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Harada, M.
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ARTINIAN RINGS RELATED TO RELATIVE ALMOST PROJECTIVITY

MANABU HARADA

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Let R be an artinian ring. We consider the following condition: if eR/A is fR/B-projective (resp. N-projective for an R-module N), then every submodule M' of eR/A is fR/B-projective (resp. N-projective), where e and f are primitive idempotents. We have shown in [7] that R satisfies the above condition for any eR/A and any fR/B if and only if R is a hereditary ring with $J^2=0$. In this paper we consider a weaker condition: if eR/A is N-projective, then M' is almost N-projective where i): N is local and ii): N is a direct sum of local modules, respectively. In the second section we shall study QF, QF-2, and QF-3 rings with the above weaker condition, respectively. We study right almost hereditary rings with $J^2=0$ in the third section.

In a forthcoming paper we shall give a charaterization of rings over which the weaker condition is satisfied when M and N are any R-modules.

1. Characterizations

We always assume that R is an associative artinian ring with identity and every module is a finitely generated and unitary right R-module. Moreover since we are interested in the structure of R, we may assume that R is basic.

Let M and N be any finitely generated R-modules. We have studied rings with the following properties (1) (4) in [3] and [7]:

(1) If M is N-projective, then M' is again N-projective for any submodule M' of M.

(2) If eR/B is fR/A-projective, then C/B is again fR/A-projective for any $C \supset B$, where e and f are primitive idempotents and $C \supset B$ (resp. A) are R-submodules of eR (resp. fR).

(3) $e=f$ in (2).

(4) If M is almost N-projective, then M' is again almost N-projective for any submodules M' of M.

Here we shall consider a weaker condition than (4).

(5) If M is N-projective, then M' is almost N-projective for any submodule M' of M.

Let R be a two-sided artinian ring. We know from [3] or [7] that the following are equivalent: i) (1) holds, ii) (2) holds and iii) R is a hereditary ring with $J^2=0$.

In this section we shall give a characterization of artinian rings over which (5) holds on local modules M and N. By $\mathrm{J}(M)$ (resp. J) we denote the Jacobson radical of M (resp. of R).

Lemma 1. *Let $fJ \supset A \supset B$ be submodules of fR such that A/B is almost fR-projective. Then there exists a submodule S^* of fR such that $A=S^*\oplus B$, where f is a primitive idempotent.*

Proof. Consider a diagram

$$\begin{array}{c} A/B \\ \downarrow h \\ fR \xrightarrow{\nu} fR/B \rightarrow 0 \end{array}$$

where h is the inclusion.

Since $h(A/B) \subset fJ/B$ and fR is indecomposable, there exists $\tilde{h}$: $A/B \rightarrow fR$ with $\nu\tilde{h}=h$, and hence $A=B\oplus\tilde{h}(A/B)$.

From now on we study (5) when M and N are local modules. We denote primitive idempotents by e, f, g, and so on.

Lemma 2. *Assume* (5) *on local modules M and N. Then for any local module L, every submodule of fR is almost L-projective.*

Proof. Since fR is L-projective, this is clear from (5).

Corollary. *Assume* (5) *on local modules M and N and $\bar{e}\bar{R}=eR/eJ$ is a simple component of* $\mathrm{Soc}(R)$. *Let x be a non-zero element in fJ with $xe=x$. Then xR is simple.*

Proof. Since $fJ/xJ \supset xR/xJ \approx \bar{e}\bar{R}$, xR/xJ is isomorphic to a submodule of some gR, and xR/xJ is almost fR-projective by Lemma 2. Hence $xR=xJ\oplus S$ and $xR=S\approx\bar{e}\bar{R}$ by Lemma 1.

Lemma 3. *Let X be an R-module such that X is isomorphic to a submodule of* $\mathrm{J}(L)$, *where L is a local R-module. If X is almost L-projective, X is quasi-projective.*

Proof. We may assume $X \subset J(L)$. Let A be any submodule of X and consider a diagram

$$\begin{array}{c} X \\ \downarrow h \\ X/A \\ \cap \\ L \overset{v}{\rightarrow} L/A \rightarrow 0, \end{array}$$

where h is the natural homomorphism of X to X/A.
Then there exists $\tilde{h}$: $X \rightarrow L$ with $v\tilde{h}=h$, and hence $\tilde{h}(X) \subset X$. Therefore X is quasi-projective.

Corollary. *Assume* (5) *on local modules M and N. Then every submodule of any indecomposable quasi-projective module is quasi-projective.*

Proof. This clear from Lemma 3.

Lemma 4. *If* (5) *holds on local modules M and N, then* $J^3=0$.

Proof. From Corollary to Lemma 3 $eJ=X_1 \oplus X_2 \oplus \cdots \oplus X_m$ for a primitive idempotent e, where the X_i are indecomposable and quasi-projective. Further $eJ^2=X_1J \oplus X_2J \oplus \cdots \oplus X_mJ$, X_i/X_iJ is simple and $X_iJ=Y_{i1} \oplus Y_{i2} \oplus \cdots \oplus Y_{in_i}$, where the Y_{ij} are indecomposable and quasi-projective. We denote this situation by the following figure:

$$(6) \qquad \begin{array}{lllll} eJ & \quad X_1 & \quad X_2 & \cdots & X_m \\ eJ^2 & Y_{11} \cdots Y_{1n_1} & Y_{21} \cdots Y_{2n_1} & \cdots & Y_{m1} \cdots Y_{mn_m} \\ eJ^3 & Z_{111} \cdots & \cdots & & \end{array}$$

We note $X_1 \cap eJ^2=X_1J$ and so on from (6). Let $X_i \approx f_iR/A_i$ and $f_iJ \approx g_{i1}R/C_{i1} \oplus \cdots \oplus g_{in_i}R/C_{in_i}$. Then since $f_iJ/A_i \approx Y_{i1} \oplus \cdots \oplus Y_{in_i}$ $(=X_iJ)$, Y_{ik} is a homomorphic image of some $g_{it}R$. Now assume $eJ^3 \neq 0$ for some e. Then we may suppose $Y_{11} \not\subset \mathrm{Soc}(eR)$. Let $X_1 \approx fR/A$ and $Y_{11} \approx gR/C$ (via θ). Then $fJ=X' \oplus \cdots$; $X' \approx gR/A'$ (via θ') from the above remark. Since Y_{11} $(\approx gR/C) \not\subset \mathrm{Soc}(eR)$, $X' \not\subset \mathrm{Soc}(fR)$ by Corollary to Lemma 2. Hence $X'J \neq 0$. eR/eJ^3 is fR/fJ^3-projective by [1], p. 22, Exercise 4, and hence $Y_{11}/Y_{11}J \approx gR/gJ$ is almost fR/fJ^3-projective (see (6)). Consider a diagram

$$\begin{array}{c} Y_{11}/Y_{11}J \\ \downarrow h \\ (X'+fJ^3)/(X'J+fJ^3)\approx gR/gJ \\ \cap \\ fR/fJ^3 \xrightarrow{\nu} (fR/fJ^3)/((X'J+fJ^3)/fJ^3) \to 0, \end{array}$$

where h is the induced isomorphism from θ and θ'.
Then there exists $\tilde{h}$: $Y_{11}/Y_{11}J \to fR/fJ^3$ with $\nu\tilde{h}=h$. Therefore $\tilde{h}(Y_{11}/Y_{11}J)+fJ^3+X'J=X'+fJ^3$, and hence $X'+fJ^3=\tilde{h}(Y_{11}/Y_{11}J)+fJ^3$. Accordingly $X'/(X'\cap fJ^3)$ $(\approx(X'+fJ^3)/fJ^3)$ is simple. On the other hand $X'\cap fJ^3=X'J^2$. Therefore $X'J=X'J^2$, and hence $X'J=0$, a contradiction.

