| Title | On generalized Siegel domains |
|---|---|
| Author(s) | Kodama, Akio |
| Citation | Osaka Journal of Mathematics. 1977, 14(2), p. 227-252 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/11686 |
| rights | |
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Osaka J. Math.
**14** (1977), 227-252

# ON GENERALIZED SIEGEL DOMAINS

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(Received April 1, 1976)

**Introduction.** In [3], Kaup, Matsushima and Ochiai defined the notion of "generalized Siegel domain with exponent $c$", which is a natural generalization of the notion of Siegel domain of the first or second kind.

In this paper we consider exclusively a generalized Siegel domain $\mathcal{D}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2. Let Aut $(\mathcal{D})$ denote the group of all holomorphic transformations of $\mathcal{D}$. It is well-known that the group Aut $(\mathcal{D})$ has the structure of real Lie group and the Lie algebra $\mathfrak{g}$ of Aut $(\mathcal{D})$ is canonically identified with the real Lie algebra $\mathfrak{g}(\mathcal{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$. Furthermore it is known that the Lie algebra $\mathfrak{g}(\mathcal{D})$ has the following graded structure [3]:

$$\mathfrak{g}(\mathcal{D})=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1\,,$$
$$[\mathfrak{g}_\lambda,\mathfrak{g}_\mu]\subset\mathfrak{g}_{\lambda+\mu},\ \text{and}\ \dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$$

for some $k$, $0\leqq k\leqq m$.

In section 2 we shall prove the following Theorem.

**Theorem 1.** *Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$, $0\leqq k\leqq m$. Let $Aut_0$ $(\mathcal{D})$ denote the identity component of Aut $(\mathcal{D})$. Then there exists a generalized Siegel domain $\tilde{\mathcal{D}}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 which is holomorphically equivalent to $\mathcal{D}$ and such that, by choosing a suitable coordinates system $(z, w_1, \cdots, w_m)$ in $\boldsymbol{C}\times\boldsymbol{C}^m$,*

(1) *the orbit $\tilde{\mathcal{D}}_0$ of $Aut_0$ $(\tilde{\mathcal{D}})$ containing the point $(\sqrt{-1}, 0, \cdots, 0)\in\tilde{\mathcal{D}}$ is the elementary Siegel domain*

$$\tilde{\mathcal{D}}_0=\{(z, w_1, \cdots, w_k, 0, \cdots, 0)\in\boldsymbol{C}\times\boldsymbol{C}^m\,|\,\mathrm{Im.}\ z-\sum_{\alpha=1}^{k}|w_\alpha|^2>0\}$$

*and*

(2) *if we put*

$$\tilde{\mathcal{D}}_{\sqrt{-1}}=\{(w_{k+1}, \cdots, w_m)\in\boldsymbol{C}^{m-k}\,|\,(\sqrt{-1}, 0, \cdots, 0, w_{k+1}, \cdots, w_m)\in\tilde{\mathcal{D}}\}\,,$$

*then $\tilde{\mathcal{D}}_{\sqrt{-1}}$ is a circular domain in $\boldsymbol{C}^{m-k}$ containing the origin 0 of $\boldsymbol{C}^{m-k}$. Moreover the domain $\tilde{\mathcal{D}}$ can be expressed by $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{D}}_{\sqrt{-1}}$ as follows:*

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$$\tilde{\mathscr{D}} = \Big\{(z, w_1, \cdots, w_m) \in \boldsymbol{C}\times\boldsymbol{C}^m \,|\, (z, w_1, \cdots, w_k, 0, \cdots, 0) \in \tilde{\mathscr{D}}_0\,,$$

$$\Big(\frac{w_{k+1}}{(\mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2)^{1/2}}, \cdots, \frac{w_m}{(\mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2)^{1/2}}\Big) \in \tilde{\mathscr{D}}_{\sqrt{-1}}\Big\}\,.$$

As a corollary of Theorem 1, we shall show that if the Lie algebra $\mathfrak{g}(\mathscr{D})$ is semi-simple, then $\mathscr{D}$ is a Siegel domain of the second kind which is holomorphically equivalent to the elementary Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$.

In section 3 we shall consider the group Aut $(\mathscr{D})$ of all holomorphic transformations of a generalized Siegel domain $\mathscr{D}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2}=2k$. By Theorem 1 we can regard $\tilde{\mathscr{D}}$ as a holomorphic fibre space over the elementary Siegel domain $\tilde{\mathscr{D}}_0$ with the projection $\pi\colon \tilde{\mathscr{D}} \to \tilde{\mathscr{D}}_0$ given by $\pi(z, w_1, \cdots, w_m) = (z, w_1, \cdots, w_k, 0, \cdots, 0)$ and the fibre $\pi^{-1}((\sqrt{-1}, 0, \cdots, 0))$ is the circular domain $\tilde{\mathscr{D}}_{\sqrt{-1}}$. In Theorem 2 we shall prove that $\mathrm{Aut}_0(\tilde{\mathscr{D}})$ is the direct product of $\mathrm{Aut}_0\,(\tilde{\mathscr{D}}_0)$ and the identity component of the isotropy subgroup of $\mathrm{Aut}_0\,(\tilde{\mathscr{D}}_{\sqrt{-1}})$ at the origin 0 of $\tilde{\mathscr{D}}_{\sqrt{-1}}$.

The author would like to express his thanks to Professor S. Murakami and Doctor Y. Sakane for their valuable advices.

## 1. Preliminaries

Throughout this paper we use the following notations. Let $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denote the field of real numbers (resp. complex numbers) as usual. Let ${}^tA$ (resp. $\mathbf{1}_l$, $\mathbf{0}_{s,t}$) denote the transpose of a matrix $A$ (resp. the unit matrix of degree $l$, $s\times t$ zero matrix) and $A^{-1}$ the inverse matrix of $A$ if $A$ is non-singular.

In this section we recall the definitions and the known results on generalized Siegel domains. We fix a coordinates system $(z_1, \cdots, z_n, w_1, \cdots, w_m)$ in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ once and for all.

A domain $\mathscr{D}$ in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ is called a *generalized Siegel domain with exponent c* if the following conditions are satisfied:

(1) $\mathscr{D}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{n+m}$ and $\mathscr{D}$ contains a point of the form $(z, 0)$ where $z\in\boldsymbol{C}^n$ and 0 denotes the origin of $\boldsymbol{C}^m$.

(2) $\mathscr{D}$ is invariant by the transformations of $\boldsymbol{C}^{n+m}$ of the following types:

(a) $(z, w) \mapsto (z+a, w)$ for all $a\in\boldsymbol{R}^n$;

(b) $(z, w) \mapsto (z, e^{\sqrt{-1}t}w)$ for all $t\in\boldsymbol{R}$;

(c) $(z, w) \mapsto (e^t z, e^{ct} w)$ for all $t\in\boldsymbol{R}$,

where $c$ is a fixed real number depending only on $\mathscr{D}$. We call $c$ the *exponent* of $\mathscr{D}$.

We denote by $\Omega$ an open convex cone in $\boldsymbol{R}^n$ not containing any full straight line. For a given convex cone $\Omega$ in $\boldsymbol{R}^n$, a mapping $F\colon \boldsymbol{C}^m\times\boldsymbol{C}^m \to \boldsymbol{C}^n$ is called an *$\Omega$-hermitian form* if

(1) $F$ is complex linear with respect to the first variable;

(2) $F(u, v)=\overline{F(v, u)}$ for any $u, v\in \boldsymbol{C}^m$;

(3) $F(u, u)\in\bar{\Omega}$ for any $u\in\boldsymbol{C}^m$ and $F(u, u)=0$ only if $u=0$, where $\bar{\Omega}$ denotes the closure of $\Omega$ in $\boldsymbol{R}^n$.

For a given convex cone $\Omega$ in $\boldsymbol{R}^n$ and an $\Omega$-hermitian form $F$: $\boldsymbol{C}^m\times\boldsymbol{C}^m\to\boldsymbol{C}^n$, the domain

$$\mathcal{D}(\Omega, F)=\{(z, w)\in\boldsymbol{C}^n\times\boldsymbol{C}^m \mid \mathrm{Im}.\, z-F(w, w)\in\Omega\}$$

in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ is called the *Siegel domain of the second kind associated with* $\Omega$ *and* $F$. If $m=0$, the domain $\mathcal{D}(\Omega, F)$ reduces to the domain

$$\mathcal{D}(\Omega)=\{z\in\boldsymbol{C}^n \mid \mathrm{Im}.\, z\in\Omega\}$$

which we call the *Siegel domain of the first kind associated with* $\Omega$. It is easy to see that if we put $c=1/2$ then the domain $\mathcal{D}(\Omega, F)$ satisfies the condition (2) of the definition of generalized Siegel domain. Moreover it is known that $\mathcal{D}(\Omega, F)$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{n+m}$ [7]. Obviously every point of the form $(\sqrt{-1}a, 0)$, $a\in\Omega$, is contained in $\mathcal{D}(\Omega, F)$ and hence the domain $\mathcal{D}(\Omega, F)$ is a generalized Siegel domain with exponent 1/2. From this fact, the notion of generalized Siegel domain may be considered as a generalization of the notion of Siegel domain of the second kind. In the following we regard $\mathcal{D}(\Omega)$ as the special case of the second kind and by a Siegel domain we mean a Siegel domain of the first or second kind.

Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent $c$. Since $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{n+m}$, by a well-known theorem of H. Cartan the group Aut$(\mathcal{D})$ has the structure of real Lie group and the Lie algebra of Aut $(\mathcal{D})$ is identified with the Lie algebra $\mathfrak{g}(\mathcal{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$ [2].

From the definition, the following holomorphic vector fields on $\mathcal{D}$ is contained in $\mathfrak{g}(\mathcal{D})$:

(a) $$\frac{\partial}{\partial z_k} \quad \text{for } k=1, 2, \cdots, n$$

(b) $$\partial'=\sqrt{-1}\sum_{\alpha=1}^{m} w_\alpha\frac{\partial}{\partial w_\alpha}$$

(c) $$\partial=\sum_{k=1}^{n} z_k\frac{\partial}{\partial z_k}+c\sum_{\alpha=1}^{m} w_\alpha\frac{\partial}{\partial w_\alpha}.$$

By Kaup, Matsushima and Ochiai [3], every vector field $X\in\mathfrak{g}(\mathcal{D})$ is a polynomial vector field, and so we can express $X$ in the follwoing form:

$$X=\sum_{k=1}^{n}\Big(\sum_{\nu,\mu\geq 0} P^k_{\nu\mu}\Big)\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^{m}\Big(\sum_{\nu,\mu\geq 0} Q^\alpha_{\nu\mu}\Big)\frac{\partial}{\partial w_\alpha}$$

where $P^k_{\nu\mu}$ and $Q^\alpha_{\nu\mu}$ are homogeneous polynomials of degrees $\nu$ in $z_l(1\leqq l\leqq n)$ and $\mu$ in $w_\beta$ $(1\leqq\beta\leqq m)$. If $\mathcal{D}$ is a generalized Siegel domain with exponent $c=1/2$, we have the following theorem on the Lie algebra $\mathfrak{g}(\mathcal{D})$.

**Theorem A** (Kaup, Matsushima and Ochiai [3]).
*Let $\mathcal{D}$ be a generalized Siegel domain in $C^n\times C^m$ with exponent 1/2. Then we have*

(1)
$$\mathfrak{g}(\mathcal{D})=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1\,,$$
$$[\mathfrak{g}_\lambda,\mathfrak{g}_\mu]\subset\mathfrak{g}_{\lambda+\mu},\ \text{where } \mathfrak{g}_\lambda=\{X\in\mathfrak{g}(\mathcal{D})\,|\,[\partial, X]=\lambda X\}\,.$$

*More precisely we can describe each subspace $\mathfrak{g}_\lambda$ as follows:*

$$\mathfrak{g}_{-1}=\left\{\sum_{k=1}^n a^k\frac{\partial}{\partial z_k}\,\middle|\, a=(a^k)\in \boldsymbol{R}^n\right\}$$
$$\mathfrak{g}_{-1/2}=\left\{\sum_{k=1}^n P^k_{0,1}\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^m Q^\alpha_{0,0}\frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(\mathcal{D})\right\}$$
$$\mathfrak{g}_0=\left\{\sum_{k=1}^n P^k_{1,0}\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^m Q^\alpha_{0,1}\frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(\mathcal{D})\right\}$$
$$\mathfrak{g}_{1/2}=\left\{\sum_{k=1}^n P^k_{1,1}\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^m (Q^\alpha_{1,0}+Q^\alpha_{0,2})\frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(\mathcal{D})\right\}$$
$$\mathfrak{g}_1=\left\{\sum_{k=1}^n P^k_{2,0}\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^m Q^\alpha_{1,1}\frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(\mathcal{D})\right\}$$

(2) *Let $\mathfrak{r}$ be the radical of $\mathfrak{g}(\mathcal{D})$. Then*

$$\mathfrak{r}=\mathfrak{r}_{-1}+\mathfrak{r}_{-1/2}+\mathfrak{r}_0,\ \textit{where}\ \mathfrak{r}_\lambda=\mathfrak{r}\cap\mathfrak{g}_\lambda\,.$$

(3) (i) $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1}=n,\ \dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}\leqq 2m\,,$

(ii) $\dim_{\boldsymbol{R}}\mathfrak{g}_{1/2}=\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}-\dim_{\boldsymbol{R}}\mathfrak{r}_{-1/2}\,,$

$\dim_{\boldsymbol{R}}\mathfrak{g}_1=n-\dim_{\boldsymbol{R}}\mathfrak{r}_{-1}\,.$

