



Title	Multiplicative P-subgroups of simple algebras
Author(s)	Hikari, Michitaka
Citation	Osaka Journal of Mathematics. 1973, 10(2), p. 369-374
Version Type	VoR
URL	https://doi.org/10.18910/11691
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

MULTIPLICATIVE P-SUBGROUPS OF SIMPLE ALGEBRAS

MICHITAKA HIKARI

(Received May 1., 1972)

Amitsur ([1]) determined all finite multiplicative subgroups of division algebras. We will try to determine, more generally, multiplicative subgroups of simple algebras. In this paper we will characterize p -groups contained in full matrix algebras $M_n(\Delta)$ of fixed degree n , where Δ are division algebras of characteristic 0.

All division algebras considered in this paper will be of characteristic 0.

Let Δ be a division algebra. We will denote by $M_n(\Delta)$ the full matrix algebra of degree n over Δ . By a subgroup of $M_n(\Delta)$ we will mean a multiplicative subgroup of $M_n(\Delta)$. Further let K be a subfield of the center of Δ and let G be a finite subgroup of $M_n(\Delta)$. Now we define $V_K(G) = \{\sum \alpha_i g_i \mid \alpha_i \in K, g_i \in G\}$. Then $V_K(G)$ is clearly a K -subalgebra of $M_n(\Delta)$ and there is a natural epimorphism $KG \rightarrow V_K(G)$ where KG denotes the group algebra of G over K . Hence $V_K(G)$ is a semi-simple K -subalgebra of $M_n(\Delta)$, which is a direct summand of KG . As usual $\mathbf{Q}, \mathbf{R}, \mathbf{C}, \mathbf{H}$ denote respectively the rational number field, the real number field, the complex number field and the quaternion algebra over \mathbf{R} .

If an abelian group G has invariants (e_1, \dots, e_n) , $e_n \neq 1$, $e_{i+1} \mid e_i$, we say briefly that G has invariants of length n .

We begin with

Proposition 1. *Let n be a fixed positive integer and let G be a finite abelian group. Then there is a division algebra Δ such that $G \subset M_n(\Delta)$ if and only if G has invariants of length $\leq n$.*

Proof. This may be well known. Here we give a proof. Suppose that there is a division algebra Δ such that $G \subset M_n(\Delta)$. An abelian group G has invariants of length $\leq n$ whenever each Sylow subgroup of G has invariants of length $\leq n$. Hence we may assume that G is a p -group ($\neq 1$). Let m be the length of invariants of G . Then G contains the elementary abelian group G_0 of

order p^m . We can write $\mathbf{Q}G_0 \cong \mathbf{Q} \oplus \overbrace{\mathbf{Q}(\varepsilon_p) \oplus \dots \oplus \mathbf{Q}(\varepsilon_p)}^{1+p+\dots+p^{m-1}}$ where ε_p denotes the primitive p -th root of unity. Since $V_{\mathbf{Q}}(G_0)$ is a direct summand of $\mathbf{Q}G_0$ and

$G_0 \subset V_{\mathbf{Q}}(G_0)$, we have $V_{\mathbf{Q}}(G_0) \cong \overbrace{\mathbf{Q}(\varepsilon_p) \oplus \dots \oplus \mathbf{Q}(\varepsilon_p)}^m$. On the other hand, since

$V_Q(G_0) \subset M_n(\Delta)$, there exist at most n orthogonal idempotents in $V_Q(G_0)$. Thus we have $m \leq n$. The converse is obvious. Q.E.D.

Proposition 2 *Let p be an odd prime and $0 < n < p$. Let P be a finite p -group. If there exists a division algebra Δ such that $P \subset M_n(\Delta)$, then P is abelian.*

Proof. Let $V_Q(P) \cong M_{p^{l_1}}(\Delta_1) \oplus \cdots \oplus M_{p^{l_t}}(\Delta_t)$ be the decomposition of $V_Q(P)$ into simple algebras where each Δ_i is a division algebra. Then we easily see that $p^{l_1} + \cdots + p^{l_t} \leq n$. Therefore, when $n < p$, we have $l_1 = \cdots = l_t = 0$. Since p is odd, each division algebra Δ_i is commutative ([3]). Hence $V_Q(P)$ is commutative. This conclude that P is abelian. Q.E.D.

