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Osaka University
ON GENERALIZED SIEGEL DOMAINS

AKIO KODAMA*)

(Received April 1, 1976)

Introduction. In [3], Kaup, Matsushima and Ochiai defined the notion of "generalized Siegel domain with exponent \( c \)”, which is a natural generalization of the notion of Siegel domain of the first or second kind.

In this paper we consider exclusively a generalized Siegel domain \( \mathcal{D} \) in \( \mathbb{C} \times \mathbb{C}^m \) with exponent 1/2. Let Aut(\( \mathcal{D} \)) denote the group of all holomorphic transformations of \( \mathcal{D} \). It is well-known that the group Aut(\( \mathcal{D} \)) has the structure of real Lie group and the Lie algebra \( \mathfrak{g} \) of Aut(\( \mathcal{D} \)) is canonically identified with the real Lie algebra \( \mathfrak{g}(\mathcal{D}) \) consisting of all complete holomorphic vector fields on \( \mathcal{D} \). Furthermore it is known that the Lie algebra \( \mathfrak{g}(\mathcal{D}) \) has the following graded structure [3]:

\[
\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1,
\]

\[
[g_{\lambda}, g_{\mu}] \subset g_{\lambda+\mu}, \text{ and } \dim_R g_{-1/2} = 2k
\]

for some \( k, 0 \leq k \leq m \).

In section 2 we shall prove the following Theorem.

Theorem 1. Let \( \mathcal{D} \) be a generalized Siegel domain in \( \mathbb{C} \times \mathbb{C}^m \) with exponent 1/2 and \( \dim_R \mathfrak{g}_{-1/2} = 2k, 0 \leq k \leq m \). Let \( \text{Aut}_0(\mathcal{D}) \) denote the identity component of Aut(\( \mathcal{D} \)). Then there exists a generalized Siegel domain \( \mathcal{D} \) in \( \mathbb{C} \times \mathbb{C}^m \) with exponent 1/2 which is holomorphically equivalent to \( \mathcal{D} \) and such that, by choosing a suitable coordinates system \( (z, w_1, \ldots, w_m) \) in \( \mathbb{C} \times \mathbb{C}^m \),

1. the orbit \( \mathcal{D}_0 \) of \( \text{Aut}_0(\mathcal{D}) \) containing the point \( (\sqrt{-1}, 0, \ldots, 0) \in \mathcal{D} \) is the elementary Siegel domain

\[
\mathcal{D}_0 = \{(z, w_1, \ldots, w_m, 0, \ldots, 0) \in \mathbb{C} \times \mathbb{C}^m | \text{Im. } z - \sum_{a=1}^{k} |w_a|^2 > 0 \}
\]

and

2. if we put

\[
\mathcal{D}_{\sqrt{-1}} = \{(w_{k+1}, \ldots, w_m) \in \mathbb{C}^{m-k} | (\sqrt{-1}, 0, \ldots, 0, w_{k+1}, \ldots, w_m) \in \mathcal{D} \},
\]

then \( \mathcal{D}_{\sqrt{-1}} \) is a circular domain in \( \mathbb{C}^{m-k} \) containing the origin 0 of \( \mathbb{C}^{m-k} \). Moreover the domain \( \mathcal{D} \) can be expressed by \( \mathcal{D}_0 \) and \( \mathcal{D}_{\sqrt{-1}} \) as follows:

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As a corollary of Theorem 1, we shall show that if the Lie algebra $\mathfrak{g}(\mathcal{D})$ is semi-simple, then $\mathcal{D}$ is a Siegel domain of the second kind which is holomorphically equivalent to the elementary Siegel domain in $\mathbb{C} \times \mathbb{C}^m$.

In section 3 we shall consider the group $\text{Aut} \,(\mathcal{D})$ of all holomorphic transformations of a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent $1/2$ and $\dim \mathfrak{g}_{1/2}=2k$. By Theorem 1 we can regard $\mathcal{D}$ as a holomorphic fibre space over the elementary Siegel domain $\mathcal{D}_0$ with the projection $\pi: \mathcal{D} \to \mathcal{D}_0$ given by $\pi(z, w_1, \ldots, w_m)=(z, 0, \ldots, 0)$ and the fibre $\pi^{-1}(\sqrt{-1}, 0, \ldots, 0)$ is the circular domain $\mathcal{D}_{\sqrt{-1}}$. In Theorem 2 we shall prove that $\text{Aut}_0(\mathcal{D})$ is the direct product of $\text{Aut}_0(\mathcal{D}_0)$ and the identity component of the isotropy subgroup of $\text{Aut}_0(\mathcal{D}_{\sqrt{-1}})$ at the origin 0 of $\mathcal{D}_{\sqrt{-1}}$.

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1. Preliminaries

Throughout this paper we use the following notations. Let $\mathbb{R}$ (resp. $\mathbb{C}$) denote the field of real numbers (resp. complex numbers) as usual. Let $^tA$ (resp. $0_{s \times t}$) denote the transpose of a matrix $A$ (resp. the unit matrix of degree $l$, $s \times t$ zero matrix) and $A^{-1}$ the inverse matrix of $A$ if $A$ is non-singular.

In this section we recall the definitions and the known results on generalized Siegel domains. We fix a coordinates system $(z_1, \ldots, z_n, w_1, \ldots, w_m)$ in $\mathbb{C} \times \mathbb{C}^m$ once and for all.

A domain $\mathcal{D}$ in $\mathbb{C}^n \times \mathbb{C}^m$ is called a generalized Siegel domain with exponent $c$ if the following conditions are satisfied:

1. $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$ and $\mathcal{D}$ contains a point of the form $(z, 0)$ where $z \in \mathbb{C}^n$ and 0 denotes the origin of $\mathbb{C}^m$.

2. $\mathcal{D}$ is invariant by the transformations of $\mathbb{C}^{n+m}$ of the following types:

   a) $(z, w) \mapsto (z+a, w)$ for all $a \in \mathbb{R}^n$;

   b) $(z, w) \mapsto (z, e^{\sqrt{-1}t}w)$ for all $t \in \mathbb{R}$;

   c) $(z, w) \mapsto (e^{c}z, e^{c}w)$ for all $t \in \mathbb{R}$,

where $c$ is a fixed real number depending only on $\mathcal{D}$. We call $c$ the exponent of $\mathcal{D}$.

We denote by $\Omega$ an open convex cone in $\mathbb{R}^n$ not containing any full straight line. For a given convex cone $\Omega$ in $\mathbb{R}^n$, a mapping $F: \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ is called an $\Omega$-hermitian form if
(1) $F$ is complex linear with respect to the first variable;

(2) $F(u, v) = F(v, u)$ for any $u, v \in \mathbb{C}^m$;

(3) $F(u, u) \in \overline{\Omega}$ for any $u \in \mathbb{C}^m$ and $F(u, u) = 0$ only if $u = 0$, where $\overline{\Omega}$ denotes the closure of $\Omega$ in $\mathbb{R}^n$.

For a given convex cone $\Omega$ in $\mathbb{R}^n$ and an $\Omega$-hermitian form $F: \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}$, the domain

$$\mathcal{D}(\Omega, F) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m | \text{Im. } z - F(w, w) \in \Omega\}$$

in $\mathbb{C}^n \times \mathbb{C}^m$ is called the Siegel domain of the second kind associated with $\Omega$ and $F$. If $m = 0$, the domain $\mathcal{D}(\Omega, F)$ reduces to the domain

$$\mathcal{D}(\Omega) = \{z \in \mathbb{C}^n | \text{Im. } z \in \Omega\}$$

which we call the Siegel domain of the first kind associated with $\Omega$. It is easy to see that if we put $c = 1/2$ then the domain $\mathcal{D}(\Omega, F)$ satisfies the condition (2) of the definition of generalized Siegel domain. Moreover it is known that $\mathcal{D}(\Omega, F)$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$ [7].

Obviously every point of the form $(\sqrt{-1}a, 0), a \in \Omega$, is contained in $\mathcal{D}(\Omega, F)$ and hence the domain $\mathcal{D}(\Omega, F)$ is a generalized Siegel domain with exponent 1/2. From this fact, the notion of generalized Siegel domain may be considered as a generalization of the notion of Siegel domain of the second kind. In the following we regard $\mathcal{D}(\Omega)$ as the special case of the second kind and by a Siegel domain we mean a Siegel domain of the first or second kind.

Let $\mathcal{D}$ be a generalized Siegel domain in $\mathbb{C}^n \times \mathbb{C}^m$ with exponent $c$. Since $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{n+m}$, by a well-known theorem of H. Cartan the group $\text{Aut}(\mathcal{D})$ has the structure of real Lie group and the Lie algebra of $\text{Aut}(\mathcal{D})$ is identified with the Lie algebra $\mathfrak{g}(\mathcal{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$ [2].

From the definition, the following holomorphic vector fields on $\mathcal{D}$ is contained in $\mathfrak{g}(\mathcal{D})$:

(a) \[ \frac{\partial}{\partial z_k} \quad \text{for } k = 1, 2, \ldots, n \]

(b) \[ \partial' = \sqrt{-1} \sum_{\alpha=1}^{n} w_{\alpha} \frac{\partial}{\partial w_{\alpha}} \]

(c) \[ \partial = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} + c \sum_{\alpha=1}^{n} w_{\alpha} \frac{\partial}{\partial w_{\alpha}} \]

By Kaup, Matsushima and Ochiai [3], every vector field $X \in \mathfrak{g}(\mathcal{D})$ is a polynomial vector field, and so we can express $X$ in the following form:

$$X = \sum_{k=1}^{n} \left( \sum_{\gamma, \delta \geq 0} P_{\gamma k} \right) \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^{n} \left( \sum_{\gamma, \delta \geq 0} Q_{\gamma \alpha} \right) \frac{\partial}{\partial w_{\alpha}}$$
where \( P^k_\mu \) and \( Q^\alpha_\mu \) are homogeneous polynomials of degrees \( \nu \) in \( z_i (1 \leq l \leq n) \) and \( \mu \) in \( w_\beta (1 \leq \beta \leq m) \). If \( \mathcal{D} \) is a generalized Siegel domain with exponent \( c = 1/2 \), we have the following theorem on the Lie algebra \( g(\mathcal{D}) \).

**Theorem A** (Kaup, Matsushima and Ochiai [3]).

Let \( \mathcal{D} \) be a generalized Siegel domain in \( C^n \times C^m \) with exponent 1/2. Then we have

\[
\begin{align*}
g(\mathcal{D}) &= g_{-1} + g_{-1/2} + g_0 + g_{1/2} + g_1, \\
[g_\lambda, g_\mu] &\subset g_{\lambda + \mu}, \text{ where } g_\lambda = \{ X \in g(\mathcal{D}) | [\partial, X] = \lambda X \}.
\end{align*}
\]

More precisely we can describe each subspace \( g_\lambda \) as follows:

\[
\begin{align*}
g_{-1} &= \left\{ \sum_{k=1}^n a_k \frac{\partial}{\partial z_k} | a = (a_k) \in \mathbb{R}^n \right\} \\
g_{-1/2} &= \left\{ \sum_{k=1}^n P^k_{0,1} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^n Q^\alpha_0 \frac{\partial}{\partial w_\alpha} \in g(\mathcal{D}) \right\} \\
g_0 &= \left\{ \sum_{k=1}^n P^k_{1,0} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^n Q^\alpha_{1,1} \frac{\partial}{\partial w_\alpha} \in g(\mathcal{D}) \right\} \\
g_{1/2} &= \left\{ \sum_{k=1}^n P^k_{1,1} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^n (Q^\alpha_{1,0} + Q^\alpha_{0,2}) \frac{\partial}{\partial w_\alpha} \in g(\mathcal{D}) \right\} \\
g_1 &= \left\{ \sum_{k=1}^n P^k_{2,0} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^n Q^\alpha_{2,1} \frac{\partial}{\partial w_\alpha} \in g(\mathcal{D}) \right\}
\end{align*}
\]

(2) Let \( r \) be the radical of \( g(\mathcal{D}) \). Then

\[
r = r_{-1} + r_{-1/2} + r_0, \text{ where } r_\lambda = r \cap g_\lambda.
\]

(3) 
(i) \( \dim_R g_{-1} = n, \) \( \dim_R g_{-1/2} \leq 2m, \)

(ii) \( \dim_R g_{1/2} = \dim_R g_{-1/2} - \dim_R r_{-1/2}, \)

\( \dim_R g_1 = n - \dim_R r_{-1/2}. \)

(4) Let \( a = g_{-1} + g_{-1/2} + g_0. \) Then \( a \) is the subalgebra of \( g(\mathcal{D}) \) corresponding to the subgroup \( \text{Aff } (\mathcal{D}) \) of \( \text{Aut } (\mathcal{D}) \) consisting of all complex affine transformations of \( C^{n+m} \) leaving invariant the domain \( \mathcal{D} \).