(4) *Let $\mathfrak{a}=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0$. Then $\mathfrak{a}$ is the subalgebra of $\mathfrak{g}(\mathcal{D})$ corresponding to the subgroup Aff $(\mathcal{D})$ of Aut $(\mathcal{D})$ consisting of all complex affine transformations of $C^{n+m}$ leaving invariant the domain $\mathcal{D}$.*

(5) *$\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ is the subalgebra corresponding to the subgroup $\{g\in Aut(\mathcal{D})\,|\,g$ leaves invariant the complex submanifold $\mathcal{D}_1\subset\mathcal{D}\}$, where $\mathcal{D}_1=\{(z,w)\in\mathcal{D}\,|\,w=0\}$ is equivalent to a Siegel domain of the first kind in $C^n$.*

By Theorem A, we can write $X\in\mathfrak{g}_{-1/2}$ in the form

$$X=\sum_{k=1}^n P^k_{0,1}(X)\frac{\partial}{\partial z_k}+\sum_{\alpha=1}^m c^\alpha(X)\frac{\partial}{\partial w_\alpha}$$

where $P^k_{0,1}(X)$ denotes a homogeneous polynomial of degree one in $w_\alpha(1\leqq\alpha\leqq m)$

depending on $X$ and $c^{\alpha}(X)$ is a constant depending on $X$. Then by a simple computation, we get

$$(1.1)\qquad ad\partial'\cdot X=\sqrt{-1}\sum_{k=1}^{n}P_{0,1}^{k}(X)\frac{\partial}{\partial z_k}-\sqrt{-1}\sum_{\alpha=1}^{m}c^{\alpha}(X)\frac{\partial}{\partial w_\alpha}.$$

Hence the endomorphism $ad\partial'$ defines a complex structure on $\mathfrak{g}_{-1/2}$. From this fact and (3) of Theorem A, we obtain the following corollary:

**Corollary.** *$dim_R\,\mathfrak{g}_{-1/2}=2k$ for some $k$, $0\leqq k\leqq m$.*

Since the group Aff $(\boldsymbol{C}^{n+m})$ of all complex affine transformations of $\boldsymbol{C}^{n+m}$ is represented as a semi-direct product $GL(n+m,\boldsymbol{C})\cdot\boldsymbol{C}^{n+m}$, we can write each element $g\in\mathrm{Aff}(\boldsymbol{C}^{n+m})$ in the form $g=(A,a)$, where $A\in GL(n+m,\boldsymbol{C})$ and $a\in\boldsymbol{C}^{n+m}$. Obviously the mapping which carries $g=(A,a)$ to the matrix $\begin{pmatrix}A & a\\ 0 & 1\end{pmatrix}$ $\in GL(n+m+1,\boldsymbol{C})$ is a faithful representation of $\mathrm{Aff}(\boldsymbol{C}^{n+m})$. Since $\mathrm{Aff}(\mathcal{D})$ is a colsed subgroup of $\mathrm{Aff}(\boldsymbol{C}^{n+m})$, we can identify $\mathrm{Aff}(\mathcal{D})$ with the closed subgroup of $GL(n+m+1,\boldsymbol{C})$, and so the Lie algebra $\mathfrak{a}$ is identified with the subalgebra of $\mathfrak{gl}(n+m+1,\boldsymbol{C})$.

Let $M$ be a hyperbolic manifold in the sense of Kobayashi [4]. It is known that the group $\mathrm{Aut}(M)$ of all holomorphic transformations of $M$ is a Lie group and its isotropy subgroup $K_p$ at a point $p$ of $M$ is compact [4]. We may identify the Lie algebra of $\mathrm{Aut}(M)$ with the Lie algebra $\mathfrak{g}(M)$ consisting of all complete holomorphic vector fields on $M$. A hyperbolic manifold $M$ is called a *hyperbolic circular domain in $\boldsymbol{C}^d$* if the following conditions are satisfied:

(1) $M$ is a domain in $\boldsymbol{C}^d$;

(2) $M$ is circular, that is, $M$ is invariant by the following global one-parameter subgroup of transformations:

$$l_t\colon (w_1,\cdots,w_d)\mapsto(e^{\sqrt{-1}t}w_1,\cdots,e^{\sqrt{-1}t}w_d),\quad t\in\boldsymbol{R}$$

where $(w_1,\cdots,w_d)$ denotes a coordinates system in $\boldsymbol{C}^d$. Let $M$ be a hyperbolic circular domain in $\boldsymbol{C}^d$ containing the origin 0 of $\boldsymbol{C}^d$. Since the one-parameter subgroup $\{l_t\,|\,t\in\boldsymbol{R}\}$ induces an element $\partial=\sqrt{-1}\sum_{\alpha=1}^{d}w_\alpha\frac{\partial}{\partial w_\alpha}$ of $\mathfrak{g}(M)$, we can show that every vector field $X\in\mathfrak{g}(M)$ is expressed in the form

$$X=\sum_{\alpha=1}^{d}\Big(\sum_{\nu\geqq 0}P_{\nu}^{\alpha}\Big)\frac{\partial}{\partial w_\alpha}$$

where $P_{\nu}^{\alpha}$ is a homogeneous polynomial of degree $\nu$ in $w_\beta$ $(1\leqq\beta\leqq d)$, by the same way as in [3]. More precisely we can show the following Theorem $B$ (cf. [8]):

**Theorem B.** *Let $M$ be a hyperbolic circular domain in $C^d$ containing the origin 0 of $C^d$. For the vector field $\partial=\sqrt{-1}\sum_{\alpha=1}^{d} w_\alpha \frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(M)$, we define an endomorphism $J$ of $\mathfrak{g}(M)$ by $J(X)=[\partial, X]$ for $X\in\mathfrak{g}(M)$. Let $\mathfrak{k}(M)$ denote the Lie subalgebra of $\mathfrak{g}(M)$ corresponding to the isotropy subgroup $K$ of $Aut(M)$ at the origin $0\in M$. Then we have*

$$(1) \qquad \mathfrak{k}(M)=\left\{\sum_{\alpha=1}^{d} P_1^\alpha \frac{\partial}{\partial w_\alpha}\,\middle|\,\sum_{\alpha=1}^{d} P_1^\alpha \frac{\partial}{\partial w_\alpha}\in\mathfrak{g}(M)\right\},$$

*which is equal to the kernel of $J$; and*

(2) *if we put* $\mathfrak{p}(M)=\{X\in\mathfrak{g}(M)\,|\,J^2(X)=-X\}$,

*then* $\mathfrak{g}(M)=\mathfrak{k}(M)+\mathfrak{p}(M)$ (direct sum).

Proof. The same way as in Lemma 3.1 of [3].

## 2. The case of a generalized Siegel domain in $C\times C^m$ with exponent 1/2.

In the following part of the paper, we consider exclusively the generalized Siegel domain $\mathcal{D}$ in $C\times C^m$ with $c=1/2$ and $\dim_R \mathfrak{g}_{-1/2}=2k$ for some $k$, $0\leqq k\leqq m$.

We may assume without loss of generality (by change of linear coordinates if necessary) that $(\sqrt{-1}, 0)\in\mathcal{D}$.

**Lemma 1.** *If $(z, w)\in\mathcal{D}$, then* $\mathrm{Im}.\, z>0$.

Proof. Suppose that there exists a point $(z_0, w_0)\in\mathcal{D}$ such that $\mathrm{Im}.\, z_0\leqq 0$. Since $\mathcal{D}$ is a domain in $C\times C^m$ and $(\sqrt{-1}, 0)\in\mathcal{D}$, there exists a continuous path $\phi$: $[0, 1]\to\mathcal{D}$ such that $\phi(0)=(z_0, w_0)$ and $\phi(1)=(\sqrt{-1}, 0)$. Put $\phi(t)=(z(t), w(t))$ for $t\in[0, 1]$. Then there exists a point $t_0\in[0, 1]$ such that $\mathrm{Im}.\, z(t_0)=0$ by our assumption. Obviously this shows that the point $(0, w(t_0))$ belongs to $\mathcal{D}$. Hence we see that $\mathcal{D}$ contains a point of the form $(0, w_1)$, $w_1\neq 0$, since $\mathcal{D}$ is open. Then, by definition, $\mathcal{D}$ also contains the set $\{(0, e^{1/2t}e^{\sqrt{-1}\theta}w_1)\,|\,t, \theta\in\boldsymbol{R}\}$, which is naturally identified with $C-\{0\}$. Thus there exists an injective holomorphic mapping $\Psi: C-\{0\}\to$a bounded subset of $C^{m+1}$, because $\mathcal{D}$ is equivalent to a bounded domain in $C^{m+1}$. Let $\Psi(z)=(f_1(z), \cdots, f_{m+1}(z))$. Then each $f_i$ is a bounded holomorphic function defined on $C-\{0\}$. Hence, by the Riemann's extension theorem, $f_i$ extends to a bounded holomorphic function on $C$ and so it is constant. In particular $\Psi$ is a constant mapping. Obviously this is a contradiction. q.e.d.

In order to prove Theorem 1 we shall consider first the case where $\dim_R \mathfrak{g}_{-1/2}=2k>0$, *i.e.*, $k\geqq 1$, in the following.

By Theorem $A$, we can write each vector field $X\in\mathfrak{g}_{-1/2}$ as follows:

$$X=\left(\sum_{\alpha=1}^{m} b_\alpha(X)w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^{m} c^\beta(X)\frac{\partial}{\partial w_\beta},$$

where $b_\alpha(X)$ and $c^\beta(X)$ are complex numbers depending on $X$. We define a linear mapping $C$: $\mathfrak{g}_{-1/2}\to \boldsymbol{C}^m$ by $C(X)=(c^1(X), \cdots, c^m(X))$. Then we have

(2.1) $C$: $\mathfrak{g}_{-1/2}\to \boldsymbol{C}^m$ is injective.

In fact, if $C(X)=0$, then it follows from (1.1) that $\sqrt{-1}X\in\mathfrak{g}(\mathcal{D})$. By a theorem of E. Cartan [1], we have that $\mathfrak{g}(\mathcal{D})\cap\sqrt{-1}\,\mathfrak{g}(\mathcal{D})=0$ and hence $X=0$.

Since $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$ by our assumption, the image $V=\{C(X)\,|\,X\in\mathfrak{g}_{-1/2}\}$ of $C$ is a complex $k$-dimensional vector subspace of $\boldsymbol{C}^m$ by (1.1) and (2.1). Fix a non-singular linear mapping $\mathcal{L}^1$: $\boldsymbol{C}^m\to\boldsymbol{C}^m$ such that

$$\mathcal{L}^1(V)=\{(d_1, \cdots, d_k, 0, \cdots, 0)\in\boldsymbol{C}^m\,|\,d=(d_i)\in\boldsymbol{C}^k\}\ .$$

**Lemma 2.** *There exists a non-singular linear mapping* $\mathcal{L}^2$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ *of the form* $\tilde{z}=z$, $\tilde{w}_\alpha=\sum_{\beta=1}^{m}A_{\alpha\beta}w_\beta$ $(1\leqq\alpha\leqq m)$ *such that*

$$\mathcal{L}^2_*\mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha=1}^{m} a_\alpha(X)\tilde{w}_\alpha\right)\frac{\partial}{\partial\tilde{z}}+\sum_{\beta=1}^{k} d_\beta(X)\frac{\partial}{\partial\tilde{w}_\beta}\,\middle|\,(d^\beta(X))\in\boldsymbol{C}^k\right\}$$

*where* $\mathcal{L}^2_*$ *denotes the differential of* $\mathcal{L}^2$.

Proof. Let $C$: $\mathfrak{g}_{-1/2}\to\boldsymbol{C}^m$ and $\mathcal{L}^1$: $\boldsymbol{C}^m\to\boldsymbol{C}^m$ be the same mappings as before. Then, for

$$X=\left(\sum_{\alpha=1}^{m} b_\alpha(X)w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^{m} c^\beta(X)\frac{\partial}{\partial w_\beta}\in\mathfrak{g}_{-1/2}\,,$$

we have $\mathcal{L}^1(C(X))=(d^1(X), \cdots, d^k(X), 0, \cdots, 0)$ for some $d^\beta(X)\in\boldsymbol{C}(1\leqq\beta\leqq k)$. Let $(1\oplus\mathcal{L}^1)(z, w)=(z, \mathcal{L}^1(w))$. If we put $\mathcal{L}^2=1\oplus\mathcal{L}^1$, then $\mathcal{L}^2$ satisfies our claim. q.e.d.