DEFINITION. Let $P_0 = \langle g \rangle$ be a cyclic group of order p . Let P, P' be finite p -groups and let P_1', P_2', \dots, P_p' be the copies of P' . We will call P a *simple (1-fold) p -extension of P'* if P is an extension of $P_1' \times P_2' \times \cdots \times P_p'$ by P_0 such that $P_1'^g = P_2', \dots, P_{p-1}'^g = P_p', P_p'^g = P_1'$. It should be remarked that this extension does not always split. More generally, a finite p -group P will be called an *n -fold p -extension of a finite p -group P'* , if there exist finite p -groups, $P_0 = P', P_1, \dots, P_{n-1}, P_n = P$ such that, for each $0 \leq i \leq n-1$, P_{i+1} is a simple p -extension of P_i .

Now we set

$$T_p^{(0)} = \begin{cases} \{\text{all cyclic } p\text{-groups}\} & \text{when } p \neq 2, \\ \{\text{all generalized quaternion 2-groups}\} & \text{when } p = 2, \end{cases}$$

and $\tilde{T}_p^{(0)} = \{\text{all cyclic } p\text{-groups}\}$ for any prime p . An element of $T_p^{(0)}$ (resp. $\tilde{T}_p^{(0)}$) is called a *p -group of 0-type (resp. \tilde{o} -type)*.

A finite p -group P is said to be of *n -type (resp. \tilde{n} -type)* if P is an n -fold p -extension of a p -group of 0-type (resp. \tilde{o} -type). We denote by $T_p^{(n)}$ (resp. $\tilde{T}_p^{(n)}$) the set of all p -groups of n -type (resp. \tilde{n} -type).

Our main result is given the following

Theorem. *Let n be a fixed positive integer and let P be a finite p -group. Then following conditions are equivalent:*

- (1) *P is a subgroup of $M_n(\mathbf{H})$ (resp. $M_n(\mathbf{C})$).*
- (2) *There is a division algebra Δ (resp. a commutative field K) such that $P \subset M_n(\Delta)$ (resp. $M_n(K)$).*
- (3) *There exist non-negative integers, t, m_0, \dots, m_t with $\sum_{i=0}^t p^i m_i \leq n$ and $P_i^{(1)}, P_i^{(2)}, \dots, P_i^{(m_i)} \in T_p^{(i)}$ (resp. $\tilde{T}_p^{(i)}$) for each $0 \leq i \leq t$ such that $P \subset \prod_{i=0}^t \prod_{j=1}^{m_i} P_i^{(j)}$.*

The following theorem plays an essential part in the proof of our main theorem.

Theorem (Witt-Roquette [3], [4]). *Let P be a p -group. Let K be a*

commutative field of characteristic 0. Suppose that one of the following hypotheses is satisfied.

- (a) $p \neq 2$,
- (b) $p=2$ and $\sqrt{-1} \in K$.
- (c) $p=2$ and P does not contain a cyclic subgroup of index 2.

Then if χ is a nonlinear irreducible faithful character of P there exists $P_0 \triangleleft P$ and a character ζ of P_0 such that $|P:P_0|=p$, $\chi=\zeta^P$ and $K(\chi)=K(\zeta)$.

From this theorem the following remark follows directly.

REMARK. If K is an algebraic number field in this theorem, each division algebra equivalent to a simple component of KP is an algebraic number field or a quaternion algebra.

Lemma 3. Let P be a finite non-abelian p -group and let Δ be a division algebra such that $P \subset M_n(\Delta)$. Suppose that $V_Q(P) = M_n(\Delta)$.

(1) Suppose that P is a 2-group which is not of type 0 and that Δ is non-commutative. Then there exists a subgroup P_0 of P of index 2 such that $V_Q(P_0) \cong M_{n/2}(\Delta) \oplus M_{n/2}(\Delta)$.

(2) Suppose that Δ is commutative. Then we have $V_C(P) = M_n(C)$ and there exists a normal subgroup P_0 of P of index p such that $V_C(P_0) \cong$