(5) \( g_{-1} + g_0 + g_1 \) is the subalgebra corresponding to the subgroup \( \{ g \in \text{Aut } (\mathcal{D}) \} \) \( | g \) leaves invariant the complex submanifold \( \mathcal{D}_1 \subset \mathcal{D}, \) where \( \mathcal{D}_1 = \{ (z,w) \in \mathcal{D} | w = 0 \} \) is equivalent to a Siegel domain of the first kind in \( C^n. \)

By Theorem A, we can write \( X \in g_{-1/2} \) in the form

\[
X = \sum_{k=1}^n P^k_{0,1}(X) \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^n c^\alpha(X) \frac{\partial}{\partial w_\alpha}
\]

where \( P^k_{0,1}(X) \) denotes a homogeneous polynomial of degree one in \( w_\alpha (1 \leq \alpha \leq m) \).
depending on $X$ and $c^*(X)$ is a constant depending on $X$. Then by a simple computation, we get

\[(1.1) \quad \text{ad} \partial' X = \sqrt{-1} \sum_{k=1}^n P_k(X) \frac{\partial}{\partial x_k} - \sqrt{-1} \sum_{a=1}^n c^*(X) \frac{\partial}{\partial w_a}.\]

Hence the endomorphism $\text{ad} \partial'$ defines a complex structure on $\mathfrak{g}_{-1/2}$. From this fact and (3) of Theorem A, we obtain the following corollary:

**Corollary.** $\dim_R \mathfrak{g}_{-1/2} = 2k$ for some $k$, $0 \leq k \leq m$.

Since the group $\text{Aff} \left( C^{n+m} \right)$ of all complex affine transformations of $C^{n+m}$ is represented as a semi-direct product $GL(n+m, C) \cdot C^{n+m}$, we can write each element $g \in \text{Aff}(C^{n+m})$ in the form $g = (A, a)$, where $A \in GL(n+m, C)$ and $a \in C^{n+m}$. Obviously the mapping which carries $g = (A, a)$ to the matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \in GL(n+m+1, C)$ is a faithful representation of $\text{Aff}(C^{n+m})$. Since $\text{Aff}(D)$ is a closed subgroup of $\text{Aff}(C^{n+m})$, we can identify $\text{Aff}(D)$ with the closed subgroup of $GL(n+m+1, C)$, and so the Lie algebra $\mathfrak{a}$ is identified with the subalgebra of $\mathfrak{gl}(n+m+1, C)$.

Let $M$ be a hyperbolic manifold in the sense of Kobayashi [4]. It is known that the group $\text{Aut}(M)$ of all holomorphic transformations of $M$ is a Lie group and its isotropy subgroup $K_p$ at a point $p$ of $M$ is compact [4]. We may identify the Lie algebra of $\text{Aut}(M)$ with the Lie algebra $\mathfrak{g}(M)$ consisting of all complete holomorphic vector fields on $M$. A hyperbolic manifold $M$ is called a hyperbolic circular domain in $C^d$ if the following conditions are satisfied:

1. $M$ is a domain in $C^d$;
2. $M$ is circular, that is, $M$ is invariant by the following global one-parameter subgroup of transformations:

$$l_t: (w_1, \ldots, w_d) \mapsto (e^{\sqrt{-1}t}w_1, \ldots, e^{\sqrt{-1}t}w_d), \quad t \in \mathbb{R}$$

where $(w_1, \ldots, w_d)$ denotes a coordinates system in $C^d$. Let $M$ be a hyperbolic circular domain in $C^d$ containing the origin $0$ of $C^d$. Since the one-parameter subgroup $\{l_t | t \in \mathbb{R}\}$ induces an element $\partial = \sqrt{-1} \sum_{a=1}^d w_a \frac{\partial}{\partial w_a}$ of $\mathfrak{g}(M)$, we can show that every vector field $X \in \mathfrak{g}(M)$ is expressed in the form

$$X = \sum_{a=1}^d \left( \sum_{\beta \geq 0} P^\beta_a \right) \frac{\partial}{\partial w_a}$$

where $P^\beta_a$ is a homogeneous polynomial of degree $\beta$ in $w_a$ ($1 \leq \beta \leq d$), by the same way as in [3]. More precisely we can show the following Theorem B (cf. [8]):
Theorem B. Let $M$ be a hyperbolic circular domain in $C^d$ containing the origin $0$ of $C^d$. For the vector field $\partial=-\sqrt{-1}\sum_{\alpha=1}^d w_\alpha \frac{\partial}{\partial w_\alpha} \in g(M)$, we define an endomorphism $J$ of $g(M)$ by $J(X)=[0, X]$ for $X \in g(M)$. Let $\mathfrak{t}(M)$ denote the Lie subalgebra of $g(M)$ corresponding to the isotropy subgroup $K$ of $\text{Aut}(M)$ at the origin $0 \in M$. Then we have

\begin{equation}
\mathfrak{t}(M) = \left\{ \sum_{\alpha=1}^d P^a \frac{\partial}{\partial w_\alpha} \bigg| \sum_{\alpha=1}^d P^a \frac{\partial}{\partial w_\alpha} \in g(M) \right\},
\end{equation}

which is equal to the kernel of $J$; and

(2) if we put $\mathfrak{p}(M) = \{X \in g(M) | J(X) = -X\}$, then $g(M) = \mathfrak{t}(M) + \mathfrak{p}(M)$ (direct sum).

Proof. The same way as in Lemma 3.1 of [3].

2. The case of a generalized Siegel domain in $C \times C^m$ with exponent $1/2$.

In the following part of the paper, we consider exclusively the generalized Siegel domain $\mathcal{D}$ in $C \times C^m$ with $c=1/2$ and $\dim_k g_{-1/2} = 2k$ for some $k$, $0 \leq k \leq m$.

We may assume without loss of generality (by change of linear coordinates if necessary) that $(\sqrt{-1}, 0) \in \mathcal{D}$.

Lemma 1. If $(z, w) \in \mathcal{D}$, then $\text{Im.} z > 0$.

Proof. Suppose that there exists a point $(z_0, w_0) \in \mathcal{D}$ such that $\text{Im.} z_0 \leq 0$. Since $\mathcal{D}$ is a domain in $C \times C^m$ and $(\sqrt{-1}, 0) \notin \mathcal{D}$, there exists a continuous path $\phi: [0, 1] \rightarrow \mathcal{D}$ such that $\phi(0) = (z_0, w_0)$ and $\phi(1) = (\sqrt{-1}, 0)$. Put $\phi(t) = (z(t), w(t))$ for $t \in [0, 1]$. Then there exists a point $t_0 \in [0, 1]$ such that $\text{Im.} z(t_0) = 0$ by our assumption. Obviously this shows that the point $(0, w(t_0))$ belongs to $\mathcal{D}$. Hence we see that $\mathcal{D}$ contains a point of the form $(0, w_0), w_0 \neq 0$, since $\mathcal{D}$ is open. Then, by definition, $\mathcal{D}$ also contains the set $\{(0, e^{1/2}e^{\sqrt{-1}w_1}) | t, \theta \in R\}$, which is naturally identified with $C - \{0\}$. Thus there exists an injective holomorphic mapping $\Psi: C - \{0\} \rightarrow a$ bounded subset of $C^{n+1}$, because $\mathcal{D}$ is equivalent to a bounded domain in $C^{n+1}$. Let $\Psi(z) = (f_1(z), \cdots, f_{n+1}(z))$. Then each $f_i$ is a bounded holomorphic function defined on $C - \{0\}$. Hence, by the Riemann’s extension theorem, $f_i$ extends to a bounded holomorphic function on $C$ and so it is constant. In particular $\Psi$ is a constant mapping. Obviously this is a contradiction.

q.e.d.

In order to prove Theorem 1 we shall consider first the case where $\dim_k g_{-1/2} = 2k > 0$, i.e., $k \geq 1$, in the following.

By Theorem A, we can write each vector field $X \in g_{-1/2}$ as follows:
where \( b_a(X) \) and \( c^\beta(X) \) are complex numbers depending on \( X \). We define a linear mapping \( C: g_{-1/2} \to \mathbb{C}^m \) by \( C(X) = (c^1(X), \ldots, c^m(X)) \). Then we have

\[
C: g_{-1/2} \to \mathbb{C}^m
\]

is injective.

In fact, if \( C(X) = 0 \), then it follows from (1.1) that \( \sqrt{-1}X \in g(\mathcal{D}) \). By a theorem of E. Cartan [1], we have that \( g(\mathcal{D}) \cap \sqrt{-1}g(\mathcal{D}) = 0 \) and hence \( X = 0 \).

Since \( \dim_k g_{-1/2} = 2k \), by our assumption, the image \( V = \{ C(X) | X \in g_{-1/2} \} \) of \( C \) is a complex \( k \)-dimensional vector subspace of \( \mathbb{C}^m \) by (1.1) and (2.1). Fix a non-singular linear mapping \( \mathcal{L}^1: \mathbb{C}^m \to \mathbb{C}^m \) such that

\[
\mathcal{L}^1(V) = \{(d_1, \ldots, d_a, 0, \ldots, 0) \in \mathbb{C}^m | d = (d_i) \in \mathbb{C}^k \}.
\]

**Lemma 2.** There exists a non-singular linear mapping \( \mathcal{L}^2: \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m \) of the form \( z = 2 + w \), \( \tilde{w}_a = \sum_{\beta=1}^m A_{a\beta} w_\beta \) \((1 \leq \alpha \leq m)\) such that

\[
\mathcal{L}_\beta^2 g_{-1/2} = \left\{ \left( \sum_{\beta=1}^m a_{a\beta}(X) \tilde{w}_\beta \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta(X) \frac{\partial}{\partial \tilde{w}_\beta} \bigg| (d^\beta(X)) \in \mathbb{C}^k \right\}
\]

where \( \mathcal{L}_\beta^2 \) denotes the differential of \( \mathcal{L}^2 \).

**Proof.** Let \( C: g_{-1/2} \to \mathbb{C}^m \) and \( \mathcal{L}^1: \mathbb{C}^m \to \mathbb{C}^m \) be the same mappings as before. Then, for

\[
X = \left( \sum_{\alpha=1}^m b_a(X) w_a \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m c^\beta(X) \frac{\partial}{\partial \tilde{w}_\beta},
\]

we have \( \mathcal{L}^1(C(X)) = (d^1(X), \ldots, d^m(X), 0, \ldots, 0) \) for some \( d^\beta(X) \in \mathbb{C}(1 \leq \beta \leq k) \). Let \( (1 \oplus \mathcal{L}^1)(z, w) = (z, \mathcal{L}^1(w)) \). If we put \( \mathcal{L}^2 = 1 \oplus \mathcal{L}^1 \), then \( \mathcal{L}^2 \) satisfies our claim.

