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## ON UNIQUENESS PROBLEM FOR LOCAL DIRICHLET FORMS

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### 1. Introduction

Let  $X$  be a locally compact separable metric space and let  $m$  be a positive Radon measure on  $X$  with everywhere dense support. Let  $(\mathcal{E}, \mathcal{F})$  be a regular Dirichlet space satisfying the strong local property, i.e.,  $\mathcal{E}(u, v) = 0$  if  $u$  is constant on a neighbourhood of the support of the measure  $|v| \cdot m$ . Then, the form  $\mathcal{E}$  can be written as

$$\mathcal{E}(u, u) = \frac{1}{2} \int_X d\mu_{\langle u \rangle}, \quad u \in \mathcal{F},$$

where  $\mu_{\langle u \rangle}$  is the *energy measure* of  $u \in \mathcal{F}$  (cf. §3.2 in [7]). We say that a function  $u$  is *locally in  $\mathcal{F}$*  ( $u \in \mathcal{F}_{loc}$  in notation) if, for any relatively compact open subset  $G$  of  $X$ , there exists a function  $w \in \mathcal{F}$  such that  $u = w$   $m$ -a.e. on  $G$ . Because of the strong locality of  $(\mathcal{E}, \mathcal{F})$ , the energy measure  $\mu_{\langle u \rangle}$  can be defined for  $u \in \mathcal{F}_{loc}$ .

A pseudo metric  $\rho$  on  $X$  associated with  $(\mathcal{E}, \mathcal{F})$  is defined by

$$(1) \quad \rho(x, y) = \sup\{u(x) - u(y) : u \in \mathcal{F}_{loc} \cap C(X), \mu_{\langle u \rangle} \leq m\},$$

where  $\mu_{\langle u \rangle} \leq m$  means that the energy measure  $\mu_{\langle u \rangle}$  is absolutely continuous with respect to  $m$  with Radon-Nikodym derivative  $\frac{d\mu_{\langle u \rangle}}{dm} \leq 1$   $m$ -a.e. The pseudo metric  $\rho$  is called *intrinsic metric* and its properties has been investigated by Biroli and Mosco [1] and Sturm [17], [18]. Now, we make the following:

**ASSUMPTION A.**  $\rho$  is a metric on  $X$  and the topology induced by it coincides with the original one. Moreover,  $(X, \rho)$  is a complete metric space.

The objective of this paper is to show the uniqueness of the extensions of  $(\mathcal{E}, \mathcal{F})$  under Assumption A. In §2, we shall prove that if  $(\mathcal{E}, \mathcal{F})$  fulfills Assumption A, then it has a unique extension in Silverstein's sense (Theorem 2.2), which was introduced in [14] in order to classify the symmetric Markov semigroups dominating

the semigroup associated with  $(\mathcal{E}, \mathcal{F})$ .

Suppose that  $X$  is a smooth manifold and the domain of the self-adjoint operator  $A$  corresponding to  $(\mathcal{E}, \mathcal{F})$  contains the space  $C_0^\infty(X)$ , the set of infinitely differentiable functions with compact support. We can then consider self-adjoint extensions of the symmetric operator  $A \upharpoonright C_0^\infty(X)$ , where  $A \upharpoonright C_0^\infty(X)$  denotes the restriction of  $A$  to  $C_0^\infty(X)$ . In §3, we shall show that if  $A$  is hypoelliptic, Assumption A implies the essential self-adjointness of  $A \upharpoonright C_0^\infty(X)$  (Theorem 3.1).

Let  $(M, g)$  be a connected, smooth Riemannian manifold and  $\Delta$  the Laplace-Beltrami operator, that is, the self-adjoint operator associated with the regular Dirichlet space

$$(2) \quad \left\{ \begin{array}{l} \mathcal{E}(u, v) = \int_M (\text{grad } u, \text{grad } v) dV_g \\ \mathcal{F} = \text{the closure of } C_0^\infty(M) \text{ with respect to } \mathcal{E} + (\cdot, \cdot)_{V_g}, \end{array} \right.$$

where  $V_g$  denotes the Riemannian volume. Then, the intrinsic metric associated with the regular Dirichlet form (2) is nothing but the Riemannian distance, and Assumption A is equivalent to the completeness of the Riemannian manifold  $(M, g)$ . Hence, Theorem 3.1 tells us that if  $(M, g)$  is complete, then the operator  $\Delta \upharpoonright C_0^\infty(M)$  has a unique self-adjoint extension. This fact is well known (see Davies [5]) and thus Theorem 3.1 is regarded as an extension of it.