$$\overbrace{M_{n/p}(C) \oplus \cdots \oplus M_{n/p}(C)}^p.$$

Proof. (a) Let M be a simple $M_n(\Delta)$ -module and let E be a splitting field of Δ . Since M is a non-linear faithful QP -module by the assumption that $V_Q(P) = M_n(\Delta)$, there exists a non-linear faithful irreducible EP -module N such that $M \otimes_Q E \cong m_Q(N)(N \oplus N^\sigma \oplus \cdots)$, $\sigma \in \text{Gal}(Q(\zeta)/Q)$, where ζ is the character of N and $m_Q(N)$ denotes the Schur index of N . Applying the Witt-Roquette's theorem to N , we can find a normal subgroup P_0 of P and an irreducible EP_0 -module N_0 with character ζ_0 such that $N_0^P \cong N$ and $Q(\zeta) = Q(\zeta_0)$. Let χ denote the character of M . Then we have $\chi = m_Q(\zeta)(\zeta + \zeta^\sigma + \cdots) = m_Q(\zeta)(\zeta_0 + \zeta_0^\sigma + \cdots) + m_Q(\zeta)(\zeta_0^\sigma + (\zeta_0^\sigma)^\sigma + \cdots)$ where $\{1, g\}$ are representatives of P/P_0 . Since $2 = m_Q(\zeta) \leq m_Q(\zeta_0) \leq 2$, we have $m_Q(\zeta) = m_Q(\zeta_0) = 2$. Let $\chi_0 = m_Q(\zeta_0)(\zeta_0 + \zeta_0^\sigma + \cdots)$. Then χ_0 is a Q -character of P_0 . Further let M_0 be the QP_0 -module corresponding to χ_0 . Then we see that $M_0 \oplus M_0^g \cong QP \otimes_{QP_0} M_0 \cong QP \otimes_{QP_0} M_0^g \cong M$ as QP -module. Since $M_0 \cong M_0^g$ as QP_0 -module, we have

$$\begin{aligned} \Delta &\cong \text{Hom}_{QP}(M, M) \\ &\cong \text{Hom}_{QP}(QP \otimes_{QP_0} M_0, QP \otimes_{QP_0} M_0) \\ &\cong \text{Hom}_{QP_0}(M_0, \text{Hom}_{QP}(QP, QP \otimes_{QP_0} M_0)) \\ &\cong \text{Hom}_{QP_0}(M_0, QP \otimes_{QP_0} M_0) \\ &\cong \text{Hom}_{QP_0}(M_0, M_0), \end{aligned}$$

and, similarly, $\Delta \cong \text{Hom}_{\mathcal{Q}P_0}(M_0^g, M_0^g)$. Clearly $\dim_{\mathcal{Q}} M_0 = \dim_{\mathcal{Q}} M_0^g = \frac{1}{2} \dim_{\mathcal{Q}} M$, and the semi-simple subalgebra $V_{\mathcal{Q}}(P_0) \subset V_{\mathcal{Q}}(P) = M_n(\Delta)$ has only two simple components corresponding to M_0, M_0^g . Thus we get $V_{\mathcal{Q}}(P_0) \cong M_{n/2}(\Delta) \oplus M_{n/2}(\Delta)$.

(b) Since Δ is commutative by the assumption, we have $\mathcal{C} \otimes_{\Delta} V_{\mathcal{Q}}(P) \cong \mathcal{C} \otimes_{\Delta} M_n(\Delta) \cong M_n(\mathcal{C})$. From this it follows directly that $V_{\mathcal{C}}(P) = M_n(\mathcal{C})$. Let M be a simple $V_{\mathcal{C}}(P)$ -($\mathcal{C}P$ -)module and let χ be the character of M . According to the Witt-Roquette's theorem, there exists a normal subgroup P_0 of P of index p and an irreducible $\mathcal{C}P_0$ -module M_0 such that $M \cong M_0^p$. Hence, along the same

line as in the case (a), we can show that $V_{\mathcal{C}}(P_0) \cong \overbrace{M_{n/p}(\mathcal{C}) + \cdots + M_{n/p}(\mathcal{C})}^p$.
Q.E.D.

Lemma 4. *Let P be a finite p -group. Suppose one of the following conditions:*

(a) $p=2$ and P is a subgroup of $M_{2^n}(\Delta)$ such that $V_{\mathcal{Q}}(P) = M_{2^n}(\Delta)$ where Δ is a quaternion algebra.

(b) P is a subgroup of $M_{p^n}(\mathcal{C})$ such that $V_{\mathcal{C}}(P) = M_{p^n}(\mathcal{C})$. Then P is a subgroup of a p -group of n -type. Further, in the case (b) P is a subgroup of a p -group of \tilde{n} -type.

Proof. We will give the proof only in the case (a), because the proof in the case (b) can be done similarly. This will be done by induction on n . In case $n=0$ this is obvious. Hence we assume that $n \geq 1$. By Lemma 3, there exists a normal subgroup P_0 of P of index 2 such that $V_{\mathcal{Q}}(P_0) = A_1 \oplus A_2$ where $A_i \cong M_{2^{n-1}}(\Delta)$. Let M_i be a simple A_i -module and let $\{1, g\}$ be representatives of P/P_0 . Then $M_2 \cong M_1^g$ as $\mathcal{Q}P_0$ -module. Let P_i denote the image of P_0 by the projection on A_i . Then $V_{\mathcal{Q}}(P_i) = M_{2^{n-1}}(\Delta)$. Hence, by induction, each P_i is a subgroup of a 2-group of $(n-1)$ -type. We regard M_i as $\mathcal{Q}P_0$ -module by the projection $P_0 \rightarrow P_i$ and so, since $M_2 \cong M_1^g$, we have $P_2 = P_1^g$ and the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{g} & P_0 \\ \downarrow & & \downarrow \\ P_1 \times P_2 & \xrightarrow{(g, g)} & P_2 \times P_1 \end{array}$$

