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Osaka University
ON GENERALIZED SIEGEL DOMAINS

AKIO KODAMA*)

(Received April 1, 1976)

Introduction. In [3], Kaup, Matsushima and Ochiai defined the notion of "generalized Siegel domain with exponent c", which is a natural generalization of the notion of Siegel domain of the first or second kind.

In this paper we consider exclusively a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent 1/2. Let $\text{Aut}(\mathcal{D})$ denote the group of all holomorphic transformations of $\mathcal{D}$. It is well-known that the group $\text{Aut}(\mathcal{D})$ has the structure of real Lie group and the Lie algebra $\mathfrak{g}$ of $\text{Aut}(\mathcal{D})$ is canonically identified with the real Lie algebra $\mathfrak{g}(\mathcal{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$. Furthermore it is known that the Lie algebra $\mathfrak{g}(\mathcal{D})$ has the following graded structure [3]:

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1/2} + \mathfrak{g}_{1/2} + \mathfrak{g}_0 + \mathfrak{g}_1,$$

with $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$, and $\dim \mathfrak{g}_{-1/2} = 2k$ for some $k$, $0 \leq k \leq m$.

In section 2 we shall prove the following Theorem.

Theorem 1. Let $\mathcal{D}$ be a generalized Siegel domain in $\mathbb{C} \times \mathbb{C}^m$ with exponent $1/2$ and $\dim \mathfrak{g}_{-1/2} = 2k$, $0 \leq k \leq m$. Let $\text{Aut}_0(\mathcal{D})$ denote the identity component of $\text{Aut}(\mathcal{D})$. Then there exists a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent 1/2 which is holomorphically equivalent to $\mathcal{D}$ and such that, by choosing a suitable coordinates system $(z, w_1, \ldots, w_m)$ in $\mathbb{C} \times \mathbb{C}^m$,

1. the orbit $\mathcal{D}_0$ of $\text{Aut}_0(\mathcal{D})$ containing the point $(\sqrt{-1}, 0, \ldots, 0) \in \mathcal{D}$ is the elementary Siegel domain

$$\mathcal{D}_0 = \{(z, w_1, \ldots, w_m, 0, \ldots, 0) \in \mathbb{C} \times \mathbb{C}^m | \text{Im. } z - \sum_{a=1}^k |w_a|^2 > 0\}$$

and

2. if we put

$$\mathcal{D}_{\sqrt{-1}} = \{(w_k+1, \ldots, w_m) \in \mathbb{C}^{m-k} | (\sqrt{-1}, 0, \ldots, 0, w_k+1, \ldots, w_m) \in \mathcal{D}\},$$

then $\mathcal{D}_{\sqrt{-1}}$ is a circular domain in $\mathbb{C}^{m-k}$ containing the origin 0 of $\mathbb{C}^{m-k}$. Moreover the domain $\mathcal{D}$ can be expressed by $\mathcal{D}_0$ and $\mathcal{D}_{\sqrt{-1}}$ as follows:

* Recent address: Akita University
A. KODAMA

As a corollary of Theorem 1, we shall show that if the Lie algebra $\mathfrak{g}(\mathcal{D})$ is semi-simple, then $\mathcal{D}$ is a Siegel domain of the second kind which is holomorphically equivalent to the elementary Siegel domain in $\mathbb{C} \times \mathbb{C}^m$.

In section 3 we shall consider the group $\text{Aut}(\mathcal{D})$ of all holomorphic transformations of a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent $1/2$ and $\dim_R \mathfrak{g}_{1/2} = 2k$. By Theorem 1 we can regard $\mathcal{D}$ as a holomorphic fibre space over the elementary Siegel domain $\mathcal{D}_0$ with the projection $\pi: \mathcal{D} \to \mathcal{D}_0$ given by $\pi(z, w_1, \ldots, w_m) = (z, w_1, \ldots, w_k, 0, \ldots, 0)$ and the fibre $\pi^{-1}(\sqrt{-1}, 0, \ldots, 0)$ is the circular domain $\mathcal{D}_{\sqrt{-1}}$. In Theorem 2 we shall prove that $\text{Aut}_0(\mathcal{D})$ is the direct product of $\text{Aut}_0(\mathcal{D}_0)$ and the identity component of the isotropy subgroup of $\text{Aut}_0(\mathcal{D}_{\sqrt{-1}})$ at the origin $0$ of $\mathcal{D}_{\sqrt{-1}}$.

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1. Preliminaries

Throughout this paper we use the following notations. Let $\mathbb{R}$ (resp. $\mathbb{C}$) denote the field of real numbers (resp. complex numbers) as usual. Let $^tA$ (resp. $0_{s \times t}$) denote the transpose of a matrix $A$ (resp. the unit matrix of degree $l$, $s \times t$ zero matrix) and $A^{-1}$ the inverse matrix of $A$ if $A$ is nonsingular.

In this section we recall the definitions and the known results on generalized Siegel domains. We fix a coordinates system $(z_1, \ldots, z_n, w_1, \ldots, w_m)$ in $\mathbb{C}^n \times \mathbb{C}^m$ once and for all.

A domain $\mathcal{D}$ in $\mathbb{C}^n \times \mathbb{C}^m$ is called a generalized Siegel domain with exponent $c$ if the following conditions are satisfied:

1. $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$ and $\mathcal{D}$ contains a point of the form $(z, 0)$ where $z \in \mathbb{C}^n$ and $0$ denotes the origin of $\mathbb{C}^m$.
2. $\mathcal{D}$ is invariant by the transformations of $\mathbb{C}^{n+m}$ of the following types:
   a) $(z, w) \mapsto (z + a, w)$ for all $a \in \mathbb{R}^n$;
   b) $(z, w) \mapsto (z, e^{it}w)$ for all $t \in \mathbb{R}$;
   c) $(z, w) \mapsto (e^{ix}, e^{ix}w)$ for all $t \in \mathbb{R}$,
where $c$ is a fixed real number depending only on $\mathcal{D}$. We call $c$ the exponent of $\mathcal{D}$.

We denote by $\Omega$ an open convex cone in $\mathbb{R}^n$ not containing any full straight line. For a given convex cone $\Omega$ in $\mathbb{R}^n$, a mapping $F: \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ is called an $\Omega$-hermitian form if

$$\mathcal{D} = \left\{ (z, w_1, \ldots, w_m) \in \mathbb{C} \times \mathbb{C}^m \mid (z, w_1, \ldots, w_k, 0, \ldots, 0) \in \mathcal{D}_0, \left( \frac{w_{k+1}}{(\text{Im. } z - \sum_{s=1}^{k} |w_s|^2)^{1/2}}, \ldots, \frac{w_m}{(\text{Im. } z - \sum_{s=1}^{n} |w_s|^2)^{1/2}} \right) \in \mathcal{D}_{\sqrt{-1}} \right\}.$$
(1) $F$ is complex linear with respect to the first variable;
(2) $F(u, v) = \overline{F(v, u)}$ for any $u, v \in \mathbb{C}^m$;
(3) $F(u, u) \in \overline{\Omega}$ for any $u \in \mathbb{C}^m$ and $F(u, u) = 0$ only if $u = 0$, where $\overline{\Omega}$ denotes the closure of $\Omega$ in $\mathbb{R}^n$.

For a given convex cone $\Omega$ in $\mathbb{R}^n$ and an $\Omega$-hermitian form $F : \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}^n$, the domain

$$\mathcal{D}(\Omega, F) = \{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m \mid \text{Im. } z - F(w, w) \in \Omega \}$$

in $\mathbb{C}^n \times \mathbb{C}^m$ is called the Siegel domain of the second kind associated with $\Omega$ and $F$. If $m = 0$, the domain $\mathcal{D}(\Omega, F)$ reduces to the domain

$$\mathcal{D}(\Omega) = \{ z \in \mathbb{C}^n \mid \text{Im. } z \in \Omega \}$$

which we call the Siegel domain of the first kind associated with $\Omega$. It is easy to see that if we put $c = 1/2$ then the domain $\mathcal{D}(\Omega, F)$ satisfies the condition (2) of the definition of generalized Siegel domain. Moreover it is known that $\mathcal{D}(\Omega, F)$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$ [7]. Obviously every point of the form $(\sqrt{-1}a, 0), a \in \Omega$, is contained in $\mathcal{D}(\Omega, F)$ and hence the domain $\mathcal{D}(\Omega, F)$ is a generalized Siegel domain with exponent $1/2$. From this fact, the notion of generalized Siegel domain may be considered as a generalization of the notion of Siegel domain of the second kind. In the following we regard $\mathcal{D}(\Omega)$ as the special case of the second kind and by a Siegel domain we mean a Siegel domain of the first or second kind.

Let $\mathcal{D}$ be a generalized Siegel domain in $\mathbb{C}^n \times \mathbb{C}^m$ with exponent $c$. Since $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$, by a well-known theorem of H. Cartan the group $\text{Aut}(\mathcal{D})$ has the structure of real Lie group and the Lie algebra of $\text{Aut}(\mathcal{D})$ is identified with the Lie algebra $\mathfrak{g}(\mathcal{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$ [2].

From the definition, the following holomorphic vector fields on $\mathcal{D}$ is contained in $\mathfrak{g}(\mathcal{D})$:

(a) $$\frac{\partial}{\partial z_k} \quad \text{for } k = 1, 2, \ldots, n$$

(b) $$\partial' = i \sum_{a=1}^m w_a \frac{\partial}{\partial w_a}$$

(c) $$\partial = \sum_{k=1}^n z_k \frac{\partial}{\partial z_k} + c \sum_{a=1}^m w_a \frac{\partial}{\partial w_a}.$$

By Kaup, Matsushima and Ochiai [3], every vector field $X \in \mathfrak{g}(\mathcal{D})$ is a polynomial vector field, and so we can express $X$ in the following form:

$$X = \sum_{k=1}^n \left( \sum_{i, j \geq 0} P^k_{i,j} \frac{\partial}{\partial z_k} + \sum_{a=1}^m \left( \sum_{i, j \geq 0} Q^a_{i,j} \right) \frac{\partial}{\partial w_a} \right).$$
where $P_{i\mu}$ and $Q_{i\mu}^{*}$ are homogeneous polynomials of degrees $\nu$ in $z_l(1 \leq l \leq n)$ and $\mu$ in $w_\beta(1 \leq \beta \leq m)$. If $\mathcal{D}$ is a generalized Siegel domain with exponent $c=1/2$, we have the following theorem on the Lie algebra $\mathfrak{g}(\mathcal{D})$.

**Theorem A** (Kaup, Matsushima and Ochiai [3]).

*Let $\mathcal{D}$ be a generalized Siegel domain in $\mathbb{C}^n \times \mathbb{C}^m$ with exponent $1/2$. Then we have*

\[
\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1,
\]

where $\mathfrak{g}_\lambda \subset \mathfrak{g}_{\lambda + \mu}$, with $\mathfrak{g}_\lambda = \{X \in \mathfrak{g}(\mathcal{D}) | [\partial, X] = \lambda X\}$.