Let $\tilde{\mathcal{D}}$ be the image of $\mathcal{D}$ under the mapping $\mathcal{L}^2$ given in Lemma 2. Then it is easy to see that $\tilde{\mathcal{D}}$ is also a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and the Lie algebra $\mathfrak{g}(\tilde{\mathcal{D}})$ coincides with $\mathcal{L}^2_*\mathfrak{g}(\mathcal{D})$. Put $\tilde{\partial}=\tilde{z}\frac{\partial}{\partial\tilde{z}}+\frac{1}{2}\sum_{\beta=1}^{m}\tilde{w}_\alpha\frac{\partial}{\partial\tilde{w}_\alpha}$. Then $\mathcal{L}^2_*\,\partial=\tilde{\partial}$. Thus it follows from Theorem A that $\mathcal{L}^2_*\,\mathfrak{g}_\lambda=\tilde{\mathfrak{g}}_\lambda$, where $\tilde{\mathfrak{g}}_\lambda=\{\tilde{X}\in\mathfrak{g}(\tilde{\mathcal{D}})\,|\,[\tilde{\partial}, \tilde{X}]=\lambda\tilde{X}\}$. In particular we have

$$\tilde{\mathfrak{g}}_{-1/2}=\left\{\left(\sum_{\alpha=1}^{m} a_\alpha\tilde{w}_\alpha\right)\frac{\partial}{\partial\tilde{z}}+\sum_{\beta=1}^{k} d^\beta\frac{\partial}{\partial\tilde{w}_\beta}\,\middle|\,d=(d^\beta)\in\boldsymbol{C}^k\right\}$$

by Lemma 2, where each $a_\alpha$ is uniquely determmined by $d=(d^\beta)$. Hence we may assume that

$$\mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha=1}^{m} a_\alpha w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^{k} d^\beta\frac{\partial}{\partial w_\beta}\,\middle|\,d=(d^\beta)\in\boldsymbol{C}^k\right\}$$

to prove Theorem 1, considering $\tilde{\mathscr{D}}$ instead of $\mathscr{D}$ if necessary. Then by using (1.1) and (2.1), we can show that each vector field $X\in\mathfrak{g}_{-1/2}$ is of the following form:

$$X=(\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}a_{\alpha\beta}\overline{c^{\beta}(X)}w_{\alpha})\frac{\partial}{\partial z}+\sum_{\beta=1}^{k}c^{\beta}(X)\frac{\partial}{\partial w_{\beta}}$$

where $c^{\beta}(X)$ is a complex number depending on $X$ and $a_{\alpha\beta}$ is a complex number depending only on $\mathfrak{g}_{-1/2}$ and hence $\mathscr{D}$ (cf.Vey [9], Lemme 5.1). Thus we get

$$\mathfrak{g}_{-1/2}=\left\{(\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}a_{\alpha\beta}\overline{c^{\beta}}w_{\alpha})\frac{\partial}{\partial z}+\sum_{\beta=1}^{k}c^{\beta}\frac{\partial}{\partial w_{\beta}}\,\middle|\,(c^{\beta})\in\boldsymbol{C}^{k}\right\}. \tag{2.2}$$

**Lemma 3.** *The matrix* $(a_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ *in* (2.2) *is non-singular skew-hermitian.*

Proof. Let $X=(\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}a_{\alpha\beta}\overline{c^{\beta}(X)}w_{\alpha})\frac{\partial}{\partial z}+\sum_{\beta=1}^{k}c^{\beta}(X)\frac{\partial}{\partial w_{\beta}}\in\mathfrak{g}_{-1/2}$.

Then, by (1.1) we get

$$[\partial', X]=\sqrt{-1}\,(\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}a_{\alpha\beta}\overline{c^{\beta}(X)}w_{\alpha})\frac{\partial}{\partial z}-\sqrt{-1}\sum_{\beta=1}^{k}c^{\beta}(X)\frac{\partial}{\partial w_{\beta}}.$$

Put $Y=[\partial', X]$. By a direct calculation we get

$$[X, Y]=2\sqrt{-1}\,(\sum_{\alpha,\beta=1}^{k}a_{\alpha\beta}c^{\alpha}(X)\overline{c^{\beta}(X)})\frac{\partial}{\partial z}.$$

Since $[X, Y]\in\mathfrak{g}_{-1}$, we see that the number $\sum_{\alpha,\beta=1}^{k}a_{\alpha\beta}c^{\alpha}(X)\overline{c^{\beta}(X)}$ is pure imaginary by (1) of Theorem A. Hence $\sum_{\alpha,\beta=1}^{k}(a_{\alpha\beta}+\overline{a_{\beta\alpha}})c^{\alpha}(X)\overline{c^{\beta}(X)}=0$. On the other hand, since the set $\{C(X)=(c^{\beta}(X))\,|\,X\in\mathfrak{g}_{-1/2}\}$ is a complex $k$-dimensional vector subspace of $\boldsymbol{C}^{m}$, we get $a_{\alpha\beta}+\overline{a_{\beta\alpha}}=0$ for $1\leqq\alpha,\ \beta\leqq k$.

We need some preparations to prove that $(a_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ is non-singular. We identify the Lie algebra $\mathfrak{a}=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_{0}$ with the subalgebra of $\mathfrak{gl}(m+2, C)$ as in §1. Thus we can represent the vector field $X\in\mathfrak{g}_{-1/2}$ by the following matrix:

$$\left(\begin{array}{c|ccc|c}
0 & \sum_{\beta=1}^{k}a_{1\beta}\overline{c^{\beta}(X)}, & \cdots\cdots, & \sum_{\beta=1}^{k}a_{m\beta}\overline{c^{\beta}(X)}, & 0 \\
\hline
0 & & & & c^{1}(X) \\
\vdots & & & & \vdots \\
\vdots & & \mathbf{0}_{m,m} & & c^{k}(X) \\
\vdots & & & & 0 \\
\vdots & & & & \vdots \\
\hline
0 & 0, & \cdots\cdots, & 0 & 0
\end{array}\right).$$

Therefore the global one-parameter subgroup exptX generated by $X$ is given by

$$\begin{pmatrix} 1 & t\sum_{\beta=1}^{k} a_{1\beta}\overline{c^{\beta}(X)}, \cdots, t\sum_{\beta=1}^{k} a_{m\beta}\overline{c^{\beta}(X)} & \frac{t^2}{2}\sum_{\alpha,\beta=1}^{k} a_{\alpha\beta}c^{\alpha}(X)\overline{c^{\beta}(X)} \\ 0 & & tc^{1}(X) \\ \vdots & & \vdots \\ \vdots & \mathbf{1}_m & tc^{k}(X) \\ \vdots & & 0 \\ \vdots & & \vdots \\ \vdots & & 0 \\ 0 & 0, \cdots\cdots, 0 & 1 \end{pmatrix}.$$

Thus the action of exptX on $\mathcal{D}$ is given by

$$(2.3)\quad \begin{cases} z \mapsto z + t\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k} a_{\alpha\beta}\overline{c^{\beta}(X)}w_{\alpha} + \frac{t^2}{2}\sum_{\alpha,\beta=1}^{k} a_{\alpha\beta}c^{\alpha}(X)\overline{c^{\beta}(X)} \\ w_{\alpha} \mapsto w_{\alpha} + tc^{\alpha}(X), \quad 1 \leqq \alpha \leqq k \\ w_{\beta} \mapsto w_{\beta} \quad , k+1 \leqq \beta \leqq m . \end{cases}$$

Now we can prove that $(a_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ is non-singular. Since $(a_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ is skew-hermitian, it is enough to show that

(2.4) $\sum_{\alpha,\beta=1}^{k} a_{\alpha\beta}c^{\alpha}\overline{c^{\beta}} \neq 0$ for any nonzero vector $c=(c^{\alpha}) \in \boldsymbol{C}^{k}$.

Suppose that there exists a nonzero vector $c_0=(c_0^1, \cdots, c_0^k)$ such that $\sum_{\alpha,\beta=1}^{k} a_{\alpha\beta}c_0^{\alpha}\overline{c_0^{\beta}} =0$. Then the vector field

$$X_{c_0} = \Big(\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k} a_{\alpha\beta}\overline{c_0^{\beta}}w_{\alpha}\Big)\frac{\partial}{\partial z} + \sum_{\beta=1}^{k} c_0^{\beta}\frac{\partial}{\partial w_{\beta}}$$

belonging to $\mathfrak{g}_{-1/2}$ generates the global one-parameter subgroup $\mathrm{expt}X_{c_0}$ which acts on $\mathcal{D}$ by

$$\begin{cases} z \mapsto z + t\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k} a_{\alpha\beta}\overline{c_0^{\beta}}w_{\alpha} \\ w_{\alpha} \mapsto w_{\alpha} + tc_0^{\alpha}, \quad 1 \leqq \alpha \leqq k \\ w_{\beta} \mapsto w_{\beta} \quad , k+1 \leqq \beta \leqq m . \end{cases}$$

Thus $\mathrm{expt}X_{c_0}\cdot(\sqrt{-1}, 0)=(\sqrt{-1}, tc_0^1, \cdots, tc_0^k, 0, \cdots, 0)$. Hence $\mathcal{D}$ must contain the set $\{(\sqrt{-1}, e^{\sqrt{-1}\theta}tc_0^1, \cdots, e^{\sqrt{-1}\theta}tc, 0, \cdots, 0) \mid t, \theta \in \boldsymbol{R}\}$, which is identified with the complex plane $\boldsymbol{C}$ since $c_0 \neq 0$ by our assumption. But this is a contradiction, because $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$. q.e.d.

**Lemma 4.** *There exists a non-singular linear mapping* $\mathcal{L}^3$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ *of the form*

(*) $\tilde{z}=z,\ \tilde{w}_\alpha=\sum_{\beta=1}^m B_{\alpha\beta}w_\beta (1\leqq\alpha\leqq m)$, *such that*

$$\mathcal{L}_*^3\mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha,\beta=1}^k d_{\alpha\beta}\overline{c^\beta}\tilde{w}_\alpha\right)\frac{\partial}{\partial\tilde{z}}+\sum_{\beta=1}^k c^\beta\frac{\partial}{\partial\tilde{w}_\beta}\ \middle|\ c=(c^\beta)\in\boldsymbol{C}^k\right\}$$

*where* $(d_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ *is a non-singular skew-hermitian matrix.*

Proof. Let $\mathcal{L}^3$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ be a non-singular linear mapping defined by (*). Then, by a simple caluclation, we have $\mathcal{L}_*^3\frac{\partial}{\partial z}=\frac{\partial}{\partial\tilde{z}}$ and $\mathcal{L}_*^3\frac{\partial}{\partial w_\alpha}=\sum_{\beta=1}^m B_{\beta\alpha}\frac{\partial}{\partial\tilde{w}_\beta}(1\leqq\alpha\leqq m)$. Put $B=(B_{\alpha\beta})_{1\leqq\alpha,\beta\leqq m}$. Let $E=(E_{\alpha\beta})=B^{-1}$. Take a vector field

$$X=\left(\sum_{\alpha=1}^m\sum_{\beta=1}^k a_{\alpha\beta}\overline{c^\beta(X)}w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^k c^\beta(X)\frac{\partial}{\partial w_\beta}$$

belonging to $\mathfrak{g}_{-1/2}$. Then we have

$$\mathcal{L}_*^3X=\left\{\sum_{\lambda=1}^m\left(\sum_{\alpha=1}^m\sum_{\beta=1}^k a_{\alpha\beta}\overline{c^\beta(X)}E_{\alpha\lambda}\right)\tilde{w}_\lambda\right\}\frac{\partial}{\partial\tilde{z}}+\sum_{\lambda=1}^m\left(\sum_{\beta=1}^k c^\beta(X)B_{\lambda\beta}\right)\frac{\partial}{\partial\tilde{w}_\lambda}.$$

Now we have to find out the matrix $B$ which satisfies the following conditions:

$$\sum_{\alpha=1}^m\sum_{\beta=1}^k a_{\alpha\beta}\overline{c^\beta(X)}E_{\alpha\lambda}=0\quad\text{for all }\lambda,\ k+1\leqq\lambda\leqq m;\tag{2.5}$$

$$\sum_{\beta=1}^k c^\beta(X)B_{\lambda\beta}=0\qquad\text{for all }\lambda,\ k+1\leqq\lambda\leqq m.\tag{2.6}$$

Since $\{C(X)=(c^\beta(X))\,|\,X\in\mathfrak{g}_{-1/2}\}=\boldsymbol{C}^k$, the conditions are equivalent to the following

$$(2.5)'\qquad \begin{pmatrix} a_{11}, & \cdots, & a_{k1}, & \cdots, & a_{m1}\\ \vdots & & \vdots & & \vdots\\ a_{1k}, & \cdots, & a_{kk}, & \cdots, & a_{mk}\end{pmatrix}\cdot{}^t\!\begin{pmatrix} E_{1,k+1}, & \cdots, & E_{m,k+1}\\ \vdots & & \vdots\\ E_{1m}, & \cdots\cdots, & E_{mm}\end{pmatrix}=\boldsymbol{0}_{k,m-k}$$

$$(2.6)'\qquad \begin{pmatrix} B_{k+1,1}, & \cdots\cdots, & B_{k+1,k}\\ \vdots & & \vdots\\ B_{m,1}, & \cdots\cdots, & B_{m,k}\end{pmatrix}=\boldsymbol{0}_{m-k,k}.$$

Put $A_1=(a_{ij})_{1\leqq i,j\leqq k}$, $A_2=(a_{st})_{k+1\leqq s\leqq m,1\leqq t\leqq k}$, $E_1=(E_{ij})_{1\leqq i\leqq k,k+1\leqq j\leqq m}$ and $E_2=(E_{st})_{k+1\leqq s,t\leqq m}$. Then, (2.5)′ can be written as ${}^tA_1E_1+{}^tA_2E_2=\boldsymbol{0}_{k,m-k}$. Since the matrix $A_1$ is non-singular by Lemma 3, we have

$$(2.5)''\qquad E_1=-{}^tA_1^{-1}\cdot{}^tA_2\cdot E_2.$$

Now we define a mapping $\mathcal{L}^3$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ by

$$\mathcal{L}^3:\begin{pmatrix}\tilde{z}\\ \tilde{w}_1\\ \vdots\\ \tilde{w}_m\end{pmatrix}=\left(\begin{array}{c|c|c}1 & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{1}_k & -{}^tA_1^{-1}{}^tA_2\\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1}_{m-k}\end{array}\right)^{-1}\cdot\begin{pmatrix}z\\ w_1\\ \vdots\\ w_m\end{pmatrix}.$$

Then $\mathcal{L}^3$ satisfies the conditions (2.5)$''$ and (2.6)$'$ and hence we have proved Lemma 4. q.e.d.