On the other hand, we can find 2-groups \tilde{P}_1, \tilde{P}_2 of $(n-1)$ -type such that $\tilde{P}_1 \cong \tilde{P}_2$. Here we may assume that the restriction of the isomorphism $\tilde{P}_1 \cong \tilde{P}_2$ on P_1 coincides with $g: P_1 \cong P_2$. We denote this isomorphism from \tilde{P}_1 onto \tilde{P}_2 by σ . Put $h = g^2$. Then the map $(1, h): \tilde{P}_2 \times \tilde{P}_1 \rightarrow \tilde{P}_2 \times \tilde{P}_1$ is an isomorphism and so $(\sigma, h\sigma^{-1}): \tilde{P}_1 \times \tilde{P}_2 \rightarrow \tilde{P}_2 \times \tilde{P}_1$ is an isomorphism, too. Since the restriction of $h\sigma^{-1}$ on P_2 coincides with $hg^{-1} = g$, we get the following commutative diagram:

$$\begin{array}{ccc}
 P_0 & \xrightarrow{g} & P_0 \\
 \downarrow & & \downarrow \\
 P_1 \times P_2 & \xrightarrow{(g, g)} & P_2 \times P_1 \\
 \downarrow & & \downarrow \\
 \tilde{P}_1 \times \tilde{P}_2 & \xrightarrow{(\sigma, h\sigma^{-1})} & \tilde{P}_2 \times \tilde{P}_1
 \end{array}$$

Let $\langle u \rangle$ be a cyclic group of order 2. The automorphism $(\sigma, h\sigma^{-1})$ and the factor set $\{(1, 1)=(u, 1)=(1, u)=1, (u, u)=h\}$ define a group \tilde{P} with normal subgroup $\tilde{P}_1 \times \tilde{P}_2$ and $\tilde{P}/\tilde{P}_1 \times \tilde{P}_2 \cong \langle u \rangle$, because $(h\sigma^{-1}, \sigma) \cdot (\sigma, h\sigma^{-1}) = (h, \sigma h\sigma^{-1}) = (h, h^{\sigma^{-1}}) = (h, h^{\sigma^{-1}}) = (h, h^{\sigma^{-1}}) = (h, h)$. Then the group \tilde{P} is clearly a 2-group of n -type which contains P . Thus the proof of the lemma is completed.

Lemma 5. *If $P \in T_2^{(n)}$ (resp. $\tilde{T}_p^{(n)}$), P is a subgroup of $M_{2^n}(\mathbf{H})$ (resp. $M_p(\mathbf{C})$) and $V_R(P) = M_{2^n}(\mathbf{H})$ (resp. $V_C(P) = M_p(\mathbf{C})$).*

Proof. We will prove this in the case $P \in T_2^{(n)}$.

(a) $n=0$. Since P is a generalized quaternion group, P is a subgroup of \mathbf{H} and $V_R(P) = \mathbf{H}$ ([1], [2]).

(b) $n>0$. We proceed by induction on n . By the definition of $T_2^{(n)}$, there exist 2-groups $P_1, P_2 \in T_2^{(n-1)}$ such that $P_1 \times P_2$ is a subgroup of P of index 2 and that $P_1^g = P_2$, where g is a representative of a generator of $P/P_1 \times P_2$. By the induction hypothesis each P_i is a subgroup of $M_{2^{n-1}}(\mathbf{H})$ and $V_R(P_i) = M_{2^{n-1}}(\mathbf{H})$. Let M_1 be a simple $V_R(P_1)$ -($\mathbf{R}P_1$)-module. Put $M = M_1 \otimes_{\mathbf{R}(P_1 \times P_2)} \mathbf{R}P$. Since $P_1^g = P_2$, M_1^g is a simple $\mathbf{R}P_2$ -module. It follows that $M_1 \cong M_1^g$ as $\mathbf{R}(P_1 \times P_2)$ -module and therefore $\text{Hom}_{\mathbf{R}P}(M, M) \cong \text{Hom}_{\mathbf{R}(P_1 \times P_2)}(M_1, M_1 \oplus M_1^g) \cong \text{Hom}_{\mathbf{R}(P_1 \times P_2)}(M_1, M_1) = \mathbf{H}$. We see that the simple component of $\mathbf{R}P$ corresponding to M is $M_{2^n}(\mathbf{H})$. Because M is a faithful $\mathbf{R}P$ -module, P is a subgroup of $M_{2^n}(\mathbf{H})$ and $V_R(P) = M_{2^n}(\mathbf{H})$.