*More precisely we can describe each subspace $\mathfrak{g}_\lambda$ as follows:*

\[
\begin{align*}
\mathfrak{g}_{-1} &= \left\{ \sum_{k=1}^n a^k \frac{\partial}{\partial z_k} \middle| a = (a^k) \in \mathbb{R}^n \right\} \\
\mathfrak{g}_{-1/2} &= \left\{ \sum_{k=1}^n P_{i,1}^k \frac{\partial}{\partial z_k} + \sum_{\sigma=1}^m Q_{\sigma,0}^{*} \frac{\partial}{\partial w_\sigma} \in \mathfrak{g}(\mathcal{D}) \right\} \\
\mathfrak{g}_0 &= \left\{ \sum_{k=1}^n P_{i,0}^k \frac{\partial}{\partial z_k} + \sum_{\sigma=1}^m Q_{\sigma,1}^{*} \frac{\partial}{\partial w_\sigma} \in \mathfrak{g}(\mathcal{D}) \right\} \\
\mathfrak{g}_{1/2} &= \left\{ \sum_{k=1}^n P_{i,1}^k \frac{\partial}{\partial z_k} + \sum_{\sigma=1}^m (Q_{\sigma,1}^{*} + Q_{i,2}^{*}) \frac{\partial}{\partial w_\sigma} \in \mathfrak{g}(\mathcal{D}) \right\} \\
\mathfrak{g}_1 &= \left\{ \sum_{k=1}^n P_{i,0}^k \frac{\partial}{\partial z_k} + \sum_{\sigma=1}^m Q_{\sigma,0}^{*} \frac{\partial}{\partial w_\sigma} \in \mathfrak{g}(\mathcal{D}) \right\}
\end{align*}
\]

\[\mathfrak{g}_\lambda \subset \mathfrak{g}_{\lambda + \mu},\]

\[
\begin{align*}
\dim_{\mathbb{R}} \mathfrak{g}_{-1} &= n, & \dim_{\mathbb{R}} \mathfrak{g}_{-1/2} &\leq 2m, \\
\dim_{\mathbb{R}} \mathfrak{g}_{1/2} &= \dim_{\mathbb{R}} \mathfrak{g}_{-1/2} - \dim_{\mathbb{R}} \mathfrak{g}_{-1/2}, \\
\dim_{\mathbb{R}} \mathfrak{g}_1 &= n - \dim_{\mathbb{R}} \mathfrak{g}_{-1}.
\end{align*}
\]

\[(3) \quad (i) \quad \dim_{\mathbb{R}} \mathfrak{g}_{-1} = n, \quad \dim_{\mathbb{R}} \mathfrak{g}_{-1/2} \leq 2m, \]

\[\quad (ii) \quad \dim_{\mathbb{R}} \mathfrak{g}_{1/2} = \dim_{\mathbb{R}} \mathfrak{g}_{-1/2} - \dim_{\mathbb{R}} \mathfrak{g}_{-1/2}, \]

\[\quad \dim_{\mathbb{R}} \mathfrak{g}_1 = n - \dim_{\mathbb{R}} \mathfrak{g}_{-1}.
\]

\[(4) \quad \mathfrak{a} = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0. \quad \text{Then $\mathfrak{a}$ is the subalgebra of $\mathfrak{g}(\mathcal{D})$ corresponding to the subgroup $\text{Aff} (\mathcal{D})$ of $\text{Aut} (\mathcal{D})$ consisting of all complex affine transformations of $\mathbb{C}^{n+m}$ leaving invariant the domain $\mathcal{D}$.}
\]

\[(5) \quad \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \text{ is the subalgebra corresponding to the subgroup $\{g \in \text{Aut} (\mathcal{D}) \mid g \text{ leaves invariant the complex submanifold } \mathcal{D}_1 \subset \mathcal{D}\}$, where } \mathcal{D}_1 = \{(z,w) \in \mathcal{D} | w = 0\} \text{ is equivalent to a Siegel domain of the first kind in } \mathbb{C}^n.
\]

By Theorem A, we can write $X \in \mathfrak{g}_{-1/2}$ in the form

\[
X = \sum_{k=1}^n P_{i,1}^k(X) \frac{\partial}{\partial z_k} + \sum_{\sigma=1}^m c^{*}(X) \frac{\partial}{\partial w_\sigma}
\]

where $P_{i,1}^k(X)$ denotes a homogeneous polynomial of degree one in $w_\alpha(1 \leq \alpha \leq m)$.
depending on \( X \) and \( c^\alpha(X) \) is a constant depending on \( X \). Then by a simple computation, we get

\[
(1.1) \quad \text{ad} \partial' \cdot X = \sqrt{-1} \sum_{k=1}^{n^*} P_{0,1}^k(X) \frac{\partial}{\partial x_k} - \sqrt{-1} \sum_{\alpha=1}^{c^\alpha(X)} \frac{\partial}{\partial w_\alpha}.
\]

Hence the endomorphism \( \text{ad} \partial' \) defines a complex structure on \( g_{-1/2} \). From this fact and (3) of Theorem A, we obtain the following corollary:

**Corollary.** \( \dim_R g_{-1/2} = 2k \) for some \( k, 0 \leq k \leq m \).

Since the group \( \text{Aff}(C^{n+m}) \) of all complex affine transformations of \( C^{n+m} \) is represented as a semi-direct product \( GL(n+m, C) \cdot C^{n+m} \), we can write each element \( g \in \text{Aff}(C^{n+m}) \) in the form \( g = (A, a) \), where \( A \in GL(n+m, C) \) and \( a \in C^{n+m} \). Obviously the mapping which carries \( g = (A, a) \) to the matrix \( (A, a) \) \( \in GL(n+m+1, C) \) is a faithful representation of \( \text{Aff}(C^{n+m}) \). Since \( \text{Aff}(\mathbb{C}) \) is a closed subgroup of \( \text{Aff}(C^{n+m}) \), we can identify \( \text{Aff}(\mathbb{C}) \) with the closed subgroup of \( GL(n+m+1, C) \), and so the Lie algebra \( \mathfrak{a} \) is identified with the subalgebra of \( \mathfrak{g}(n+m+1, C) \).

Let \( M \) be a hyperbolic manifold in the sense of Kobayashi [4]. It is known that the group \( \text{Aut}(M) \) of all holomorphic transformations of \( M \) is a Lie group and its isotropy subgroup \( K_p \) at a point \( p \) of \( M \) is compact [4]. We may identify the Lie algebra of \( \text{Aut}(M) \) with the Lie algebra \( \mathfrak{g}(M) \) consisting of all complete holomorphic vector fields on \( M \). A hyperbolic manifold \( M \) is called a hyperbolic circular domain in \( C^d \) if the following conditions are satisfied:

1. \( M \) is a domain in \( C^d \);
2. \( M \) is circular, that is, \( M \) is invariant by the following global one-parameter subgroup of transformations:

\[
l_t: (w_1, \ldots, w_d) \mapsto (e^{-t} w_1, \ldots, e^{-t} w_d), \quad t \in \mathbb{R}
\]

where \( (w_1, \ldots, w_d) \) denotes a coordinates system in \( C^d \). Let \( M \) be a hyperbolic circular domain in \( C^d \) containing the origin 0 of \( C^d \). Since the one-parameter subgroup \( \{l_t \mid t \in \mathbb{R} \} \) induces an element \( \partial = \sqrt{-1} \sum_{\alpha=1}^{d} w_\alpha \frac{\partial}{\partial w_\alpha} \) of \( \mathfrak{g}(M) \), we can show that every vector field \( X \in \mathfrak{g}(M) \) is expressed in the form

\[
X = \sum_{\alpha=1}^{d} \left( \sum_{\beta \geq 0} P_\beta^{w_\alpha} \right) \frac{\partial}{\partial w_\alpha}
\]

where \( P_\beta^{w_\alpha} \) is a homogeneous polynomial of degree \( \beta \) in \( w_\alpha \) (1 \( \leq \beta \leq d \)), by the same way as in [3]. More precisely we can show the following Theorem B (cf. [8]):
Theorem B. Let $M$ be a hyperbolic circular domain in $\mathbb{C}^d$ containing the origin $0$ of $\mathbb{C}^d$. For the vector field $\partial = -\sqrt{-1} \sum_{a=1}^{d} w_a \frac{\partial}{\partial w_a} \in \mathfrak{g}(M)$, we define an endomorphism $J$ of $\mathfrak{g}(M)$ by $J(X) = [0, X]$ for $X \in \mathfrak{g}(M)$. Let $\mathfrak{t}(M)$ denote the Lie subalgebra of $\mathfrak{g}(M)$ corresponding to the isotropy subgroup $K$ of $\text{Aut}(M)$ at the origin $0 \in M$. Then we have

\begin{align*}
\mathfrak{t}(M) &= \left\{ \sum_{a=1}^{d} P_a \frac{\partial}{\partial w_a} \mid \sum_{a=1}^{d} P_a \frac{\partial}{\partial w_a} \in \mathfrak{g}(M) \right\},
\end{align*}

which is equal to the kernel of $J$; and

\begin{align*}
\text{if we put } \mathfrak{p}(M) &= \{ X \in \mathfrak{g}(M) \mid J(X) = -X \},
\text{then } \mathfrak{g}(M) &= \mathfrak{t}(M) + \mathfrak{p}(M) \quad \text{(direct sum)}.
\end{align*}

Proof. The same way as in Lemma 3.1 of [3].

2. The case of a generalized Siegel domain in $\mathbb{C} \times \mathbb{C}^m$ with exponent $1/2$.

In the following part of the paper, we consider exclusively the generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with $c=1/2$ and $\dim_{\mathbb{R}} g_{-1/2} = 2k$ for some $k$, $0 \leq k \leq m$.

We may assume without loss of generality (by change of linear coordinates if necessary) that $(\sqrt{-1}, 0) \in \mathcal{D}$.

Lemma 1. If $(z, w) \in \mathcal{D}$, then $\text{Im.} \ z > 0$.

Proof. Suppose that there exists a point $(x_0, w_0) \in \mathcal{D}$ such that $\text{Im.} \ x_0 \leq 0$. Since $\mathcal{D}$ is a domain in $\mathbb{C} \times \mathbb{C}^m$ and $(\sqrt{-1}, 0) \in \mathcal{D}$, there exists a continuous path $\phi: [0, 1] \rightarrow \mathcal{D}$ such that $\phi(0) = (x_0, w_0)$ and $\phi(1) = (\sqrt{-1}, 0)$. Put $\phi(t) = (z(t), w(t))$ for $t \in [0, 1]$. Then there exists a point $t_0 \in [0, 1]$ such that $\text{Im.} \ z(t_0) = 0$ by our assumption. Obviously this shows that the point $(0, w(t_0))$ belongs to $\mathcal{D}$. Hence we see that $\mathcal{D}$ contains a point of the form $(0, w_i), w_i \neq 0$, since $\mathcal{D}$ is open. Then, by definition, $\mathcal{D}$ also contains the set $\{(0, e^{1/2 \theta} \sqrt{-1} w_1) \mid t, \theta \in \mathbb{R} \}$, which is naturally identified with $\mathbb{C} - \{0\}$. Thus there exists an injective holomorphic mapping $\Psi: \mathbb{C} - \{0\} \rightarrow \text{a bounded subset of } \mathbb{C}^{m+1}$, because $\mathcal{D}$ is equivalent to a bounded domain in $\mathbb{C}^{m+1}$. Let $\Psi(z) = (f_1(z), \ldots, f_{m+1}(z))$. Then each $f_i$ is a bounded holomorphic function defined on $\mathbb{C} - \{0\}$. Hence, by the Riemann's extension theorem, $f_i$ extends to a bounded holomorphic function on $\mathbb{C}$ and so it is constant. In particular $\Psi$ is a constant mapping. Obviously this is a contradiction.

q.e.d.

In order to prove Theorem 1 we shall consider first the case where $\dim_{\mathbb{R}} g_{-1/2} = 2k > 0$, i.e., $k \geq 1$, in the following.

By Theorem A, we can write each vector field $X \in g_{-1/2}$ as follows:
where \( b_a(X) \) and \( c^\beta(X) \) are complex numbers depending on \( X \). We define a linear mapping \( C: g_{-1/2} \rightarrow C^m \) by \( C(X)=(c^1(X), \ldots, c^m(X)) \). Then we have

\[
(2.1) \quad C: g_{-1/2} \rightarrow C^m \text{ is injective.}
\]

In fact, if \( C(X)=0 \), then it follows from (1.1) that \( \sqrt{-1}X \in g(\mathcal{D}) \). By a theorem of E. Cartan [1], we have that \( g(\mathcal{D}) \cap \sqrt{-1}g(\mathcal{D})=0 \) and hence \( X=0 \).

Since \( \dim_R g_{-1/2}=2k \) by our assumption, the image \( V=\{C(X)|X \in g_{-1/2}\} \) of \( C \) is a complex \( k \)-dimensional vector subspace of \( C^m \) by (1.1) and (2.1). Fix a non-singular linear mapping \( \mathcal{L}^1: C^m \rightarrow C^m \) such that

\[
\mathcal{L}^1(V) = \{(d_1, \ldots, d_k, 0, \ldots, 0) \in C^m | d = (d_i) \in C^k \}.
\]

**Lemma 2.** There exists a non-singular linear mapping \( \mathcal{L}^2: C \times C^m \rightarrow C \times C^m \) of the form \( 2=z, \tilde{w}_a=\sum_{\beta=1}^m A_{a\beta}w_\beta \) (\( 1 \leq \alpha \leq m \)) such that

\[
\mathcal{L}^2_\beta g_{-1/2} = \left\{ \sum_{\alpha=1}^m a_{a\alpha}(X)w_a \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta(X) \frac{\partial}{\partial \tilde{w}_\beta} \big| (d^\beta(X)) \in C^k \right\}
\]

where \( \mathcal{L}^2_\beta \) denotes the differential of \( \mathcal{L}^2 \).