We emphasize that if a regular Dirichlet form is given, its extensions in Silverstein's sense always can be considered. Accordingly, Theorem 2.2 applies to singular Dirichlet forms as given in §4.

## 2. Uniqueness of extension in Silverstein's sense

For any Dirichlet space  $(\mathcal{E}, \mathcal{F})$  on  $L^2(X, m)$ , denote by  $\mathcal{F}_b$  the set of essentially bounded functions in  $\mathcal{F}$ . Then the space  $\mathcal{F}_b$  is an algebra over the real field  $\mathbb{R}$  (cf. [6] or A.4 in [7]). The following class of extensions was introduced by M. Silverstein [14]:

$$\mathcal{A}(\mathcal{E}, \mathcal{F}) = \left\{ \begin{array}{l} (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \text{ is a symmetric Dirichlet space on } L^2(X; m), \\ (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}): \tilde{\mathcal{F}} \supset \mathcal{F}, \tilde{\mathcal{E}}(u, u) = \mathcal{E}(u, u) \text{ for } u \in \mathcal{F}, \text{ and} \\ u \cdot v \in \mathcal{F} \text{ for } \forall u \in \tilde{\mathcal{F}}_b, \forall v \in \mathcal{F}_b \text{ (ideal property).} \end{array} \right.$$

We call an element of  $\mathcal{A}(\mathcal{E}, \mathcal{F})$  an *extension of  $(\mathcal{E}, \mathcal{F})$  in Silverstein's sense*. For the meanings of the extension in Silverstein's sense, see Theorem 20.1 in [14] or A.4.4 in [7].

Let  $\mathcal{F}^{ref}$  be the function space defined by

$$\mathcal{F}^{ref} = \{u \in L^2(X; m); u^{(n)} = (-n \vee u) \wedge n \in \mathcal{F}_{loc} \text{ for } \forall n > 0, \sup_n \mu_{\langle u^{(n)} \rangle}(X) < \infty\},$$

and set

$$\mathcal{E}^{ref}(u, u) = \lim_{n \rightarrow \infty} \frac{1}{2} \mu_{\langle u^{(n)} \rangle}(X) \quad \text{for } u \in \mathcal{F}^{ref}.$$

The above form  $(\mathcal{E}^{ref}, \mathcal{F}^{ref})$  is said to be the *reflected Dirichlet space* and was introduced by Z. Q. Chen [3]. We then have

**Theorem 2.1.** *For any  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \in \mathcal{A}(\mathcal{E}, \mathcal{F})$*

$$\tilde{\mathcal{F}} \subset \mathcal{F}^{ref}, \quad \tilde{\mathcal{E}}(u, u) \geq \mathcal{E}^{ref}(u, u) \quad u \in \tilde{\mathcal{F}}.$$

The above theorem was obtained and a short proof was given in [21]; however, we give a full proof for the reader's convenience. In order to do so, we need the fact shown in [20]. Let  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \in \mathcal{A}(\mathcal{E}, \mathcal{F})$  and let  $(X', m', \mathcal{E}', \mathcal{F}', \Phi)$  be its regular representaion, i.e.,  $(\mathcal{E}', \mathcal{F}')$  is a regular Dirichlet form on  $L^2(X'; m')$  and  $\Phi$  is an isometrically isomorphic map between two Dirichlet rings  $\tilde{\mathcal{F}}_b$  and  $\mathcal{F}'_b$  (see A.4 in [7] for detail). The map  $\Phi$  is constructed through the Gel'fand representation of a certain closed subalgebra  $L$  of  $L^\infty(X; m)$  satisfying

- (L.1)  $L$  is countably generated.
- (L.2)  $\tilde{\mathcal{F}} \cap L$  is dense both in  $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}}_1)$  and in  $(L, \|\cdot\|_\infty)$ .
- (L.3)  $L^1(X; m) \cap L$  is dense in  $(L, \|\cdot\|_\infty)$ .