Now by Lemma 4, we may assume to prove Theorem 1 that

$$\text{(2.7)}\qquad \mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha,\beta=1}^{k} d_{\alpha\beta}\overline{c^\beta}w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^{k}c^\beta\frac{\partial}{\partial w_\beta}\,\middle|\,(c^\beta)\in\boldsymbol{C}^k\right\}.$$

**Lemma 5.** *There exists a non-singular linear mapping* $\mathcal{L}^4$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ *of the form*

$$\tilde{z}=z,\ \tilde{w}_\alpha=\sum_{\lambda=1}^{k}c_{\alpha\lambda}w_\lambda\ (1\leqq\alpha\leqq k)\ \textit{and}\ \tilde{w}_\beta=w_\beta\ (k+1\leqq\beta\leqq m)$$

*such that*

$$\mathcal{L}^4_*\mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha=1}^{k} d_\alpha\overline{c^\alpha}\tilde{w}_\alpha\right)\frac{\partial}{\partial\tilde{z}}+\sum_{\beta=1}^{k}c^\beta\frac{\partial}{\partial\tilde{w}_\beta}\,\middle|\,(c^\beta)\in\boldsymbol{C}^k\right\}$$

*where each* $d_\alpha$ *is a nonzero purely imaginary number depending only on* $\mathcal{D}$.

Proof. By Lemma 4, the matrix $D=(d_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$ in (2.7) is non-singular and skew-hermitian. Hence $D$ can be diagonalized by a suitable unitary matrix $U=(u_{\alpha\beta})_{1\leqq\alpha,\beta\leqq k}$. Put $U^{-1}\cdot D\cdot U=\text{diag.}(d_1,\cdots,d_k)$, where diag. $(d_1,\cdots,d_k)$ denotes the diagonal matrix whose $(l, l)$-component is $d_l$. Then, since $D$ is non-singular and skew-hermitian, each $d_l$ is a nonzero purely imaginary number. Now define a non-singular linear mapping $\mathcal{L}^4$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ by $\tilde{z}=z$, $\tilde{w}_\alpha=\sum_{\lambda=1}^{k}u_{\lambda\alpha}w_\lambda$ $(1\leqq\alpha\leqq k)$ and $\tilde{w}_\beta=w_\beta$ $(k+1\leqq\beta\leqq m)$.

Then it is easy to see that the mapping $\mathcal{L}^4$ satisfies our conditions. q.e.d.

**Proof of Theorem 1**: Suppose first $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k>0$. By Lemma 5 we may assume that

$$\mathfrak{g}_{-1/2}=\left\{\left(\sum_{\alpha=1}^{k} d_\alpha\overline{c^\alpha}w_\alpha\right)\frac{\partial}{\partial z}+\sum_{\beta=1}^{k}c^\beta\frac{\partial}{\partial w_\beta}\,\middle|\,(c_\beta)\in\boldsymbol{C}^k\right\}.$$

Note that each $d_\alpha$ is a nonzero purely imaginary number. For the sake of simplicity, we denote $(w_1, \cdots, w_k)$ and $(w_{k+1}, \cdots, w_m)$ by $w'$ and $w''$, respectively. For $a\in\boldsymbol{R}$ (resp. $t\in\boldsymbol{R}$) we denote by $T_a$ (resp. $\Psi_t$) the holomorphic transformation $(z, w)\mapsto(z+a, w)$ (resp. $(z, w)\mapsto(e^t z, e^{1/2t}w)$) of $C^{m+1}$. Now we define a mapping $\Phi$: $C^k\times C^k\to C$ by

$$\Phi(u, v)=\frac{1}{2\sqrt{-1}}\sum_{\alpha=1}^{k}d_\alpha u^\alpha\overline{v^\alpha}\qquad \text{for } u=(u^\alpha),\ v=(v^\alpha)\in C^k .$$

Then each vector field belonging to $\mathfrak{g}_{-1/2}$ is expressed in the from $2\sqrt{-1}\Phi(w', c)\frac{\partial}{\partial z}+\sum_{\alpha=1}^{k}c^\alpha\frac{\partial}{\partial w_\alpha}$. Since this vector field is determined completely by $c=(c^\alpha)\in C^k$, we write it by $X_c$. By (2.3) the vector field $X_c$ generates the global one-parameter subgroup $\exp tX_c$:

$$(z, w', w'')\mapsto(z+2\sqrt{-1}\Phi(w', tc)+\sqrt{-1}\Phi(tc, tc), w'+tc, w'') .$$

Now we claim that

$$\Phi(c, c)\geqq 0\qquad \text{for all } c\in C^k . \tag{2.8}$$

Suppose that there exists a nonzero vector $c_0\in C^k$ such that $\Phi(c_0, c_0)<0$. Then, for a point $(z_0, 0)\in\mathcal{D}$, we have

$$\exp tX_{c_0}\cdot(z_0, 0)=(z_0+\sqrt{-1}\Phi(tc_0, tc_0), tc_0, 0)$$

for any $t\in R$. Thus, by Lemma 1, $\mathrm{Im}.z_0+\Phi(tc_0, tc_0)>0$ for any $t\in R$. This is impossible since $\Phi(c_0, c_0)<0$. Therefore we get (2.8). In particular, we see that each number $\lambda_\alpha:=d_\alpha/2\sqrt{-1}\,(1\leqq\alpha\leqq k)$ is positive. Now we define a linear mapping $\mathcal{L}^5$: $C\times C^m\to C\times C^m$ by $\tilde{z}=z$, $\tilde{w}_\alpha=\sqrt{\lambda_\alpha}w_\alpha$ $(1\leqq\alpha\leqq k)$ and $\tilde{w}_\beta=w_\beta$ $(k+1\leqq\beta\leqq m)$. Then it is easy to see that

$$\mathcal{L}^5_*\mathfrak{g}_{-1/2}=\left\{2\sqrt{-1}\Big(\sum_{\alpha=1}^{k}\overline{c^\alpha}\tilde{w}_\alpha\Big)\frac{\partial}{\partial\tilde{z}}+\sum_{\alpha=1}^{k}c^\alpha\frac{\partial}{\partial\tilde{w}_\alpha}\,\Big|\,(c^\alpha)\in C^k\right\} .$$

Hence, by considering the image $\tilde{\mathcal{D}}=\mathcal{L}^5(\mathcal{D})$ if necessary, we may assume that

$$\mathfrak{g}_{-1/2}=\left\{2\sqrt{-1}\Big(\sum_{\alpha=1}^{k}\overline{c^\alpha}w_\alpha\Big)\frac{\partial}{\partial z}+\sum_{\alpha=1}^{k}c^\alpha\frac{\partial}{\partial w_\alpha}\,\Big|\,(c^\alpha)\in C^k\right\} .$$

Define a mapping $F$: $C^k\times C^k\to C$ by

$$F(u, v)=\sum_{\alpha=1}^{k}u^\alpha\overline{v^\alpha}\qquad \text{for any } u=(u^\alpha),\ v=(v^\alpha)\in C^k .$$

Then the domain

$$\mathcal{E}=\{(z, w', 0)\in C\times C^m\,|\,\mathrm{Im}.z-F(w', w')>0\}$$

is an elementary Siegel domain. Now we put

$$\mathcal{D}_{\sqrt{-1}}=\{w''\in C^{m-k}\,|\,(\sqrt{-1}, 0, w'')\in\mathcal{D}\} .$$

We shall show that $\mathcal{D}_{\sqrt{-1}}$ is connected. Take two points $P_0=(\sqrt{-1}, 0, w_0'')$ and $P_1=(\sqrt{-1}, 0, w_1'')$ of $\mathcal{D}$. Then there exists a continuous path $\Gamma$: $[0, 1] \to \mathcal{D}$ such that $\Gamma(0)=P_0$ and $\Gamma(1)=P_1$. For any $t\in[0, 1]$, we put $\Gamma(t)=(z(t), w'(t), w''(t))$, where $z(t)\in \boldsymbol{C}$, $w'(t)\in \boldsymbol{C}^k$ and $w''(t)\in \boldsymbol{C}^{m-k}$. Since

$$\begin{aligned} &T_{-Re.z(t)}\cdot \exp X_{-w'(t)}\cdot(z(t), w'(t), w''(t)) \\ &= (\sqrt{-1}(\mathrm{Im}.z(t)-F(w'(t), w'(t))), 0, w''(t)), \end{aligned}$$

we see that $\mathrm{Im}.z(t)-F(w'(t), w'(t))>0$ for any $t\in[0, 1]$ by Lemma 1. Thus we can define a continous function $l(t)$ on $[0,1]$ by $l(t)=log(\mathrm{Im}.z(t)-F(w'(t), w'(t)))$. Then it is obvious that $l(0)=l(1)=0$ and $e^{l(t)}=\mathrm{Im}.z(t)-F(w'(t), w'(t))$ for any $t\in[0, 1]$. Thus the point

$$(\sqrt{-1}, 0, e^{-1/2l(t)}w''(t)) = (e^{-l(t)}e^{l(t)}\cdot\sqrt{-1}, 0, e^{-1/2l(t)}w''(t))$$

belongs to $\mathcal{D}$ by the definition of $\mathcal{D}$. Put $g(t)=e^{-1/2l(t)}w''(t)$. Then $g(t)\in\mathcal{D}_{\sqrt{-1}}$ for nay $t\in[0, 1]$, $g(0)=w_0''$ and $g(1)=w_1''$. Thus $\mathcal{D}_{\sqrt{-1}}$ is connected. It is obvious that $\mathcal{D}_{\sqrt{-1}}$ is a circular domain in $\boldsymbol{C}^{m-k}$ containing the origin 0 by the definition of the generalized Siegel domain. Let $(z,w',w'')$ be a point of $\mathcal{D}$. Then there exists a real number $t_0$ such that $e^{t_0}=\mathrm{Im}.z-F(w', w')$, because $T_{-Re.z}\cdot\exp X_{-w'}\cdot(z, w', w'')=(\sqrt{-1}(\mathrm{Im}.z-F(w', w')), 0, w'')$ belongs to $\mathcal{D}$ and hence $\mathrm{Im}.z-F(w', w')>0$ by Lemma 1. Thus we have $\Psi_{-t_0}\cdot T_{-Re.z}\cdot\exp X_{-w'}\cdot (z, w', w'')=(\sqrt{-1}, 0, e^{-t_0/2}w'')$. Hence $(\mathrm{Im}.z-F(w', w'))^{-1/2}\cdot w''\in\mathcal{D}_{\sqrt{-1}}$, and so $\mathcal{D}$ is contained in the set

$$\{(z, w', w'')\in \boldsymbol{C}\times \boldsymbol{C}^m \,|\, \mathrm{Im}.z-F(w', w')>0,\ (\mathrm{Im}.z-F(w', w'))^{-1/2}\cdot w''\in\mathcal{D}_{\sqrt{-1}}\}\ .$$

Conversely, take a point $(z, w', w'')\in \boldsymbol{C}\times \boldsymbol{C}^m$ such that $\mathrm{Im}.z-F(w', w')>0$ and $(\mathrm{Im}.z-F(w', w'))^{-1/2}\cdot w''\in\mathcal{D}_{\sqrt{-1}}$. Then, by the same way as above, we can show that there exists a real number $t_0$ such that $e^{t_0}=\mathrm{Im}.z-F(w', w')$ and

$$T_{Re.z}\cdot\exp X_{w'}\cdot\Psi_{t_0}\cdot(\sqrt{-1}, 0, e^{-t_0/2}w'') = (z, w', w'')\ .$$

This shows that $(z, w', w'')\in\mathcal{D}$, since $(\sqrt{-1}, 0, e^{-t_0/2}w'')\in\mathcal{D}$ by the definition of $\mathcal{D}_{\sqrt{-1}}$. Therefore

$$\begin{aligned} \mathcal{D} = \{&(z, w', w'')\in \boldsymbol{C}\times \boldsymbol{C}^m \,|\, \mathrm{Im}.z-F(w', w')>0\,, \\ &(\mathrm{Im}.z-F(w', w'))^{-1/2}\cdot w''\in\mathcal{D}_{\sqrt{-1}}\}\ . \end{aligned}$$

Now we shall show that the orbit $\mathcal{D}_0$ of $\mathrm{Aut}_0(\mathcal{D})$ containing the point $(\sqrt{-1}, 0)\in\mathcal{D}$ coincides with the elementary Siegel domain $\mathcal{E}$. Let $(z, w', 0)\in\mathcal{E}$. Since $\mathrm{Im}.z-F(w', w')>0$, there exists a real number $t_0$ such that $e^{t_0}=\mathrm{Im}.z-F(w', w')$. Then it is easy to see that $T_{Re.z}\cdot\exp X_{w'}\cdot\Psi_{t_0}\cdot(\sqrt{-1}, 0)=(z, w', 0)$, and so $\mathcal{E}\subset\mathrm{Aut}_0(\mathcal{D})\cdot(\sqrt{-1}, 0)=\mathcal{D}_0$. We claim that $\mathcal{D}_0\subset\mathcal{E}$. Let $G$

be the identity component $\mathrm{Aut}_0(\mathcal{D})$ of $\mathrm{Aut}(\mathcal{D})$, $K$ the isotropy subgroup of $G$ at $(\sqrt{-1}, 0)$ and $G_a$ the identity component of $\mathrm{Aff}(\mathcal{D})$. Put $K_a=G_a\cap K$. Then we can show that $G/K=G_a/K_a$ by the same way as Lemma 2.3. of Nakajima [5]. Therefore it is enough to see that $G_a\cdot(\sqrt{-1}, 0)\subset\mathcal{E}$. Let $P(\mathcal{D})$ (resp. $GL_0(\mathcal{D})$) be the analytic subgroup of $G_a$ generated by the subalgebra $\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}$ (resp. $\mathfrak{g}_0$.) Then we have $G_a=P(\mathcal{D})\cdot GL_0(\mathcal{D})$ (semi-direct product), because $P(\mathcal{D})\cdot GL_0(\mathcal{D})$ is an abstract subgroup of $G_a$ and contains an open neighborhood of the identity element of $G_a$. Since $GL_0(\mathcal{D})\cdot(\sqrt{-1}, 0)\subset\mathcal{D}_1$ by (5), of Theorem $A$ and obviously $P(\mathcal{D})\cdot\mathcal{E}\subset\mathcal{E}$, we get $G_a\cdot(\sqrt{-1}, 0)\subset\mathcal{E}$. Therefor $G\cdot(\sqrt{-1}, 0)=G_a\cdot(\sqrt{-1}, 0)=\mathcal{E}$. This completes the first case where $k>0$.