Proof. Let \( C: g_{-1/2} \rightarrow C^m \) and \( \mathcal{L}^1: C^m \rightarrow C^m \) be the same mappings as before. Then, for

\[
X = \left( \sum_{\alpha=1}^m b_a(X)w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta(X) \frac{\partial}{\partial \tilde{w}_\beta} \in g_{-1/2},
\]

we have \( \mathcal{L}^1(C(X))=(d^1(X), \ldots, d^k(X), 0, \ldots, 0) \) for some \( d^\beta(X) \in C(1 \leq \beta \leq k) \). Let \( (1 \oplus \mathcal{L}^1)(z, w)=(z, \mathcal{L}^1(w)) \). If we put \( \mathcal{L}^2=1 \oplus \mathcal{L}^1 \), then \( \mathcal{L}^2 \) satisfies our claim. q.e.d.

Let \( \mathcal{D} \) be the image of \( \mathcal{D} \) under the mapping \( \mathcal{L}^2 \) given in Lemma 2. Then it is easy to see that \( \mathcal{D} \) is also a generalized Siegel domain in \( C \times C^m \) with exponent 1/2 and the Lie algebra \( g(\mathcal{D}) \) coincides with \( \mathcal{L}^2 g(\mathcal{D}) \). Put \( \tilde{\partial}=-\frac{z}{2} \frac{\partial}{\partial z} + \frac{1}{2} \sum_{\alpha=1}^m \tilde{w}_a \frac{\partial}{\tilde{w}_a} \). Then \( \mathcal{L}^2 \tilde{\partial}=\tilde{\partial} \). Thus it follows from Theorem A that \( \mathcal{L}^2 \tilde{g}_\lambda=\tilde{g}_\lambda \), where \( \tilde{g}_\lambda=\{X \in g(\mathcal{D})| [\tilde{\partial}, X]=\lambda X \} \). In particular we have

\[
\tilde{g}_{-1/2} = \left\{ \left( \sum_{\alpha=1}^m a_{a\alpha}(X)w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta \frac{\partial}{\partial \tilde{w}_\beta} \big| d = (d^\beta) \in C^k \right\}
\]

by Lemma 2, where each \( a_a \) is uniquely determined by \( d=(d^\beta) \). Hence we may assume that

\[
g_{-1/2} = \left\{ \left( \sum_{\alpha=1}^m a_{a\alpha}(X)w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta \frac{\partial}{\partial \tilde{w}_\beta} \big| d = (d^\beta) \in C^k \right\}
\]
to prove Theorem 1, considering $\mathcal{D}$ instead of $\mathcal{D}$ if necessary. Then by using (1.1) and (2.1), we can show that each vector field $X \in \mathfrak{g}_{-1/2}$ is of the following form:

$$X = \left( \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial \omega_{\beta}}$$

where $e^{\theta}(X)$ is a complex number depending on $X$ and $a_{\alpha \beta}$ is a complex number depending only on $\mathfrak{g}_{-1/2}$ and hence $\mathcal{D}$ (cf.Vey [9], Lemme 5.1). Thus we get

$$\mathfrak{g}_{-1/2} = \left\{ \left( \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial \omega_{\beta}} \mid (c^{\beta}) \in \mathbb{C}^{k} \right\} .$$

**Lemma 3.** The matrix $(a_{\alpha \beta})_{1 \leq \alpha, \beta \leq k}$ in (2.2) is non-singular skew-hermitian.

**Proof.** Let $X = \left( \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial \omega_{\beta}} \in \mathfrak{g}_{-1/2}.$

Then, by (1.1) we get

$$[\theta', X] = \sqrt{-1} \left( \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) w_{\alpha} \right) \frac{\partial}{\partial z} - \sqrt{-1} \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial \omega_{\beta}} .$$

Put $Y = [\theta', X]$. By a direct calculation we get

$$[X, Y] = 2\sqrt{-1} \left( \sum_{\alpha, \beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) e^{\theta}(X) \right) \frac{\partial}{\partial z} .$$

Since $[X, Y] \in \mathfrak{g}_{-1}$, we see that the number $\sum_{\alpha, \beta=1}^{k} a_{\alpha \beta} e^{\theta}(X) c^{\beta}(X)$ is pure imaginary by (1) of Theorem A. Hence $\sum_{\alpha, \beta=1}^{k} (a_{\alpha \beta} + a_{\beta \alpha}) e^{\theta}(X) c^{\beta}(X) = 0$. On the other hand, since the set $\{C(X) = (e^{\theta}(X)) \mid X \in \mathfrak{g}_{-1/2}\}$ is a complex $k$-dimensional vector subspace of $\mathbb{C}^{m}$, we get $a_{\alpha \beta} + a_{\beta \alpha} = 0$ for $1 \leq \alpha, \beta \leq k$.

We need some preparations to prove that $(a_{\alpha \beta})_{1 \leq \alpha, \beta \leq k}$ is non-singular. We identify the Lie algebra $\mathfrak{a} = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_{0}$ with the subalgebra of $\mathfrak{gl}(m+2, \mathbb{C})$ as in §1. Thus we can represent the vector field $X \in \mathfrak{g}_{-1/2}$ by the following matrix:
Therefore the global one-parameter subgroup \( \exp tX \) generated by \( X \) is given by

\[
\begin{pmatrix}
1 & t \sum_{\beta=1}^{n} a_{\alpha\beta} \bar{c}^\beta(X), & \cdots, & t \sum_{\beta=1}^{n} a_{m\beta} \bar{c}^\beta(X) \\
0 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1_m \\
\end{pmatrix}
\]

Thus the action of \( \exp tX \) on \( \mathcal{D} \) is given by

\[
\begin{align*}
&z \mapsto z + t \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \bar{c}^\beta(X)w_\alpha + \frac{t^2}{2} \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta} c^\alpha(X) \bar{c}^\beta(X) \\
&w_\alpha \mapsto w_\alpha + tc^\alpha(X), \quad 1 \leq \alpha \leq k \\
&w_\beta \mapsto w_\beta, \quad k+1 \leq \beta \leq m.
\end{align*}
\]

(2.3)

Now we can prove that \( (\Lambda^\alpha, \Lambda^\beta)_{\alpha, \beta} \in \mathbb{F} \) is non-singular. Since \( (a_{\alpha\beta})_{\alpha, \beta} \) is skew-hermitian, it is enough to show that

\[
(2.4) \quad \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta} c^\alpha \bar{c}^\beta \neq 0
\]

for any nonzero vector \( c=(c^\alpha) \in \mathbb{C}^k \).

Suppose that there exists a nonzero vector \( c_0=(c_0^\alpha, \cdots, c_0^k) \) such that \( \sum_{\alpha, \beta=1}^{n} a_{\alpha\beta} c_0^\alpha \bar{c}_0^\beta = 0 \). Then the vector field

\[
X_{c_0} = \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \bar{c}_0^\beta w_\alpha \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{n} c_0^\beta \frac{\partial}{\partial w_\beta}
\]

belonging to \( g_{-1/2} \) generates the global one-parameter subgroup \( \exp X_{c_0} \) which acts on \( \mathcal{D} \) by

\[
\begin{align*}
&z \mapsto z + t \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \bar{c}_0^\beta w_\alpha \\
&w_\alpha \mapsto w_\alpha + tc_0^\alpha, \quad 1 \leq \alpha \leq k \\
&w_\beta \mapsto w_\beta, \quad k+1 \leq \beta \leq m.
\end{align*}
\]

Thus \( \exp X_{c_0} \cdot (\sqrt{-1}, 0) = (\sqrt{-1}, tc_0^\alpha, \cdots, tc_0^k, 0, \cdots, 0) \). Hence \( \mathcal{D} \) must contain the set \( \{ (\sqrt{-1}, e^{i\theta} tc_0^1, \cdots, e^{i\theta} tc_0^k, 0, \cdots, 0) | t, \theta \in \mathbb{R} \} \), which is identified with the complex plane \( \mathbb{C} \) since \( c_0 \neq 0 \) by our assumption. But this is a contradiction, because \( \mathcal{D} \) is holomorphically equivalent to a bounded domain in \( \mathbb{C}^{m+1} \). q.e.d.
Lemma 4. There exists a non-singular linear mapping \( \mathcal{L}^3 : \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m \) of the form

\[
\begin{align*}
&(*) \quad z = z, \quad \bar{w}_a = \sum_{\beta=1}^m B_{a\beta} \bar{w}_\beta (1 \leq \alpha \leq m), \text{ such that} \\
&\mathcal{L}^3 \bar{w}_{-1/2} = \left\{ \left( \sum_{a=1}^k d_{a\beta} \bar{c}^\beta w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta \frac{\partial}{\partial \bar{w}_\beta} \right\} \left( c^\alpha \in \mathbb{C}^k \right)
\end{align*}
\]

where \((d_{a\beta})_{1 \leq a, \beta \leq k}\) is a non-singular skew-hermitian matrix.

Proof. Let \( \mathcal{L}^3 : \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m \) be a non-singular linear mapping defined by (*). Then, by a simple calculation, we have \( \mathcal{L}^3 \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \) and \( \mathcal{L}^3 \frac{\partial}{\partial \bar{w}_a} = \sum_{\beta=1}^m B_{a\beta} \frac{\partial}{\partial \bar{w}_\beta} (1 \leq \alpha \leq m) \). Put \( B = (B_{a\beta})_{1 \leq a, \beta \leq m} \). Let \( E = (E_{a\beta}) = B^{-1} \). Take a vector field

\[
X = \left( \sum_{\beta=1}^m \sum_{\alpha=1}^k a_{a\beta} \bar{c}^\beta (X \bar{w}_a) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta (X) \frac{\partial}{\partial \bar{w}_\beta} \right)
\]

belonging to \( g_{-1/2} \). Then we have

\[
\mathcal{L}^3 \bar{w}_{-1/2} = \left\{ \left( \sum_{\alpha=1}^k \sum_{\beta=1}^m a_{a\beta} \bar{c}^\beta (X \bar{w}_a) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta (X) B_{a\beta} \frac{\partial}{\partial \bar{w}_\alpha} \right) \right\}.
\]

Now we have to find out the matrix \( B \) which satisfies the following conditions:

\[
(2.5) \quad \sum_{\alpha=1}^k \sum_{\beta=1}^m a_{a\beta} \bar{c}^\beta (X) E_{a\lambda} = 0 \quad \text{for all } \lambda, \quad k+1 \leq \lambda \leq m;
\]

\[
(2.6) \quad \sum_{\beta=1}^m c^\beta (X) B_{\lambda \bar{\beta}} = 0 \quad \text{for all } \lambda, \quad k+1 \leq \lambda \leq m.
\]

Since \( \{C(X) = (c^\beta (X)) \mid X \in g_{-1/2}\} = \mathbb{C}^k \), the conditions are equivalent to the following

\[
(2.5)' \quad \begin{pmatrix} a_{i1} & \cdots & a_{ik} \\ \vdots & \ddots & \vdots \\ a_{1k} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} E_{1,1} & \cdots & E_{1,k+1} \\ \vdots & \ddots & \vdots \\ E_{k,1} & \cdots & E_{kk} \end{pmatrix} = 0_{k, m-k}
\]

\[
(2.6)' \quad \begin{pmatrix} B_{k+1,1} & \cdots & B_{k+1,k} \\ \vdots & \ddots & \vdots \\ B_{m,1} & \cdots & B_{m,k} \end{pmatrix} = 0_{m-k, k}.
\]

Put \( A_1 = (a_{i1})_{1 \leq i, j \leq k} \), \( A_2 = (a_{i1})_{k+1 \leq i, j \leq m, 1 \leq k \leq k} \), \( E_1 = (E_{ij})_{1 \leq i \leq k, k+1 \leq j \leq m} \) and \( E_2 = (E_{ij})_{k+1 \leq i \leq m, 1 \leq j \leq m} \). Then, (2.5)' can be written as \( 'A_1 E_1 + 'A_2 E_2 = 0_{k, m-k} \). Since the matrix \( A_1 \) is non-singular by Lemma 3, we have

\[
(2.5)'' \quad E_1 = - 'A_1^{-1} 'A_2 E_2.
\]
Now we define a mapping $\mathcal{L}^3: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m$ by

$$
\mathcal{L}^3: \begin{pmatrix} z \\ \bar{w}_1 \\ \vdots \\ \bar{w}_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -iA_i^{-1}A_j \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1_{m-k} \end{pmatrix} \begin{pmatrix} z \\ w_1 \\ \vdots \\ w_m \end{pmatrix}.
$$

Then $\mathcal{L}^3$ satisfies the conditions (2.5)$''$ and (2.6)$'$ and hence we have proved Lemma 4. q.e.d.