For the existence of such a subalgebra  $L$ , see Theorem A.4.1 in [7]. By considering  $\mathcal{F} \cap C_0(X)$  if necessary, we can assume that

$$(3) \quad C_0(X) \subset L.$$

**Lemma 2.1.** *For  $u, v, w \in C_0(X)$  such that  $\text{supp}[u] \cap \text{supp}[v] = \emptyset$  and  $w = k$  (constant) on a neighbourhood of  $\text{supp}[u]$ ,*

- (i)  $\text{supp}[\Phi(u)] \cap \text{supp}[\Phi(v)] = \emptyset$
- (ii)  $\Phi(w) = k$  on a neighbourhood of  $\text{supp}[\Phi(u)]$ .

**Proof.** (i) Take  $f, g \in C_0(X)$  such that  $\text{supp}[f] \cap \text{supp}[g] = \emptyset$  and  $f$  and  $g$  are equal to 1 on  $\text{supp}[u]$  and  $\text{supp}[v]$ , respectively. Then, since  $\Phi(u) = \Phi(fu) = \Phi(f)\Phi(u)$ ,  $\Phi(f) = 1$  on  $\{x \in X' : \Phi(u)(x) \neq 0\}$ . On account of (3),  $\Phi(f)$  is a continuous function on  $X'$  (cf. Lemma A.4.3 in [7]). Hence,  $\text{supp}[\Phi(u)]$  is included in the open set  $\{\Phi(f) > 0\}$ , and by the same reason,  $\text{supp}[\Phi(v)]$  is included in the open set  $\{\Phi(g) > 0\}$ .  $\{\Phi(f) > 0\} \cap \{\Phi(g) > 0\} = \emptyset$  because  $\Phi(f)\Phi(g) = \Phi(fg) = \Phi(0) = 0$ , so  $\text{supp}[\Phi(u)] \cap \text{supp}[\Phi(v)] = \emptyset$ .

(ii) Suppose that  $w=k$  on an open set  $U (\supset \text{supp}[u])$  and take  $f \in C_0(X)$  such that  $f=1$  on  $\text{supp}[u]$  and  $\text{supp}[f] \subset U$ . Then,  $\Phi(w)=k$  on  $\{\Phi(f)>0\}$  because  $\Phi(w)\Phi(f)=\Phi(wf)=\Phi(kf)=k\Phi(f)$ .  $\square$

According to the Beurling-Deny formula, the regular Dirichlet form  $(\mathcal{E}', \mathcal{F}')$  can be decomposed as

$$\mathcal{E}'(u, v) = \mathcal{E}'^c(u, v) + \int_{X' \times X' - d} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))J'(dxdy) + \int_{X'} \tilde{u}(x)\tilde{v}(x)k'(dx),$$

for  $u, v \in \mathcal{F}'$

where  $\tilde{u}$  and  $\tilde{v}$  mean quasi continuous versions of  $u$  and  $v$ . Let us define Radon measures  $J$  on  $X \times X - d$  and  $k$  on  $X$  as follows: for  $f, g \in C_0(X)$  with  $\text{supp}[f] \cap \text{supp}[g] = \emptyset$

$$(4) \quad \int_{X \times X} f(x)g(y)J(dxdy) = \int_{X' \times X'} \Phi(f)(x)\Phi(g)(y)J'(dxdy)$$

and for  $f \in C_0(X)$

$$(5) \quad \int_X f(x)k(dx) = \int_{X'} \Phi(f)(x)k'(dx).$$

Note that  $J$  and  $k$  are well defined in view of Lemma 2.1. Finally, define the form  $\tilde{\mathcal{E}}^c(u, v)$  on  $\mathcal{F} \cap C_0(X)$  by

$$(6) \quad \tilde{\mathcal{E}}^c(u, v) = \mathcal{E}'^c(\Phi(u), \Phi(v)).$$

By Lemma 2.1 (ii),  $\tilde{\mathcal{E}}^c$  becomes a local form. We then see that the Dirichlet form  $\tilde{\mathcal{E}}$  can be decomposed as, for  $u, v \in \mathcal{F} \cap C_0(X)$