It remains the case where $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=0$, *i.e.*, $k=0$. But in this case Theorem 1 is now obvious from the proof of the case where $k>0$. q.e.d.

Corollaries of Theorem 1: As an immediate consequence of Theorem 1 we obtain the following corollary.

**Corollary 1.** *Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2m$. Then $\mathcal{D}$ is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain*

$$\mathcal{E}=\{(z, w_1, \cdots, w_m)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid \mathrm{Im}.z-\sum_{\alpha=1}^{m}|w_\alpha|^2>0\}\ .$$

**Corollary 2.** *There exists no generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent* 1/2 *such that* $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2m-2$.

Proof. Suppose that there exists a generalized Siegel domain $\mathcal{D}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2m-2$. Then, by Theorem 1 there exists a generalized Siegel domain $\tilde{\mathcal{D}}$ with exponent 1/2 which is holomorphically equivalent to $\mathcal{D}$ and is expressed in the following form with respect to a suitable coordinates system $(z, w_1, \cdots, w_m)$ in $\boldsymbol{C}\times\boldsymbol{C}^m$:

$$\tilde{\mathcal{D}}=\{(z, w_1, \cdots, w_m)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid \mathrm{Im}.z-\sum_{\alpha=1}^{m-1}|w_\alpha|^2>0\ ,$$

$$(\mathrm{Im}.z-\sum_{\alpha=1}^{m-1}|w_\alpha|^2)^{-1/2}\cdot w_m\in\tilde{\mathcal{D}}_{\sqrt{-1}}\}$$

where $\tilde{\mathcal{D}}_{\sqrt{-1}}$ is a circular domain in $\boldsymbol{C}$ containing the origin of $\boldsymbol{C}$. Since $\tilde{\mathcal{D}}_{\sqrt{-1}}$ is given by $\tilde{\mathcal{D}}_{\sqrt{-1}}=\{w_m\in\boldsymbol{C} \mid |w_m|<R\}$ for some positive number $R$,

$$\tilde{\mathcal{D}}=\{(z, w_1, \cdots, w_m)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid \mathrm{Im}.z-(\sum_{\alpha=1}^{m-1}|w_\alpha|^2+R^{-2}|w_m|^2)>0\}\ .$$

Thus $\tilde{\mathcal{D}}$ is a Siegel domain of the second kind in $\boldsymbol{C}\times\boldsymbol{C}^m$. Then we see that $\dim_{\boldsymbol{R}}\tilde{\mathfrak{g}}_{-1/2}=2m$ in the decomposition of $\mathfrak{g}(\tilde{\mathcal{D}})$ as in Theorem A. But this is a contradiction since $\dim_{\boldsymbol{R}}\tilde{\mathfrak{g}}_{-1/2}=\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2m-2$ by our assumption. q.e.d.

**Corollary 3.** *Let* $\tilde{\mathcal{D}}$ *and* $\tilde{\mathcal{D}}_0$ *be the same domains as in Theorem* 1 *and* $\prod$: $\mathfrak{g}(\tilde{\mathcal{D}})\to\mathfrak{g}(\tilde{\mathcal{D}}_0)$ *the homomorphism induced by the Lie group homomorphism of* $Aut_0(\tilde{\mathcal{D}})$ *to* $Aut_0(\tilde{\mathcal{D}}_0)$ *defined by* $g\mapsto g|\tilde{\mathcal{D}}_0$, *where* $g|\tilde{\mathcal{D}}_0$ *denotes the restriction of* $g$ *to* $\tilde{\mathcal{D}}_0$. *Then* $\prod$ *is surjective.*

Proof. Note that $\tilde{\mathcal{D}}_0$ is the $\mathrm{Aut}_0(\tilde{\mathcal{D}})$-orbit. Let $(z, w_1, \cdots, w_m)$ be the coordinates system in $\boldsymbol{C}\times\boldsymbol{C}^m$ as in Theorem 1. Let $\mathfrak{g}(\tilde{\mathcal{D}})=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ (resp. $\mathfrak{g}(\tilde{\mathcal{D}}_0)=\mathfrak{g}^o_{-1}+\mathfrak{g}^o_{-1/2}+\mathfrak{g}^o_0+\mathfrak{g}^o_{1/2}+\mathfrak{g}^o_1$) be the decomposition of $\mathfrak{g}(\tilde{\mathcal{D}})$ (resp. $\mathfrak{g}(\tilde{\mathcal{D}}_0)$) as in Theorem A. Since $\tilde{\mathcal{D}}_0$ is an elementary Sigel domain, $\mathfrak{g}(\tilde{\mathcal{D}}_0)$ is simple. In particular, we have

$$(2.9)\qquad \begin{aligned} \mathfrak{g}^o_0 &= [\mathfrak{g}^o_{-1/2}, \mathfrak{g}^o_{1/2}]+[\mathfrak{g}^o_{-1}, \mathfrak{g}^o_1] \text{ and} \\ \mathfrak{g}^o_{1/2} &= [\mathfrak{g}^o_{-1/2}, \mathfrak{g}^o_1]\,. \end{aligned}$$

Put $\partial^o = z\dfrac{\partial}{\partial z}+\dfrac{1}{2}\sum_{\alpha=1}^{k} w_\alpha \dfrac{\partial}{\partial w_\alpha}$. Then it is obvious that $\prod(\partial)=\partial^o$. Hence the homomorphism $\prod$ preserves the gradition, *i.e.*, $\prod(\mathfrak{g}_\lambda)\subset\mathfrak{g}^o_\lambda$. Now we shall show that $\prod$ is injective on $\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}+\mathfrak{g}_1$. Since $\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}=\mathfrak{g}^o_{-1}+\mathfrak{g}^o_{-1/2}$, it is sufficient to show that $\prod$ is injective on $\mathfrak{g}_{1/2}+\mathfrak{g}_1$. Let $X_1\in\mathfrak{g}_1$ such that $\prod(X_1)=0$. Then $\prod\left(\left[\dfrac{\partial}{\partial z}, \left[\dfrac{\partial}{\partial z}, X_1\right]\right]\right)=0$. Since $\left[\dfrac{\partial}{\partial z}, \left[\dfrac{\partial}{\partial z}, X_1\right]\right]\in\mathfrak{g}_{-1}$ and $\prod$ is identity on $\mathfrak{g}_{-1}$, we have $\left[\dfrac{\partial}{\partial z}, \left[\dfrac{\partial}{\partial z}, X_1\right]\right]=0$. On the other hand, it is known that the endomorphism $\left(ad\left(\dfrac{\partial}{\partial z}\right)\right)^2$: $\mathfrak{g}_1\to\mathfrak{g}_{-1}$ is injective (cf. [9]). Thus we get $X_1=0$. Therefore $\prod$ is injective on $\mathfrak{g}_1$. Analogously we can show that $\prod$ is injective on $\mathfrak{g}_{1/2}$ by using the injectiveity of $ad\left(\dfrac{\partial}{\partial z}\right)$: $\mathfrak{g}_{1/2}\to\mathfrak{g}_{-1/2}$. Note that the subalgebra $\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ corresponds to the subgroup leaving the upper half plane $\mathcal{D}_1=\{(z, 0)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid \mathrm{Im}.z>0\}$ invariant. Now we claim that each element of $\mathrm{Aut}_0(\mathcal{D}_1)$ can be extended to an element of $\mathrm{Aut}_0(\tilde{\mathcal{D}})$. We identify $\mathrm{Aut}_0(\mathcal{D}_1)$ with $SL(2, \boldsymbol{R})/\{\pm 1_2\}$. Since each element $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2, \boldsymbol{R})$ acts on $\mathcal{D}_1$ by a holomorphic transformation $l_\gamma$: $z\mapsto(az+b)\,(cz+d)^{-1}$, we can define a mapping $\tilde{l}_\gamma$: $\mathcal{D}_1\times\boldsymbol{C}^m\to\mathcal{D}_1\times\boldsymbol{C}^m$ by $\tilde{l}_\gamma(z, w)=(l_\gamma(z), (cz+d)^{-1}w)$. Since $\tilde{l}_{\gamma_1\cdot\gamma_2}=\tilde{l}_{\gamma_1}\cdot\tilde{l}_{\gamma_2}$ for any $\gamma_1, \gamma_2\in SL(2, \boldsymbol{R})$, $\tilde{l}_\gamma$ induces a holomorphic transformation of $\tilde{\mathcal{D}}$ if

$$(2.10)\qquad \tilde{l}_\gamma(\tilde{\mathcal{D}})\subset\tilde{\mathcal{D}}\,.$$

Put $w'=(w_1, \cdots, w_k)$, $w''=(w_{k+1}, \cdots, w_m)$ and $\|w'\|=(\sum_{\alpha=1}^{k}|w_\alpha|^2)^{1/2}$ for any $w=(w_1, \cdots, w_m)\in\boldsymbol{C}^m$. Then

$$(2.11)\qquad \mathrm{Im}.\ l_\gamma(z)-\|(cz+d)^{-1}w'\|^2 = |cz+d|^{-2}(\mathrm{Im}.z-\|w'\|^2)>0$$

for any $(z, w', w'')\in\tilde{\mathscr{D}}$. Since

$$\begin{aligned}&\mathrm{Im}.\ l_\gamma(z)-||(cz+d)^{-1}w'||^2)^{-1/2}\cdot(cz+d)^{-1}\cdot w''\\&=e^{\sqrt{-1}\theta(z,\gamma)}(\mathrm{Im}.z-||w'||^2)^{-1/2}\cdot w'',\end{aligned}$$

where $\theta(z,\gamma)=-\arg.(cz+d)$, and $e^{\sqrt{-1}\theta(z,\gamma)}(\mathrm{Im}.z-||w'||^2)^{-1/2}w''\in\tilde{\mathscr{D}}_{\sqrt{-1}}$, we have

$$(2.12)\quad (\mathrm{Im}.\ l_\gamma(z)-||(cz+d)^{-1}w'||^2)^{-1/2}\cdot(cz+d)^{-1}\cdot w''\in\tilde{\mathscr{D}}_{\sqrt{-1}}\,.$$

By (2) of Theorem 1, (2.11) and (2.12) imply (2.10). Hence we get $\mathfrak{g}_1\neq 0$ and hence $\Pi(\mathfrak{g}_1)\neq 0$. We now prove that $\Pi$ is surjective. Since $\dim_{\boldsymbol{R}}\mathfrak{g}_1^o=1$ and $\Pi(\mathfrak{g}_1)\neq 0$, we get $\Pi(\mathfrak{g}_1)=\mathfrak{g}_1^o$. Therefore it follows that $\mathfrak{g}_{1/2}^o=[\mathfrak{g}_{-1/2}^o, \mathfrak{g}_1^o]=\Pi([\mathfrak{g}_{-1/2}, \mathfrak{g}_1])\subset\Pi(\mathfrak{g}_{1/2})$, and so $\Pi(\mathfrak{g}_{1/2})=\mathfrak{g}_{1/2}^o$. Then $\mathfrak{g}_0^o=[\mathfrak{g}_{-1/2}^o, \mathfrak{g}_{1/2}^o]+[\mathfrak{g}_{-1}^o, \mathfrak{g}_1^o]=\Pi([\mathfrak{g}_{-1/2}, \mathfrak{g}_{1/2}]+[\mathfrak{g}_{-1}, \mathfrak{g}_1])\subset\Pi(\mathfrak{g}_0)$, and so $\Pi(\mathfrak{g}_0)=\mathfrak{g}_0^o$. Therefore $\Pi$ is surjective. q.e.d.