Now by Lemma 4, we may assume to prove Theorem 1 that

$$
\mathfrak{g}_{-1/2} = \left\{ \left( \sum_{a=1}^{k} d_a e^a w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{m} e^{\beta} \frac{\partial}{\partial w_{\beta}} \left| (c^\beta) \in C^k \right. \right\}.
$$

**Lemma 5.** There exists a non-singular linear mapping $\mathcal{L}^4: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m$ of the form

$$
z = z, \quad \bar{w}_a = \sum_{\lambda=1}^{k} c_{a\lambda} w_{\lambda} \quad (1 \leq \alpha \leq k) \quad \text{and} \quad \bar{w}_b = w_{b} \quad (k+1 \leq \beta \leq m)
$$

such that

$$
\mathcal{L}^4 |_{\mathfrak{g}_{-1/2}} = \left\{ \left( \sum_{a=1}^{k} d_a e^a w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{m} e^{\beta} \frac{\partial}{\partial w_{\beta}} \left| (c^\beta) \in C^k \right. \right\}
$$

where each $d_a$ is a nonzero purely imaginary number depending only on $D$.

**Proof.** By Lemma 4, the matrix $D=(d_{ab})_{1 \leq a, b \leq k}$ in (2.7) is non-singular and skew-hermitian. Hence $D$ can be diagonalized by a suitable unitary matrix $U=(u_{ab})_{1 \leq a, b \leq k}$. Put $U^{-1}D U = \text{diag.} (d_1, \ldots, d_k)$, where $\text{diag.} (d_1, \ldots, d_k)$ denotes the diagonal matrix whose $(i, i)$-component is $d_i$. Then, since $D$ is non-singular and skew-hermitian, each $d_i$ is a nonzero purely imaginary number. Now define a non-singular linear mapping $\mathcal{L}^4: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m$ by $z = z, \quad \bar{w}_a = \sum_{\lambda=1}^{k} u_{a\lambda} w_{\lambda} \quad (1 \leq \alpha \leq k)$ and $\bar{w}_b = w_{b} \quad (k+1 \leq \beta \leq m)$.

Then it is easy to see that the mapping $\mathcal{L}^4$ satisfies our conditions. q.e.d.

**Proof of Theorem 1:** Suppose first $\text{dim}_R \mathfrak{g}_{-1/2} = 2k > 0$. By Lemma 5 we may assume that

$$
\mathfrak{g}_{-1/2} = \left\{ \left( \sum_{a=1}^{k} d_a e^a w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{m} e^{\beta} \frac{\partial}{\partial w_{\beta}} \left| (c^\beta) \in C^k \right. \right\}.
$$

Note that each $d_a$ is a nonzero purely imaginary number. For the sake of simplicity, we denote $(w_1, \ldots, w_k)$ and $(w_{k+1}, \ldots, w_m)$ by $w'$ and $w''$, respectively.

For $a \in R$ (resp. $t \in R$) we denote by $T_a$ (resp. $\Psi_t$) the holomorphic transforma-
tion \((z, w)\rightarrow (z+a, w)\) (resp. \((z, w)\rightarrow(e^{iz}, e^{iz}w)\)) of \(\mathbb{C}^{m+1}\). Now we define a mapping \(\Phi: \mathbb{C}^k \times \mathbb{C}^k \rightarrow \mathbb{C}\) by

\[
\Phi(u, v) = \frac{1}{2\sqrt{-1}} \sum_{a=1}^k d_a u^a v^a \quad \text{for} \quad u = (u^a), v = (v^a) \in \mathbb{C}^k.
\]

Then each vector field belonging to \(g_{-1/2}\) is expressed in the form \(2\sqrt{-1}\Phi(w', c)\)

\[
\frac{\partial}{\partial z} + \sum_{a=1}^k e^a \frac{\partial}{\partial w_a}.
\]

Since this vector field is determined completely by \(c=(c^a) \in \mathbb{C}^k\), we write it by \(X_c\). By (2.3) the vector field \(X_c\) generates the global one-parameter subgroup \(\text{exp}X_c\):

\[
(z, w', w'') \mapsto (z+2\sqrt{-1}\Phi(w', tc)+\sqrt{-1}\Phi(tc, tc), w'+tc, w'').
\]

Now we claim that

\[(2.8) \quad \Phi(c, c) \geq 0 \quad \text{for all} \quad c \in \mathbb{C}^k.
\]

Suppose that there exists a nonzero vector \(c_0 \in \mathbb{C}^k\) such that \(\Phi(c_0, c_0) < 0\). Then, for a point \((x_0, 0) \in \mathbb{D}\), we have

\[
\text{exp}X_{c_0}(x_0, 0) = (x_0 + \sqrt{-1}\Phi(tc_0, tc_0), tc_0, 0)
\]

for any \(t \in \mathbb{R}\). Thus, by Lemma 1, \(\text{Im}x_0 + \Phi(tc_0, tc_0) > 0\) for any \(t \in \mathbb{R}\). This is impossible since \(\Phi(c_0, c_0) < 0\). Therefore we get (2.8). In particular, we see that each number \(\lambda_{a} := d_a / 2\sqrt{-1} (1 \leq \alpha \leq k)\) is positive. Now we define a linear mapping \(L^5: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m\) by \(z = z\), \(w_a = \sqrt{\lambda_a w_a} (1 \leq \alpha \leq k)\) and \(\bar{w}_a = w_a (k+1 \leq \beta \leq m)\). Then it is easy to see that

\[
-L_{(5)g_{-1/2}} = \left\{2\sqrt{-1} \left(\sum_{a=1}^k \bar{c}^a w_a\right) \frac{\partial}{\partial z} + \sum_{a=1}^k c^a \frac{\partial}{\partial w_a} \right| (c^a) \in \mathbb{C}^k\right\}.
\]

Hence, by considering the image \(\mathbb{D} = L^5(\mathbb{D})\) if necessary, we may assume that

\[
g_{-1/2} = \left\{2\sqrt{-1} \left(\sum_{a=1}^k \bar{c}^a w_a\right) \frac{\partial}{\partial z} + \sum_{a=1}^k c^a \frac{\partial}{\partial w_a} \right| (c^a) \in \mathbb{C}^k\right\}.
\]

Define a mapping \(F: \mathbb{C}^k \times \mathbb{C}^k \rightarrow \mathbb{C}\) by

\[
F(u, v) = \sum_{a=1}^k u^a v^a \quad \text{for any} \quad u = (u^a), v = (v^a) \in \mathbb{C}^k.
\]

Then the domain

\[
\mathcal{C} = \{(z, w', 0) \in \mathbb{C} \times \mathbb{C}^m \mid \text{Im}z - F(w', w') > 0\}
\]

is an elementary Siegel domain. Now we put

\[
\mathcal{D}_{\sqrt{-1}} = \{w'' \in \mathbb{C}^{m-k} \mid (\sqrt{-1}, 0, w'') \in \mathbb{D}\}.
\]
We shall show that $\mathcal{D}_{\sqrt{-1}}$ is connected. Take two points $P_0=\left(\sqrt{-1}, 0, w_0''\right)$ and $P_1=\left(\sqrt{-1}, 0, w_1''\right)$ of $\mathcal{D}$. Then there exists a continuous path $\Gamma: [0, 1] \to \mathcal{D}$ such that $\Gamma(0)=P_0$ and $\Gamma(1)=P_1$. For any $t \in [0, 1]$, we put $\Gamma(t)=\left(\gamma(t), w'(t), w''(t)\right)$, where $\gamma(t) \in C$, $w'(t) \in C^k$ and $w''(t) \in C^{m-k}$. Since

$$T_{-Re.\psi(t)} \cdot \exp X_{-w'(t)} \cdot \left(\gamma(t), w'(t), w''(t)\right) = \left(\sqrt{-1}(\text{Im}.z(t)-F(w'(t), w'(t))), 0, w''(t)\right),$$

we see that $\text{Im}.z(t)-F(w'(t), w'(t))>0$ for any $t \in [0, 1]$ by Lemma 1. Thus we can define a continuous function $l(t)$ on $[0,1]$ by $l(t)=\log(\text{Im}.z(t)-F(w'(t), w'(t)))$. Then it is obvious that $l(0)=l(1)=0$ and $e^{l(t)}=\text{Im}.z(t)-F(w'(t), w'(t))$ for any $t \in [0, 1]$. Thus the point

$$\left(\sqrt{-1}, 0, e^{-1/2l(t)}w''(t)\right) = \left(e^{-l(t)}x(t), \sqrt{-1}, 0, e^{-1/2l(t)}w''(t)\right)$$

belongs to $\mathcal{D}$ by the definition of $\mathcal{D}$. Put $g(t)=e^{-1/2l(t)}w''(t)$. Then $g(t) \in \mathcal{D}_{\sqrt{-1}}$ for any $t \in [0, 1]$, $g(0)=w_0''$ and $g(1)=w_1''$. Thus $\mathcal{D}_{\sqrt{-1}}$ is connected. It is obvious that $\mathcal{D}_{\sqrt{-1}}$ is a circular domain in $C^{m-k}$ containing the origin 0 by the definition of the generalized Siegel domain. Let $(z,w',w'')$ be a point of $\mathcal{D}$. Then there exists a real number $t_0$ such that $e^{t_0}=\text{Im}.z-F(w', w')$, because

$$T_{-Re.\psi(t)} \cdot \exp X_{-w'} \cdot (z, w', w'') = \left(\sqrt{-1}(\text{Im}.z-F(w', w')), 0, w''\right)$$

hence $\text{Im}.z-F(w', w')>0$ by Lemma 1. Thus we have $\Psi_{-t_0} \cdot T_{-Re.\psi} \cdot \exp X_{-w'} \cdot (z, w', w'') = \left(\sqrt{-1}, 0, e^{-t_0/2}w''\right)$. Hence $\left(\text{Im}.z-F(w', w')\right)^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}$, and so $\mathcal{D}$ is contained in the set

$$\{ (z, w', w'') \in C \times C^m \mid \text{Im}.z-F(w', w')>0, (\text{Im}.z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}} \}.$$ 

Conversely, take a point $(z,w',w'') \in C \times C^m$ such that $\text{Im}.z-F(w', w')>0$ and $(\text{Im}.z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}$. Then, by the same way as above, we can show that there exists a real number $t_0$ such that $e^{t_0}=\text{Im}.z-F(w', w')$ and

$$T_{Re.\psi} \cdot \exp X_{w'} \cdot \psi_{t_0} \cdot \left(\sqrt{-1}, 0, e^{-t_0/2}w''\right) = (z, w', w'').$$

This shows that $(z,w',w'') \in \mathcal{D}$, since $\left(\sqrt{-1}, 0, e^{-t_0/2}w''\right) \in \mathcal{D}$ by the definition of $\mathcal{D}_{\sqrt{-1}}$. Therefore

$$\mathcal{D} = \left\{ (z, w', w'') \in C \times C^m \mid \text{Im}.z-F(w', w')>0, \right. \left. (\text{Im}.z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}} \right\}.$$

Now we shall show that the orbit $\mathcal{D}_0$ of $\text{Aut}_0(\mathcal{D})$ containing the point $(\sqrt{-1}, 0) \in \mathcal{D}$ coincides with the elementary Siegel domain $\mathcal{E}$. Let $(z,w',0) \in \mathcal{E}$. Since $\text{Im}.z-F(w', w')>0$, there exists a real number $t_0$ such that $e^{t_0}=\text{Im}.z-F(w', w')$. Then it is easy to see that $T_{Re.\psi} \cdot \exp X_{w'} \cdot \psi_{t_0} \cdot \left(\sqrt{-1}, 0\right) = (z, w', 0)$, and so $\mathcal{E} \subset \text{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0) = \mathcal{D}_0$. We claim that $\mathcal{D}_0 \subset \mathcal{E}$. Let $G$
be the identity component \( \text{Aut}_0(\mathcal{D}) \) of \( \text{Aut}(\mathcal{D}) \), \( K \) the isotropy subgroup of \( G \) at \((\sqrt{-1}, 0)\) and \( G_x \) the identity component of \( \text{Aff}(\mathcal{D}) \). Put \( K_x = G_x \cap K \). Then we can show that \( G/K = G_x/K_x \) by the same way as Lemma 2.3. of Nakajima [5]. Therefore it is enough to see that \( G_x \cdot (\sqrt{-1}, 0) \subseteq \mathcal{E} \). Let \( P(\mathcal{D}) \) (resp. \( \text{GL}_0(\mathcal{D}) \)) be the analytic subgroup of \( G_x \) generated by the subalgebra \( g_{-i} + g_{-1/2} \) (resp. \( g_0 \)). Then we have \( G_x = P(\mathcal{D}) \cdot \text{GL}_0(\mathcal{D}) \) (semi-direct product), because \( P(\mathcal{D}) \cdot \text{GL}_0(\mathcal{D}) \) is an abstract subgroup of \( G_x \) and contains an open neighborhood of the identity element of \( G_x \). Since \( \text{GL}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0) \subseteq \mathcal{D}_1 \) by (5), of Theorem A and obviously \( P(\mathcal{D}) \cdot \mathcal{E} \subseteq \mathcal{E} \), we get \( G_x \cdot (\sqrt{-1}, 0) \subseteq \mathcal{E} \). Therefore \( G \cdot (\sqrt{-1}, 0) = G_x \cdot (\sqrt{-1}, 0) = \mathcal{E} \). This completes the first case where \( k > 0 \).