We will omit the proof in the case $P \in \tilde{T}_p^{(n)}$, because we can prove it along the same line as in the case $P \in T_2^{(n)}$. Q.E.D.

Now we give the proof of our main theorem.

Proof of the main theorem. The implication (1) \Rightarrow (2) is obvious and therefore it suffices to show the implications (2) \Rightarrow (3) \Rightarrow (1).

(a) (2) \Rightarrow (3). Assume $P \subset M_n(\Delta)$. Let $V_Q(P) \cong M_{p^{i_1}}(\Delta_s) \oplus \cdots \oplus M_{p^{i_s}}(\Delta_s)$ be the decomposition of $V_Q(P)$ into simple algebras where each Δ_i is a division algebra. Here we easily see that $p^{i_1} + \cdots + p^{i_s} \leq n$. Let P_i be the image of P by the projection to $M_{p^{i_i}}(\Delta_i)$, for each $1 \leq i \leq s$. Then P can be identified with a subgroup of $\prod_{i=1}^s P_i$ and, for each $1 \leq i \leq s$, $V_Q(P_i) \cong M_{p^{i_i}}(\Delta_i)$. According to the

remark on the Witt-Roquette's theorem, Δ_i is a quaternion algebra or an algebraic number field. Further if Δ_i is a quaternion algebra for some $1 \leq i \leq s$, $p=2$ ([3]). If Δ_i is an algebraic number field, by Lemma 3 (2) $V_C(P_i) \cong M_{p^{i_i}}(\mathbf{C})$. Applying Lemma 4, it follows that each P_i is a subgroup of a p -group of i_i -type. Here (3) is concluded in this case.

Assume $P \subset M_n(K)$. Let L be the algebraic closure of K and let $L' = \mathbf{C} \cap L$. Since K is commutative, we have $L \otimes_K M_n(K) \cong M_n(L)$. From this it follows directly that $V_{L'}(P) \subset M_n(L)$. In addition, each division algebra equivalent to a simple component of $L'P$ coincides with $L'([3])$. Let $V_{L'}(P) \cong M_{p^{i_1}}(L') \oplus \cdots \oplus M_{p^{i_s}}(L')$ be the decomposition of $V_{L'}(P)$ into simple algebras. Then $p^{i_1} + \cdots + p^{i_s} \leq n$. If P_i is the image of P by the projection to $M_{p^{i_i}}(L')$, P_i is a subgroup of $M_{p^{i_i}}(\mathbf{C}) \cong M_{p^{i_i}}(L') \otimes_{L'} \mathbf{C}$ and $V_C(P_i) \cong M_{p^{i_i}}(\mathbf{C})$. It follows from Lemma 4 that P_i is a subgroup of \tilde{i}_i -type. On the other hand P can be identified with a subgroup of $\prod_{i=1}^s P_i$ and so we conclude (3).

(b) (3) \Rightarrow (1). Since $P_i^{(j)}$ is a p -group of i -type (resp. \tilde{i} -type), by Lemma 5, $P_i^{(j)}$ is a subgroup of $M_{p^i}(\mathbf{H})$ (resp. $M_{p^i}(\mathbf{C})$) and so $\prod_i \prod_{j=1}^{m_i} P_i^{(j)} \subset \sum_{i,j}^{\oplus} M_{p^i}(\mathbf{H}) \subset M_n(\mathbf{H})$ (resp. $\prod_i \prod_{j=1}^{m_i} P_i^{(j)} \subset M_n(\mathbf{C})$) by $\sum_{i=0}^t p^i m_i \leq n$. Since P is a subgroup of $\prod_i \prod_{j=1}^{m_i} P_i^{(j)}$, P is a subgroup of $M_n(\mathbf{H})$ (resp. $M_n(\mathbf{C})$). Q.E.D.

TOKYO UNIVERSITY OF EDUCATION

References

- [1] S. Amitsur: *Finite subgroups of division rings*, Trans. Amer. Math. Soc. **80** (1955), 361–386.
- [2] I.N. Herstein: *Finite multiplicative subgroups in division rings*, Pacific J. Math. **1** (1953), 121–126.
- [3] P. Roquette: *Realisierung von Darstellungen endlicher nilpotenter Gruppen*, Arch. Math. **9** (1958), 241–250.
- [4] E. Witt: *Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zalenkörper*, J. Reine Angew. Math. **190** (1952), 231–245.