Let \( \mathcal{D} \) be the image of \( \mathcal{D} \) under the mapping \( \mathcal{L}^2 \) given in Lemma 2. Then it is easy to see that \( \mathcal{D} \) is also a generalized Siegel domain in \( \mathbb{C} \times \mathbb{C}^m \) with exponent \( 1/2 \) and the Lie algebra \( g(\mathcal{D}) \) coincides with \( \mathcal{L}_\beta^2 g \). Put \( \tilde{\vartheta} = \frac{2}{1} \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta(X) \frac{\partial}{\partial \tilde{w}_\beta} \). Then \( \mathcal{L}_\beta^2 \vartheta = \tilde{\vartheta} \). Thus it follows from Theorem A that \( \mathcal{L}_\beta^2 g_{\lambda} = \tilde{g}_{\lambda} \), where \( \tilde{g}_{\lambda} = \{ X \in g(\mathcal{D}) | [\tilde{\vartheta}, X] = \lambda X \} \). In particular we have

\[
\tilde{g}_{-1/2} = \left\{ \left( \sum_{\alpha=1}^m a_{a\beta}(X) \tilde{w}_\beta \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta \frac{\partial}{\partial \tilde{w}_\beta} \bigg| d = (d^\beta) \in \mathbb{C}^k \right\}
\]

by Lemma 2, where each \( a_{a\beta} \) is uniquely determined by \( d = (d^\beta) \). Hence we may assume that

\[
g_{-1/2} = \left\{ \left( \sum_{\alpha=1}^m a_{a\beta}(X) w_\beta \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^m d^\beta \frac{\partial}{\partial \tilde{w}_\beta} \bigg| d = (d^\beta) \in \mathbb{C}^k \right\}
\]
to prove Theorem 1, considering \( \mathcal{D} \) instead of \( \mathcal{D}_0 \) if necessary. Then by using (1.1) and (2.1), we can show that each vector field \( X \in g_{-1/2} \) is of the following form:

\[
X = \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} a_{\alpha \beta} \underaccent{\bar}{c}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial w_{\beta}}
\]

where \( c^\alpha(X) \) is a complex number depending on \( X \) and \( a_{\alpha \beta} \) is a complex number depending only on \( g_{-1/2} \) and hence \( \mathcal{D} \) (cf. Vey [9], Lemma 5.1). Thus we get

\[
(2.2) \quad g_{-1/2} = \left\{ \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} a_{\alpha \beta} \overline{c}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial w_{\beta}} \right\} \mathbb{C}^k.
\]

**Lemma 3.** The matrix \( (a_{\alpha \beta})_{1 \leq \alpha, \beta \leq k} \) in (2.2) is non-singular skew-hermitian.

**Proof.** Let \( X = \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} a_{\alpha \beta} \overline{c}(X) w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial w_{\beta}} \in g_{-1/2} \).

Then, by (1.1) we get

\[
[X', Y] = \sqrt{-1} \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} a_{\alpha \beta} \overline{c}(X) w_{\alpha} \right) \frac{\partial}{\partial z} - \sqrt{-1} \sum_{\beta=1}^{k} c^{\beta}(X) \frac{\partial}{\partial w_{\beta}}.
\]

Put \( Y = [X', X] \). By a direct calculation we get

\[
[X, Y] = 2\sqrt{-1} \left( \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k} a_{\alpha \beta} \overline{c}(X) \right) \frac{\partial}{\partial z}.
\]

Since \( [X, Y] \in g_{-1} \), we see that the number \( \sum_{\alpha, \beta=1}^{k} a_{\alpha \beta} \overline{c}(X) \) is pure imaginary by (1) of Theorem A. Hence \( \sum_{\alpha, \beta=1}^{k} (a_{\alpha \beta} + a_{\beta \alpha}) c^{\alpha}(X) c^{\beta}(X) = 0 \). On the other hand, since the set \( \{ C(X) = (c^{\alpha}(X)) \mid X \in g_{-1/2} \} \) is a complex \( k \)-dimensional vector subspace of \( C^m \), we get \( a_{\alpha \beta} + a_{\beta \alpha} = 0 \) for \( 1 \leq \alpha, \beta \leq k \).

We need some preparations to prove that \( (a_{\alpha \beta})_{1 \leq \alpha, \beta \leq k} \) is non-singular. We identify the Lie algebra \( a = g_{-1} + g_{-1/2} + g_0 \) with the subalgebra of \( gl(m+2, \mathbb{C}) \) as in §1. Thus we can represent the vector field \( X \in g_{-1/2} \) by the following matrix:

\[
\begin{pmatrix}
0 & \sum_{\beta=1}^{k} a_{\alpha \beta} c^\beta(X) & \ldots & \sum_{\beta=1}^{k} a_{m \beta} c^\beta(X) & 0 \\
0 & \vdots & \ddots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ldots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]
Therefore the global one-parameter subgroup \( \text{expt}X \) generated by \( X \) is given by

\[
\begin{pmatrix}
1 & t \sum_{\beta=1}^{k} a_{\alpha\beta} \bar{c}^{\beta}(X), \cdots, t \sum_{\beta=1}^{k} a_{m\beta} \bar{c}^{\beta}(X) & \frac{t^2}{2} \sum_{\alpha,\beta=1}^{k} a_{\alpha\beta} c^{\alpha}(X) c^{\beta}(X) \\
0 & \vdots & \vdots \\
0 & \cdots & 1_m \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\end{pmatrix}
\]

Thus the action of \( \text{expt}X \) on \( \mathcal{D} \) is given by

\[
\begin{align*}
\zeta & \mapsto \zeta + t \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} a_{\alpha\beta} \bar{c}^{\beta}(X) w_{\alpha} + \frac{t^2}{2} \sum_{\alpha,\beta=1}^{k} a_{\alpha\beta} c^{\alpha}(X) c^{\beta}(X) \\
w_{a} & \mapsto w_{a} + t c^{a}(X), \quad 1 \leq \alpha \leq k \\
w_{\beta} & \mapsto w_{\beta}, \quad k+1 \leq \beta \leq m.
\end{align*}
\]

(2.3)

Now we can prove that \( (\Lambda^{\alpha}_{\beta})_{\substack{1 \leq \alpha \leq m \leq k}} \) is non-singular. Since \( (a_{\alpha\beta})_{\substack{1 \leq \alpha \leq m \leq k}} \) is skew-hermitian, it is enough to show that

\[
(2.4) \quad \sum_{\alpha,\beta=1}^{k} a_{\alpha\beta} c^{\alpha} \bar{c}^{\beta} = 0
\]

for any nonzero vector \( c=(c^{\alpha}) \in \mathbb{C}^k \).

Suppose that there exists a nonzero vector \( c_0=(c_{0}^{\alpha}, \cdots, c_{0}^{k}) \) such that

\[
\sum_{\alpha,\beta=1}^{k} a_{\alpha\beta} c^{\alpha}_0 \bar{c}^{\beta}_0 = 0.
\]

Then the vector field

\[
X_{c_0} = \left( \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} a_{\alpha\beta} \bar{c}^{\beta}_0 w_{\alpha} \right) \frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c_{0}^{\beta} \frac{\partial}{\partial w_{\beta}}
\]

belonging to \( g_{-1/2} \) generates the global one-parameter subgroup \( \text{expt}X_{c_0} \) which acts on \( \mathcal{D} \) by

\[
\begin{align*}
\zeta & \mapsto \zeta + t \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} a_{\alpha\beta} \bar{c}^{\beta}_0 w_{\alpha} \\
w_{a} & \mapsto w_{a} + t c^{a}_0, \quad 1 \leq \alpha \leq k \\
w_{\beta} & \mapsto w_{\beta}, \quad k+1 \leq \beta \leq m.
\end{align*}
\]

Thus \( \text{expt}X_{c_0}(\sqrt{-1}, 0)=(\sqrt{-1}, t c_0^\alpha, \cdots, t c_0^\beta, 0, \cdots, 0) \). Hence \( \mathcal{D} \) must contain the set \( \{ (\sqrt{-1}, e^{\sqrt{-1} \theta} t c_0^\alpha, \cdots, e^{\sqrt{-1} \theta} t c_0^\beta, 0, \cdots, 0) \mid t, \theta \in \mathbb{R} \} \), which is identified with the complex plane \( \mathbb{C} \) since \( c_0 \neq 0 \) by our assumption. But this is a contradiction, because \( \mathcal{D} \) is holomorphically equivalent to a bounded domain in \( \mathbb{C}^{m+1} \). q.e.d.
Lemma 4. There exists a non-singular linear mapping \( \mathcal{L}^3: \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m \) of the form

\[
(*) \quad \bar{z} = z, \quad \bar{w}_\alpha = \sum_{\beta=1}^{m} B_{\alpha \beta} w_\beta (1 \leq \alpha \leq m), \quad \text{such that}
\]

\[
\mathcal{L}_3^* \mathbf{g}_{1/2} = \left\{ \left( \sum_{\alpha=1}^{k} d_{\alpha} \bar{z} \frac{\partial}{\partial z} + \sum_{\beta=1}^{m} e_{\beta} \frac{\partial}{\partial w_\beta} \right) \bigg| c = (c^\theta) \in \mathbb{C}^k \right\}
\]

where \((d_{\alpha}, e_{\beta})_{1 \leq \alpha, \beta \leq k}\) is a non-singular skew-hermitian matrix.

Proof. Let \( \mathcal{L}^3: \mathbb{C} \times \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^m \) be a non-singular linear mapping defined by \((*)\). Then, by a simple calculation, we have \( \mathcal{L}_3^* \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \) and \( \mathcal{L}_3^* \frac{\partial}{\partial w_\alpha} = \sum_{\beta=1}^{m} B_{\alpha \beta} \frac{\partial}{\partial w_\beta} \) \((1 \leq \alpha \leq m)\). Put \( B = (B_{\alpha \beta})_{1 \leq \alpha, \beta \leq m} \). Let \( E = (E_{\alpha \beta}) = B^{-1} \). Take a vector field

\[
X = \left( \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m} a_{\alpha \beta} e^\theta(X) w_\beta \right) \frac{\partial}{\partial z} + \sum_{\alpha=1}^{m} e^\theta(X) \frac{\partial}{\partial w_\alpha}
\]

belonging to \( \mathfrak{g}_{1/2} \). Then we have

\[
\mathcal{L}_3^* \mathbf{g}_{1/2} = \left\{ \left( \sum_{\lambda=1}^{k} \left( \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} a_{\alpha \beta} e^\theta(X) E_{\lambda \alpha} \right) \bar{w}_\lambda \right) \frac{\partial}{\partial z} + \sum_{\lambda=1}^{m} \left( \sum_{\alpha=1}^{m} e^\theta(X) B_{\lambda \alpha} \right) \frac{\partial}{\partial w_\lambda} \right\}
\]

Now we have to find out the matrix \( B \) which satisfies the following conditions:

\[ (2.5) \quad \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} a_{\alpha \beta} e^\theta(X) E_{\alpha \lambda} = 0 \quad \text{for all } \lambda, k+1 \leq \lambda \leq m; \]

\[ (2.6) \quad \sum_{\beta=1}^{m} e^\theta(X) B_{\lambda \beta} = 0 \quad \text{for all } \lambda, k+1 \leq \lambda \leq m. \]

Since \( \{C(X) = (e^\theta(X)) \mid X \in \mathfrak{g}_{1/2}\} = \mathbb{C}^k \), the conditions are equivalent to the following

\[ (2.5)' \quad \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} E_{1,1} & \cdots & E_{1,k+1} \\ \vdots & \ddots & \vdots \\ E_{m,1} & \cdots & E_{m,m} \end{pmatrix} = 0_{k,m-k} \]

\[ (2.6)' \quad \begin{pmatrix} B_{k+1,1} & \cdots & B_{k+1,k} \\ \vdots & \ddots & \vdots \\ B_{m,1} & \cdots & B_{m,k} \end{pmatrix} = 0_{m-k,k} \]

Put \( A_1 = (a_{ij})_{1 \leq i, j \leq k} \), \( A_2 = (a_{ij})_{k+1 \leq i \leq m, 1 \leq j \leq k} \), \( E_1 = (E_{ij})_{1 \leq i \leq k, k+1 \leq j \leq m} \) and \( E_2 = (E_{ij})_{k+1 \leq i \leq m, 1 \leq j \leq m} \). Then, (2.5)' can be written as \( 'A_1 E_1 + 'A_2 E_2 = 0_{k,m-k} \). Since the matrix \( A_1 \) is non-singular by Lemma 3, we have

\[ (2.5)'\quad E_1 = -'^{A_1} A_2 E_2. \]
Now we define a mapping \( \mathcal{L}^3: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m \) by
\[
\mathcal{L}^3: \begin{pmatrix} z \\ \bar{w}_1 \\ \vdots \\ \bar{w}_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_k & -iA^{-1}A_z \\ 0 & 0 & 1_{m-k} \end{pmatrix} \begin{pmatrix} z \\ w_1 \\ \vdots \\ w_m \end{pmatrix}.
\]

Then \( \mathcal{L}^3 \) satisfies the conditions (2.5)' and (2.6)' and hence we have proved Lemma 4. q.e.d.