It remains the case where \( \dim_{\mathbb{R}} g_{-1/2} = 0 \), i.e., \( k = 0 \). But in this case Theorem 1 is now obvious from the proof of the case where \( k > 0 \). q.e.d.

Corollaries of Theorem 1: As an immediate consequence of Theorem 1 we obtain the following corollary.

**Corollary 1.** Let \( \mathcal{D} \) be a generalized Siegel domain in \( C \times C^m \) with exponent \( 1/2 \) and \( \dim_{\mathbb{R}} g_{-1/2} = 2m \). Then \( \mathcal{D} \) is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain

\[
\mathcal{E} = \{(z, w_1, \ldots, w_m) \in C \times C^m \mid \text{Im.} z - \sum_{a=1}^{2m} |w_a|^2 > 0 \}. 
\]

**Corollary 2.** There exists no generalized Siegel domain in \( C \times C^m \) with exponent \( 1/2 \) such that \( \dim_{\mathbb{R}} g_{-1/2} = 2m - 2 \).

**Proof.** Suppose that there exists a generalized Siegel domain \( \mathcal{D} \) in \( C \times C^m \) with exponent \( 1/2 \) and \( \dim_{\mathbb{R}} g_{-1/2} = 2m - 2 \). Then, by Theorem 1 there exists a generalized Siegel domain \( \mathcal{D} \) with exponent \( 1/2 \) which is holomorphically equivalent to \( \mathcal{D} \) and is expressed in the following form with respect to a suitable coordinates system \( (z, w_1, \ldots, w_m) \) in \( C \times C^m \):

\[
\mathcal{D} = \{(z, w_1, \ldots, w_m) \in C \times C^m \mid \text{Im.} z - \sum_{a=1}^{2m} |w_a|^2 > 0 \},
\]

\[
(\text{Im.} z - \sum_{a=1}^{2m} |w_a|^2)^{-1/2} \cdot w_m \in \mathcal{D}_{\sqrt{\sqrt{-1}}}
\]

where \( \mathcal{D}_{\sqrt{\sqrt{-1}}} \) is a circular domain in \( C \) containing the origin of \( C \). Since \( \mathcal{D}_{\sqrt{\sqrt{-1}}} \) is given by \( \mathcal{D}_{\sqrt{\sqrt{-1}}} = \{w_m \in C \mid |w_m| < R \} \) for some positive number \( R \),

\[
\mathcal{D} = \{(z, w_1, \ldots, w_m) \in C \times C^m \mid \text{Im.} z - \sum_{a=1}^{2m} |w_a|^2 + R^{-2} |w_m|^2 > 0 \}.
\]

Thus \( \mathcal{D} \) is a Siegel domain of the second kind in \( C \times C^m \). Then we see that \( \dim_{\mathbb{R}} g_{-1/2} = 2m \) in the decomposition of \( g(\mathcal{D}) \) as in Theorem A. But this is a contradiction since \( \dim_{\mathbb{R}} g_{-1/2} = 2m - 2 \) by our assumption. q.e.d.
Corollary 3. Let $\tilde{\Omega}$ and $\tilde{\Omega}_0$ be the same domains as in Theorem 1 and $g(\tilde{\Omega}) \rightarrow g(\tilde{\Omega}_0)$ the homomorphism induced by the Lie group homomorphism of $\text{Aut}_0(\tilde{\Omega})$ to $\text{Aut}_0(\tilde{\Omega}_0)$ defined by $g \mapsto g|\tilde{\Omega}_0$, where $g|\tilde{\Omega}_0$ denotes the restriction of $g$ to $\tilde{\Omega}_0$. Then $\Pi$ is surjective.

Proof. Note that $\tilde{\Omega}_0$ is the $\text{Aut}_0(\tilde{\Omega})$-orbit. Let $(z, w_1, \cdots, w_m)$ be the coordinates system in $\mathbb{C} \times \mathbb{C}^m$ as in Theorem 1. Let $g(\tilde{\Omega}) = g - \frac{1}{2} + g_0 + g_{1/2} + g_i$ (resp. $g(\tilde{\Omega}_0) = g_{e^{-\frac{1}{2}} + g_{1/2} + g_0 + g_i}$) be the decomposition of $g(\tilde{\Omega})$ (resp. $g(\tilde{\Omega}_0)$) as in Theorem A. Since $\tilde{\Omega}_0$ is an elementary Siegel domain, $g(\tilde{\Omega}_0)$ is simple. In particular, we have

\[ g^e = [g_{-\frac{1}{2}}, [g_{1/2}, g_i]] \text{ and } g_{1/2} = [g_{-1}, g_i]. \]

Put $\partial^e = z \frac{\partial}{\partial z} + \frac{1}{2} \sum_{a=1}^m w_a \frac{\partial}{\partial w_a}$. Then it is obvious that $\Pi(\partial^e) = \partial^e$. Hence the homomorphism $\Pi$ preserves the gradition, i.e., $\Pi(g_0) \subset g^e$. Now we shall show that $\Pi$ is injective. Let $g - \frac{1}{2} + g_0 + g_{1/2} + g_i$ and $g_{e^{-\frac{1}{2}} + g_{1/2} + g_0 + g_i}$ be simple. In particular, we have

\[ g^e = [g_{-\frac{1}{2}}, [g_{1/2}, g_i]] \text{ and } g_{1/2} = [g_{-1}, g_i]. \]

Therefore $\Pi$ is injective on $g_{-1}$. Analogously we can show that $\Pi$ is injective on $g_{1/2}$ by using the injectivity of $\text{ad}(\frac{\partial}{\partial z}) : g_{1/2} \rightarrow g_{-1/2}$. Note that the subalgebra $g_{-1} + g_0 + g_i$ corresponds to the subgroup leaving the upper half plane $\Omega_1=\{(z, 0) \in \mathbb{C} \times \mathbb{C}^m | \text{Im} z > 0\}$ invariant. Now we claim that each element of $\text{Aut}_0(\Omega_1)$ can be extended to an element of $\text{Aut}_0(\tilde{\Omega})$. We identify $\text{Aut}_0(\Omega_1)$ with $SL(2, \mathbb{R})/\{\pm 1_2\}$. Since each element $\gamma = (a \ b \ c \ d) \in SL(2, \mathbb{R})$ acts on $\Omega_1$ by a holomorphic transformation $l_\gamma : z \mapsto (az + b)(cz + d)^{-1}$, we can define a mapping $\tilde{l}_\gamma : \Omega_1 \times \mathbb{C}^m \rightarrow \Omega_1 \times \mathbb{C}^m$ by $\tilde{l}_\gamma(x, w) = (l_\gamma(x), (cz + d)^{-1}w)$.

(2.10) $\Pi(\Omega_1) \subset \tilde{\Omega}.$

Put $w = (w_1, \cdots, w_k)$, $w' = (w_{k+1}, \cdots, w_m)$ and $||w'|| = (\sum_{a=1}^m |w_a|^2)^{1/2}$ for any $w = (w_1, \cdots, w_m) \in \mathbb{C}^m$. Then

(2.11) $\text{Im} l_\gamma(x) - ||(cz + d)^{-1}w'||^2 = |cz + d|^2 (\text{Im} x - ||w'||^2) > 0$
for any \((z, w', w') \in \mathcal{D}\). Since

\[
\text{Im. } l_{\gamma}(z) - ||(cz+d)^{-1}w'||^2 - 1 \cdot w'' = e^{\theta(z, \gamma)}(\text{Im. } z - ||w'||^2 - 1 \cdot w'')
\]

where \(\theta(z, \gamma) = -\arg(z^2 + d)\), and 

\[
e^{\theta(z, \gamma)}(\text{Im. } z - ||w'||^2 - 1 \cdot w'') \in \mathcal{D} \setminus \mathcal{I} \text{, we have}
\]

\[
(2.12) \quad \text{Im. } l_{\gamma}(z) - ||(cz+d)^{-1}w'||^2 - 1 \cdot w'' \in \mathcal{I} \setminus \mathcal{I}.
\]

By (2) of Theorem 1, (2.11) and (2.12) imply (2.10). Hence we get \(g_0 = 0\) and hence \(\Pi(g_0) = 0\). We now prove that \(\Pi\) is surjective. Since \(\dim \mathfrak{g} = 1\) and \(\Pi(g_0) = 0\), we get \(\Pi(g_0) = \mathfrak{g}^0\). Therefore it follows that \(g_0^0 = [g_0^1, g_0^2] = \Pi([g_0^1, g_0^2]) \subseteq \Pi(g_0^0)\), and so \(\Pi(g_0^0) = g_0^0\). Then \(g_0^0 = [g_0^1, g_0^2] + [g_0^2, g_0^3] = \Pi([g_0^1, g_0^2] + [g_0^2, g_0^3]) \subseteq \Pi(g_0)\), and so \(\Pi(g_0) = g_0^0\). Therefore \(\Pi\) is surjective.

**q.e.d.**

**Corollary 4.** Let \(\mathcal{D}\) be a generalized Siegel domain in \(C^m\times C^m\) with exponent 1/2. If the Lie algebra \(g(\mathcal{D})\) is semi-simple, then \(\mathcal{D}\) is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain

\[
\mathcal{E} = \{(z, w_1, \ldots, w_m) \in C^m \times C^m | \text{Im. } z - \sum_{a=1}^{m} |w_a|^2 > 0\}.
\]

**Proof.** We claim that \(\dim_R \mathfrak{g} = 2m\), i.e., \(k = m\). Then our assertion is obvious by Corollary 1. We may assume \(\mathcal{D} = \mathcal{D}_0\) in Theorem 1 without loss of generality. Suppose that \(k \geq m\). We consider first the case where \(k > 0\). Let \(\Pi: g(\mathcal{D}) \rightarrow (\mathcal{D}_0)\) be the homomorphism defined in Corollary 3. Then \(\Pi\) is surjective by Corollary 3. Put \(\mathfrak{s}_0 = \Pi(g_0)\). Then \(\mathfrak{s}_0\) is a semi-simple ideal of the semi-simple Lie algebra \(g(\mathcal{D})\). Thus there exists a semi-simple ideal \(\mathfrak{s}_1\) such that \(g(\mathcal{D}) = \mathfrak{s}_0 + \mathfrak{s}_1\) (direct sum). Since \(\mathfrak{s}_0\) is isomorphic to \(g(\mathcal{D})\), \(\mathfrak{s}_1\) is simple. Since \(\Pi\) is injective on \(g_1 \oplus g_1 \oplus g_1 \oplus g_1\) by the proof of Corollary 3, \(\mathfrak{s}_1\) is contained in \(g_0\). Let \(B\) denote the Killing form of \(g(\mathcal{D})\). Put \(g_0^0 = \{X \in g_0 | B(X, \mathfrak{s}_0) = 0\}\). Noting that the ideal \(\mathfrak{s}_1\) is a graded Lie subalgebra, it is easy to see that \(g_0 = g_1^0 + \mathfrak{s}_1\) and \(g_0 = g_1 + g_0 + g_0 + g_1\) and \(g_0 = [g_1, g_1 + g_1]\). Since \(\mathfrak{s}_2 = \Pi(\mathfrak{s}_2) \subset g_0\), every vector field \(X \in \mathfrak{s}_2\) is given by \(X = \sum_{a=1}^{m} O_{a, 1} \frac{\partial}{\partial w_a}\) in Theorem A. Thus it can be expressed by the matrix

\[
(2.13) \quad X = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & C \\
0 & 0 & 0
\end{pmatrix}.
\]