Now $J^3=0$ from Lemma 4. We denote an indecomsable and projective module P with $PJ^2\neq 0$ (resp. $PJ^2=0$, $PJ\neq 0$) by eR (resp. fR). From Corollary to Lemma 3 we suppose $eJ=X_1\oplus X_2\oplus\cdots\oplus X_s\oplus S_1\oplus\cdots\oplus S_t$, where $X_i\approx h_iR/A_i$; $A_i\neq h_iJ$ and $S_j\approx g_jR/g_jJ$; the h_k and g_m are primitive idempotents.

Lemma 5. *Assume* (5) *on local modules M and N and eJ is as above. Then X_i is projective and uniserial, and hence $X_i\approx f_iR$ for some f_i.*

Proof. Let $X_1\approx h_1R/A_1$. Suppose $h_1R=e_1R$, i.e. $h_1J^2\neq 0$. Then $A_1\neq 0$; θ: $e_1R/A_1\approx X_1$. Let $e_1J=X'_1\oplus\cdots\oplus X'_{s'}\oplus S'_1\oplus\cdots$ similar to eJ above (note $X'_1\neq 0$). Since $\theta(e_1J/A_1)\subset X_1J=\mathrm{Soc}(X_1)$, $A_1\supset X'_1\oplus\cdots\oplus X'_{s'}$ by Corollary to Lemma 2. If $\{S'_i\}=\phi$, $A_1=eJ$, a contradiction. Hence assume $\{S'_i\}\neq\phi$. Then since $A_1\neq e_1J$, there exists S'_1 such that $S'_1\not\subset A_1$. Being a submodule of eR, e_1R/A_1 is almost e_1R/S'_1-projective by Lemma 2. However A_1 is characteristic by Corollary to Lemma 3 and $S'_1\not\subset A_1$, $S'_1\not\supset A_1$, because $A_1\supset X'_1$, and hence $e_1R/A_1\oplus e_1R/S'_1$ does not have LPSM, a contradiction to [4], Proposition 4. Therefore $h_1R=fR$, i.e, $h_1J^2=0$ and $h_1J\neq 0$. The above argument shows us $A_1=0$, since fJ is semisimple. Next we shall show that $X_1=f_1R$ is uniserial. Suppose $f_1J=A\oplus B\oplus\cdots$, where A, B are non-zero simple modules. Now $\theta(eJ)=0$ for any θ in $\mathrm{Hom}_R(eR,f_1R)$. Hence eR/A is f_1R/B-projective. Accordingly f_1R/A is almost f_1R/B-projective, and $f_1R/A\oplus f_1R/B$ has LPSM and hence $A=B$ by [9], Lemma 1. Therefore f_1J is simple.

From Lemmas 4 and 5 we have

$$(7)\qquad \begin{array}{l} eJ\approx f_1R\oplus f_2R\oplus\cdots\oplus f_sR\oplus S_1\oplus\cdots\oplus S_k;\quad f_iR \text{ is uniserial} \\ (e'J\approx f'_1R\oplus\cdots\oplus f'_{s'}R\oplus S'_1\oplus\cdots) \end{array}$$

Since f_iR is projective, we have

Lemma 6. *Let R be any artinian ring. If eJ and $e'J$ have the above structure*

(7) (where f_iR *need not be uniserial*), *then for any non-isomorophic homomorphism* θ: $eR \to e'R$, $\theta(eJ)=0$.

Lemma 7. *Assume* (5) *on local modules M and N. If* $eR \not\approx e'R$ *in* (7), $f_iR \not\approx f'_jR$ *for any i and j.*

Proof. Assume $fR \approx f'R$. Now eR/fJ is $e'R$-projective by Lemma 6. As a consequence $fR/fJ \approx f'R/f'J$ is almost $e'R$-projective, which is a contradiction from Lemma 1.

We can express (7) as follows:

$$(7')\qquad \begin{array}{l} eR \supset eJ \approx \Sigma_{i=1}^{s} \oplus (f_iR)^{(n_i)} \oplus \Sigma_{j=1}^{t} \oplus S_j, \quad \text{where the } f_iR \text{ are} \\ \text{uniserial (and } e'R \supset e'J \approx \Sigma_{i=1}^{s'} \oplus (f'_iR)^{(n'_i)} \oplus \Sigma_{j=1}^{t'} \oplus S'_j). \end{array}$$

We put $P_i=(f_iR)^{(n_i)}$ and $P=\Sigma_{i=1}^{s}\oplus P_i$. Let π_i: $P \to P_i$ be the projection of P onto P_i. We shall regard $(f_iR)^{(n_i)}$ as a submodule of eJ.

Lemma 8. *Suppose that* (5) *holds on local modules M and N. Let eR and P be as above. Let S be a simple submodule of P. Then* $eReS=\Sigma_{i\in I}\oplus \mathrm{Soc}(P_i)$, *where I is a subset of* $\{1,2,\cdots,s\}$.

Proof. Let first $S=\mathrm{Soc}(f_1R)$ and $S^*=eReS$. If $S^* \not\supset \mathrm{Soc}(P_1)$, then there exists $f_{1i}R$ such that $f_{1i}R \cap S^*=0$; $f_{1i}R=f_1R$ which is the ith component of P_1. Since eR/S is eR/S^*-projective, f_1R/S is almost eR/S^*-projective. From the diagram

$$\begin{array}{c} f_1R/S \\ \wr\wr \\ f_{1i}R/S_i \\ \wr\wr \\ (S^*\oplus f_{1i}R)/(S^*\oplus S_i) \\ \cap \\ eR/S^* \to eR/(S^*\oplus S_i) \to 0, \text{ where } S_i=\mathrm{Soc}(f_{1i}R). \end{array}$$

we obtain a contradiciton. Therefore $S^* \supset \mathrm{Soc}(P_1)$. Next assume that S is any simple submodule of P. Since eR/S^* is eR/S^*-projective, P/S^* is quasi-projective by Corollary to Lemma 3. Further $S^* \subset \mathrm{Soc}(P)=\mathrm{J}(P)$, and hence P is a projective cover of P/S^*. Accordingly $S^* \supset \pi_{ij}(S^*)$, where π_{ij}: $P \to f_{ij}R$ is the projection. Moreover $\pi_i(S^*) \supset \pi_i(S) \neq 0$ implies $\pi_{ij}(S^*)=\mathrm{Soc}(f_{ij}R) \subset S^*$ for some j, and hence $S^* \supset \mathrm{Soc}(P_i)$ from the initial part. Let $I=\{i_j\in\{1,\cdots,s\} \mid \pi_{i_j}(S)\neq 0\}$. Then we have shown $S^* \supset \Sigma_I\oplus\mathrm{Soc}(P_{i_j})$. On the other hand $S\subset \Sigma_I\oplus\mathrm{Soc}(P_{i_j})$, and hence $S^* \subset \Sigma_I\oplus\mathrm{Soc}(P_{i_j})$ for $eRe\,\mathrm{Soc}(P_{i_j})=\mathrm{Soc}(P_{i_j})$ by Corollary to Lemma 2.

Next we assume that (5) holds whenever M is local and N is any finite direct sum of local modules. By P(Soc(R)) we denote the projective cover of Soc(R).