**Corollary 4.** *Let $\mathscr{D}$ be a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2. If the Lie algebra $\mathfrak{g}(\mathscr{D})$ is semi-simple, then $\mathscr{D}$ is a Siegel domain which is holomorphically equivalent to the elementary Siegel domain*

$$\mathscr{E}=\{(z, w_1, \cdots, w_m)\in\boldsymbol{C}\times\boldsymbol{C}^m\,|\,\mathrm{Im}.z-\sum_{\alpha=1}^{m}|w_\alpha|^2>0\}\,.$$

Proof. We claim that $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2m$, *i.e.*, $k=m$. Then our assertion is obvious by Corollary 1. We may assume $\mathscr{D}=\tilde{\mathscr{D}}$ in Theorem 1 without loss of generality. Suppose that $k\lneqq m$. We consider first the case where $k>0$. Let $\Pi\colon \mathfrak{g}(\tilde{\mathscr{D}})\to(\tilde{\mathscr{D}}_0)$ be the homomorphism defined in Corollary 3. Then $\Pi$ is surjective by Corollary 3. Put $\mathfrak{s}_2=\mathrm{Ker}\,\Pi$. Then $\mathfrak{s}_2$ is a semi-simple ideal of the semi-simple Lie algebra $\mathfrak{g}(\tilde{\mathscr{D}})$. Thus there exists a semi-simple ideal $\mathfrak{s}_1$ such that $\mathfrak{g}(\tilde{\mathscr{D}})=\mathfrak{s}_1+\mathfrak{s}_2$ (direct sum). Since $\mathfrak{s}_1$ is isomorphic to $\mathfrak{g}(\tilde{\mathscr{D}}_0)$, $\mathfrak{s}_1$ is simple. Since $\Pi$ is injective on $\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ by the proof of Corollary 3, $\mathfrak{s}_2$ is contained in $\mathfrak{g}_0$. Let $B$ denote the Killing form of $\mathfrak{g}(\tilde{\mathscr{D}})$. Put $\mathfrak{g}_0^1=\{X\in\mathfrak{g}_0\,|\,B(X, \mathfrak{s}_2)=0\}$. Noting that the ideal $\mathfrak{s}_1$ is a graded Lie subalgebra, it is easy to see that $\mathfrak{g}_0=\mathfrak{g}_0^1+\mathfrak{s}_2$, $\mathfrak{s}_1=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0^1+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ and $\mathfrak{g}_0^1=[\mathfrak{g}_{-1/2}, \mathfrak{g}_{1/2}]$. Since $\mathfrak{s}_2=\mathrm{Ker}\,\Pi\subset\mathfrak{g}_0$, every vector field $X\in\mathfrak{s}_2$ is given by $X=\sum_{\alpha=k+1}^{m}Q_{0,1}^{\alpha}\dfrac{\partial}{\partial w_\alpha}$ in Theorem A. Thus it can be expressed by the matrix

$$(2.13)\qquad X=\left(\begin{array}{c|c|c}0 & \boldsymbol{0} & \boldsymbol{0}\\\hline \boldsymbol{0} & \boldsymbol{0}_{k,k} & C\\\hline \boldsymbol{0} & \boldsymbol{0}_{m-k,k} & D\end{array}\right).$$

Now we claim that $C=\boldsymbol{0}_{k,m-k}$ in (2.13). Let $S_1$ (resp. $S_2$) be the analytic subgroup of $\mathrm{Aut}_0(\tilde{\mathscr{D}})$ corresponding to $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$). Obviously

$$(2.14) \qquad g_1 \cdot g_2 = g_2 \cdot g_1 \qquad \text{for any } g_1 \in S_2 \text{ and } g_2 \in S_2.$$

Let $X_c (c \in \boldsymbol{C}^k)$ be the vector field belonging to $\mathfrak{g}_{-1/2}$ defined in the proof of Theorem 1. Put $g_1 = \exp X_c$ and

$$g_2 = \exp X = \left(\begin{array}{c:c:c} 1 & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{1}_k & A \\ \hdashline \mathbf{0} & \mathbf{0} & E \end{array}\right).$$

It is easy to see that if $A = \mathbf{0}_{k,m-k}$, then $C = \mathbf{0}$. By a routine calculation, we get

$$g_1 \cdot g_2 \cdot (z, w', w'') = (z + 2\sqrt{-1} F(w' + Aw'', c) + \sqrt{-1} F(c, c), w' + Aw'' + c, Ew'')$$

and

$$g_2 \cdot g_1 (z, w', w'') = (z + 2\sqrt{-1} F(w', c) + \sqrt{-1} F(c, c), w' + c + Aw'', Ew'')$$

for any $(z, w', w'') \in \tilde{\mathscr{D}}$. By (2.14), we get $F(w' + Aw'', c) = F(w', c)$ and hence $F(Aw'', c) = 0$. Since $c$ is arbitrary, we get $Aw'' = 0$ for any element $w''$ of an open subset of $\boldsymbol{C}^{m-k}$. Thus $A = \mathbf{0}$. Therefore we get

$$(2.15) \qquad \mathfrak{s}_2 = \left\{ \left(\begin{array}{c:c} \mathbf{0}_{k+1,k+1} & \mathbf{0} \\ \hdashline \mathbf{0} & * \end{array}\right) \right\} \text{ and } S_2 = \left\{ \left(\begin{array}{c:c} \mathbf{1}_{k+1} & \mathbf{0} \\ \hdashline \mathbf{0} & * \end{array}\right) \right\}.$$

Since $\tilde{\mathscr{D}}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$ and any bounded domain in $\boldsymbol{C}^{m+1}$ is hyperbolic in the sense of Kobayashi [4], $\tilde{\mathscr{D}}$ is hyperbolic. Since $\tilde{\mathscr{D}}_{\sqrt{-1}}$ is a complex submanifold of $\tilde{\mathscr{D}}$, it is also hyperbolic. Thus $\tilde{\mathscr{D}}_{\sqrt{-1}}$ is a hyperbolic circular domain in $\boldsymbol{C}^{m-k}$ containing the origin 0. By §.1, we have that $\mathrm{Aut}_0(\tilde{\mathscr{D}}_{\sqrt{-1}})$ is a Lie group and its isotropy subgroup $K_{\sqrt{-1}}$ at $0 \in \tilde{\mathscr{D}}_{\sqrt{-1}}$ is compact. Moreover $K_{\sqrt{-1}}$ is a subgroup of $GL(m-k, \boldsymbol{C})$ by Theorem B. Let $\mathfrak{k}_{\sqrt{-1}}$ be the subalgebra of $\mathfrak{g}(\tilde{\mathscr{D}}_{\sqrt{-1}})$ corresponding to $K_{\sqrt{-1}}$. Now we claim that $\mathfrak{k}_{\sqrt{-1}}$ can be identified with $\mathfrak{s}_2$. By (2.15) we can identify $S_2$ with a subgroup of $K_{\sqrt{-1}}$. Conversely, let $K^0{}_{\sqrt{-1}}$ be the identity component of $K_{\sqrt{-1}}$ and take an arbitrary element $g \in K^0{}_{\sqrt{-1}}$. Put $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$, where $1 = \mathbf{1}_{k+1}$. Then we can easily see that $\tilde{g}$ leaves $\tilde{\mathscr{D}}$ invariant by (2) of Theorem 1, and hence $\tilde{g}$ defines a holomorphic transformation of $\tilde{\mathscr{D}}$ and $\tilde{g} \in S_2$ by (2.15). Thus $K^0{}_{\sqrt{-1}}$ can be identified with $S_2$ in a natural way. In particular, $\mathfrak{k}_{\sqrt{-1}}$ is a semi-simple Lie algebra. On the other hand, $\mathfrak{k}_{\sqrt{-1}}$ contains a nonzero element $\partial'' = \sqrt{-1} \sum_{\alpha=k+1}^{m} w_\alpha \frac{\partial}{\partial w_\alpha}$ induced by the global one-parameter subgroup $w'' \mapsto e^{\sqrt{-1}t} w''$ $(t \in \boldsymbol{R})$ and obviously $\partial''$ belongs to the center of $\mathfrak{k}_{\sqrt{-1}}$. This is a contradiction.

Suppose next $k=0$. Then we can show as above that the Lie algebra $\mathfrak{k}_{\sqrt{-1}}$ is identified with the semi-simple Lie algebra

$$\mathrm{Ker}\ \Pi = \left\{\left(\begin{array}{c|c} 0 & \mathbf{0}_{1,m} \\ \hline \mathbf{0}_{m,1} & * \end{array}\right)\right\}.$$

On the other hand, $\mathfrak{k}_{\sqrt{-1}}$ contains a nonzero element $\partial'=\sqrt{-1}\sum_{\alpha=1}^{m} w_\alpha \frac{\partial}{\partial w_\alpha}$ belonging to the center. This is a contradiction.

Therefore $k=m$, and we complete the proof. q.e.d.

## 3. The structure of Aut ($\mathcal{D}$)

The purpose of this section is to consider the structure of the group of all holomorphic transformations of a generalized Siegel domain $\mathcal{D}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2}= 2k$ for some $k$, $0\leqq k\leqq m$.

In this section we use the following notations. For a point

$$\mathfrak{z} = {}^t(z^1, \cdots, z^{k+1})\in \boldsymbol{C}^{k+1}, \text{ define } ||\mathfrak{z}|| = (\sum_{j=1}^{k+1} |z^j|^2)^{1/2}.$$

Put

$$U(k+1, 1) = \left\{g\in GL(k+2, \boldsymbol{C})\,|\,{}^t\bar{g}\cdot\left(\begin{array}{c|c} \mathbf{1}_{k+1} & 0 \\ \hline 0 & -1 \end{array}\right)\cdot g = \left(\begin{array}{c|c} \mathbf{1}_{k+1} & 0 \\ \hline 0 & -1 \end{array}\right)\right\}$$

and

$$SU(k+1, 1) = U(k+1, 1)\cap SL(k+2, \boldsymbol{C}).$$

For each element $\gamma=\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in SU(k+1, 1)$, where $A=(a_{ij})_{1\leqq i,j\leqq k+1}$, $\mathfrak{b}={}^t(b_1, \cdots, b_{k+1})$ and $\mathfrak{c}=(c_1, \cdots, c_{k+1})$, we put

$$(3.1)\qquad \begin{cases} L_j(\gamma) = (a_{j1}+b_j, 2a_{j2}, 2a_{j3}, \cdots, 2a_{j,k+1}); \\ C(\gamma) = (c_1+d, 2c_2, 2c_3, \cdots, 2c_{k+1}); \\ B_j(\gamma) = \sqrt{-1}(b_j-a_{j1}) \text{ and } D(\gamma) = \sqrt{-1}(d-c_1) \end{cases}$$

for $j=1, 2, \cdots, k+1$.

It is easy to see that $U(k+1, 1)$ coincides with all matrices $\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in GL(k+2,\boldsymbol{C})$ of the form ${}^t\bar{A}A-{}^t\bar{\mathfrak{c}}\mathfrak{c}=\mathbf{1}_{k+1}$, ${}^t\bar{\mathfrak{b}}\mathfrak{b}-|d|^2=-1$ and ${}^t\bar{\mathfrak{b}}A-\bar{d}\mathfrak{c}=0_{1,k+1}$. From this, we get

$$(3.2)\qquad |\mathfrak{c}\mathfrak{z}+d|^2-||A\mathfrak{z}+\mathfrak{b}||^2 = 1-||\mathfrak{z}||^2$$

for any $\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in U(k+1, 1)$ and any $\mathfrak{z}\in\boldsymbol{C}^{k+1}$, by an easy computation.

Now we consider the group Aut ($\mathcal{E}$) of all holomorphic transformations of the elementary Siegel domain

$$\mathcal{E}= \{(z, w_1, \cdots, w_k)\in \boldsymbol{C}\times \boldsymbol{C}^k \mid \mathrm{Im}.z-\sum_{\alpha=1}^{k} |w_\alpha|^2>0\} .$$

The elementary Siegel domain $\mathcal{E}$ is holomorphically equivalent to the unit open ball $\mathscr{B}=\{\mathfrak{z}={}^t(z^1, \cdots, z^{k+1})\in \boldsymbol{C}^{k+1} \mid \|\mathfrak{z}\|<1\}$. In fact, the biholomorphic isomorphism $\phi$: $\mathcal{E}\to\mathscr{B}$ is given by

$$(3.3) \qquad z^1 = (z-\sqrt{-1})(z+\sqrt{-1})^{-1},\ z^j = 2w_{j-1}(z+\sqrt{-1})^{-1}$$

for $j=2, 3, \cdots, k+1$. It is well-known that the group $\mathrm{Aut}_0(\mathscr{B})$ can be identified with the simple Lie group $SU(k+1,1)$ and each element $\gamma=\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in SU(k+1,1)$ acts on $\mathscr{B}$ by the holomorphic transformation $\sigma_\gamma$: $\mathfrak{z}\mapsto(A\mathfrak{z}+\mathfrak{b})(\mathfrak{c}\mathfrak{z}+d)^{-1}$. Define $\Psi^0_\gamma=\phi^{-1}\cdot\sigma_\gamma\cdot\phi$ for each $\gamma\in SU(k+1, 1)$. Then it is obvious that $\Psi^0_\gamma$ defines a holomorphic transformation of $\mathcal{E}$. By a direct calculation, we see that the action of $\Psi^0_\gamma$ on $\mathcal{E}$ is given by

$$\begin{cases} z \mapsto \sqrt{-1}\,\dfrac{1+(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))} \\ w_j \mapsto \sqrt{-1}\,\dfrac{(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_{j+1}(\gamma)Z+B_{j+1}(\gamma))}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))} \end{cases}$$

for $j=1, 2, \cdots, k$, where $Z={}^t(z, w_1, \cdots, w_k)\in\mathcal{E}$ and $C(\gamma)$, $L_j(\gamma)$, $B_j(\gamma)$, $D(\gamma)$ are defined by (3.1).

Let $K^0{}_{\sqrt{-1}}$ be the identity component of the isotropy subgroup of $\mathrm{Aut}(\tilde{\mathscr{D}}_{\sqrt{-1}})$ at the origin $0\in\tilde{\mathscr{D}}_{\sqrt{-1}}$. We define a mapping $\Psi_{\gamma,K}$: $\tilde{\mathscr{D}}_0\times \boldsymbol{C}^{m-k}\to\tilde{\mathscr{D}}_0\times \boldsymbol{C}^{m-k}$ for each $\gamma\in SU(k+1, 1)$ and $K\in K^0{}_{\sqrt{-1}}$ as follows:

$$\Psi_{\gamma,K}\colon \begin{cases} z \mapsto \sqrt{-1}\,\dfrac{1+(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))} \\ w_j \mapsto \sqrt{-1}\,\dfrac{(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_{j+1}(\gamma)Z+B_{j+1}(\gamma))}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))} \\ \qquad\qquad \text{for } j=1, 2, \cdots, k\,. \\ W \mapsto K\cdot \dfrac{2\sqrt{-1}\,(C(\gamma)Z+D(\gamma))^{-1}}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))}\cdot W \end{cases}$$

for $Z={}^t(z, w_1, \cdots, w_k)\in\tilde{\mathscr{D}}_0$ and $W={}^t(w_{k+1}, \cdots, w_m)\in \boldsymbol{C}^{m-k}$. Since $\tilde{\mathscr{D}}_0=\{(z, w_1, \cdots, w_k, 0, \cdots, 0)\in \boldsymbol{C}\times \boldsymbol{C}^m \mid \mathrm{Im}.z-\sum_{\alpha=1}^{k} |w_\alpha|^2>0\}=\mathcal{E}$, $\Psi_{\gamma,K}$ is a well-defined holomorphic mapping of $\tilde{\mathscr{D}}_0\times \boldsymbol{C}^{m-k}$ into itself.