Now we claim that \(C = 0_{k, m - k}\) in (2.13). Let \(S_1\) (resp. \(S_2\)) be the analytic sub-
group of Aut(\(\tilde{D}\)) corresponding to \(\mathfrak{s}_1\) (resp. \(\mathfrak{s}_2\)). Obviously

\[
g_1 \cdot g_2 = g_2 \cdot g_1 \quad \text{for any } g_1 \in S_2 \text{ and } g_2 \in S_2.
\]

Let \(X_c(c \in C^k)\) be the vector field belonging to \(g_{-1/2}\) defined in the proof of Theorem 1. Put \(g_1 = \exp X_c\) and

\[
g_2 = \exp X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_k & A \\ 0 & 0 & E \end{pmatrix}.
\]

It is easy to see that if \(A = 0\), then \(C = 0\). By a routine calculation, we get

\[
g_1 \cdot g_2 \cdot (z, w', w'') = (z + 2\sqrt{-1}F(w' + Aw'', c) + \sqrt{-1}F(c, c), w' + Aw'' + c, Ew'')
\]

and

\[
g_2 \cdot g_1 (z, w', w'') = (z + 2\sqrt{-1}F(w', c) + \sqrt{-1}F(c, c), w' + c + Aw'', Ew'')
\]

for any \((z, w', w'') \in \tilde{D}\). By (2.14), we get \(F(w' + Aw'', c) = F(w', c)\) and hence \(F(Aw'', c) = 0\). Since \(c\) is arbitrary, we get \(Aw'' = 0\) for any element \(w''\) of an open subset of \(C^{m-k}\). Thus \(A = 0\). Therefore we get

\[
\mathfrak{s}_2 = \left\{ \begin{pmatrix} 0_{k+1, k+1} & 0 \\ 0 & * \end{pmatrix} \right\} \quad \text{and} \quad S_2 = \left\{ \begin{pmatrix} 1_{k+1} & 0 \\ 0 & * \end{pmatrix} \right\}.
\]

Since \(\tilde{D}\) is holomorphically equivalent to a bounded domain in \(C^{m+1}\) and any bounded domain in \(C^{m+1}\) is hyperbolic in the sense of Kobayashi [4], \(\tilde{D}\) is hyperbolic. Since \(\tilde{D}_\sqrt{-1}\) is a complex submanifold of \(\tilde{D}\), it is also hyperbolic. Thus \(\tilde{D}_\sqrt{-1}\) is a hyperbolic circular domain in \(C^{m-k}\) containing the origin 0. By §.1, we have that Aut(\(\tilde{D}_\sqrt{-1}\)) is a Lie group and its isotropy subgroup \(K_{\sqrt{-1}}\) at \(0 \in \tilde{D}_\sqrt{-1}\) is compact. Moreover \(K_{\sqrt{-1}}\) is a subgroup of GL\((m-k, C)\) by Theorem B. Let \(\mathfrak{f}_\sqrt{-1}\) be the subalgebra of \(g(\tilde{D}_\sqrt{-1})\) corresponding to \(K_{\sqrt{-1}}\). Now we claim that \(\mathfrak{f}_\sqrt{-1}\) can be identified with \(\mathfrak{s}_2\). By (2.15) we can identify \(S_2\) with a subgroup of \(K_{\sqrt{-1}}\). Conversely, let \(K^0_{\sqrt{-1}}\) be the identity component of \(K_{\sqrt{-1}}\) and take an arbitrary element \(g \in K^0_{\sqrt{-1}}\). Put \(\check{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\), where \(1 = 1_{k+1}\). Then we can easily see that \(\check{g}\) leaves \(\tilde{D}\) invariant by (2) of Theorem 1, and hence \(\check{g}\) defines a holomorphic transformation of \(\tilde{D}\) and \(\check{g} \in S_2\) by (2.15). Thus \(K^0_{\sqrt{-1}}\) can be identified with \(S_2\) in a natural way. In particular, \(\mathfrak{f}_\sqrt{-1}\) is a semi-simple Lie algebra. On the other hand, \(\mathfrak{f}_\sqrt{-1}\) contains a nonzero element \(\partial'' = \sqrt{-1} \sum_{a \in \mathbb{Z}^k} w_a \frac{\partial}{\partial w_a}\) induced by the global one-parameter subgroup \(w' \mapsto e^{-t/2}w'\) \((t \in \mathbb{R})\) and obviously \(\partial''\) belongs to the center of \(\mathfrak{f}_\sqrt{-1}\). This is a contradiction.
Suppose next $k=0$. Then we can show as above that the Lie algebra $\mathfrak{f}_{-1}$ is identified with the semi-simple Lie algebra

$$\text{Ker } \Pi = \left\{ \begin{pmatrix} 0 & 0_{1,m} \\ 0_{m,1} & * \end{pmatrix} \right\}.$$ 

On the other hand, $\mathfrak{f}_{-1}$ contains a nonzero element $\partial' = \sqrt{-1} \sum_{a=1}^{n} \frac{\partial}{\partial w_a}$ belonging to the center. This is a contradiction. Therefore $k=m$, and we complete the proof. q.e.d.

3. The structure of Aut ($\mathcal{D}$)

The purpose of this section is to consider the structure of the group of all holomorphic transformations of a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent $1/2$ and $\dim \mathcal{D} = 2k$ for some $0 \leq k \leq m$.

In this section we use the following notations. For a point $\delta = (\xi, \eta) \subset \mathcal{D}$, define $||\delta|| = (\sum_{j=1}^{k+1} |z_j|^2)^{1/2}$.

Put

$$U(k+1, 1) = \left\{ g \in GL(k+2, \mathbb{C}) \mid g \cdot \begin{pmatrix} 1_{k+1} & 0 \\ 0 & -1 \end{pmatrix} \cdot g = \begin{pmatrix} 1_{k+1} & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and

$$SU(k+1, 1) = U(k+1, 1) \cap SL(k+2, \mathbb{C}).$$

For each element $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1, 1)$, where $A = (a_{ij})_{1 \leq i,j \leq k+1}$, $b = (b_1, \ldots, b_{k+1})$ and $c = (c_1, \ldots, c_{k+1})$, we put

\begin{align*}
L_j(\gamma) &= (a_{j1} + b_j, 2a_{j2}, 2a_{j3}, \ldots, 2a_{jk+1}); \\
C(\gamma) &= (c_1 + d, 2c_2, 2c_3, \ldots, 2c_{k+1}); \\
B_j(\gamma) &= \sqrt{-1}(b_j - a_{j1}) \quad \text{and} \quad D(\gamma) = \sqrt{-1}(d - c_1)
\end{align*}

for $j = 1, 2, \ldots, k+1$.

It is easy to see that $U(k+1, 1)$ coincides with all matrices $\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in GL(k+2, \mathbb{C})$ of the form $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$, where $\gamma = 1_{k+1}$, $|d|^2 = -1$ and $\gamma = 0_{1,k+1}$. From this, we get

\begin{align*}
|c\delta + d|^2 - ||A\delta + b||^2 = 1 - ||\delta||^2
\end{align*}

for any $\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in U(k+1, 1)$ and any $\delta \in \mathcal{C}^{k+1}$, by an easy computation.
Now we consider the group $\text{Aut}(\mathcal{E})$ of all holomorphic transformations of the elementary Siegel domain

$$\mathcal{E} = \{(z, w_1, \ldots, w_k) \in \mathbb{C} \times \mathbb{C}^k | \text{Im} \cdot z - \sum_{a=1}^{k} |w_a|^2 > 0\}.$$  

The elementary Siegel domain $\mathcal{E}$ is holomorphically equivalent to the unit open ball $\mathcal{B} = \{z = (z^1, \ldots, z^{k+1}) \in \mathbb{C}^{k+1} | ||z|| < 1\}$. In fact, the biholomorphic isomorphism $\phi: \mathcal{E} \to \mathcal{B}$ is given by

$$\phi(z) = (z - \sqrt{-1})(z + \sqrt{-1})^{-1}, \quad \phi^j = 2w_j.(z + \sqrt{-1})^{-1}$$

for $j = 2, 3, \ldots, k+1$. It is well-known that the group $\text{Aut}(\mathcal{B})$ can be identified with the simple Lie group $SU(k+1, 1)$ and each element $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1, 1)$ acts on $\mathcal{B}$ by the holomorphic transformation $\tau_\gamma: z \mapsto (A \cdot z + b)(cz + d)^{-1}$. Define $\Psi_\gamma = \phi^{-1} \cdot \tau_\gamma \cdot \phi$ for each $\gamma \in SU(k+1, 1)$. Then it is obvious that $\Psi_\gamma$ defines a holomorphic transformation of $\mathcal{E}$. By a direct calculation, we see that the action of $\Psi_\gamma$ on $\mathcal{E}$ is given by

$$\begin{cases} 
  z \mapsto \sqrt{-1} \cdot \frac{1 + (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))} \\
  w_j \mapsto \sqrt{-1} \cdot \frac{(C(\gamma)Z + D(\gamma))^{-1} \cdot (L_{j+i}(\gamma)Z + B_{j+i}(\gamma))}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))}
\end{cases}$$

for $j = 1, 2, \ldots, k$, where $Z = (z, w_1, \ldots, w_k) \in \mathcal{E}$ and $C(\gamma), L_i(\gamma), B_i(\gamma), D(\gamma)$ are defined by (3.1).

Let $K^0_{\sqrt{-1}}$ be the identity component of the isotropy subgroup of $\text{Aut}(\mathcal{D}_{\sqrt{-1}})$ at the origin $0 \in \mathcal{D}_{\sqrt{-1}}$. We define a mapping $\Psi_{\gamma, K}: \mathcal{D}_{\sqrt{-1}} \times \mathbb{C}^{m-k} \to \mathcal{D}_{\sqrt{-1}} \times \mathbb{C}^{m-k}$ for each $\gamma \in SU(k+1, 1)$ and $K \in K^0_{\sqrt{-1}}$ as follows:

$$\Psi_{\gamma, K}: \begin{cases} 
  z \mapsto \sqrt{-1} \cdot \frac{1 + (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))} \\
  w_j \mapsto \sqrt{-1} \cdot \frac{(C(\gamma)Z + D(\gamma))^{-1} \cdot (L_{j+i}(\gamma)Z + B_{j+i}(\gamma))}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))}
\end{cases}$$

for $j = 1, 2, \ldots, k$.

$$W \mapsto K \cdot \frac{2\sqrt{-1} \cdot (C(\gamma)Z + D(\gamma))^{-1}}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_i(\gamma)Z + B_i(\gamma))} \cdot W$$

for $Z = (z, w_1, \ldots, w_k) \in \mathcal{D}_0$ and $W = (w_{k+1}, \ldots, w_m) \in \mathbb{C}^{m-k}$. Since $\mathcal{D}_0 = \{(z, w_1, \ldots, w_k, 0, \ldots, 0) \in \mathbb{C} \times \mathbb{C}^m | \text{Im} \cdot z - \sum_{a=1}^{k} |w_a|^2 > 0\} = \mathcal{E}$, $\Psi_{\gamma, K}$ is a well-defined holomorphic mapping of $\mathcal{D}_0 \times \mathbb{C}^{m-k}$ into itself.