Lemma 9. *Let R be as above. Then* P(Soc(R)) *is a direct sum of uniserial modules.*

Proof. Let $\bar{g}R=gR/gJ$ be isomorphic to a simple component of Soc(R) and $gJ\neq 0$. Take two submodules A_1, A_2 of gJ such that $gJ^j\supset A_i\supset gJ^{j+1}$ and A_i/gJ^{j+1} is simple ($i=1,2$ and $j=1,2$). Since $\bar{g}R$ is isomorphic to a proper submodule of some hR, $\bar{g}R$ is almost $(gR/A_1\oplus gR/A_2)$-projective by assumption. Assume that gJ^j/gJ^{j+1} is not simple, and $A_1\neq A_2$, $A_i\neq gJ^j$. Then $\bar{g}R$ is not gR/A_i-projective, and hence $gR/A_1\oplus gR/A_2$ has LPSM by [6], Theorem. Therefore $A_1=A_2$ by [9], Lemma 1, a contradiction. As a consequence gR is uniserial.

We consider a direct sum $M=M_1\oplus M_2$. Let π_i be the projection of M onto M_i for $i=1,2$. For any submodule A of M we put

(8) $$A_i=A\cap P_i \text{ and } A^i=\pi_i(A) \text{ for } i=1,2.$$

We use the following trivial lemma (see. [5], p.449)

Lemma 10. *Let M and A be as above. Then θ: $A^1/A_1\approx A^2/A_1$ and $A=\{m_1+m_2|m_i\in A^i$ and $\theta(m_1+A_1)=m_2+A_2\}$.*

Finally we obtain the main theorem.

Theorem 1. *Let R be an artinian ring. (5) holds on local modules M and N, if and only if i): $J^3=0$ and eJ has the structure (7′) with f_iR uniserial, ii) if $eR\not\approx e'R$, then $f_iR\not\approx f_jR$ for all i and j in (7′) and iii) $\bar{f}_i\bar{R}$ in (7′) is never isomorphic to any simple component of* Soc(R), *and iv) the condition in Lemma 8, $eReS=\Sigma_I\oplus\mathrm{Soc}(P_i)$ for any simple submodule S in P, is satisfied, where e, e' are any primitive idempotents with $eJ^2\neq 0$ and $e'J^2\neq 0$.*

Proof. Suppose that (5) holds. Then we have i)$\sim$iv) by Corollary to Lemma 2 and Lemmas 4, 5, 7 and 8. Conversely we assume 1)$\sim$iv). First we study a structure of submodule B/A of eR/A. We take the decomposition (7′): $eJ=P_1\oplus\cdots\oplus P_s\oplus S_1\oplus\cdots\oplus S_t$. Put $P=\Sigma_{i=1}^s\oplus P_i$ and $\tilde{S}=\Sigma_{j=1}^t\oplus S_j$, and hence $eJ=P\oplus\tilde{S}$. We apply Lemma 10 to this decomposition $eJ=P\oplus\tilde{S}$ and the submodule A of eJ. Then there exists an isomorphism θ: $A^1/A_1\approx A^2/A_2$. Since any simple sub-factor module of $P/\mathrm{Soc}(P)$ is never isomorphic to any one of $\tilde{S}$ (and hence any one of A^2/A_2) by iii), $A^1/A_1\subset(\mathrm{Soc}(P)+A_1)/A_1\approx \mathrm{Soc}(P)/(\mathrm{Soc}(P)\cap A_1)$. Accordingly there exists a submodule K_1 of Soc(P) such that

$A^1/A_1=(K_1\oplus A_1)/A_1$. A^2 being semisimple, we obtain $A^2=A_2\oplus K_2$ for some K_2 in A^2, and clearly θ: $K_1\approx K_2$. Therefore $A=A_1\oplus A_2\oplus K_2(\theta^{-1})$ by Lemma 10, where $A_1\subset P$ and A_2, K_2 are contained in $\tilde{S}$. Since $\tilde{S}$ is semisimple, $\tilde{S}=A_2\oplus K_2\oplus K_2'$ for some K_2'. Then $eJ=P\oplus A_2\oplus K_2(\theta^{-1})\oplus K_2'$, and putting $\tilde{S}'=A_2\oplus K_2(\theta^{-1})\oplus K_2'$, we obtain

$$A=A\cap P\oplus A\cap\tilde{S}'\ (eJ=P\oplus\tilde{S}'). \tag{9}$$

Next let $eJ\supset B\supset A$. Then we obtain from the above observation (take first the decomposition of B and use the above argument on A)

$$\begin{aligned} eJ&=P\oplus\tilde{S}_a\oplus\tilde{S}_b\oplus\tilde{S}_c\supset\\ B&=B_1\oplus\tilde{S}_a\oplus\tilde{S}_b\supset\\ A&=A_1\oplus\tilde{S}_a,\end{aligned} \tag{10}$$

where $B_1=B\cap P$, $A_1=A\cap P$ and the $\tilde{S}_a$, $\tilde{S}_b$ and $\tilde{S}_c$ are contained in Soc(eJ). From (10) we may study the structure of B_1/A_1. Hence we assume $P\supset B_1=P\cap B\supset A_1=P\cap A$. Since P is projective, considering first the decomposition of A, we obtain

$$\begin{aligned} P&=P_1\oplus P_2\oplus P_3\supset\\ B_1&=P_1\oplus P_2\oplus B_1\cap P_3\supset\\ A_1&=P_1\oplus A_1\cap(P_2\oplus P_3),\end{aligned} \tag{11}$$

where the P_i are isomorphic to direct sums of some copies of $\{f_{i1}R,\cdots,f_{iq}R\}$ and $B_1\cap P_3$, $A_1\cap(P_2\oplus P_3)$ are semisimple modules

(12) whose simple components are isomorphic to those of Soc(eJ).

Since $A_1\cap(P_2\oplus P_3)\subset P_2\oplus B_1\cap P_3$ and $A_1\ \cap(P_2\oplus P_3)$, $B_1\cap P_3$ are semisimple, we obtain a new decomposition: $P_2\oplus B_1\cap P_3\ =P_2\oplus V'$ such that $A\supset A_1\cap(P_2\oplus P_3)$ $=A_2\oplus A_3$ and $A_2\subset \mathrm{J}(P_2)$, $A_3\subset V'$, which is a semisimple module as (12). Therefore $B_1/A_1\approx P_2/A_2\oplus V$. Let $P_2\approx\Sigma_{I'}\oplus(f_iR)^{(m_i)}$; $m_i\leqq n_i$, where $I'\subset\{1,2,\cdots,s\}$ and I the subset of I' such that $k\in I$ if and only if $\pi_k(A_2)\neq 0$, where π_k: $P\to(f_kR)^{(m_i)}$ is the projection. Then

$$B_1/A_1\approx\Sigma_I\oplus(f_iR)^{(m_i)}/A_2\oplus\Sigma_{I'-I}\oplus(f_tR)^{(m_t)}\oplus V, \tag{13}$$

where $A\supset A_1\supset A_2$ and V is a semisimple module as (12).

We resume to prove the converse. We shall show first that

a) Soc(R) is almost L-projective for any local module $L=gR/D$.