Now by Lemma 4, we may assume to prove Theorem 1 that
\[
(2.7) \quad g_{-1/2} = \left\{ \left( \sum_{a=1}^k d_a c^a \bar{w}_a, \sum_{b=1}^k c^b \bar{w}_b \right) \left| (c^b) \in \mathcal{C}^k \right. \right\}.
\]

**Lemma 5.** There exists a non-singular linear mapping \( \mathcal{L}^4: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m \) of the form
\[
z = z, \, \bar{w}_a = \sum_{\lambda=1}^k d_{a\lambda} w_\lambda \quad (1 \leq \alpha \leq k) \quad \text{and} \quad \bar{w}_b = w_{b+1} \quad (k+1 \leq \beta \leq m)
\]
such that
\[
\mathcal{L}^4 g_{-1/2} = \left\{ \left( \sum_{a=1}^k d_a c^a \bar{w}_a, \sum_{b=1}^k c^b \bar{w}_b \right) \left| (c^b) \in \mathcal{C}^k \right. \right\}
\]
where each \( d_a \) is a nonzero purely imaginary number depending only on \( D \).

**Proof.** By Lemma 4, the matrix \( D=(d_{ab})_{1 \leq a,b \leq k} \) in (2.7) is non-singular and skew-hermitian. Hence \( D \) can be diagonalized by a suitable unitary matrix \( U=(u_{ab})_{1 \leq a,b \leq k} \). Put \( U^{-1}D \cdot U = \text{diag.} \quad (d_1, \ldots, d_k) \), where \( \text{diag.} \quad (d_1, \ldots, d_k) \) denotes the diagonal matrix whose \((l,l)\)-component is \( d_l \). Then, since \( D \) is non-singular and skew-hermitian, each \( d_l \) is a nonzero purely imaginary number. Now define a non-singular linear mapping \( \mathcal{L}^4: \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C} \times \mathbb{C}^m \) by \( z = z, \bar{w}_a = \sum_{\lambda=1}^k u_{a\lambda} w_\lambda \quad (1 \leq \alpha \leq k) \) and \( \bar{w}_b = w_{b+1} \quad (k+1 \leq \beta \leq m) \).

Then it is easy to see that the mapping \( \mathcal{L}^4 \) satisfies our conditions. q.e.d.

**Proof of Theorem 1:** Suppose first \( \dim_{\mathbb{R}} g_{-1/2} = 2k > 0 \). By Lemma 5 we may assume that
\[
\mathcal{L}^4 g_{-1/2} = \left\{ \left( \sum_{a=1}^k d_a c^a \bar{w}_a, \sum_{b=1}^k c^b \bar{w}_b \right) \left| (c^b) \in \mathcal{C}^k \right. \right\}.
\]

Note that each \( d_a \) is a nonzero purely imaginary number. For the sake of simplicity, we denote \((w_1, \ldots, w_k)\) and \((w_{k+1}, \ldots, w_m)\) by \( \bar{w} \) and \( \bar{w}' \), respectively. For \( a \in \mathbb{R} \) (resp. \( t \in \mathbb{R} \)) we denote by \( T_a \) (resp. \( \Psi_t \)) the holomorphic transforma-
tion \((z, w)\mapsto(z+a, w)\) (resp. \((z, w)\mapsto(e^iz, e^{i\beta}w)\)) of \(C^{m+1}\). Now we define a mapping \(\Phi: C^k\times C^k\to C\) by

\[
\Phi(u, v) = \frac{1}{2\sqrt{-1}} \sum_{a=1}^{k} d_{u^a}v^a\quad\text{for } u = (u^a), v = (v^a) \in C^k.
\]

Then each vector field belonging to \(g_{-1/2}\) is expressed in the form \(2\sqrt{-1}\Phi(w', c)\frac{\partial}{\partial z} + \sum_{a=1}^{k} c^a\frac{\partial}{\partial w_a}\). Since this vector field is determined completely by \(c = (c^a) \in C^k\), we write it by \(X_c\). By (2.3) the vector field \(X_c\) generates the global one-parameter subgroup \(\text{expt}X_c:\)

\[
(z, w', w'') \mapsto (z + 2\sqrt{-1}\Phi(w', tc) + \sqrt{-1}\Phi(tc, tc), w' + tc, w'').
\]

Now we claim that

\[\Phi(c, c) \geq 0 \quad \text{for all } c \in C^k.\]

Suppose that there exists a nonzero vector \(c_0 \in C^k\) such that \(\Phi(c_0, c_0) < 0\). Then, for a point \((z_0, 0) \in \mathcal{D}\), we have

\[
\text{expt}X_{c_0}(z_0, 0) = (z_0 + \sqrt{-1}\Phi(tc_0, tc_0), tc_0, 0)
\]

for any \(t \in \mathbb{R}\). Thus, by Lemma 1, \(\text{Im}z_0 + \Phi(tc_0, tc_0) > 0\) for any \(t \in \mathbb{R}\). This is impossible since \(\Phi(c_0, c_0) < 0\). Therefore we get (2.8). In particular, we see that each number \(\lambda_a := d_a/2\sqrt{-1} (1 \leq \alpha \leq k)\) is positive. Now we define a linear mapping \(\mathcal{L}^k: C^k \times C^m \to C^k \times C^m\) by \(z = z, \quad \bar{w}_a = \sqrt{-1}\lambda_a w_a (1 \leq \alpha \leq k)\) and \(\bar{w}_\beta = w_\beta (k + 1 \leq \beta \leq m)\). Then it is easy to see that

\[
\mathcal{L}^k g_{-1/2} = \left\{2\sqrt{-1}\left(\sum_{a=1}^{k} c^a\bar{w}_a\right)\frac{\partial}{\partial z} + \sum_{a=1}^{k} c^a\frac{\partial}{\partial w_a}\right\} (c^a) \in C^k\right\}.
\]

Hence, by considering the image \(\tilde{\mathcal{D}} = \mathcal{L}^k(\mathcal{D})\) if necessary, we may assume that

\[
g_{-1/2} = \left\{2\sqrt{-1}\left(\sum_{a=1}^{k} c^a\bar{w}_a\right)\frac{\partial}{\partial z} + \sum_{a=1}^{k} c^a\frac{\partial}{\partial w_a}\right\} (c^a) \in C^k\right\}.
\]

Define a mapping \(F: C^k \times C^k \to C\) by

\[
F(u, v) = \sum_{a=1}^{k} u^a v^a \quad \text{for any } u = (u^a), v = (v^a) \in C^k.
\]

Then the domain

\[\mathcal{D} \setminus \mathcal{D}_{-1} = \left\{w' \in C^{m-k}| (\sqrt{-1}, 0, w') \in \mathcal{D}\right\}.
\]
We shall show that $\mathcal{D}_{\sqrt{-1}}$ is connected. Take two points $P_0=(\sqrt{-1}, 0, w_o''')$ and $P_1=(\sqrt{-1}, 0, w_1''')$ of $\mathcal{D}$. Then there exists a continuous path $\Gamma: [0, 1] \rightarrow \mathcal{D}$ such that $\Gamma(0)=P_0$ and $\Gamma(1)=P_1$. For any $t \in [0, 1]$, we put $\Gamma(t)=(z(t), w'(t), w''(t))$, where $z(t) \in \mathbb{C}$, $w'(t) \in \mathbb{C}^k$ and $w''(t) \in \mathbb{C}^{m-k}$. Since
\[
T_{-Re(z(t))} \cdot \exp X_{-w'(t)} \cdot (z(t), w'(t), w''(t))
= (\sqrt{-1}(\text{Im} \cdot z(t)-F(w'(t), w'(t))), 0, w''(t)),
\]
we see that $\text{Im} \cdot z(t)-F(w'(t), w'(t))>0$ for any $t \in [0, 1]$ by Lemma 1. Thus we can define a continuous function $l(t)$ on $[0, 1]$ by $l(t)=\log(\text{Im} \cdot z(t)-F(w'(t), w'(t)))$. Then it is obvious that $l(0)=l(1)=0$ and $e^{l(t)}=\text{Im} \cdot z(t)-F(w'(t), w'(t))$ for any $t \in [0, 1]$. Thus the point
\[
(\sqrt{-1}, 0, e^{-1/2l(t)}w''(t)) = (e^{-l(t)}e^{l(t)}, \sqrt{-1}, 0, e^{-1/2l(t)}w''(t))
\]
belongs to $\mathcal{D}$ by the definition of $\mathcal{D}$. Put $g(t)=e^{-1/2l(t)}w''(t)$. Then $g(t) \in \mathcal{D}_{\sqrt{-1}}$, for any $t \in [0, 1]$, $g(0)=w_o'''$ and $g(1)=w_1'''$. Thus $\mathcal{D}_{\sqrt{-1}}$ is connected. It is obvious that $\mathcal{D}_{\sqrt{-1}}$ is a circular domain in $\mathbb{C}^{m-k}$ containing the origin 0 by the definition of the generalized Siegel domain. Let $(z, w', w'')$ be a point of $\mathcal{D}$. Then there exists a real number $t_0$ such that $e^{t_0}=\text{Im} \cdot z-F(w', w')$, because $T_{-Re.z} \cdot \exp X_{-w'} \cdot (z, w', w'')=(\sqrt{-1}(\text{Im} \cdot z-F(w', w'))$, $0, w'')$ belongs to $\mathcal{D}$ and hence $\text{Im} \cdot z-F(w', w')>0$ by Lemma 1. Thus we have $\Psi_t \circ T_{-Re.z} \circ \exp X_{-w'} \cdot (z, w', w'')=(\sqrt{-1}, 0, e^{-t_0}w'')$. Hence $(\text{Im} \cdot z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}$, and so $\mathcal{D}$ is contained in the set
\[
\{(z, w', w'') \in \mathbb{C} \times \mathbb{C}^m | \text{Im} \cdot z-F(w', w')>0, (\text{Im} \cdot z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}\}.
\]
Conversely, take a point $(z, w', w'') \in \mathbb{C} \times \mathbb{C}^m$ such that $\text{Im} \cdot z-F(w', w')>0$ and $(\text{Im} \cdot z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}$. Then, by the same way as above, we can show that there exists a real number $t_0$ such that $e^{t_0}=\text{Im} \cdot z-F(w', w')$ and
\[
T_{Re.z} \circ \exp X_{w'} \circ \Psi_{t_0} \cdot (\sqrt{-1}, 0, e^{-t_0}w'') = (z, w', w'').
\]
This shows that $(z, w', w'') \in \mathcal{D}$, since $(\sqrt{-1}, 0, e^{-t_0}w'') \in \mathcal{D}$ by the definition of $\mathcal{D}_{\sqrt{-1}}$. Therefore
\[
\mathcal{D} = \{(z, w', w'') \in \mathbb{C} \times \mathbb{C}^m | \text{Im} \cdot z-F(w', w')>0, (\text{Im} \cdot z-F(w', w'))^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}\}.
\]
Now we shall show that the orbit $\mathcal{D}_0$ of $\text{Aut}_0(\mathcal{D})$ containing the point $(\sqrt{-1}, 0) \in \mathcal{D}$ coincides with the elementary Siegel domain $\mathcal{E}$. Let $(z, w', 0) \in \mathcal{E}$. Since $\text{Im} \cdot z-F(w', w')>0$, there exists a real number $t_0$ such that $e^{t_0}=\text{Im} \cdot z-F(w', w')$. Then it is easy to see that $T_{Re.z} \cdot \exp X_{w'} \cdot \Psi_{t_0} \cdot (\sqrt{-1}, 0)=(z, w', 0)$, and so $\mathcal{E} \subset \text{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0)=\mathcal{D}_0$. We claim that $\mathcal{D}_0 \subset \mathcal{E}$. Let $G$
be the identity component $\text{Aut}_0(\mathfrak{D})$ of $\text{Aut}(\mathfrak{D})$, $K$ the isotropy subgroup of $G$ at $(\sqrt{-1}, 0)$ and $G_s$ the identity component of $\text{Aff}(\mathfrak{D})$. Put $K_s=G_s\cap K$. Then we can show that $G/K=G_s/K_s$ by the same way as Lemma 2.3. of Nakajima [5]. Therefore it is enough to see that $G_s\cdot(\sqrt{-1},0)\subset\mathcal{E}$. Let $P(\mathfrak{D})$ (resp. $GL_0(\mathfrak{D})$) be the analytic subgroup of $G_s$ generated by the subalgebra $g_{-1}+g_{-1/2}$ (resp. $g_{0}$). Then we have $G_s=P(\mathfrak{D})\cdot GL_0(\mathfrak{D})$ (semi-direct product), because $P(\mathfrak{D})\cdot GL_0(\mathfrak{D})$ is an abstract subgroup of $G_s$ and contains an open neighborhood of the identity element of $G_s$. Since $GL_0(\mathfrak{D})\cdot(\sqrt{-1},0)\subset\mathcal{D}_1$ by (5), of Theorem A and obviously $P(\mathfrak{D})\cdot\mathcal{E}\subset\mathcal{E}$, we get $G_s\cdot(\sqrt{-1},0)\subset\mathcal{E}$. Therefor $G\cdot(\sqrt{-1},0)=G_s\cdot(\sqrt{-1},0)=\mathcal{E}$. This completes the first case where $k>0$.

