. Atiyah:   K-theory, Benjamin 1967.

[4]   M.F. Atiyah:   *Bott periodicity and the index of elliptic operators,* Quart. J. Math. Oxford (2), **19** (1968), 113–140.

[5]   M.F. Atiyah and G.B. Segal:   *Equivariant K-theory and completions,* J. Differential Geom. **3** (1969), 1–18.

[6]   M.F. Atiyah and I.M. Singer:   *The index of elliptic operators* IV, Ann. of Math. **87** (1968), 484–530.

[7]   J.C. Becker and D.H. Gottlieb:   *The transfer map and fibre bundles,* Topology **14** (1975), 1–12.

[8]   T. tom Dieck:   *Faserbündel mit Gruppenoperation* Arch. Math. **20** (1969), 136–143.

[9]   C. Kosniowski:   *Equivariant cohomology and stable cohomotopy,* Math. Ann. **210** (1974), 83–104.

[10]   G. Nishida:   *The transfer homomorphism in equivariant generalised cohomology,* J. Math. Kyoto Univ. **18** (1978), 435–451.

[11]   G.B. Segal:   *The representation ring of a compact Lie group,* Inst. Hautes Études Sci. Publ. Math. **34** (1968), 113–128.

[12]   G.B. Segal:   *Equivariant K-theory,* Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129–151.

[13]   G.B. Segal:   *Categories and cohomology theories,* Topology **13** (1974), 293–312.

[14]   D. Stasheff:   *A classification theorem for fibre spaces,* Topology **2** (1963), 239–246.

[15]   J.P. May:   $E_\infty$ ring spaces and $E_\infty$ ring spectra, Lecture Notes in Math. 577, Springer.

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