Let S be a simple component of Soc(R) and consider a diagram:

$$(14)\qquad \begin{array}{c} S \\ \downarrow h \\ gR/D \xrightarrow{v} gR/C \to 0. \end{array}$$

If h is an epimorphism, h is an isomorphism. Hence putting $\tilde{h}=h^{-1}v$, we have $h\tilde{h}=v$. Accordingly we assume that h is not an epimorphism, i.e., $h(S)\subset gJ/C$. If $gR=fR$ $(fJ^2=0)$, fJ/D is semisimple, and hence we obtain $\tilde{h}$: $S\to fJ/D\subset fR/D$ with $v\tilde{h}=h$. Next assume $gR=eR$ $(eJ^2\neq 0)$. Then we may consider the following diagram instead of (14)

$$(14')\qquad \begin{array}{c} S \\ \downarrow h \\ eJ/D \xrightarrow{v} eJ/C \to 0. \end{array}$$

Let $S\approx \bar{k}R$ for a primitive idempotent k and $h(S)=(xR+C)/C$; $xk=x\in eJ$. Then $x\in \mathrm{Soc}(eJ)$ by iii), and hence $xR=xR/xJ$ is simple. Accordingly $h(S)=(xR+C)/C \approx xR$. Since $xR\cap D\subset xR\cap C=0$, we obtain an isomorphism $\tilde{h}$: $S\to xR\subset eJ/D$ with $v\tilde{h}=h$. Thus we have shown a).
Now let $M=gR/A$, $N=pR/D$ and M be N-projective. Take any diagram for any submodule M' of

$$(15)\qquad \begin{array}{c} M' \\ \downarrow h \\ pR/D \xrightarrow{v} pR/C \to 0 \end{array}$$

α) $M=fR/A$ $(fJ^2=0,\ fJ\neq 0)$.

Then any proper submodule M' of M is contained in Soc(R). Hence M' is almost pR/D-projective by a). Next assume

β) $M=eR/A$ $(eJ^2\neq 0)$ and $N=fR/D$.

From (10) and (13) M' is a direct sum of the following submodules:
1) $S\approx \mathrm{Soc}(f_iR)$ or $\approx S_j$, 2) $\Sigma_I\oplus(f_iR)^{(m_i)}/A_2$, where $\pi_i(A_2)\neq 0$ for $i\in I$, and 3) f_jR.
In the cases 1) and 3), M' is almost N-projective by a). Hence we may assume $M'=\Sigma_I\oplus(f_iR)^{(m_i)}/A_2$.

If $fR\not\approx f_iR$ for all i in 2), $\mathrm{Hom}_R(M',fR)=0$ by iii). Hence M' is trivially N-projective. If $fR\approx f_iR$ for some i, fR is uniserial and $fJ^2=0$. Then $fR\to fR/fJ\to 0$ is only a non-trivial exact seuqnce. Therefore M' is almost fR/D-projective (note that fR is projective). Assume

γ) $M=eR/A$ and $N=e'R/D$; $e'R\not\approx eR$.

Since eR/A is eR/D-projective, $eReA \subset D$. Further $0 \neq \pi_i(A_2)$ implies $\mathrm{Soc}(P_i) \subset eReA_2 \subset eReA \subset D \subset C$ by iii) and iv) (we note that if $P = P'_1 \oplus P'_2 \oplus \cdots \oplus P'_s$, where $P'_i \approx (f_i R)^{(p_i)}$, then $P_i = P'_i$ by iii)). We put $eJ = X \oplus Y$, where $X = \Sigma_I \oplus P_i$ and $Y = \Sigma_{j \notin I} \oplus P_j \oplus \tilde{S}$. Then from Lemma 10 and iii) $D = D \cap X \oplus D \cap Y \subset C = C \cap X \oplus C \cap Y$. As a consequence we obtain from (15)

$$\begin{array}{ccc} & & M' \\ & & \downarrow h \\ eJ/D = X/(D \cap X) \oplus Y/(D \cap Y) & \rightarrow & X/(C \cap X) \oplus Y/(C \cap Y) \rightarrow 0 \end{array}$$

Since $\mathrm{Hom}_R(P_i, P_j) = 0$ for $j \notin I$ and $\mathrm{Hom}_R(P_i, \tilde{S}) = 0$, $h(M') \subset X/(C \cap X)$. Further $X/(D \cap X)$ is semisimple for $D \cap X \supset \mathrm{Soc}(X)$, and hence we obtain $\tilde{h}$: $M' \rightarrow eJ/D$ with $v\tilde{h} = h$.

Next we consider (5) when N is a finite direct sum of local modules.

Theorem 2. *Let R be as above. Then* (5) *holds whenever M is local and N is a finite direct sum of local modules if and only if i)$\sim$iv) in Theorem* 1 *and v) the condition in Lemma* 9, $\mathrm{P}(\mathrm{Soc}(R))$ *is a direct sum of uniserial modules, are satisfied.*

Proof. "Only if" is given by Theorem 1 and Lemma 9. Conversely we assume i)$\sim$v). We use the same argument as given in the proof of Thorem 1. Let $N = \Sigma \oplus h_j R/B_j$, where the h_j are primitive idempotents and M $(= gR/A)$ be N-projective. Then M is $h_j R/B_j$-projective. Take any submodule of M' in M. We know from the proof of Theorem 1 that if M' is almost $h_j R/B_j$-projective, but not $h_j R/B_j$-projective, then M' is simple or $M' \approx \Sigma_I \oplus (f_i R)^{(m_i)}/A_2$ (see a), α) and β) in the proof of Theorem 1). In this case $h_j R$ is uniserial by v) and [4], Theorem 1. Hence M' is almost N-projective by [6], Theorem.

In a forthcoming paper we shall study (5) when N (resp. M) is any R-module.

2. Several rings with (5)

If gR is uniform for every primitive idempotent g, then we call R a *right QF-2 ring*. If E(R), the injective hull of R, is projective, than we call R a *QF-3 ring*. In this section we shall study QF, QF-2 and QF-3 rings with (5), respectively.

Proposition 1. *Assume that R is either local or* QF, *then* (5) *holds on local modules M and N if and only if $J^2 = 0$.*

Proof. If (5) holds, then there are no eR with $eJ^2 \neq 0$ from the assumption and Corollary to Lemma 2. The converse is clear from [7], Proposition 7.

Lemma 11. *Assume* (5) *on local modules* M *and* N. *If* hR *is uniform, then* hR *is uniserial, where* h *is a primitive idempotent.*

Proof. This is clear from Corollary to Lemma 3.

Proposition 2. R *is a right* QF-2 *ring over which* (5) *holds on local modules* M *and* N *if and only if* R *is a right serial ring with* $J^3=0$ *such that* 1) *if* $eJ^2\neq 0$, eJ/eJ^2 *is never monomorphic to* Soc(R), *and* 2) *if* $e_iJ^2\neq 0$ *for* $i=1,2$ *and* $e_1J/e_1J^2\approx e_2J/e_2J^2$, *then* $e_1R\approx e_2R$.

Proof. Assume (5) on local modules M and N. Then R is a right serial ring with 1) and 2) by Theorem 1 and Lemma 11. Conversely 1) implies that eJ is projective (cf. Lemma 14 below). Hence (5) holds by Theorem 1.

Next we study left QF-2 rings with (5) as right R-modules.

Lemma 12. *Let* R *be a ring with* $J^3=0$. *Assume that* eR *has the structure* (7′) *if* $eJ^2\neq 0$ (*where* f_iR *need not be uniserial*). *Let* θ *be a homomorphism of* hR *to* $h'R$. *If* $\theta(hJ)\neq 0$, θ *is monomorphic, where* e, h *and* h' *are primitive idempotents.*

Proof. Suppose that θ is not isomorphic. Since $\theta(hJ)\neq 0$, $\theta(hR)\not\subset \mathrm{Soc}(h'R)$. Hence $h'J^2\neq 0$. If $hJ^2\neq 0$, θ is isomorphic by Lemma 6. Hnece $hJ^2=0$, and θ is monomorphic from (7′).