It remains the case where $\dim_R g_{-1/2}=0$, i.e., $k=0$. But in this case Theorem 1 is now obvious from the proof of the case where $k>0$. q.e.d.

Corollaries of Theorem 1: As an immediate consequence of Theorem 1 we obtain the following corollary.

**Corollary 1.** Let $\mathcal{D}$ be a generalized Siegel domain in $C\times C^m$ with exponent $1/2$ and $\dim_R g_{-1/2}=2m$. Then $\mathcal{D}$ is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain

$$\mathcal{E} = \{(z, w_1, \ldots, w_m) \in C \times C^m | \text{Im}.z - \sum_{a=1}^m |w_a|^2 > 0\}.$$  

**Corollary 2.** There exists no generalized Siegel domain in $C\times C^m$ with exponent $1/2$ such that $\dim_R g_{-1/2}=2m-2$.

Proof. Suppose that there exists a generalized Siegel domain $\mathcal{D}$ in $C\times C^m$ with exponent $1/2$ and $\dim_R g_{-1/2}=2m-2$. Then, by Theorem 1 there exists a generalized Siegel domain $\mathfrak{D}$ with exponent $1/2$ which is holomorphically equivalent to $\mathcal{D}$ and is expressed in the following form with respect to a suitable coordinates system $(z, w_1, \ldots, w_m)$ in $C\times C^m$:

$$\mathfrak{D} = \{(z, w_1, \ldots, w_m) \in C \times C^m | \text{Im}.z - \sum_{a=1}^m |w_a|^2 > 0, \text{Im}.z - \sum_{a=1}^m |w_a|^2 > 0\}.$$  

where $\mathfrak{D}_{\sqrt{-1}}$ is a circular domain in $C$ containing the origin of $C$. Since $\mathfrak{D}_{\sqrt{-1}}$ is given by $\mathfrak{D}_{\sqrt{-1}} = \{w_m \in C | |w_m| < R\}$ for some positive number $R$,

$$\mathfrak{D} = \{(z, w_1, \ldots, w_m) \in C \times C^m | \text{Im}.z - \sum_{a=1}^m |w_a|^2 + R^2 |w_m| > 0\}.$$  

Thus $\mathfrak{D}$ is a Siegel domain of the second kind in $C\times C^m$. Then we see that $\dim_R g_{-1/2}=2m$ in the decomposition of $g(\mathfrak{D})$ as in Theorem A. But this is a contradiction since $\dim_R g_{-1/2}=\dim_R g_{-1/2}=2m-2$ by our assumption. q.e.d.
Corollary 3. Let $\mathcal{D}$ and $\mathcal{D}_0$ be the same domains as in Theorem 1 and $\Pi \colon g(\mathcal{D}) \to g(\mathcal{D}_0)$ the homomorphism induced by the Lie group homomorphism of $\text{Aut}_0(\mathcal{D})$ to $\text{Aut}_0(\mathcal{D}_0)$ defined by $g \mapsto g|\mathcal{D}_0$, where $g|\mathcal{D}_0$ denotes the restriction of $g$ to $\mathcal{D}_0$. Then $\Pi$ is surjective.

Proof. Note that $\mathcal{D}_0$ is the $\text{Aut}_0(\mathcal{D})$-orbit. Let $(z, w_1, \ldots, w_m)$ be the coordinates system in $\mathbb{C} \times \mathbb{C}^m$ as in Theorem 1. Let $g(\mathcal{D})=g_{-1}+g_{-1/2}+g_{0}+g_{1/2}+g_i$ (resp. $g(\mathcal{D}_0)=g_{-1}+g_{-1/2}+g_{0}+g_{1/2}+g_i$) be the decomposition of $g(\mathcal{D})$ (resp. $g(\mathcal{D}_0)$) as in Theorem A. Since $\mathcal{D}_0$ is an elementary Siegel domain, $g(\mathcal{D}_0)$ is simple. In particular, we have

$$
g_i^0 = \left[g_{-i/2}, g_{i/2}^0\right] + \left[g_{-1}, g_i\right]
g_i^0 = \left[g_{-i/2}, g_{i/2}\right].$$

Put $\beta = z \frac{\partial}{\partial z} + \frac{1}{2} \sum_{a=1}^4 w_a \frac{\partial}{\partial w_a}$. Then it is obvious that $\Pi(\beta) = \beta$. Hence the homomorphism $\Pi$ preserves the gradation, i.e., $\Pi(g_i) \subset g_i^0$. Now we shall show that $\Pi$ is injective on $g_{-1}+g_{-1/2}+g_{1/2}+g_i$. Since $g_{-1}+g_{1/2} = g_{-1} + g_{1/2}$, it is sufficient to show that $\Pi$ is injective on $g_{1/2}+g_i$. Let $X_i \in g_i$ such that $\Pi(X_i) = 0$. Then $\Pi\left(\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, X_i\right]\right) = 0$. Since $\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, X_1\right] \in \mathfrak{g}_{-1}$ and $\Pi$ is identity on $g_{-1}$, we have $\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, X_1\right] = 0$. On the other hand, it is known that the endomorphism $(\text{ad}(\frac{\partial}{\partial z}))^2 : g_i \to g_{-1}$ is injective (cf. [9]). Thus we get $X_i = 0$. Therefore $\Pi$ is injective on $g_i$. Analogously we can show that $\Pi$ is injective on $g_{1/2}$ by using the injectivity of $\text{ad}(\frac{\partial}{\partial z}) : g_{1/2} \to g_{-1/2}$. Note that the subalgebra $g_{-1}+g_{1/2}+g_i$ corresponds to the subgroup leaving the upper half plane $\mathcal{D}_1 = \{(z, 0) \in \mathbb{C} \times \mathbb{C}^m : \text{Im}.z > 0\}$ invariant. Now we claim that each element of $\text{Aut}_0(\mathcal{D}_1)$ can be extended to an element of $\text{Aut}_0(\mathcal{D})$. We identify $\text{Aut}_0(\mathcal{D}_1)$ with $\text{SL}(2, \mathbb{R})/\{\pm 1\}$. Since each element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ acts on $\mathcal{D}_1$ by a holomorphic transformation $l_{\gamma} : z \mapsto (az+b)(cz+d)^{-1}$, we can define a mapping $\tilde{l}_{\gamma} : \mathcal{D}_1 \times \mathbb{C}^m \to \mathcal{D}_1 \times \mathbb{C}^m$ by $\tilde{l}_{\gamma}(z, w) = (l_{\gamma}(z), (cz+d)^{-1}w)$. Since $\tilde{l}_{\gamma_1 \cdot \gamma_2} = \tilde{l}_{\gamma_1} \cdot \tilde{l}_{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \text{SL}(2, \mathbb{R})$, $\tilde{l}_{\gamma}$ induces a holomorphic transformation of $\mathcal{D}$ if

$$
\tilde{l}_{\gamma}(\mathcal{D}) \subset \mathcal{D}.
$$

Put $w' = (w_1, \ldots, w_k), w'' = (w_{k+1}, \ldots, w_m)$ and $||w'|| = \left(\sum_{a=1}^k |w_a|^2\right)^{1/2}$ for any $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$. Then

$$
\text{Im}. l(z) - ||(cz+d)^{-1}w'||^2 = \left|cz+d\right|^2 \text{Im}.z - ||w'||^2 > 0
$$
for any \((z, w', w'') \in \mathcal{D}\). Since
\[
\text{Im. } l_{\gamma}(z) = -\|(cz+d)^{-1}w'\|^2 \geq (cz+d)^{-1}\cdot w''
\]
where \(\theta(z, \gamma) = -\arg((cz+d), e^{\frac{1}{2}}(cz+d)^{-1}w'' \in \mathcal{D}\).

By (2) of Theorem 1, (2.11) and (2.12) imply (2.10). Hence we get \(g_1 = 0\) and hence \(\Pi (g_1) = 0\). We now prove that \(\Pi\) is surjective. Since \(\dim \mathfrak{g}_1 = 1\) and
\[
\Pi (g_1) = 0,
\]
we get \(\Pi (g_1) = 0\). Therefore it follows that \(g_{1/2} = [g_{-1/2}, g_1] = \Pi ([g_{-1/2}, g_1]) \subset \mathfrak{g}(g_0)\), and so \(\Pi (g_{1/2}) = 0\). Then \(g_0 = [g_{-1/2}, g_0] + [g_{-1}, g_0] = \Pi ([g_{-1/2}, g_0/2] + [g_{-1}, g_1]) \subset \Pi (g_0)\), and so \(\Pi (g_0) = 0\). Therefore \(\Pi\) is surjective.

Corollary 4. Let \(\mathcal{D}\) be a generalized Siegel domain in \(\mathbb{C} \times \mathbb{C}^m\) with exponent \(1/2\). If the Lie algebra \(\mathfrak{g}(\mathcal{D})\) is semi-simple, then \(\mathcal{D}\) is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain
\[
\mathcal{E} = \{(z, w_1, \ldots, w_m) \in \mathbb{C} \times \mathbb{C}^m | \text{Im. } z - \sum_{\sigma=1}^m |w_\sigma|^2 > 0\}.
\]

Proof. We claim that \(\dim \mathfrak{g}_1 = 2m\), i.e., \(k = m\). Then our assertion is obvious by Corollary 1. We may assume \(\mathcal{D} = \mathcal{D}\) in Theorem 1 without loss of generality. Suppose that \(k \geq m\). We consider first the case where \(k > 0\). Let \(\Pi: \mathfrak{g}(\mathcal{D}) \rightarrow (\mathcal{D}, 0)\) be the homomorphism defined in Corollary 3. Then \(\Pi\) is surjective by Corollary 3. Put \(\mathfrak{g}_2 = \ker \Pi\). Then \(\mathfrak{g}_2\) is a semi-simple ideal of the semi-simple Lie algebra \(\mathfrak{g}(\mathcal{D})\). Thus there exists a semi-simple ideal \(\mathfrak{g}_1\) such that \(\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_1 + \mathfrak{g}_2\) (direct sum). Since \(\mathfrak{g}_2\) is isomorphic to \(\mathfrak{g}(\mathcal{D})\), \(\mathfrak{g}_2\) is simple. Since \(\Pi\) is injective on \(\mathfrak{g}_1 + \mathfrak{g}_2\), \(\mathfrak{g}_1\) is contained in \(\mathfrak{g}_0\). Let \(B\) denote the Killing form of \(\mathfrak{g}(\mathcal{D})\). Put \(\mathfrak{g}_0 = \{X \in \mathfrak{g}_0 | B(X, \mathfrak{g}_2) = 0\}\). Noting that the ideal \(\mathfrak{g}_2\) is a graded Lie subalgebra, it is easy to see that \(\mathfrak{g}_0 = \mathfrak{g}_1 + \mathfrak{g}_2\), \(\mathfrak{g}_1 = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2\) and \(\mathfrak{g}_0 = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2\). Since \(\mathfrak{g}_2 = \ker \Pi \subset \mathfrak{g}_0\), every vector field \(X \in \mathfrak{g}_2\) is given by \(X = \sum_{\sigma=1}^m Q_\sigma \frac{\partial}{\partial w_\sigma}\) in Theorem A. Thus it can be expressed by the matrix
\[
X = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & C \\
0 & 0 & 0
\end{pmatrix}.
\]