Now we can state Theorem 2.
Theorem 2. Let $\Psi_{\gamma,K} : \mathcal{D}_0 \times \mathbb{C}^{m-k} \to \mathcal{D}_0 \times \mathbb{C}^{m-k}$ be the holomorphic mapping defined as above. Then $\Psi_{\gamma,K}$ induces a holomorphic transformation of $\mathcal{D}_0$, and moreover any holomorphic transformation of $\mathcal{D}_0$ belonging to the identity component of $\text{Aut}(\mathcal{D}_0)$ is of this form, i.e.,

$$\text{Aut}_0(\mathcal{D}_0) = \{ \Psi_{\gamma,K} \mid \gamma \in SU(k+1,1), K \in K_0^{0,1} \}.$$ 

Proof. Let $(x, w_1, \cdots, w_m)$ be the coordinates system in $\mathbb{C} \times \mathbb{C}^m$ defined in Theorem 1. We put $w'=(w_1, \cdots, w_k), w''=(w_{k+1}, \cdots, w_m)$ and $||w'||=(\sum_{s=1}^k |w_s|^2)^{1/2}$ as before. First we claim that each element $\Psi_{\gamma}^0 \in \text{Aut}_0(\mathcal{E})=\text{Aut}_0(\mathcal{D}_0)$ can be extended to a holomorphic transformation of $\mathcal{D}_0$. We consider the following mappings:

$$w_s \mapsto \omega_s = \frac{2 \sqrt{-1} (C(\gamma)Z+D(\gamma))^{-1} w_s}{1-(C(\gamma)Z+D(\gamma))^{-1} \cdot (L_s(\gamma)Z+B_s(\gamma))}$$

for $s=k+1, k+2, \cdots, m$. Put $\Psi_{\gamma}^0=(\Psi_{\gamma}^{0,1}, \cdots, \Psi_{\gamma}^{0,k+1})$. We shall show that

$$(3.4) \quad \left((\Psi_{\gamma}^0(Z)), \omega_{k+1}, \cdots, \omega_m\right) \in \mathcal{D}_0$$

for any $(z, w)=(Z, w_{k+1}, \cdots, w_m) \in \mathcal{D}_0$.

Put $(\Psi_{\gamma}^0(Z))_{\omega}=(\Psi_{\gamma}^{0,1}(Z), \cdots, \Psi_{\gamma}^{0,k+1}(Z))$. If we show the following two conditions

$$(3.5) \quad \text{Im. } \Psi_{\gamma}^{0,1}(Z) - \| (\Psi_{\gamma}^0(Z))_{\omega} \|^2 > 0 \quad \text{and}$$

$$(3.6) \quad (\text{Im. } \Psi_{\gamma}^{0,1}(Z) - \| (\Psi_{\gamma}^0(Z))_{\omega} \|^2)^{-1/2} \cdot \omega'' \in \mathcal{D}_0^{0,1},$$

where $\omega''=(\omega_{k+1}, \cdots, \omega_m)$, then (3.4) will follow from Theorem 1. The condition (3.5) is obvious, since $\Psi_{\gamma}^0$ is a holomorphic transformation of $\mathcal{D}_0$. By routine calculations, we get

$$\text{Im. } \Psi_{\gamma}^{0,1}(Z) - \| (\Psi_{\gamma}^0(Z))_{\omega} \|^2$$

$$= \frac{1-\sum_{j=1}^k |(C(\gamma)Z+D(\gamma))^{-1} \cdot (L_j(\gamma)Z+B_j(\gamma))|^2}{1-(C(\gamma)Z+D(\gamma))^{-1} \cdot (L_s(\gamma)Z+B_s(\gamma))} \cdot \omega_s,$$

and hence

$$\text{Im. } \Psi_{\gamma}^{0,1}(Z) - \| (\Psi_{\gamma}^0(Z))_{\omega} \|^2 = \frac{2e^{\sqrt{-1} \theta(Z,\gamma)\omega_s}}{\sqrt{1-(C(\gamma)Z+D(\gamma))^{-1} \cdot (L_s(\gamma)Z+B_s(\gamma))}},$$

where

$$\theta(Z, \gamma) = -\text{arg. } \{1-(C(\gamma)Z+D(\gamma))^{-1} \cdot (L_s(\gamma)Z+B_s(\gamma))\}$$

and

$$= -\text{arg. } (C(\gamma)Z+D(\gamma)) + \pi/2.$$

Let $\phi$ be the biholomorphic isomorphism defined in (3.3) and put $\zeta=\phi(Z) \in \mathcal{B}$. 

Then we get
\[ C(\gamma)Z + D(\gamma) = (z + \sqrt{-1})(c_3 + d) \quad \text{and} \]
\[ \sum_{j=1}^{k+1}(C(\gamma)Z + D(\gamma))^{-1}(L_j(\gamma)Z + B_j(\gamma)) \mid ^2 = \mid (A_3 + b) \cdot (c_3 + d)^{-1} \mid ^2. \]
Hence it follows from (3.2) that
\[ \frac{2w_s}{\mid C(\gamma)Z + D(\gamma) \mid \cdot (1 - \sum_{j=1}^{k+1}(C(\gamma)Z + D(\gamma))^{-1}(L_j(\gamma)Z + B_j(\gamma)) \mid ^2)^{1/2}} \]
\[ = \frac{2w_s}{\mid z + \sqrt{-1} \mid \cdot (1 - \mid \delta \mid ^2)^{1/2}}. \]
Moreover it is easy to check that
\[ 1 - \mid \delta \mid ^2 = 4 \mid z + \sqrt{-1} \mid ^{-2}(\text{Im.}z - \mid w' \mid ^2). \]
Thus we get
\[ (\text{Im.}\Psi_\gamma^0(Z) - \mid (\Psi_\gamma^0(Z))_w \mid ^2)^{-1/2} \cdot \tilde{w}_z = e^{\sqrt{-1}(Z, \gamma)}(\text{Im.}z - \mid w' \mid ^2)^{-1/2} \cdot w_z, \]
and hence
\[ (\text{Im.}\Psi_\gamma^0(Z) - \mid (\Psi_\gamma^0(Z))_w \mid ^2)^{-1/2} \cdot \tilde{w}'' = e^{\sqrt{-1}(Z, \gamma)}(\text{Im.}z - \mid w' \mid ^2)^{-1/2} \cdot w''. \]
Since \((\text{Im.}z - \mid w' \mid ^2)^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}} \) and \( \mathcal{D}_{\sqrt{-1}} \) is circular, we get \((\text{Im.}\Psi_\gamma^0(Z) - \mid (\Psi_\gamma^0(Z))_w \mid ^2)^{-1/2} \cdot \tilde{w}'' \in \mathcal{D}_{\sqrt{-1}}. \) Therefore we have (3.4). By (3.4), we can define a mapping \( \Psi_\gamma: \mathcal{D} \rightarrow \mathcal{D} \) by
\[ (3.7) \quad \Psi_\gamma: (Z, w'') \mapsto ((\Psi_\gamma^0(Z)), \tilde{w}''). \]
It is easy to see that this mapping \( \Psi_\gamma \) is an extension of \( \Psi_\gamma^0 \) if we verify the following relation
\[ (3.8) \quad \Psi_{\gamma_2} \cdot \Psi_{\gamma_1} = \Psi_{\gamma_2 \cdot \gamma_1} \quad \text{for any } \gamma_1, \gamma_2 \in SU(k+1, 1). \]
For this, consider a mapping \( \tilde{\phi}: \{z \in \mathbb{C} | \text{Im.}z > 0\} \times \mathcal{C}^m \rightarrow \mathcal{C}^{m+1} \) defined by
\[ (3.9) \quad z^1 = (z - \sqrt{-1})(z + \sqrt{-1})^{-1}, \quad z^j = 2w_{j-1}(z + \sqrt{-1})^{-1} \]
for \( j = 2, 3, \ldots, m+1. \) Note that the restriction \( \tilde{\phi}: \mathcal{D}_0 \rightarrow \mathcal{C}^{m+1} \) is nothing but the biholomorphic isomorphism \( \phi: \mathcal{D}_0 \rightarrow \mathcal{B} \) defined in (3.3). Since \( \text{Im.}z > 0 \) if \( (z, w) \in \mathcal{D} \) by Lemma 1, it is easy to check that \( \tilde{\phi} \) is injective and holomorphic on \( \mathcal{D}. \) Thus \( \tilde{\phi} \) defines a biholomorphic isomorphism of \( \mathcal{D} \) onto the image domain \( \mathcal{B}: = \tilde{\phi}(\mathcal{D}) \) in \( \mathcal{C}^{m+1}. \) Now we define a holomorphic mapping \( \sigma_\gamma: \mathcal{B} \times \mathcal{C}^{m-k} \rightarrow \mathcal{C}^{m+1} \) for each \( \gamma = (A_3 \ b \ c \ d) \in SU(k+1, 1) \) by
\[ \sigma_\gamma: \begin{cases} \delta \mapsto (A_3 + b) \cdot (c_3 + d)^{-1} \\ \delta' \mapsto (c_3 + d)^{-1} \delta' \end{cases} \]
where \( s \in \mathcal{B} \) and \( s' = \langle x^{k+1}, \ldots, x^{m+1} \rangle \in \mathcal{C}^{m-k} \). Then by direct calculations we get

\[
\phi(\Psi_\gamma(z, w)) = \sigma_{\gamma}(\phi(z, w)) \quad \text{for all } (z, w) \in \mathcal{D}.
\]

From this fact, the verification of (3.8) has reduced to verify the following relation

\[
(3.10) \quad \sigma_{\gamma_2} \circ \sigma_{\gamma_1} = \sigma_{\gamma_2 \gamma_1} \quad \text{for any } \gamma_1, \gamma_2 \in SU(k+1, 1).
\]

But (3.10) follows from the relation \( \Delta A - \Delta c = 1_{k+1}, \Delta b - |d|^2 = -1 \) and \( \Delta A - \Delta c = 0 \), which is satisfied for any \( \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in U(k+1, 1) \). Therefore we have showed that each element \( \Psi_\gamma \in \text{Aut}_0(\mathcal{D}_0) \) can be extended to the element \( \Psi_\gamma \in \text{Aut}_0(\mathcal{D}) \) defined by (3.7). Next, taking an element \( K \in K^{0, \sqrt{-1}} \), we define a mapping \( \Psi_{\gamma, K} : \mathcal{D}_0 \times \mathcal{C}^{m-k} \rightarrow \mathcal{D}_0 \times \mathcal{C}^{m-k} \) by

\[
\Psi_{\gamma, K} : (\langle Z, w' \rangle) \mapsto (\langle \Psi_\gamma^0(Z), K w' \rangle)
\]

which is nothing but the mapping \( \Psi_{\gamma, K} \) defined as before. Then, by using the expression of \( \mathcal{D} \) as in Theorem 1, we can see easily that \( \Psi_{\gamma, K} \) defines a holomorphic transformation of \( \mathcal{D} \). Moreover the subset \( \{ \Psi_{\gamma, K} | \gamma \in SU(k+1, 1), K \in K^{0, \sqrt{-1}} \} \) of \( \text{Aut}_0(\mathcal{D}) \) has the structure of real Lie transformation group of \( \mathcal{D} \) with dimension equal to \( \dim SU(k+1, 1) + \dim K^{0, \sqrt{-1}} \). It remains to show that this Lie group coincides with \( \text{Aut}_0(\mathcal{D}) \). We denote by \( \mathfrak{s}u(k+1, 1) \) (resp. \( \mathfrak{f}_{\sqrt{-1}} \)) the Lie algebra of \( SU(k+1, 1) \) (resp. of \( K^{0, \sqrt{-1}} \)). We claim the following equality

\[
(3.11) \quad \dim \mathfrak{g}(\mathcal{D}) = \dim \mathfrak{s}u(k+1, 1) + \dim \mathfrak{f}_{\sqrt{-1}}.
\]

If we show (3.11), then it is obvious that \( \text{Aut}_0(\mathcal{D}) = \{ \Psi_{\gamma, K} | \gamma \in SU(k+1, 1), K \in K^{0, \sqrt{-1}} \} \). Let \( \Pi : g(\mathcal{D}) \rightarrow g(\mathcal{D}_0) \) be the homomorphism defined in Corollary 3. Let \( g(\mathcal{D}) = \mathfrak{s} + \mathfrak{r} \) be a Levi-decomposition of \( g(\mathcal{D}) \), where \( \mathfrak{r} \) denotes the radical of \( g(\mathcal{D}) \) and \( \mathfrak{s} \) denotes a maximal semi-simple subalgebra of \( g(\mathcal{D}) \). Put \( \mathfrak{s}_2 = \ker \Pi \cap \mathfrak{s} \). Then \( \mathfrak{s}_2 \) is an ideal of \( \mathfrak{s} \). Thus there exists an ideal \( \mathfrak{s}_1 \) of \( \mathfrak{s} \) such that \( \mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 \) (direct sum). Since \( g(\mathcal{D}_0) \) is a simple Lie algebra isomorphic to \( \mathfrak{s}u(k+1, 1) \) and \( \Pi \) is surjective, it follows that \( \Pi(\mathfrak{r}) = 0 \), i.e., \( \mathfrak{r} \subset \ker \Pi \). Hence we get \( g(\mathcal{D}) = \mathfrak{s} + \ker \Pi \) (direct sum) and \( \mathfrak{s}_1 \) is isomorphic to \( \mathfrak{s}u(k+1, 1) \). Since \( \ker \Pi \subset g_0 \) by the proof of Corollary 3, we see that \( [g_{-1} + g_{-1/2}, \ker \Pi] = 0 \). From this fact we can show in the same way as in the proof of Corollary 4 that \( \ker \Pi \) is identified with \( \mathfrak{f}_{\sqrt{-1}} \). Thus we get the equality (3.11) and Theorem 2 is proved.

\[q.e.d.\]

4. Examples and remarks

Given an integer \( k \) such that \( 0 \leq k \leq m, k \neq m-1 \), there is an example of the generalized Siegel domain \( \mathcal{D} \) in \( \mathcal{C} \times \mathcal{C}^m \) with exponent 1/2 and \( \dim_R \mathfrak{g}_{-1/2} = 2k \).
Indeed we have the following examples.