Lemma 13. *Let* R *be left* QF-2. *Assume that* $J^3=0$ *and* eJ *has the structure in* (7′) *if* $eJ^2\neq 0$ (*where* f_iR *need not be uniserial*). *Then* 1) *Let* S_i *be a proper simple submodule of* g_iR *for* $i=1,2$ *and* θ: $S_1\to S_2$ *isomorphic. Then* θ *is extensible to an element in* $\mathrm{Hom}_R(g_1R,g_2R)$ *or in* $\mathrm{Hom}_R(g_2R,g_1R)$. 2) *Let* f_iR *be contained in* eR *as in* (7′). *Then* $\bar{f}_iR$ *is never monomorphic to* Soc(R). 3) $f_iR(\subset eR)\not\approx f_jR(\subset e'R)$ *if* $eR\not\approx e'R$. 4) *For any simple submodule* A *of* $P_i=(f_iR)^{(n_i)}\subset eR$, $eReA\supset \mathrm{Soc}(P_i)$, *where the* g_i *are primitive idempotents.*

Proof. 1). Put $S_i=x_iR\subset g_iJ$ with $x_2=\theta(x_1)$ and $S_i\approx \bar{h}R$. Then we can assume $g_ix_ih=x_i$ for $i=1,2$. Since Rh is uniform, put $\mathrm{Soc}(Rh)=R\bar{k}$, where k is a primitive idempotent. Then Rx_i containing Soc(Rh), there exists z_i in kRg_i such that $0\neq z_1x_1=z_2x_2$. Hence from Lemma 12 we have

(17) $\qquad g_1R\approx kR$ or $g_iR\subset kR$ via z_{il} (isomorphically),

where z_{il} is the left-sided multiplication of z_i.

i) z_{1l}: $g_1R\approx kR$.

Then there exists z'_l: $kR\to g_1R$ such that $z'z_1=g_1$. Hence $x_1=(z'z_2)x_2$ and θ^{-1}

is extensible to $(z'z_2)_l \in \mathrm{Hom}_R(g_2R,g_1R)$.
Here we assume 2).

ii) z_{1l}: $g_1R \to kR$ and z_{2l}: $g_2R \to kR$ are monomorphic (not isomorphic). Then $kJ^2 \neq 0$. In order to show 1) we may assume, in this case, $kR=eR$, $g_1R=f_iR$ and $g_2R=f_{i'}R$ in (7′), i.e., $S_1 \subset f_iR \subset eR$, $S_2 \subset f_{i'}R \subset eR$ and θ: $S_1 \to S_2$, and we give the extension of θ (or θ^{-1}) in $\mathrm{Hom}_R(f_iR,f_{i'}R)$ (or in $\mathrm{Hom}_R(f_{i'}R,f_iR)$). Hence since $S_i \subset eR$, we first consider the case $g_1=g_2=e$. Since $eJ^2 \neq 0$, we obtain the case i) from (17). Hence there exists a unit z in eRe such that z_l is an extension of θ. As a consequence $(f_iR)^{(n_i)}$ being characteristic, $f_iR=f_{i'}R$. Put $(f_iR)^{(n_i)}=\Sigma_{j \leq n_i} \oplus u_jf_iR$, where $u_j=u_jf_i$ and $u_jf_iR \approx f_iR$ for all j. Then we may assume $x_1=u_1r$, $x_2=u_1r'$; r, $r' \in f_iJ$. Now $z_l(u_1f_i)=\Sigma u_jw_j$ and the w_j are units in f_iRf_i or zero by the assumption 2). Since $\Sigma u_jw_jr=z_l(u_1r)=zx_1=x_2=u_1r'$, $z_l(u_1)=u_1w_1 \in u_1f_iR$, because $w_jr \in f_iR$, $f_iR \approx u_jf_iR$ and w_j is a unit or zero. Hence θ is extensible to $(z_l|u_1R) \in \mathrm{Hom}_R(f_iR,f_iR)$.

2) Let $eR \supset f_1R$ be as (7′) and S a simple component of $\mathrm{Soc}(tR)$, where t is a primitive idempotent with $tJ \neq 0$. Suppose $S \approx f_1R/f_1J$. Then there exist x_1 in f_1R-f_1J and x_2 in S such that $ex_1f_1=x_1$, $tx_2f_1=x_2$. Since $eJ^2 \neq 0$, from the similar argument to the initial part in 1)-i) we obtain $eR \approx kR$ as in 1)-i) and $x_1=zx_2$ for some $z \in eRt$, which is a contradiction, since $x_1 \notin \mathrm{Soc}(eR)$.

3) This is clear from 1) and Lemma 6.

4) Since $A \approx \mathrm{Soc}(f_iR)$. we obtain 4) from 1).

Corollary. *Let R be as in Lemma* 13. *If g_1R and g_2R have mutually isomorphic simple submodules, then $g_1R \approx g_2R$ or one of $\{g_1R,g_2R\}$ contains isomorphically the other.*

Proof. This is clear from lemmas 12 and 13.

Proposition 3. *Let R be a left* QF-2 *ring. Then* (5) *on local modules M and N holds as right R-modules if and only if i) $J^3=0$ and eJ has the structure* (7′), *provided $eJ^2 \neq 0$, (where f_iR is uniserial).*

Proof. Let $eR \supset eJ=\Sigma \oplus P_i \oplus \Sigma \oplus S_j$, where $P_i=(f_iR)^{(m_i)}$. Then every simple sub-factor module of P_i is not isomorphic to any one of P_j for $i \neq j$. Hence the proposition is clear from Theorem 1 and Lemma 13.

Corollary. *Let R be a right and left* QF-2 *ring. If* (5) *holds on local modules M and N, then R is serial, where g and g′ are primitive idempotents.*

Proof. We may show from Proposition 2 and [13], Lemma 4.3 that every isomorphism θ: $gJ/gJ^2 \approx g'J/g'J^2$ is liftable to an element in $\mathrm{Hom}_R(gR,g'R)$.

α) $gR=eR$ and $g'R=e'R$ ($eJ^2\neq 0$ and $e'J^2\neq 0$).

Then $e=e'$ by ii) of Proposition 2. Since eJ is projective, θ is given by an element θ' in $\mathrm{Hom}_R(eJ,eJ)$. Let $eJ=xR$, $xh=x$ for a primitive idempotent h and $\theta'(x)=x'$. Since Rh is uniform, there exist a primitive idempotent k and z, z' in kRe such that $zx=z'x'\neq o$. If $z\in J$, $z_l(eJ)=0$ by Lemma 6. Hence $k=e$ and z, z' are units in eRe. As a consequence θ is liftable.

β) $gR=eR$ and $g'R=fR$ ($fJ\neq 0$).

We do not have this case by i) of Proposition 2.

γ) $gR=fR$ and $g'R=f'R$.

Then θ is liftable by Lemma 13.

We shall study serial rings with (5) in the next proposition.