Now we claim that \(C = 0_{k, m-k}\) in (2.13). Let \(S_1\) (resp. \(S_2\)) be the analytic sub-
group of $\text{Aut}_o(\mathcal{D})$ corresponding to $8_1$ (resp. $8_2$). Obviously

\[
(2.14) \quad g_1 \cdot g_2 = g_2 \cdot g_1 \quad \text{for any } g_1 \in S_2 \text{ and } g_2 \in S_2.
\]

Let $X_c(c \in \mathbb{C}^k)$ be the vector field belonging to $g_{-1/2}$ defined in the proof of Theorem 1. Put $g_1 = \exp X$ and

\[
g_2 = \exp X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_k & A \\ 0 & 0 & E \end{pmatrix}.
\]

It is easy to see that if $A = 0_{k,m-k}$ then $C = 0$. By a routine calculation, we get

\[
g_1 \cdot g_2 \cdot (z, w', w'') = (z + 2\sqrt{-1}F(w' + Aw'', c) + \sqrt{-1}F(c, c), w' + Aw'' + c, Ew'')
\]

and

\[
g_2 \cdot g_1 (z, w', w'') = (z + 2\sqrt{-1}F(w', c) + \sqrt{-1}F(c, c), w' + c + Aw'', Ew'')
\]

for any $(z, w', w'') \in \mathcal{D}$. By (2.14), we get $F(w' + Aw'', c) = F(w', c)$ and hence $F(Aw'', c) = 0$. Since $c$ is arbitrary, we get $Aw'' = 0$ for any element $w''$ of an open subset of $\mathbb{C}^{m-k}$. Thus $A = 0$. Therefore we get

\[
(2.15) \quad 8_1 = \begin{pmatrix} 0_{k+1,k+1} & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1_{k+1} & 0 \\ 0 & E \end{pmatrix}.
\]

Since $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^{m+1}$ and any bounded domain in $\mathbb{C}^{m+1}$ is hyperbolic in the sense of Kobayashi [4], $\mathcal{D}$ is hyperbolic. Since $\mathcal{D}_{\sqrt{\mathcal{C}}} = \mathcal{D}$ is a complex submanifold of $\mathcal{D}$, it is also hyperbolic. Thus $\mathcal{D}_{\sqrt{\mathcal{C}}} = \mathcal{D}$ is a hyperbolic circular domain in $\mathbb{C}^{m-k}$ containing the origin 0. By §.1, we have that $\text{Aut}_o(\mathcal{D}_{\sqrt{\mathcal{C}}})$ is a Lie group and its isotropy subgroup $K_{\sqrt{\mathcal{C}}}$ at $0 \in \mathcal{D}_{\sqrt{\mathcal{C}}}$ is compact. Moreover $K_{\sqrt{\mathcal{C}}}$ is a subgroup of $GL(m-k, \mathbb{C})$ by Theorem B. Let $\mathfrak{t}_{\sqrt{\mathcal{C}}}$ be the subalgebra of $\mathfrak{g}(\mathcal{D}_{\sqrt{\mathcal{C}}})$ corresponding to $K_{\sqrt{\mathcal{C}}}$. Now we claim that $\mathfrak{t}_{\sqrt{\mathcal{C}}}$ can be identified with $\mathfrak{s}_2$. By (2.15) we can identify $S_2$ with a subgroup of $K_{\sqrt{\mathcal{C}}}$. Conversely, let $K^0_{\sqrt{\mathcal{C}}}$ be the identity component of $K_{\sqrt{\mathcal{C}}}$ and take an arbitrary element $g \in K^0_{\sqrt{\mathcal{C}}}$. Put $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$, where $1_{k+1}$. Then we can easily see that $\tilde{g}$ leaves $\mathcal{D}$ invariant by (2) of Theorem 1, and hence $\tilde{g}$ defines a holomorphic transformation of $\mathcal{D}$ and $\tilde{g} \in S_2$ by (2.15). Thus $K^0_{\sqrt{\mathcal{C}}}$ can be identified with $S_2$ in a natural way. In particular, $\mathfrak{t}_{\sqrt{\mathcal{C}}}$ is a semi-simple Lie algebra. On the other hand, $\mathfrak{t}_{\sqrt{\mathcal{C}}}$ contains a nonzero element $\theta'' = \sqrt{-1} \sum_{a=k+1} \frac{\partial}{\partial w_a}$ induced by the global one-parameter subgroup $w'' \mapsto e^{\sqrt{-1}t w''}$ ($t \in \mathbb{R}$) and obviously $\theta''$ belongs to the center of $\mathfrak{t}_{\sqrt{\mathcal{C}}}$. This is a contradiction.
Suppose next $k=0$. Then we can show as above that the Lie algebra $\mathfrak{f}_{-1}$ is identified with the semi-simple Lie algebra

$$\text{Ker } \Pi = \left\{ \begin{pmatrix} 0 & 0_m \\ 0_{m,1} & * \end{pmatrix} \right\}.$$

On the other hand, $\mathfrak{f}_{-1}$ contains a nonzero element $\partial' = \sqrt{-1} \sum_{a=1}^m \partial_a \partial_a$ belonging to the center. This is a contradiction.

Therefore $k=m$, and we complete the proof. q.e.d.

3. The structure of Aut ($\mathcal{D}$)

The purpose of this section is to consider the structure of the group of all holomorphic transformations of a generalized Siegel domain $\mathcal{D}$ in $\mathbb{C} \times \mathbb{C}^m$ with exponent 1/2 and $\dim_{\mathbb{R}} \mathfrak{g}_{-1/2} = 2k$ for some $k$, $0 \leq k \leq m$.

In this section we use the following notations. For a point $\delta \in (\mathbb{C}^2)^{k+1}$, define $||\delta|| = \left( \sum_{j=1}^{k+1} |z_j|^2 \right)^{1/2}$.

Put

$$U(k+1, 1) = \left\{ g \in GL(k+2, \mathbb{C}) \mid g \cdot \begin{pmatrix} \mathbf{1}_{k+1} & 0 \\ 0 & -1 \end{pmatrix} \cdot g = \begin{pmatrix} \mathbf{1}_{k+1} & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and

$$SU(k+1, 1) = U(k+1, 1) \cap SL(k+2, \mathbb{C}).$$

For each element $\gamma = (A \ b) \in SU(k+1, 1)$, where $A = (a_{ij})_{1 \leq i, j \leq k+1}$, $b = (b_1, \ldots, b_{k+1})$ and $c = (c_1, \ldots, c_{k+1})$, we put

$$\begin{cases} L_j(\gamma) = (a_{j1} + b_j, 2a_{j2}, 2a_{j3}, \ldots, 2a_{j,k+1}); \\ C(\gamma) = (c_1 + d, 2c_2, 2c_3, \ldots, 2c_{k+1}); \\ B_j(\gamma) = \sqrt{-1}(b_j - a_{j1}) \text{ and } D(\gamma) = \sqrt{-1}(d - c_1) \end{cases}$$

for $j = 1, 2, \ldots, k+1$.

It is easy to see that $U(k+1, 1)$ coincides with all matrices $\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in GL(k+2, \mathbb{C})$ of the form $\begin{pmatrix} A \ b \\ c & d \end{pmatrix} = 1_{k+1}$, $\begin{pmatrix} b \ c \\ d & e \end{pmatrix}$, and $\begin{pmatrix} a \ b \\ c & d \end{pmatrix} = 0_{k+1}$. From this, we get

$$|c\delta + d|^2 = \|A\delta + b\|^2 = 1 - ||\delta||^2$$

for any $\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in U(k+1, 1)$ and any $\delta \in \mathbb{C}^{k+1}$, by an easy computation.
Now we consider the group Aut (E) of all holomorphic transformations of the elementary Siegel domain

$$E = \{(x, w_1, \cdots, w_k)\in \mathbb{C} \times \mathbb{C}^k \mid \text{Im}.x - \sum_{a=1}^k |w_a|^2 > 0\}.$$ 

The elementary Siegel domain E is holomorphically equivalent to the unit open ball $\mathcal{B} = \{\bar{z} = (z^1, \cdots, z^{k+1}) \in \mathbb{C}^{k+1} \mid ||\bar{z}|| < 1\}$. In fact, the biholomorphic isomorphism $\phi: E \to \mathcal{B}$ is given by

$$z^i = (x - \sqrt{-1}) (x + \sqrt{-1})^{-1}, \quad z^j = 2w_j - (x + \sqrt{-1})^{-1}$$

for $j = 2, 3, \ldots, k + 1$. It is well-known that the group Aut (E) can be identified with the simple Lie group $SU(k+1, 1)$ and each element $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1, 1)$ acts on $\mathcal{B}$ by the holomorphic transformation $\tau_\gamma: \bar{z} \mapsto (A\bar{z} + b)(c\bar{z} + d)^{-1}$. Define $\Psi_\gamma = \phi^{-1} \cdot \tau_\gamma \cdot \phi$ for each $\gamma \in SU(k+1, 1)$. Then it is obvious that $\Psi_\gamma$ defines a holomorphic transformation of E. By a direct calculation, we see that the action of $\Psi_\gamma$ on E is given by

$$\Psi_\gamma: \begin{cases} 
  z \mapsto \frac{1}{\sqrt{-1}} \left( \frac{1}{1 - (C(\gamma)Z + D(\gamma))^{-1}} \cdot (L_i(\gamma)Z + B_i(\gamma)) \\
  w_j \mapsto \frac{1}{\sqrt{-1}} \left( \frac{1}{1 - (C(\gamma)Z + D(\gamma))^{-1}} \cdot (L_j(\gamma)Z + B_j(\gamma)) \right) 
\end{cases}$$

for $j = 1, 2, \ldots, k$, where $Z = (z, w_1, \cdots, w_k) \in E$ and $C(\gamma)$, $L_i(\gamma)$, $B_i(\gamma)$, $D(\gamma)$ are defined by (3.1).

Let $K_{\sqrt{-1}}^k$ be the identity component of the isotropy subgroup of Aut $E_{\sqrt{-1}}$ at the origin $0 \in E_{\sqrt{-1}}$. We define a mapping $\Psi_{\gamma, K}: \tilde{E}_0 \times \mathbb{C}^{m-k} \to \tilde{E}_0 \times \mathbb{C}^{m-k}$ for each $\gamma \in SU(k+1, 1)$ and $K \in K_{\sqrt{-1}}^k$ as follows:

$$\Psi_{\gamma, K}: \begin{cases} 
  z \mapsto \frac{1}{\sqrt{-1}} \left( \frac{1}{1 - (C(\gamma)Z + D(\gamma))^{-1}} \cdot (L_i(\gamma)Z + B_i(\gamma)) \\
  w_j \mapsto \frac{1}{\sqrt{-1}} \left( \frac{1}{1 - (C(\gamma)Z + D(\gamma))^{-1}} \cdot (L_j(\gamma)Z + B_j(\gamma)) \right) 
\end{cases}$$

for $j = 1, 2, \ldots, k$. Since $\tilde{E}_0 = \{(z, w_1, \cdots, w_k, 0, \cdots, 0) \in \mathbb{C} \times \mathbb{C}^m \mid \text{Im}.z - \sum_{a=1}^k |w_a|^2 > 0\} = E$, $\Psi_{\gamma, K}$ is a well-defined holomorphic mapping of $\tilde{E}_0 \times \mathbb{C}^{m-k}$ into itself.