Now we can state Theorem 2.

**Theorem 2.** *Let $\Psi_{\gamma,K}: \tilde{\mathscr{D}}_0\times C^{m-k}\to\tilde{\mathscr{D}}_0\times C^{m-k}$ be the holomorphic mapping defined as above. Then $\Psi_{\gamma,K}$ induces a holomorphic transformation of $\tilde{\mathscr{D}}$, and moreover any holomorphic transformation of $\tilde{\mathscr{D}}$ belonging to the identity component of Aut $(\tilde{\mathscr{D}})$ is of this form, i.e.,*

$$\mathrm{Aut}_0(\tilde{\mathscr{D}}) = \{\Psi_{\gamma,K}\,|\,\gamma\in SU(k+1, 1),\ K\in K^0{}_{\sqrt{-1}}\}\ .$$

Proof. Let $(z, w_1, \cdots, w_m)$ be the coordinates system in $C\times C^m$ defined in Theorem 1. We put $w'=(w_1, \cdots, w_k)$, $w''=(w_{k+1}, \cdots, w_m)$ and $||w'||=(\sum_{\alpha=1}^{k}|w_\alpha|^2)^{1/2}$ as before. First we claim that each element $\Psi_\gamma^0\in\mathrm{Aut}_0(\mathscr{E})=\mathrm{Aut}_0(\mathscr{D}_0)$ can be extended to a holomorphic transformation of $\tilde{\mathscr{D}}$. We consider the following mappings:

$$w_s\mapsto\tilde{w}_s:=\frac{2\sqrt{-1}\,(C(\gamma)Z+D(\gamma))^{-1}w_s}{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))}$$

for $s=k+1, k+2, \cdots, m$. Put $\Psi_\gamma^0={}^t(\Psi_\gamma^{0,1}, \cdots, \Psi_\gamma^{0,k+1})$. We shall show that

$$({}^t(\Psi_\gamma^0(Z)),\ \tilde{w}_{k+1}, \cdots, \tilde{w}_m)\in\tilde{\mathscr{D}} \tag{3.4}$$

for any $(z, w)=({}^tZ, w_{k+1}, \cdots, w_m)\in\tilde{\mathscr{D}}$.

Put $(\Psi_\gamma^0(Z))_w=(\Psi_\gamma^{0,2}(Z), \cdots, \Psi_\gamma^{0,k+1}(Z))$. If we show the following two conditions

$$\mathrm{Im.}\ \Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2>0 \quad\text{and} \tag{3.5}$$

$$(\mathrm{Im.}\ \Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2)^{-1/2}\cdot\tilde{w}''\in\tilde{\mathscr{D}}_{\sqrt{-1}}\ , \tag{3.6}$$

where $\tilde{w}''=(\tilde{w}_{k+1}, \cdots, \tilde{w}_m)$, then (3.4) will follow from Theorem 1. The condition (3.5) is obvious, since $\Psi_\gamma^0$ is a holomorphic transformation of $\tilde{\mathscr{D}}_0$. By routine calculations, we get

$$\mathrm{Im.}\ \Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2 =\frac{1-\sum_{j=1}^{k+1}|(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_j(\gamma)Z+B_j(\gamma))|^2}{|1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))|^2}\ ,$$

and hence

$$(\mathrm{Im.}\ \Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2)^{-1/2}\cdot\tilde{w}_s =\frac{2e^{\sqrt{-1}\theta(Z,\gamma)}w_s}{|C(\gamma)Z+D(\gamma)|\cdot(1-\sum_{j=1}^{k+1}|(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_j(\gamma)Z+B_j(\gamma))|^2)^{1/2}}$$

where
$$\theta(Z,\gamma)=-\arg.\ \{1-(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_1(\gamma)Z+B_1(\gamma))\} -\arg.\ (C(\gamma)Z+D(\gamma))+\pi/2\ .$$

Let $\phi$ be the biholomorphic isomorphism defined in (3.3) and put $\mathfrak{z}=\phi(Z)\in\mathscr{B}$.

Then we get

$$C(\gamma)Z+D(\gamma)=(z+\sqrt{-1})(\mathfrak{c}\mathfrak{z}+d) \text{ and}$$
$$\sum_{j=1}^{k+1}|(C(\gamma)Z+D(\gamma))^{-1}(L_j(\gamma)Z+B_j(\gamma))|^2=||(A\mathfrak{z}+\mathfrak{b})\cdot(\mathfrak{c}\mathfrak{z}+d)^{-1}||^2.$$

Hence it follows from (3.2) that

$$\frac{2w_s}{|C(\gamma)Z+D(\gamma)|\cdot(1-\sum_{j=1}^{k+1}|(C(\gamma)Z+D(\gamma))^{-1}\cdot(L_j(\gamma)Z+B_j(\gamma))|^2)^{1/2}}$$
$$=\frac{2w_s}{|z+\sqrt{-1}|\cdot(1-||\mathfrak{z}||^2)^{1/2}}.$$

Moreover it is easy to check that $1-||\mathfrak{z}||^2=4|z+\sqrt{-1}|^{-2}(\mathrm{Im}.z-||w'||^2)$. Thus we get

$$(\mathrm{Im}.\Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0((Z))_w||^2)^{-1/2}\cdot\tilde{w}_s=e^{\sqrt{-1}\theta(Z,\gamma)}(\mathrm{Im}.z-||w'||^2)^{-1/2}\cdot w_s,$$

and hence

$$(\mathrm{Im}.\Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2)^{-1/2}\cdot\tilde{w}''=e^{\sqrt{-1}\theta(Z,\gamma)}(\mathrm{Im}.z-||w'||^2)^{-1/2}\cdot w''.$$

Since $(\mathrm{Im}.z-||w'||^2)^{-1/2}\cdot w''\in\tilde{\mathcal{D}}_{\sqrt{-1}}$ and $\tilde{\mathcal{D}}_{\sqrt{-1}}$ is circular, we get $(\mathrm{Im}.\Psi_\gamma^{0,1}(Z)-||(\Psi_\gamma^0(Z))_w||^2)^{-1/2}\cdot\tilde{w}''\in\tilde{\mathcal{D}}_{\sqrt{-1}}$. Therefore we have (3.4). By (3.4), we can define a mapping $\Psi_\gamma\colon\tilde{\mathcal{D}}\to\tilde{\mathcal{D}}$ by

$$\Psi_\gamma\colon({}^tZ,w'')\mapsto({}^t(\Psi_\gamma^0(Z)),\tilde{w}''). \tag{3.7}$$

It is easy to see that this mapping $\Psi_\gamma$ is an extension of $\Psi_\gamma^0$ if we verify the follwiwng relation

$$\Psi_{\gamma_2}\cdot\Psi_{\gamma_1}=\Psi_{\gamma_2\cdot\gamma_1}\qquad\text{for any }\gamma_1,\gamma_2\in SU(k+1,1). \tag{3.8}$$

For this, consider a mapping $\tilde{\phi}\colon\{z\in\boldsymbol{C}\mid\mathrm{Im}.z>0\}\times\boldsymbol{C}^m\to\boldsymbol{C}^{m+1}$ defined by

$$z^1=(z-\sqrt{-1})(z+\sqrt{-1})^{-1},\ z^j=2w_{j-1}(z+\sqrt{-1})^{-1} \tag{3.9}$$

for $j=2,3,\cdots,m+1$. Note that the restriction $\tilde{\phi}\colon\tilde{\mathcal{D}}_0\to\boldsymbol{C}^{m+1}$ is nothing but the biholomorphic isomorphism $\phi\colon\tilde{\mathcal{D}}_0\to\mathcal{B}$ defined in (3.3). Since $\mathrm{Im}.z>0$ if $(z,w)\in\tilde{\mathcal{D}}$ by Lemma 1, it is easy to check that $\tilde{\phi}$ is injective and holomorphic on $\tilde{\mathcal{D}}$. Thus $\tilde{\phi}$ defines a biholomorphic isomorphism of $\tilde{\mathcal{D}}$ onto the image domain $\tilde{\mathcal{B}}:=\tilde{\phi}(\tilde{\mathcal{D}})$ in $\boldsymbol{C}^{m+1}$. Now we define a holomorphic mapping $\tilde{\sigma}_\gamma\colon\mathcal{B}\times\boldsymbol{C}^{m-k}\to\boldsymbol{C}^{m+1}$ for each $\gamma=\begin{pmatrix}A&\mathfrak{b}\\\mathfrak{c}&d\end{pmatrix}\in SU(k+1,1)$ by

$$\tilde{\sigma}_\gamma\colon\begin{cases}\mathfrak{z}\mapsto(A\mathfrak{z}+\mathfrak{b})\cdot(\mathfrak{c}\mathfrak{z}+d)^{-1}\\\mathfrak{z}'\mapsto(\mathfrak{c}\mathfrak{z}+d)^{-1}\mathfrak{z}'\end{cases}$$

where $\mathfrak{z}\in\mathscr{B}$ and $\mathfrak{z}'={}^t(z^{k+1}, \cdots, z^{m+1})\in \boldsymbol{C}^{m-k}$. Then by direct calculations we get

$$\tilde{\phi}(\Psi_\gamma(z, w)) = \tilde{\sigma}_\gamma(\tilde{\phi}(z, w)) \qquad \text{for all } (z, w)\in\tilde{\mathscr{D}} .$$

From this fact, the verification of (3.8) has reduced to verify the following relation

$$(3.10) \qquad \tilde{\sigma}_{\gamma_2}\cdot\tilde{\sigma}_{\gamma_1} = \tilde{\sigma}_{\gamma_2\cdot\gamma_1} \qquad \text{for any } \gamma_1, \gamma_2\in SU(k+1, 1) .$$

But (3.10) follows from the relation ${}^t\bar{A}A-{}^t\bar{\mathfrak{c}}\mathfrak{c}=1_{k+1}$, ${}^t\bar{\mathfrak{b}}\mathfrak{b}-|d|^2=-1$ and ${}^t\bar{\mathfrak{b}}A-\bar{d}\mathfrak{c}=0$, which is satisfied for any $\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d \end{pmatrix}\in U(k+1, 1)$. Therefore we have showed that each element $\Psi^0_\gamma\in \mathrm{Aut}_0(\tilde{\mathscr{D}}_0)$ can be extended to the element $\Psi_\gamma\in\mathrm{Aut}_0(\tilde{\mathscr{D}})$ defined by (3.7). Next, taking an element $K\in K^0{}_{\sqrt{-1}}$, we define a mapping $\Psi_{\gamma,K}$: $\tilde{\mathscr{D}}_0\times \boldsymbol{C}^{m-k}\rightarrow\tilde{\mathscr{D}}_0\times \boldsymbol{C}^{m-k}$ by

$$\Psi_{\gamma,K}\colon ({}^tZ, w'') \mapsto ({}^t(\Psi^0_\gamma(Z)), K\tilde{w}'')$$

which is nothing but the mapping $\Psi_{\gamma,K}$ defined as before. Then, by using the expression of $\tilde{\mathscr{D}}$ as in Theorem 1, we can see easily that $\Psi_{\gamma,K}$ defines a holomorphic transformation of $\tilde{\mathscr{D}}$. Moreover the subset $\{\Psi_{\gamma,K}\,|\,\gamma\in SU(k+1, 1), K\in K^0{}_{\sqrt{-1}}\}$ of $\mathrm{Aut}_0(\tilde{\mathscr{D}})$ has the structure of real Lie transformation group of $\tilde{\mathscr{D}}$ with dimension equal to dim $SU(k+1, 1)+\dim K^0{}_{\sqrt{-1}}$. It remains to show that this Lie group coincides with $\mathrm{Aut}_0(\tilde{\mathscr{D}})$. We denote by $\mathfrak{su}(k+1, 1)$ (resp. $\mathfrak{k}_{\sqrt{-1}}$) the Lie algebra of $SU(k+1, 1)$ (resp. of $K^0{}_{\sqrt{-1}}$). We claim the following equality

$$(3.11) \qquad \dim \mathfrak{g}(\tilde{\mathscr{D}}) = \dim \mathfrak{su}(k+1, 1)+\dim \mathfrak{k}_{\sqrt{-1}} .$$

If we show (3.11), then it is obvious that $\mathrm{Aut}_0(\tilde{\mathscr{D}})=\{\Psi_{\gamma,K}\,|\,\gamma\in SU(k+1, 1), K\in K^0{}_{\sqrt{-1}}\}$. Let $\Pi\colon \mathfrak{g}(\tilde{\mathscr{D}})\rightarrow\mathfrak{g}(\tilde{\mathscr{D}}_0)$ be the homomorphism defined in Corollary 3. Let $\mathfrak{g}(\tilde{\mathscr{D}})=\mathfrak{s}+\mathfrak{r}$ be a Levi-decomposition of $\mathfrak{g}(\tilde{\mathscr{D}})$, where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(\tilde{\mathscr{D}})$ and $\mathfrak{s}$ denotes a maximal semi-simple subalgebra of $\mathfrak{g}(\tilde{\mathscr{D}})$. Put $\mathfrak{s}_2=\mathrm{Ker}\,\Pi\cap\mathfrak{s}$. Then $\mathfrak{s}_2$ is an ideal of $\mathfrak{s}$. Thus there exists an ideal $\mathfrak{s}_1$ of $\mathfrak{s}$ such that $\mathfrak{s}=\mathfrak{s}_1+\mathfrak{s}_2$ (direct sum). Since $\mathfrak{g}(\tilde{\mathscr{D}}_0)$ is a simple Lei algebra isomorphic to $\mathfrak{su}(k+1, 1)$ and $\Pi$ is surjective, it follows that $\Pi(\mathfrak{r})=0$, *i.e.*, $\mathfrak{r}\subset\mathrm{Ker}\,\Pi$. Hence we get $\mathfrak{g}(\tilde{\mathscr{D}})=\mathfrak{s}_1+\mathrm{Ker}\,\Pi$ (direct sum) and $\mathfrak{s}_1$ is isomorphic to $\mathfrak{su}(k+1, 1)$. Since $\mathrm{Ker}\,\Pi\subset\mathfrak{g}_0$ by the proof of Corollary 3, we see that $[\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}, \mathrm{Ker}\,\Pi]=0$. From this fact we can show in the same way as in the proof of Corollary 4 that $\mathrm{Ker}\,\Pi$ is identified with $\mathfrak{k}_{\sqrt{-1}}$. Thus we get the equality (3.11) and Theorem 2 is proved. q.e.d.