Lemma 14. *Let R be a serial ring with* $J^3=0$. *Then the following are equivalent*:

1) *If* $eJ^2\neq 0$, *eJ is projective.*

2) *If* $eJ^2\neq 0$, eJ/eJ^2 *is not monomorphic to* Soc(R), *where e runs over all the primitive idempotents.*

Proof. 1)→2). Suppose $eJ/eJ^2\approx \mathrm{Soc}(gR)$ for a primitive idempotent g. If $gJ^2\neq 0$, gJ is projective by 1). Let $gJ\approx hR$. Then since $\mathrm{Soc}(gR)\approx hJ=hJ/hJ^2\approx eJ/eJ^2$, $hR\approx eR$ by [13], Lemma 4.3, a contradiction. We obtain the same result if $gJ^2=0$,

2)→1). If eJ is not projective, $eJ\approx gR/gJ^2$ and $gJ^2\neq 0$. Hence $\mathrm{Soc}(eJ)\approx gJ/gJ^2$, a contradiction.

Proposition 4. *Let R be a* QF-3 *ring. Then the following are equivalant*:

1) (5) *holds on local modules M and N.*

2) *R is a serial ring with* $J^3=0$ *such that if* $eJ^2\neq 0$, eJ/eJ^2 *is not monomorphic to* Soc(R).

2') *R is serial ring with* $J^3=0$ *such that eJ is projective, if* $eJ^2\neq 0$.

3) *R is a serial ring with* $J^3=0$ *such that if* $J^2e\neq 0$, Je/J^2e *is not monomorphic to* Soc($_RR$).

4) (5) *holds on any finitely generated R-modules M and N as right R-modules as well as left R-modules.*

Proof. 1)→2). Assume that R is a QF-3 ring and (5) holds on local modules M and N. Then $J^3=0$ by Lemma 4. Next we shall show that R is a right serial ring. Let $\mathrm{E}(R)\approx\Sigma\oplus(h_iR)^{(p_i)}$, where the h_iR are indecomposable, injective and projective. We know from Lemma 11 that the h_iR are uniserial. Suppose gR is not injective for a primitive idempotent g such that $gJ\neq 0$. Then considering the projection of $E(R)$ to h_iR, we have $gR\subset J(E(R))$, since gR is not injective. Since h_iJ is projective by Lemma 5 if $h_iJ^2\neq 0$, $gR\approx h_iJ$ for some j. Therefore R is a right serial ring with $J^3=0$. The property in 2) is given by Proposition 2. We shall show that R is left serial. If $e_1J^2\neq 0$, e_1R is injective for $J^3=0$. Supposeθ: $e_1J/e_1J^2\approx e_2J/e_2J^2$ for any primitve idempotent e_2. Then $e_2J^2\neq 0$ by 1) in Proposition 2 and $e_1R\approx e_2R$ by 2) in Proposition 2. e_1J being projective from Lemma 5, θ is given by an isomorphism θ' of e_1J onto e_2J. Since e_1R is injective, θ' is extesible to an element in $\mathrm{Hom}_R(e_1R,e_2R)$. Suppose $e_1J^2=0$, then $e_2J^2=0$ as above. Hence e_1R and e_2R are contained in some injective eR for $\mathrm{Soc}(e_1R)\approx\mathrm{Soc}(e_2R)$. Hence θ is extensible to an element in $\mathrm{Hom}_R(e_1R,e_2R)$. Therefore R is serial ring by [13], Lemma 4.3.

2)$\rightarrow$1). This is clear from Proposition 2 and [13], Lemma 4.3.

2)$\leftrightarrow$2′). This is clear from Lemma 14.

1)$\rightarrow$4). Let $M=\Sigma\oplus e_iR/A_i$ be $N=\Sigma\oplus h_jR/B_j$-projective (see [12]). Take a submodule M' of M; $M'=\Sigma\oplus f_kR/C_k$. Then being uniserial, f_kR/C_k is isomorphic to a submodule of some e_iR/A_i. Since e_iR/A_i is h_jR/B_j-projective for all j, f_kR/C_k is almost h_jR/B_j-projective, and hence f_kR/C_k is almost N-projective by [6], Theorem. Hence (5) holds.

2)$\rightarrow$3). Suppose $J^2e_i\neq 0$ for $i=1,2$ and $Je_1/J^2e_1\approx J^2e_2$. Then there exists e_i' such that $(e_i'R,Re_i)$ is the injective pair for $i=1,2$ by [2], Theorem 3.1. Then $e_1'J/e_1'J^2\approx e_2'R/e_2'J$ by [2], Theorem 2.4 for $J^3=0$, and hence $e_1'J\approx e_2'R/e_2'J^2$. As a consequence $e_1'J^2\approx e_2'J/e_2'J^2$, a contradiction. Next assume $Je_1/J^2e_1\approx Jf\approx R\bar{g}$, where $J^2f=0$. If Rf is injective, gR is injective by [2], Theorem 3.1 and $e_1'J\approx gR$, a contradiction. If Rf is not injective, $E(Rf)\approx Re'$, which is again a contradiction from the initial. Then since Je_1/J^2e_1 is clearly not projective, Je_1/J^2e_1 is never monomorphic to $\mathrm{Soc}(_RR)$.

The remaining implications are clear.

3. Almost hereditary rings with $J^2=0$

We studied almost hereditary rings with $J^2=0$ in [7]. In this section we shall investigate again those rings. First we shall study a very special almost hereditary ring.

Proposition 5. *Every finitely generated R-module is almost projective if and only if R is a serial ring with* $J^2=0$.

Proof. Suppose that R is a serial ring with $J^2=0$. Then every indecomposable R-module is either eR or eR/eJ, where e is any primitive idempotent. If $eJ\neq 0$, eR is injective and hence eR/eJ is almost projective by [11], Theorem 1. Therefore every R-module is almost projective by [12]. The converse is clear from [7], Proposition 7 and [9], Corollary to Theorem 1.

Proposition 6. *Let R be an artinian ring with $J^2=0$.* Then the following are equivalent:

1) *R is right almost hereditary.*
2) *(5) holds when M is local.*
3) *(5) holds for any finitely generated R-modules M and N.*

Proof. 1)$\rightarrow$3). Assume that R is right almost hereditary. Then J is semisimple and almost projective. We quote here the argument in the proof of [7], Theorem 1. Let P be a projective cover of M; $0\rightarrow Q\rightarrow P\rightarrow M\rightarrow 0$, and M' a submodule of M. Then $M'=P'/Q$ for some submodule P' of P and $P=P_1\oplus P_2$ such that $P'\supset P_1$ and $P'\cap P_2$ is small in P. Put $Q_1=Q\cap P_1$ and $Q_2=Q\cap P_2$. Then since $P'\cap P_2$ is semisimple, we have $M'=P'/Q\approx P_1/Q_1\oplus Q^*/Q_2$, where $(P\cap P_2)/Q_2=Q^2/Q_1\oplus Q^*/Q_2$, and P_1 is a projective cover of P_1/Q_1. Suppose that M is N-projective. Then P_1/Q_1 is N-projective and $Q^*/Q_2\subset J(P)/Q_2$, Q^*/Q_2 is almost projective. Therefore M' is almost N-projective.

2)$\rightarrow$1). Since eR is N-projective for any R-module N, eJ is almost N-projective by (5). Hence eJ is almost projective.

3)$\rightarrow$1). This is clear.