Now we can state Theorem 2.
Theorem 2. Let $\Psi_{\gamma,K}: \mathcal{D}_0 \times \mathbb{C}^{n-k} \rightarrow \mathcal{D}_0 \times \mathbb{C}^{n-k}$ be the holomorphic mapping defined as above. Then $\Psi_{\gamma,K}$ induces a holomorphic transformation of $\mathcal{D}$, and moreover any holomorphic transformation of $\mathcal{D}$ belonging to the identity component of $\text{Aut}(\mathcal{D})$ is of this form, i.e.,

$$\text{Aut}_0(\mathcal{D}) = \{ \Psi_{\gamma,K} | \gamma \in SU(k+1,1), K \in K_0^{\sqrt{-1}} \}.$$ 

Proof. Let $(z, w_1, \cdots, w_m)$ be the coordinates system in $\mathbb{C} \times \mathbb{C}^m$ defined in Theorem 1. We put $w'=(w_1, \cdots, w_k)$, $w''=(w_{k+1}, \cdots, w_m)$ and $||w'||=(\sum_{s=1}^{k} |w_s|^2)^{1/2}$ as before. First we claim that each element $\Psi_{\gamma} \in \text{Aut}_0(\mathcal{D})=\text{Aut}_0(\mathcal{D}_0)$ can be extended to a holomorphic transformation of $\mathcal{D}$. We consider the following mappings:

$$w_s \mapsto \tilde{w}_s := \frac{2\sqrt{-1} - 1}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_s(\gamma)Z + B_s(\gamma))} \cdot w_s$$

for $s=k+1, k+2, \cdots, m$. Put $\Psi_{\gamma} = (\Psi_{\gamma}^0, \cdots, \Psi_{\gamma}^{k+1})$. We shall show that

$$(\Psi_{\gamma}(Z), \tilde{w}_{k+1}, \cdots, \tilde{w}_m) \in \mathcal{D}$$

for any $(z, w)=(Z, w_{k+1}, \cdots, w_m) \in \mathcal{D}$.

Put $(\Psi_{\gamma}^0(Z))_{w} = (\Psi_{\gamma}^0(Z), \cdots, \Psi_{\gamma}^{k+1}(Z))$. If we show the following two conditions

$$(3.5) \quad \text{Im. } \Psi_{\gamma}^0(Z) - ||(\Psi_{\gamma}^0(Z))_{w}||^2 > 0 \quad \text{and}$$

$$(3.6) \quad (\text{Im. } \Psi_{\gamma}^0(Z) - ||(\Psi_{\gamma}^0(Z))_{w}||^2)^{-1/2} \cdot \tilde{w}'' \in \mathcal{D},$$

where $\tilde{w}''=(\tilde{w}_{k+1}, \cdots, \tilde{w}_m)$, then (3.4) will follow from Theorem 1. The condition (3.5) is obvious, since $\Psi_{\gamma}^0$ is a holomorphic transformation of $\mathcal{D}_0$. By routine calculations, we get

$$\text{Im. } \Psi_{\gamma}^0(Z) - ||(\Psi_{\gamma}^0(Z))_{w}||^2 = \frac{1 - \sum_{j=1}^{k} ||(C(\gamma)Z + D(\gamma))^{-1} \cdot (L_j(\gamma)Z + B_j(\gamma))||^2}{1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_s(\gamma)Z + B_s(\gamma))},$$

and hence

$$\left( \text{Im. } \Psi_{\gamma}^0(Z) - ||(\Psi_{\gamma}^0(Z))_{w}||^2 \right)^{-1/2} \cdot \tilde{w}_s = \frac{2e^{\theta(Z, w_s)}}{|C(\gamma)Z + D(\gamma)| \cdot (1 - \sum_{j=1}^{k} ||(C(\gamma)Z + D(\gamma))^{-1} \cdot (L_j(\gamma)Z + B_j(\gamma))||^2)^{1/2}}$$

where

$$\theta(Z, \gamma) = -\arg. \left\{ 1 - (C(\gamma)Z + D(\gamma))^{-1} \cdot (L_s(\gamma)Z + B_s(\gamma)) \right\}$$

$$-\arg. \left( C(\gamma)Z + D(\gamma) \right) + \pi/2.$$ 

Let $\phi$ be the biholomorphic isomorphism defined in (3.3) and put $\zeta=\phi(Z) \in \mathcal{B}$. 

Then we get
\[ C(y)Z + D(y) = (z + \sqrt{-1}) (c_3 + d) \] and
\[ \sum_{j=1}^{k-1} |(C(y)Z + D(y))^{-1} (L_j(y)Z + B_j(y))|^2 = ||(A_3 + b) \cdot (c_3 + d)^{-1}||^2. \]

Hence it follows from (3.2) that
\[ \frac{2w}{|z + \sqrt{-1}| \cdot (1 - ||\delta||^2)^{1/2}}. \]

Moreover it is easy to check that \(1 - ||\delta||^2 = 4|z + \sqrt{-1}|^{-2} (\text{Im}.z - ||w'||^2).\) Thus we get
\[ (\text{Im}.\Psi_0^0(Z) - ||(\Psi_0^0(Z))_w||^2)^{-1/2} \cdot \bar{w}_z = e^{\sqrt{-1} \theta(z, \gamma)} (\text{Im}.z - ||w'||^2)^{-1/2} \cdot w_z, \]
and hence
\[ (\text{Im}.\Psi_0^0(Z) - ||(\Psi_0^0(Z))_w||^2)^{-1/2} \cdot \bar{w}' = e^{\sqrt{-1} \theta(z, \gamma)} (\text{Im}.z - ||w'||^2)^{-1/2} \cdot w'. \]

Since \((\text{Im}.z - ||w'||^2)^{-1/2} \cdot w' \in \mathcal{D}_{\sqrt{-1}}\) and \(\mathcal{D}_{\sqrt{-1}} \) is circular, we get \((\text{Im}.\Psi_0^0(Z) - ||(\Psi_0^0(Z))_w||^2)^{-1/2} \cdot \bar{w}' \in \mathcal{D}_{\sqrt{-1}}.\) Therefore we have (3.4). By (3.4), we can define a mapping \(\Psi_\gamma: \mathcal{D} \to \tilde{\mathcal{D}}\) by
\[ (3.7) \quad \Psi_\gamma: (Z, w') \mapsto ((\Psi_\gamma^0(Z)), \bar{w}''). \]

It is easy to see that this mapping \(\Psi_\gamma\) is an extension of \(\Psi_0^0\) if we verify the following relation
\[ (3.8) \quad \Psi_{\gamma_2} \cdot \Psi_{\gamma_1} = \Psi_{\gamma_2 \cdot \gamma_1}, \quad \text{for any } \gamma_1, \gamma_2 \in SU(k+1, 1). \]

For this, consider a mapping \(\tilde{\phi}: \{z \in \mathcal{C} | \text{Im}.z > 0\} \times \mathcal{C} \to \mathcal{C} \) defined by
\[ (3.9) \quad z^j = (z - \sqrt{-1}) (z + \sqrt{-1})^{-1}, \quad z' = 2w_{j-1}(z + \sqrt{-1})^{-1} \]
for \(j = 2, 3, \ldots, m+1.\) Note that the restriction \(\tilde{\phi}: \mathcal{D}_0 \to \mathcal{C}^{m+1}\) is nothing but the biholomorphic isomorphism \(\phi: \mathcal{D} \to \mathcal{D}\) defined in (3.3). Since \(\text{Im}.z > 0\) if \((z, w) \in \mathcal{D}\) by Lemma 1, it is easy to check that \(\tilde{\phi}\) is injective and holomorphic on \(\mathcal{D}\). Thus \(\tilde{\phi}\) defines a biholomorphic isomorphism of \(\mathcal{D}\) onto the image domain \(\mathcal{D} = \tilde{\phi}(\mathcal{D})\) in \(\mathcal{C}^{m+1}.\) Now we define a holomorphic mapping \(\sigma_\gamma: \mathcal{B} \times \mathcal{C}^{m-k} \to \mathcal{C}^{m+1}\) for each \(\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1, 1)\) by
\[ \sigma_\gamma: \begin{pmatrix} \delta \\ \delta' \end{pmatrix} \mapsto (A_3 + b) \cdot (c_3 + d)^{-1} \begin{pmatrix} \delta \\ \delta' \end{pmatrix} \]
where \( g \in \mathcal{B} \) and \( g' = (x^{k+1}, \ldots, z^{m+1}) \in \mathbb{C}^{m-k} \). Then by direct calculations we get
\[
\bar{\phi}(\Psi(g, w)) = \sigma_y(\bar{\phi}(g, w)) \quad \text{for all } (g, w) \in \mathcal{D}.
\]

From this fact, the verification of (3.8) has reduced to verify the following relation
\[
(3.10) \quad \sigma_{\gamma_2} \circ \sigma_{\gamma_1} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1} \quad \text{for any } \gamma_1, \gamma_2 \in SU(k+1, 1).
\]

But (3.10) follows from the relation \( \bar{A}A - \bar{c}c = 1_{k+1}, \bar{b}b - |\bar{d}|^2 = -1 \) and \( \bar{A}A - \bar{d}c = 0 \), which is satisfied for any \( \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in U(k+1, 1) \). Therefore we have showed that each element \( \Psi \in \text{Aut}_0(\mathcal{D}) \) can be extended to the element \( \Psi \in \text{Aut}_0(\mathcal{D}) \) defined by (3.7). Next, taking an element \( K \in \mathbb{K}^{0, \sqrt{-1}} \), we define a mapping \( \Psi_{y,k}: \mathcal{D}_y \times \mathbb{C}^{m-k} \to \mathcal{D}_y \times \mathbb{C}^{m-k} \) by
\[
\Psi_{y,k}: (Z, w') \mapsto (\Psi_y(Z), K \bar{w}')
\]
which is nothing but the mapping \( \Psi_{y,k} \) defined as before. Then, by using the expression of \( \mathcal{D} \) as in Theorem 1, we can see easily that \( \Psi_{y,k} \) defines a holomorphic transformation of \( \mathcal{D} \). Moreover the subset \( \{ \Psi_{y,k} \mid \gamma \in SU(k+1, 1), K \in \mathbb{K}^{0, \sqrt{-1}} \} \) of \( \text{Aut}_0(\mathcal{D}) \) has the structure of real Lie transformation group of \( \mathcal{D} \) with dimension equal to \( \dim SU(k+1, 1) + \dim \mathbb{K}^{0, \sqrt{-1}} \). It remains to show that this Lie group coincides with \( \text{Aut}_0(\mathcal{D}) \). We denote by \( \mathfrak{su}(k+1, 1) \) (resp. \( \mathfrak{f}_{\sqrt{-1}} \)) the Lie algebra of \( SU(k+1, 1) \) (resp. of \( \mathbb{K}^{0, \sqrt{-1}} \)). We claim the following equality
\[
(3.11) \quad \dim \mathfrak{g}(\mathcal{D}) = \dim \mathfrak{su}(k+1, 1) + \dim \mathfrak{f}_{\sqrt{-1}}.
\]
If we show (3.11), then it is obvious that \( \text{Aut}_0(\mathcal{D}) = \{ \Psi_{y,k} \mid \gamma \in SU(k+1, 1), K \in \mathbb{K}^{0, \sqrt{-1}} \} \). Let \( \Pi: \mathfrak{g}(\mathcal{D}) \to \mathfrak{g}(\mathcal{D}) \) be the homomorphism defined in Corollary 3. Let \( \mathfrak{g}(\mathcal{D}) = \mathfrak{s} + \mathfrak{r} \) be a Levi-decomposition of \( \mathfrak{g}(\mathcal{D}) \), where \( \mathfrak{s} \) denotes the radical of \( \mathfrak{g}(\mathcal{D}) \) and \( \mathfrak{s} \) denotes a maximal semi-simple subalgebra of \( \mathfrak{g}(\mathcal{D}) \). Put \( \mathfrak{s}_2 = \text{Ker} \Pi \cap \mathfrak{s} \). Then \( \mathfrak{s}_2 \) is an ideal of \( \mathfrak{s} \). Thus there exists an ideal \( \mathfrak{s}_1 \) of \( \mathfrak{s} \) such that \( \mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 \) (direct sum). Since \( \mathfrak{g}(\mathcal{D}_y) \) is a simple Lie algebra isomorphic to \( \mathfrak{su}(k+1, 1) \) and \( \Pi \) is surjective, it follows that \( \Pi(\mathfrak{r}) = 0 \), i.e., \( \mathfrak{r} \subset \text{Ker} \Pi \). Hence we get \( \mathfrak{g}(\mathcal{D}) = \mathfrak{s} + \text{Ker} \Pi \) (direct sum) and \( \mathfrak{s} \) is isomorphic to \( \mathfrak{su}(k+1, 1) \). Since \( \text{Ker} \Pi \subset \mathfrak{g}_0 \) by the proof of Corollary 3, we see that \( [\mathfrak{g}_1 + \mathfrak{g}_{-1/2}, \text{Ker} \Pi] = 0 \). From this fact we can show in the same way as in the proof of Corollary 4 that \( \text{Ker} \Pi \) is identified with \( \mathfrak{f}_{\sqrt{-1}} \). Thus we get the equality (3.11) and Theorem 2 is proved.

q.e.d.