**Examples.** Let \(k\) be an integer as above and \(p\) a positive integer different from 2. Put

\[
D_{\sqrt{\frac{1}{2}}} = \{(w_{k+1}, \ldots, w_m) \in \mathbb{C}^{m-k} \mid |w_{k+1}|^p + \cdots + |w_m|^p < 1\}.
\]

Obviously \(D_{\sqrt{\frac{1}{2}}}\) is a bounded Reinhardt domain in \(\mathbb{C}^{m-k}\). For this domain \(D_{\sqrt{\frac{1}{2}}}\), we define a domain \(D\) in \(\mathbb{C}^2\) as follows:

\[
D = \{(z, w_1, \ldots, w_m) \in \mathbb{C} \times \mathbb{C}^m \mid \text{Im.} \, z - \sum_{a=1}^{k} |w_a|^2 > 0, \text{Im.} \, z - \sum_{a=1}^{k} |w_a|^2 - \left( \sum_{\beta=1}^{m} |w_{\beta}|^p \right)^{2/p} > 0\}.
\]

We shall show that \(D\) is a desired example. It is easy to see that \(D\) satisfies the condition (2) of the definition of the generalized Siegel domain with exponent \(1/2\). Moreover the mapping \(\tilde{\Phi}\) defined in (3.9) gives a biholomorphic isomorphism of \(D\) onto the bounded Reinhardt domain

\[
\mathcal{R} = \{(z^1, \ldots, z^{k+1}, u^1, \ldots, u^{m-k}) \in \mathbb{C}^{m+1} \mid \sum_{a=1}^{k} |z_a|^2 + \left( \sum_{\beta=k+1}^{m} |u_{\beta}|^p \right)^{2/p} < 1\}
\]

in \(\mathbb{C}^{m+1}\). Thus \(D\) is a generalized Siegel domain in \(\mathbb{C} \times \mathbb{C}^m\) with exponent \(1/2\). Now we show that \(\dim_{\mathbb{R}} \mathfrak{g}_{-1/2} = 2k\). First we recall that the group \(\text{Aut}_0(\mathcal{R})\) consists of all transformations of the following type (cf. [6], [8]):

\[
\begin{align*}
\tilde{z} &\mapsto (A\tilde{z} + b) (c\tilde{z} + d)^{-1} \\
u^\beta &\mapsto (c\tilde{z} + d)^{-1} e^{\sqrt{-1} \theta_{\beta}} \cdot u^\beta, \quad 1 \leq \beta \leq m-k
\end{align*}
\]

where \( \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in U(k+1, 1), \theta_{\beta} \in R \) and \( \tilde{z} = (z^1, \ldots, z^{k+1}) \). Note that we can replace \(U(k+1, 1)\) by \(SU(k+1, 1)\) in (4.1), because any element \(g \in U(k+1, 1)\) can be written in the form \(g = e^{\sqrt{-1} \theta} g_0\) for suitable \(\theta \in R\) and \(g_0 \in SU(k+1, 1)\). Hence we get

\[
\text{Aut}_0(\mathcal{R}) \cdot 0 = \{(z^1, \ldots, z^{k+1}, 0, \ldots, 0) \in \mathbb{C}^{m+1} \mid \sum_{j=1}^{k+1} |z_j|^2 < 1\}.
\]

Since \(\text{Aut}_0(\mathcal{D}) = \tilde{\Phi}^{-1} \cdot \text{Aut}_0(\mathcal{R}) \cdot \tilde{\Phi}\), (4.2) implies that

\[
\text{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0) = \{(z, w_1, \ldots, w_k, 0, \ldots, 0) \in \mathbb{C} \times \mathbb{C}^m \mid \text{Im.} \, z - \sum_{a=1}^{k} |w_a|^2 > 0\}.
\]

From this fact, we can conclude that \(\dim_{\mathbb{R}} \mathfrak{g}_{-1/2} = 2k\).
Remark 1. In the case where \( n \geq 2 \), the analogy of Theorem 1 is not true in general. In fact we have the following example. Let

\[ \mathcal{D} = \{ (z_1, z_2, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid \text{Im.} z_1 - |w_1|^2 - |w_2|^2 > 0, \text{Im.} z_2 - \text{Re}(w_1 w_2) > 0 \} . \]

Then \( \mathcal{D} \) is a generalized Siegel domain in \( \mathbb{C}^2 \times \mathbb{C}^2 \) with exponent \( 1/2 \) and \( \dim_R \mathfrak{g}_{-1/2} = 2 \), more precisely

\[ \mathfrak{g}_{-1/2} = \left\{ 2 \sqrt{-1} \overline{c} w_1 \frac{\partial}{\partial z_1} + \sqrt{-1} \overline{c} w_2 \frac{\partial}{\partial z_2} + c \frac{\partial}{\partial w_1} \mid c \in \mathbb{C} \right\} . \]

We shall sketch the proof of this fact. First \( \mathcal{D} \) is a generalized Siegel domain with exponent \( 1/2 \). In fact, \( \mathcal{D} \) is contained in the domain

\[ \mathcal{D}' = \{ (z_1, z_2, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid \text{Im.} z_1 - |w_1|^2 - |w_2|^2 > 0, 2\text{Im.} z_1 + \text{Im.} z_2 > 0 \} \]

and \( \mathcal{D}' \) is holomorphically equivalent to a bounded domain in \( \mathbb{C}^4 \). Next we shall show that \( \dim_R \mathfrak{g}_{-1/2} = 2 \). For given \( c \in \mathbb{C} \), \( \text{Aut}_c(\mathcal{D}) \) contains the global one-parameter subgroup

\[ (z_1, z_2, w_1, w_2) \mapsto (z_1 + 2 \sqrt{-1} \overline{c} w_1 + \sqrt{-1} |tc|^2, z_2 + \sqrt{-1} \overline{c} w_2, w_1 + tc, w_2), t \in \mathbb{R} . \]

This global one-parameter subgroup induces a holomorphic vector field

\[ X_c = 2 \sqrt{-1} \overline{c} w_1 \frac{\partial}{\partial z_1} + \sqrt{-1} \overline{c} w_2 \frac{\partial}{\partial z_2} + c \frac{\partial}{\partial w_1} \]

belonging to \( \mathfrak{g}_{-1/2} \). Thus \( \dim_R \mathfrak{g}_{-1/2} \geq 2 \). Suppose that \( \dim_R \mathfrak{g}_{-1/2} = 4 \). Then we can show in the same way as in the proof of Proposition 5.1 of Vey [9] that \( \mathcal{D} \) is a Siegel domain of the second kind, and \( \mathcal{D} \) can be expressed as follows:

\[ \mathcal{D} = \{ (z_1, z_2, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid \text{Im.} z_1 - F_1(w, w) > 0, \text{Im.} z_2 - F_2(w, w) > 0 \} \]

where \( w = (w_1, w_2) \) and \( F = (F_1, F_2) \) is a \( \{ x \in \mathbb{R} \mid x > 0 \} \times \{ x \in \mathbb{R} \mid x > 0 \} \) — hermitian form. Hence \( F_1(w, w) \geq 0 \) and \( F_2(w, w) \geq 0 \) for any \( w \in \mathbb{C}^2 \). On the other hand, if we take a point \( (3, 0, -1, 1) \in \mathcal{D} \), then \( \text{Im.} 0 - F_1((-1, 1), (-1, 1)) > 0 \) and hence \( F_2((-1, 1), (-1, 1)) < 0 \). This is a contradiction. Thus we get \( 2 \leq \dim_R \mathfrak{g}_{-1/2} = 4 \). Hence \( \dim_R \mathfrak{g}_{-1/2} = 2 \). By (4.3), we can see that there exists no non-singular linear mapping \( \mathcal{L}^3: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2 \) satisfying the conditions stated in Lemma 4.

Remark 2. Let \((z, w)\) be a coordinates system in \( \mathbb{C} \times \mathbb{C} \) and \( \mathcal{D} \) a generalized Siegel domain in \( \mathbb{C} \times \mathbb{C} \) with exponent \( c > 0 \). Then we can show in the same way as in the proof of Theorem 1 that \( \mathcal{D} \) can be expressed as follows:

\[ \mathcal{D} = \{ (z, w) \in \mathbb{C} \times \mathbb{C} \mid \text{Im.} z - A |w|^{1/c} > 0 \} \]

where \( A \) is a positive real number depending only on \( \mathcal{D} \).

Remark 3. Let \( \mathcal{D} \) be a generalized Siegel domain in \( \mathbb{C} \times \mathbb{C}^m \) with exponent
1/2 and $\dim_{\mathbb{R}} g_{-1/2} = 2k$, $0 \leq k \leq m$. Then there is a natural $\text{Aut}_0(\mathcal{D})$-equivariant holomorphic imbedding of $\mathcal{D}$ into the complex projective space $P_{m+1}(\mathbb{C})$.

In order to show this fact, we may identify $\mathcal{D}$ with the generalized Siegel domain $\tilde{\mathcal{D}}$ as in Theorem 1. Let $\tilde{\phi}: \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ be the biholomorphic isomorphism defined in (3.9). Then $\tilde{\mathcal{D}}$ is a domain in $\mathbb{C}^{m+1}$ and the group $\text{Aut}_0(\tilde{\mathcal{D}})$ consists of all holomorphic transformations of the following type:

$$\Psi_{\gamma,K}: \begin{cases} \delta' \mapsto (A\delta + b)(c\delta + d)^{-1}, \\ \delta' \mapsto K \cdot (c\delta + d)^{-1} \cdot \delta'. \end{cases}$$

where $\delta = (z^1, \ldots, z^k)$, $\delta' = (z^{k+2}, \ldots, z^{m+1})$, $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1,1)$ and $K \in K^0_{\sqrt{-1}}$. Note that $K^0_{\sqrt{-1}}$ is a subgroup of $GL(m-k, \mathbb{C})$. By using a homogeneous coordinate of $P_{m+1}(\mathbb{C})$, we define a holomorphic imbedding $\tilde{i}: \mathbb{C}^{m+1} \subset P_{m+1}(\mathbb{C})$ by

$$\tilde{i}: (z^1, \ldots, z^{k+1}, z^{k+2}, \ldots, z^{m+1}) \mapsto (z^1, \ldots, z^{k+1}, 1, z^{k+2}, \ldots, z^{m+1}).$$

Then it is easy to see that the restriction $\tilde{i}': \tilde{\mathcal{D}} \to P_{m+1}(\mathbb{C})$ defines an $\text{Aut}_0(\tilde{\mathcal{D}})$-equivariant holomorphic imbedding of $\tilde{\mathcal{D}}$ into $P_{m+1}(\mathbb{C})$, where the holomorphic transformation $\Psi_{\gamma,K}$ of $\tilde{\mathcal{D}}$ is extended to a projective transformation $\Psi_{\gamma,K}$ of $P_{m+1}(\mathbb{C})$ induced by the matrix

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ K \end{pmatrix} \in GL(m+2, \mathbb{C}).$$

Putting $\iota = \tilde{i} \circ \tilde{\phi}$, we get a desired $\text{Aut}_0(\mathcal{D})$-equivariant holomorphic imbedding $\iota: \mathcal{D} \subset P_{m+1}(\mathbb{C})$.

**References**