Next we shall study the condition (4). Here we shall give the structure of right almost hereditary ring. From [8], Theorem 2 we know that every right almost hereditary ring is a direct sum of hereditary rings, serial rings and rings of a form

$$R=\begin{pmatrix} T_1 & X_2 & X_3 & \cdots X_m \\ 0 & S_2 & 0 & \cdots 0 \\ & & S_3 & 0\ 0 \\ & & & \ddots\ S_m \end{pmatrix}$$

where T_1 is a hereditary ring, the S_i are serial rings in the first category and the X_i is a left T_1-right S_i-module for each $i>1$. Without loss of generality, we may assume $S_i=0$ for all $i\geqq 2$. Hence in this note we assume

$$R=\begin{pmatrix} T_1 & X \\ 0 & S_2 \end{pmatrix}. \tag{18}$$

We study right almost hereditary rings of the form (18), i.e., S_2 is a serial ring in the first category and we may assume

$$S_2=\begin{pmatrix} \Delta & \Delta\cdots\Delta & 0 & \\ & \Delta\cdots\Delta & 0 & \\ & 0 & \vdots & \Delta \\ & & \ddots & \Delta \end{pmatrix}$$

where Δ is a division ring.
By h_i, f_i we denote matrix unite e_{ii} in T_1 and S_2, respectively. Then h_iX is a direct sum of copies of f_1R/B_1, where $B_1=(0\ 0\cdots 0\ \Delta\cdots\Delta\ 0\cdots 0)=f_1R(f_k+f_{k+1}+\cdots)\neq 0$; $k\geqq 2$.

If (4) holds for local modules M and N, then $J^2=0$ by [7], Proposition 7. Hence we assume $J^2=0$ in the above. Then $k=2$, i.e.,

$$h_iX=0 \quad \text{or} \quad h_iX=(f_1R/f_1J)^{(p_i)}. \tag{19}$$

We fix such a ring R and study structures of R-modules. Take a projective module $P=P_1\oplus P_2$, where $P_1\approx\Sigma\oplus(h_iR)^{(t_i)}$, $P_2\approx\Sigma\oplus(f_jR)^{(s_j)}$ and $Q\subset J(P)$. $J(P_1)$ and $J(P_2)$ do not contain a common isomorphic sub-factor module from (19). Therefore $Q=Q\cap P_1\oplus Q\cap P_2$ (put $Q_i=Q\cap P_i$). By $M_{(k)}$ we denote an R-module of the form P_k/Q_k ($k=$ 1,2). Then $M=M_{(1)}\oplus M_{(2)}$.

We put $Y=R-\begin{pmatrix} T_1 & X \\ 0 & \Delta\end{pmatrix}$ and $Z=\begin{matrix} T_1 & X \\ 0 & 0 \\ & 0 & 0\end{matrix}$.

Then Y, Z are ideals in R and R/Y is hereditary, R/Z is serial. Further the structure of R-module $M_{(1)}$ (resp. $M_{(2)}$) is the same as the structure of R/Y-module (resp. R/Z-module). (We note $\mathrm{Hom}_R(M_{(1)},M_{(2)})=0$ but $\mathrm{Hom}_R(M_{(2)},M_{(1)})\neq 0$ for some M.)

Lemma 15. *Let R be a right almost hereditary ring with $J^2=0$ as* (18). *If the hereditary ring $\tilde{R}(=R/Y)=\begin{pmatrix} T_1 & X \\ 0 & \Delta\end{pmatrix}$ satisfies* (4) (*resp.* (4) *where M is of special type*), *then R does the same.*

Proof. We use the same notations as after (19). Let M be any finitely generated R-module and M' a submodule of M. Then from the argument before Lemma 15 we obtain direct decompositions $M=M_{(1)}\oplus M_{(2)}$ and $M'=M'_{(1)}\oplus M'_{(2)}$. Since $M'_{(1)}\approx(\Sigma\oplus(h_kR^{(s'_k)})/A'$, $M_{(2)}=(\Sigma\oplus(f_jR^{(t_j)})/B$ and $\mathrm{Hom}_R(hR, fR)=0$, $\mathrm{Hom}_R(M'_{(1)},M_{(2)})=0$. Hence $M'_{(1)}\subset M_{(1)}$. Since R/Z is serial, f_iR is R/Z-injective, provided $f_iJ\neq 0$. Further f_iR is injective as R-modules from (18). Hence $M'_{(2)}$ is almost projective by [11], Theorem 1. Suppose that N is local; i) $N=hR/C$ or ii) $N=fR/D$, and M is almost N-projectve.

i) Since $M_{(1)}$ is almost N-projective as R-modules, we have same as $\tilde{R}$-module (and vice versa). Hence $M'_{(1)}$ is almost N-projective by assumption and the fact: $M'_{(1)} \subset M_{(1)}$. Further since $M'_{(2)}$ is almost projective, M' is almost N-projective.
ii) Since $\mathrm{Hom}_R(M'_{(1)}, fR/D)=0$ for any D in fR, $M'_{(1)}$ is (almost) N-projective. Hence we have shown
a) M' is almost N-projective provided N is local.
Now let $N=\Sigma\oplus N_i$; the N_i are indecomposable. We can find an integer k such that M is almost N_i-projective but not N_i-projective for all $i\leqq k$ and M is N_j-projective for all $j>k$. Then $\Sigma_{i\leqq k}\oplus N_i$ has LPSM by [6], Theorem and the N_i are local for $i\leqq k$ by [4], Theorem 1. Put $N^1=\Sigma_{i\leqq k}\oplus N_i$, $N^2=\Sigma_{j>k}\oplus N_j$. Noting that M is N^2-projective and Y is almost projective from the proof of Proposition 6. Further X is almost N_i-projective for all $i\leqq k$ by a). Hence since X is N^2-projective, X is almost N-projective by [6], Theorem. Therefore Y being almost projective, M' is almost N-projective.

REMARK. By the argument after the above a) we have shown that if (4) holds when N is local, then (4) holds for any R-module N.

Lemma 16. *Let R be a hereditary ring with $J^2=0$. Then* (4) *holds when M is a finite direct sum of local modules.*

Proof. Let M be almost N-projective for R-modules M and N, and M' a submodule of M. In order to show that M' is almost N-projective we may assume that N is local from the above remark. Let A be a submodule of gR, where g is a primitive idempotent. Assume that M is almost gR/A-projective and $M=\Sigma_{i\leqq n}\oplus M_i$; the M_i are local, i.e. $M_i=g_iR/D_i$ for all $i\leqq n$. We can suppose that M_i is almost gR/A-projective for all $j>m$. Since M_i is local and is almost gR/A-projective but not gR/A-projective, gR/A is M_i-projective for $i\leqq m$ by [4], Proposition 5. Put $L_1=\Sigma_{i\leqq m}\oplus M_i$ and $L_2=\Sigma_{j>m}\oplus M_j$, i.e., $M=L_1\oplus L_2$. Let π_i: $M\rightarrow L_i$ be the projection of M onto L_i for $i=1,2$. Now we shall show that M' is almost gR/A-projective for any submodule M' of M. Put $M'=T$ and take any diagram

$$\begin{array}{c} T \\ \downarrow h \\ gR/A \overset{\nu}{\rightarrow} gR/B \rightarrow 0 \end{array}$$

We may assume from [10], Theorem 1 that $im\,h$ is simple. If h is not an epimorphism, then we obtain μ: $im\,h\rightarrow gR/A$ with $\nu\mu=1_{im\,h}$, since gJ is semisimple. Hence we obtain $\tilde{h}=\mu h$: $T\rightarrow gR/A$ with $\nu\tilde{h}=h$. Assume that h is an epimorphism. Then $B=gJ$ and we obtain the isomorphism $\bar{h}$: $T/T_0\rightarrow gR/gJ$ induced from h, where $T_0=h^{-1}(0)$. Put $\bar{h}^{-1}(\bar{g})=t+T_0$ $(t=tg)$ and $t=t_1+t_2$;