4. Examples and remarks

Given an integer \( k \) such that \( 0 \leq k \leq m, k \neq m-1 \), there is an example of the generalized Siegel domain \( \mathcal{D} \) in \( \mathbb{C} \times \mathbb{C}^m \) with exponent 1/2 and \( \dim_{\mathbb{R}} \mathfrak{g}_{-1/2} = 2k \).
Indeed we have the following examples.

**Examples.** Let \( k \) be an integer as above and \( p \) a positive integer different from 2. Put

\[
\mathcal{D}_{\sqrt{1}} = \{ (w_{k+1}, \ldots, w_m) \in C^{m-k} | |w_{k+1}|^p + \cdots + |w_m|^p < 1 \}.
\]

Obviously \( \mathcal{D}_{\sqrt{1}} \) is a bounded Reinhardt domain in \( C^{m-k} \). For this domain \( \mathcal{D}_{\sqrt{1}} \), we define a domain \( \mathcal{D} \) in \( C \times C^m \) as follows:

\[
\mathcal{D} = \{ (z, w_1, \ldots, w_m) \in C \times C^m \mid \text{Im.} \, z - \sum_{a=1}^{k} |w_a|^2 > 0, \\
\text{where } w'' = (w_{k+1}, \ldots, w_m). \}
\]

We shall show that \( \mathcal{D} \) is a desired example. It is easy to see that \( \mathcal{D} \) satisfies the condition (2) of the definition of the generalized Siegel domain with exponent 1/2. Moreover the mapping \( \Phi \) defined in (3.9) gives a biholomorphic isomorphism of \( \mathcal{D} \) onto the bounded Reinhardt domain

\[
\mathcal{R} = \{ (z^1, \ldots, z^{k+1}, u^1, \ldots, u^{m-k}) \in C^{m+1} \mid \sum_{j=1}^{k+1} |z_j|^2 + (\sum_{\beta=k+1}^{m} |u_\beta|^p)^{2/p} < 1 \}
\]

in \( C^{m+1} \). Thus \( \mathcal{D} \) is a generalized Siegel domain in \( C \times C^m \) with exponent 1/2. Now we show that \( \dim \mathcal{R} g_{-1/2} = 2k \). First we recall that the group \( \text{Aut}_0(\mathcal{R}) \) consists of all transformations of the following type (cf. [6], [8]):

\[
\begin{align*}
\mathcal{R} & \mapsto (Az+b) (cz+d)^{-1} \\
u^\beta & \mapsto (cz+d)^{-1} e^{-\sqrt{-1} \theta \beta} \cdot u^\beta, \ 1 \leq \beta \leq m-k
\end{align*}
\]

where \((A \ b)^{T} \in U(k+1, 1), \theta_\beta \in R \) and \( z' = (z^1, \ldots, z^{k+1}) \). Note that we can replace \( U(k+1, 1) \) by \( SU(k+1, 1) \) in (4.1), because any element \( g \in U(k+1, 1) \) can be written in the form \( g = e^{\sqrt{-1} \theta} \cdot g_0 \) for suitable \( \theta \in R \) and \( g_0 \in SU(k+1, 1) \). Hence we get

\[
\text{Aut}_0(\mathcal{R}) \circ 0 = \{ (z^1, \ldots, z^{k+1}, 0, \ldots, 0) \in C^{m+1} \mid \sum_{j=1}^{k+1} |z_j|^2 < 1 \}.
\]

Since \( \text{Aut}_0(\mathcal{D}) = \Phi^{-1} \circ \text{Aut}_0(\mathcal{R}) \circ \Phi \), (4.2) implies that

\[
\text{Aut}_0(\mathcal{D}) \circ (\sqrt{-1}, 0) = \{ (z, w_1, \ldots, w_k, 0, \ldots, 0) \in C \times C^m \mid \text{Im.} \, z - \sum_{a=1}^{k} |w_a|^2 > 0 \}.
\]

From this fact, we can conclude that \( \dim \mathcal{R} g_{-1/2} = 2k \).
REMARK 1. In the case where \( n \geq 2 \), the analogy of Theorem 1 is not true in general. In fact we have the following example. Let

\[ \mathcal{D} = \{(x_1, x_2, w_1, w_2) \in C^2 \times C^2 | \text{ Im.} x_1 - |w_1|^2 - |w_2|^2 > 0, \text{ Im.} x_2 - Re(w_1 w_2) > 0 \} . \]

Then \( \mathcal{D} \) is a generalized Siegel domain in \( C^2 \times C^2 \) with exponent \( 1/2 \) and \( \dim \mathcal{R} g_{-1/2} = 2 \), more precisely

\[ (4.3) \quad g_{-1/2} = \left\{ 2\sqrt{-1} \epsilon w_1 \frac{\partial}{\partial x_1} + \sqrt{-1} \epsilon w_2 \frac{\partial}{\partial x_2} + \epsilon \frac{\partial}{\partial w_1} \mid c \in C \right\} . \]

We shall sketch the proof of this fact. First \( \mathcal{D} \) is a generalized Siegel domain with exponent \( 1/2 \). In fact, \( \mathcal{D} \) is contained in the domain

\[ \mathcal{D}' = \{(x_1, x_2, w_1, w_2) \in C^2 \times C^2 | \text{ Im.} x_1 - |w_1|^2 - |w_2|^2 > 0, 2\text{ Im.} x_1 + \text{ Im.} x_2 > 0 \} \]

and \( \mathcal{D}' \) is holomorphically equivalent to a bounded domain in \( C^4 \). Next we shall show that \( \dim \mathcal{R} g_{-1/2} = 2 \). For given \( c \in C \), \( \text{ Aut}_c (\mathcal{D}) \) contains the global one-parameter subgroup \( (x, w) \mapsto (x + 2\sqrt{-1} \epsilon w_1 + \sqrt{-1} |w_1|^2, x_2 + \sqrt{-1} \epsilon w_2, w_1 + \epsilon w_2, w_2) \), \( t \in R \).

This global one-parameter subgroup induces a holomorphic vector field \( X_c = 2\sqrt{-1} \epsilon w_1 \frac{\partial}{\partial x_1} + \sqrt{-1} \epsilon w_2 \frac{\partial}{\partial x_2} + \epsilon \frac{\partial}{\partial w_1} \) belonging to \( g_{-1/2} \). Thus \( \dim \mathcal{R} g_{-1/2} \geq 2 \). Suppose that \( \dim \mathcal{R} g_{-1/2} = 4 \). Then we can show in the same way as in the proof of Proposition 5.1 of Vey [9] that \( \mathcal{D} \) is a Siegel domain of the second kind, and \( \mathcal{D} \) can be expressed as follows:

\[ \mathcal{D} = \{(x_1, x_2, w_1, w_2) \in C^2 \times C^2 | \text{ Im.} x_1 - F_1(w, w) > 0, \text{ Im.} x_2 - F_2(w, w) > 0 \} \]

where \( w = (w_1, w_2) \) and \( F = (F_1, F_2) \) is a \( \{ x \in R \mid x > 0 \} \times \{ x \in R \mid x > 0 \} \) — hermitian form. Hence \( F_1(w, w) \geq 0 \) and \( F_2(w, w) \geq 0 \) for any \( w \in C^2 \). On the other hand, if we take a point \( (3, 0, -1, 1) \in \mathcal{D} \), then \( \text{ Im.} 0 - F_1((-1, 1), (-1, 1)) > 0 \) and hence \( F_2((-1, 1), (-1, 1)) < 0 \). This is a contradiction. Thus we get \( 2 \leq \dim \mathcal{R} g_{-1/2} \neq 4 \). Hence \( \dim \mathcal{R} g_{-1/2} = 2 \). By (4.3), we can see that there exists no non-singular linear mapping \( L_3: C^2 \times C^2 \to C^2 \times C^2 \) satisfying the conditions stated in Lemma 4.

REMARK 2. Let \( (z, w) \) be a coordinates system in \( C \times C \) and \( \mathcal{D} \) a generalized Siegel domain in \( C \times C \) with exponent \( c > 0 \). Then we can show in the same way as in the proof of Theorem 1 that \( \mathcal{D} \) can be expressed as follows:

\[ \mathcal{D} = \{(z, w) \in C \times C | \text{ Im.} z - A \cdot |w|^{1/c} > 0 \} \]

where \( A \) is a positive real number depending only on \( \mathcal{D} \).

REMARK 3. Let \( \mathcal{D} \) be a generalized Siegel domain in \( C \times C^m \) with exponent
1/2 and \( \dim_R \mathfrak{g}_{-1/2} = 2k, \) \( 0 \leq k \leq m. \) Then there is a natural \( \text{Aut}_0(\mathcal{D}) \)-equivariant holomorphic imbedding of \( \mathcal{D} \) into the complex projective space \( P_{m+1}(\mathbb{C}). \)

In order to show this fact, we may identify \( \mathcal{D} \) with the generalized Siegel domain \( \tilde{\mathcal{D}} \) as in Theorem 1. Let \( \tilde{\phi}: \tilde{\mathcal{D}} \to \mathbb{C} \) be the biholomorphic isomorphism defined in (3.9). Then \( \mathcal{D} \) is a domain in \( \mathbb{C}^{m+1} \) and the group \( \text{Aut}_0(\tilde{\mathcal{D}}) \) consists of all holomorphic transformations of the following type:

\[
\begin{align*}
\Psi_{\gamma,K} & : \begin{cases} 
3 \mapsto (A_3 + b)(c_3 + d)^{-1} \\
3' \mapsto K \cdot (c_3 + d)^{-1} \cdot 3'
\end{cases}
\end{align*}
\]

where \( 3 = (z^1, \ldots, z^{k+1}), 3' = (z^{k+2}, \ldots, z^{m+1}), \gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(k+1,1) \) and \( K \in K^0 \backslash \mathbb{R} \). Note that \( K^0 \backslash \mathbb{R} \) is a subgroup of \( GL(m-k, \mathbb{C}) \). By using a homogeneous coordinate of \( P_{m+1}(\mathbb{C}) \), we define a holomorphic imbedding \( \tilde{\iota}: \mathcal{D} \hookrightarrow P_{m+1}(\mathbb{C}) \) by

\[
\begin{align*}
\tilde{\iota}: \begin{pmatrix} z^1, \ldots, z^{k+1}, z^{k+2}, \ldots, z^{m+1} \end{pmatrix} & \mapsto \begin{pmatrix} \gamma^1, \ldots, \gamma^{k+1}, 1, \gamma^{k+2}, \ldots, \gamma^{m+1} \end{pmatrix}.
\end{align*}
\]

Then it is easy to see that the restriction \( \tilde{\iota}: \tilde{\mathcal{D}} \hookrightarrow P_{m+1}(\mathbb{C}) \) defines an \( \text{Aut}_0(\tilde{\mathcal{D}}) \)-equivariant holomorphic imbedding of \( \mathcal{D} \) into \( P_{m+1}(\mathbb{C}) \), where the holomorphic transformation \( \Psi_{\gamma,K} \) of \( \tilde{\mathcal{D}} \) is extended to a projective transformation \( \Psi_{\gamma,K} \) of \( P_{m+1}(\mathbb{C}) \) induced by the matrix

\[
\begin{pmatrix}
A & b \\
c & d \\
0 & K
\end{pmatrix} \in GL(m+2, \mathbb{C}).
\]

Putting \( \iota = \tilde{\iota} \cdot \tilde{\phi} \), we get a desired \( \text{Aut}_0(\mathcal{D}) \)-equivariant holomorphic imbedding \( \iota: \mathcal{D} \hookrightarrow P_{m+1}(\mathbb{C}). \)

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References