## 4. Examples and remarks

Given an integer $k$ such that $0\leqq k\leqq m$, $k\neq m-1$, there is an example of the generalized Siegel domain $\mathscr{D}$ in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2 and $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$.

Indeed we have the following examples.

EXAMPLES. Let $k$ be an integer as above and $p$ a positive integer different from 2. Put

$$\mathcal{D}_{\sqrt{-1}} = \{(w_{k+1}, \cdots, w_m) \in \boldsymbol{C}^{m-k} \mid |w_{k+1}|^p + \cdots + |w_m|^p < 1\} .$$

Obviously $\mathcal{D}_{\sqrt{-1}}$ is a bounded Reinhardt domain in $\boldsymbol{C}^{m-k}$. For this domain $\mathcal{D}_{\sqrt{-1}}$, we define a domain $\mathcal{D}$ in $\boldsymbol{C} \times \boldsymbol{C}^m$ as follows:

$$\mathcal{D} = \{(z, w_1, \cdots, w_m) \in \boldsymbol{C} \times \boldsymbol{C}^m \mid \mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2 > 0 ,$$

$$(\mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2)^{-1/2} \cdot w'' \in \mathcal{D}_{\sqrt{-1}}\} ,$$

where $w'' = (w_{k+1}, \cdots, w_m)$. The domain $\mathcal{D}$ is also expressed as follows:

$$\mathcal{D} = \{(z, w_1, \cdots, w_m) \in \boldsymbol{C} \times \boldsymbol{C}^m \mid \mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2 - (\sum_{\beta=k+1}^{m} |w_\beta|^p)^{2/p} > 0\} .$$

We shall show that $\mathcal{D}$ is a desired example. It is easy to see that $\mathcal{D}$ satisfies the condition (2) of the definition of the generalized Siegel domain with exponent 1/2. Moreover the mapping $\tilde{\phi}$ defined in (3.9) gives a biholomorphic isomorphism of $\mathcal{D}$ onto the bounded Reinhardt domain

$$\mathcal{R} = \{(z^1, \cdots, z^{k+1}, u^1, \cdots, u^{m-k}) \in \boldsymbol{C}^{m+1} \mid \sum_{\alpha=1}^{k+1} |z^\alpha|^2 + (\sum_{\beta=1}^{m-k} |u^\beta|^p)^{2/p} < 1\}$$

in $\boldsymbol{C}^{m+1}$. Thus $\mathcal{D}$ is a generalized Siegel domain in $\boldsymbol{C} \times \boldsymbol{C}^m$ with exponent 1/2. Now we show that $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2} = 2k$. First we recall that the group $\mathrm{Aut}_0(\mathcal{R})$ consists of all transformations of the following type (cf. [6], [8]):

$$(4.1) \qquad \begin{cases} \mathfrak{z} \mapsto (A\mathfrak{z} + \mathfrak{b})(\mathfrak{c}\mathfrak{z} + d)^{-1} \\ u^\beta \mapsto (\mathfrak{c}\mathfrak{z} + d)^{-1} e^{\sqrt{-1}\theta_\beta} \cdot u^\beta, \ 1 \leqq \beta \leqq m-k \end{cases}$$

where $\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d \end{pmatrix} \in U(k+1, 1)$, $\theta_\beta \in R$ and $\mathfrak{z} = {}^t(z^1, \cdots, z^{k+1})$. Note that we can replace $U(k+1, 1)$ by $SU(k+1, 1)$ in (4.1), because any element $g \in U(k+1, 1)$ can be written in the form $g = e^{\sqrt{-1}\theta} \cdot g_0$ for suitable $\theta \in \boldsymbol{R}$ and $g_0 \in SU(k+1, 1)$. Hence we get

$$(4.2) \quad \mathrm{Aut}_0(\mathcal{R}) \cdot 0 = \{(z^1, \cdots, z^{k+1}, 0, \cdots, 0) \in \boldsymbol{C}^{m+1} \mid \sum_{j=1}^{k+1} |z^j|^2 < 1\} .$$

Since $\mathrm{Aut}_0(\mathcal{D}) = \tilde{\phi}^{-1} \cdot \mathrm{Aut}_0(\mathcal{R}) \cdot \tilde{\phi}$, (4.2) implies that

$$\mathrm{Aut}_0(\mathcal{D}) \cdot (\sqrt{-1}, 0) = \{(z, w_1, \cdots, w_k, 0, \cdots, 0) \in \boldsymbol{C} \times \boldsymbol{C}^m \mid \mathrm{Im}.\, z - \sum_{\alpha=1}^{k} |w_\alpha|^2 > 0\} .$$

From this fact, we can conclude that $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2} = 2k$.

REMARK 1. In the case where $n\geqq 2$, the analogy of Theorem 1 is not true in general. In fact we have the following example. Let

$$\mathcal{D}=\{(z_1, z_2, w_1, w_2)\in \boldsymbol{C}^2\times\boldsymbol{C}^2 \mid \mathrm{Im}.z_1-|w_1|^2-|w_2|^2>0,\ \mathrm{Im}.z_2-Re(\overline{w}_1 w_2)>0\}\ .$$

Then $\mathcal{D}$ is a generalized Siegel domain in $\boldsymbol{C}^2\times\boldsymbol{C}^2$ with exponent 1/2 and $\dim_{\boldsymbol{R}}$ $\mathfrak{g}_{-1/2}=2$, more precisely

$$(4.3)\quad \mathfrak{g}_{-1/2}=\left\{2\sqrt{-1}\,\bar{c}w_1\frac{\partial}{\partial z_1}+\sqrt{-1}\,\bar{c}w_2\frac{\partial}{\partial z_2}+c\frac{\partial}{\partial w_1}\,\middle|\, c\in\boldsymbol{C}\right\}\ .$$

We shall sketch the proof of this fact. First $\mathcal{D}$ is a generalized Siegel domain with exponent 1/2. In fact, $\mathcal{D}$ is contained in the domain

$$\mathcal{D}'=\{(z_1, z_2, w_1, w_2)\in \boldsymbol{C}^2\times\boldsymbol{C}^2 \mid \mathrm{Im}.z_1-|w_1|^2-|w_2|^2>0,\ 2\mathrm{Im}.z_1+\mathrm{Im}.z_2>0\}$$

and $\mathcal{D}'$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^4$. Next we shall show that $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2$. For given $c\in\boldsymbol{C}$, $\mathrm{Aut}_0(\mathcal{D})$ contains the global one-parameter subgroup

$$(z_1, z_2, w_1, w_2)\mapsto(z_1+2\sqrt{-1}\,t\bar{c}w_1+\sqrt{-1}\,|tc|^2,\ z_2+\sqrt{-1}\,t\bar{c}w_2,\ w_1+tc,\ w_2),\ t\in\boldsymbol{R}\ .$$

This global one-parameter subgroup induces a holomorphic vector field $X_c=2\sqrt{-1}\,\bar{c}w_1\frac{\partial}{\partial z_1}+\sqrt{-1}\,\bar{c}w_2\frac{\partial}{\partial z_2}+c\frac{\partial}{\partial w_1}$ belonging to $\mathfrak{g}_{-1/2}$. Thus $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}\geqq 2$. Suppose that $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=4$. Then we can show in the same way as in the proof of Proposition 5.1 of Vey [9] that $\mathcal{D}$ is a Siegel domain of the second kind, and $\mathcal{D}$ can be expressed as follows:

$$\mathcal{D}=\{(z_1, z_2, w_1, w_2)\in \boldsymbol{C}^2\times\boldsymbol{C}^2 \mid \mathrm{Im}.z_1-F_1(w, w)>0,\ \mathrm{Im}.z_2-F_2(w, w)>0\}$$

where $w=(w_1, w_2)$ and $F=(F_1, F_2)$ is a $\{x\in\boldsymbol{R}\mid x>0\}\times\{x\in\boldsymbol{R}\mid x>0\}$ — hermitian form. Hence $F_1(w, w)\geqq 0$ and $F_2(w, w)\geqq 0$ for any $w\in\boldsymbol{C}^2$. On the other hand, if we take a point $(3, 0, -1, 1)\in\mathcal{D}$, then $\mathrm{Im}.0-F_2((-1, 1), (-1, 1))>0$ and hence $F_2((-1, 1), (-1, 1))<0$. This is a contradiction. Thus we get $2\leqq\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}\neq 4$. Hence $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2$. By (4.3), we can see that there exists no non-singular linear mapping $\mathcal{L}^3$: $\boldsymbol{C}^2\times\boldsymbol{C}^2\to\boldsymbol{C}^2\times\boldsymbol{C}^2$ satisfying the conditions stated in Lemma 4.

REMARK 2. Let $(z, w)$ be a coordinates system in $\boldsymbol{C}\times\boldsymbol{C}$ and $\mathcal{D}$ a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}$ with exponent $c>0$. Then we can show in the same way as in the proof of Theorem 1 that $\mathcal{D}$ can be expressed as follows:

$$\mathcal{D}=\{(z, w)\in\boldsymbol{C}\times\boldsymbol{C}\mid \mathrm{Im}.z-A|w|^{1/c}>0\}$$

where $A$ is a positive real number depending only on $\mathcal{D}$.

REMARK 3. Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent

1/2 and $\dim_R \mathfrak{g}_{-1/2}=2k$, $0\leqq k\leqq m$. Then there is a natural $\mathrm{Aut}_0(\mathscr{D})$-equivariant holomorphic imbedding of $\mathscr{D}$ into the complex projective space $P_{m+1}(\boldsymbol{C})$.

In order to show this fact, we may identify $\mathscr{D}$ with the generalized Siegel domain $\tilde{\mathscr{D}}$ as in Theorem 1. Let $\tilde{\phi}$: $\tilde{\mathscr{D}}\to\tilde{\mathscr{B}}$ be the biholomorphic isomorphism defined in (3.9). Then $\tilde{\mathscr{B}}$ is a domain in $\boldsymbol{C}^{m+1}$ and the group $\mathrm{Aut}_0(\tilde{\mathscr{B}})$ consists of all holomorphic transformations of the following type:

$$\tilde{\Psi}_{\gamma,K}\colon \begin{cases} \mathfrak{z}\mapsto (A\mathfrak{z}+\mathfrak{b})\,(\mathfrak{c}\mathfrak{z}+d)^{-1} \\ \mathfrak{z}'\mapsto K\cdot(\mathfrak{c}\mathfrak{z}+d)^{-1}\cdot\mathfrak{z}' \end{cases}$$

where $\mathfrak{z}={}^t(z^1,\cdots,z^{k+1})$, $\mathfrak{z}'={}^t(z^{k+2},\cdots,z^{m+1})$, $\gamma=\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in SU(k+1,1)$ and $K\in K^0{}_{\sqrt{-1}}$. Note that $K^0{}_{\sqrt{-1}}$ is a subgroup of $GL(m-k,\boldsymbol{C})$. By using a homogeneous coordinate of $P_{m+1}(\boldsymbol{C})$, we define a holomorphic imbedding $\tilde{\iota}$: $\boldsymbol{C}^{m+1}\hookrightarrow P_{m+1}(\boldsymbol{C})$ by

$$\tilde{\iota}\colon {}^t(z^1,\cdots,z^{k+1},z^{k+2},\cdots,z^{m+1})\mapsto {}^t(z^1,\cdots,z^{k+1},1,z^{k+2},\cdots,z^{m+1}).$$

Then it is easy to see that the restriction $\tilde{\iota}$: $\tilde{\mathscr{B}}\hookrightarrow P_{m+1}(\boldsymbol{C})$ defines an $\mathrm{Aut}_0(\tilde{\mathscr{B}})$-equivariant holomorphic imbedding of $\tilde{\mathscr{B}}$ into $P_{m+1}(\boldsymbol{C})$, where the holomorphic transformation $\tilde{\Psi}_{\gamma,K}$ of $\tilde{\mathscr{B}}$ is extended to a projective transformation $\overline{\Psi_{\gamma,K}}$ of $P_{m+1}(\boldsymbol{C})$ induced by the matrix

$$\left(\begin{array}{cc|c} A & \mathfrak{b} & \multirow{2}{*}{$\mathbf{0}$} \\ \mathfrak{c} & d & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & K \end{array}\right)\in GL(m+2,\boldsymbol{C}).$$

Putting $\iota=\tilde{\iota}\cdot\tilde{\phi}$, we get a desired $\mathrm{Aut}_0(\mathscr{D})$-equivariant holomorphic imbedding $\iota$: $\mathscr{D}\hookrightarrow P_{m+1}(\boldsymbol{C})$.

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