$t_i=\pi_i(t)$. First we assume $\pi_2(T)=\pi_2(T_0)$. Then we may suppose $t_2=0$, and hence $t=t_1\in L_1$. T/T_0 being simple, $T/T_0\approx tR/(T_0\cap tR)$ and we obtain a diagram

$$\begin{array}{ccc} & gR/A & \\ & \downarrow v & \\ & gR/gJ & \\ & \wr\!\wr\ \tilde{h}^{-1} & \\ tR & \overset{v_{tR}}{\to} tR/(T_0\cap tR) & \to 0 \\ \cap & \cap & \\ L_1 & \to L_1/(T_0\cap tR) & \to 0, \end{array}$$

where $h|tR=\tilde{h}v_{tR}$.
Since gR/A is L_1-projective, we obtain $\tilde{h}$: $gR/A\to tR\subset T$ with $v=\tilde{h}v_{tR}\tilde{h}=h\tilde{h}$. Next suppose $\pi_2(T)\neq\pi_2(T_0)$ and $t=t_1+t_2$; we may assume $t_2\notin\pi_2(T_0)$ from the above argument. Then T/T_0 being simple, $T/T_0\approx\pi_2(T)/\pi_2(T_0)$. Since $\pi_2(T)\subset L_2$, $\pi_2(T)$ is gR/A-projective from [7], Theorem 1. Consider the diagram

$$\begin{array}{c} \pi_2(T) \\ \downarrow \rho_2 \\ \pi_2(T)/\pi_2(T_0) \\ \downarrow h' \\ gR/A \overset{v}{\to} gR/gJ \to 0 \end{array}$$

where $h'(t_2+\pi_2(T_0))=\bar{g}$ (note $t_2g=t_2$).
Then there exists $\tilde{h}'$: $\pi_2(T)\to gR/A$ with $v\tilde{h}'=h'\rho_2$. Put $\tilde{h}=\tilde{h}'\pi_2$. For any y in T $h(y)=\bar{h}(y+T_0)=\bar{h}(tr+T_0)=\bar{g}r$ for some r in R. On the other hand, since $y=tr+t_0$; $t_0\in T_0$, $y=t_1r+\pi_1(t_0)+t_2r+\pi_2(t_0)$. Hence $v\tilde{h}(y)=v\tilde{h}'\pi_2(y)=h'\rho_2\pi_2(y)$ $=h'(t_2r+\pi_2(T_0))=\bar{g}r=h(y)$.
Hence $v\tilde{h}=h$.

Proposition 7. *Let R be an artinian ring. Then the following are equivalent*:

1) (4) *holds when M is local.*
2) (4) *holds when M is a finite direct sum of local modules.*
3) *Any proper submodule of every local module is almost projective.*
4) *R is a right almost hereditary ring with* $J^2=0$.

Proof. 1) → 4). This is clear from the definition and [5], Proposition 7.

4) → 3). Let $M=gR/A$. Every proper submodule M' of M is contained in gJ/A. Since gJ is semisimple, gJ/A is isomorphic to a direct summand of gJ, which is almost projective. Hence (4) holds when M is local.

3) $\rightarrow$ 1). This is trivial.
1) $\leftrightarrow$ 2). This is clear from Lemmas 15 and 16.

Corresponding to Theorems 1 and 2

Corollary. *Let R be as above. Then*

1) (4) *holds when M and N are local if and only if $J^2=0$.*

2) (4) *holds when M is local and N is a direct sum of local modules if and only if $J^2=0$ and the projective cover of* Soc(R) *is a direct sum of uniserial modules.*

3) (4) *holds when M is local if and only if $J^2=0$ and R is right almost hereditary.*

Proof. Since (5) is a generalization of (4), this is clear from Theorem 2 and Proposition 7.

4. Examples

Let $L \supset K$ be fields and σ an automorphism of K.

1.

$$R_1=\begin{pmatrix} K & K & {}_\sigma K & K \oplus {}_\sigma K \\ 0 & K & 0 & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix},$$

where (kk' in R_1)=($\sigma(k)k'$ in K) for any $k \in K$ and $k' \in {}_\sigma K$.
Then $R=R_1$ is a hereditary ring, and putting $e_{ii}=e_i$, we have $e_1R \supset e_1J \approx e_2R \oplus e_3R$ and $\mathrm{Soc}(e_2R) \approx \mathrm{Soc}(e_3R)$. Since every simple submodule S in $\mathrm{Soc}(e_2R \oplus e_3R)$ ($\subset e_1J$) is of a form $S=\{k+\theta(k) | \in \mathrm{Soc}(e_2R)\} \subset e_1J$ for some isomorphism θ of $\mathrm{Soc}(e_2R)$ onto $\mathrm{Soc}(e_3R)$, $e_1Re_1S=\mathrm{Soc}(e_1R)$. Hence we know from Theorem 2 that (5) holds on local module M and a direct sum of local modues N, and R is (almost) hereditary. If we replace K_σ with K in the above ring, then this ring has the same structure of R except iv) in Theorem 1, and (5) does not hold on this ring.

2.

$$R_2=\begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix},$$ which satisfies all conditions in Theorem 1 except i).

However R_2 satisfies (5) as left R-modules when M and N are local.

3.

$$R_3=\begin{pmatrix} K & 0 & K & K \\ 0 & K & K & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix},$$ which satisfies all conditions in Theorem 1 except ii).

4. $R_4=eK\oplus fK\oplus aK\oplus bK\oplus abK$, where $\{e,f\}$ is the set of mutually orthogonal primitive idempotents with $1=e+f$, $a=eaf$ and $b=fbf$. Then R_4 satisfies all conditions in Theorem 1 except iii)

References

[1] T. Albu and C. Nastasescu: Relative finiteness in module theory, Monographs Textbooks Pure and Appl. 84, Marcel Dekker Inc. New York and Basel.

[2] K.R. Fuller: *On indecomposable injectives over artinian rings*, Pacific J. Math. **29** (1969), 115–135.

[3] ————: *Relative projectivity and injectivity classes by simple modules*, J. London Math. Soc. **5** (1972), 423–431.

[4] M. Harada and T. Mabuchi: *On almost relative projectives*, Osaka J. Math. **26** (1989), 837–848.

[5] M. Harada and A. Tozaki: *Almost M-projectives and right Nakayama rings*, J. Algebra **122** (1989), 447–474.

[6] M. Harada: *On almost relative projectives over perfect rings*, Osaka J. Math. **27** (1990), 655–665.

[7] ————: *Hereditary rings and relative projectives*, Osaka J. Math. **28** (1991), 811–827.

[8] ————: *Almost hereditary rings*, Osaka J. Math. **28** (1991), 793–809.

[9] ————: *Characterizations of right Nakayama rings*, Glasgow Math. J. **34** (1992), 91–102.

[10] ————: *Note on almost relative projectives and almost relative injectives*, Osaka J. Math. **29** (1992), 435–446.

[11] ————: *Almost projective modules*, J. Algebra **159** (1993), 150–157.

[12] T. Nakayama: *On Frobenius algebra II*, Ann. of Math. **42** (1941), 1–21.

[13] T. Sumioka: *On artinian ring of right local type*, Math J. Okayama Univ. **29** (1987), 127–146.

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Osakabe, Yao 581
Japan