$$\begin{aligned} (7) \quad \tilde{\mathcal{E}}(u, v) &= \mathcal{E}'(\Phi(u), \Phi(v)) \\ &= \mathcal{E}'^c(\Phi(u), \Phi(v)) + \int_{X' \times X' - d} (\Phi(u)(x) - \Phi(u)(y))(\Phi(v)(x) - \Phi(v)(y))J'(dxdy) \\ &\quad + \int_{X'} \tilde{u}(x)\tilde{v}(x)k'(dx) \\ &= \tilde{\mathcal{E}}^c(u, v) + \int_{X \times X - d} (u(x) - u(y))(v(x) - v(y))J(dxdy) + \int_X u(x)v(x)k(dx). \end{aligned}$$

On the other hand,  $\tilde{\mathcal{E}} = \mathcal{E}$  on  $\mathcal{F} \cap C_0(X)$ . Hence,  $J=0$  and  $k=0$  on account of the regularity and strong locality of  $(\mathcal{E}, \mathcal{F})$ . As a result, we have  $\mathcal{E}'(\Phi(f), \Phi(g)) = \mathcal{E}'^c(\Phi(f), \Phi(g))$  for  $f, g \in \mathcal{F} \cap C_0(X)$ , and thus

$$(8) \quad \mathcal{E}'(\Phi(f), \Phi(g)) = \mathcal{E}'^c(\Phi(f), \Phi(g)) \quad \text{for } f, g \in \mathcal{F},$$

by virtue of the regularity of  $(\mathcal{E}, \mathcal{F})$ .

Proof of Theorem 2.1. Let  $u \in \tilde{\mathcal{F}}_b$ . Then the function  $u$  is an element of  $\mathcal{F}_{loc}$  by the ideal property of  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ , so the energy measure  $\mu_{\langle u \rangle}$  can be defined.

Let  $\{\Omega_n\}_{n=1}^\infty$  be a sequence of relatively compact open sets such that  $\Omega_1 \subset \bar{\Omega}_1 \subset \cdots \subset \Omega_n \subset \bar{\Omega}_n \subset \cdots$ ,  $\Omega_n \uparrow X$ . Let  $\varphi_n \in \mathcal{F} \cap C_0(X)$  be functions satisfying

$$0 \leq \varphi_n \leq 1, \quad \varphi_n = \begin{cases} 1 & \text{on } \Omega_n \\ 0 & \text{on } X \setminus \bar{\Omega}_{n+1}. \end{cases}$$

We then have from the derivation property of  $\mu_{\langle u \rangle}$  (see Lemma 3.2.5 in [7])

$$\mu_{\langle u \rangle}(X) = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu_{\langle u \rangle} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_X \varphi_n d\mu_{\langle \varphi_m u \rangle}.$$

Since  $\varphi_m u$  belongs to  $\mathcal{F}$ ,

$$(9) \quad \begin{aligned} \int_X \varphi_n d\mu_{\langle \varphi_m u \rangle} &= 2\mathcal{E}(\varphi_n \varphi_m u, \varphi_m u) - \mathcal{E}(\varphi_n, (\varphi_m u)^2) \\ &= 2\tilde{\mathcal{E}}(\varphi_n \varphi_m u, \varphi_m u) - \tilde{\mathcal{E}}(\varphi_n, (\varphi_m u)^2). \end{aligned}$$

Let  $(X', m', \mathcal{E}', \mathcal{F}', \Phi)$  be the regular representation of  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$  stated above. Then, by virtue of (8), the right hand side is equal to

$$(10) \quad \begin{aligned} &2\mathcal{E}'(\Phi(\varphi_n)\Phi(\varphi_m)\Phi(u), \Phi(\varphi_m)\Phi(u)) - \mathcal{E}'(\Phi(\varphi_n), (\Phi(\varphi_m)\Phi(u))^2) \\ &= 2\mathcal{E}'^c(\Phi(\varphi_n)\Phi(\varphi_m)\Phi(u), \Phi(\varphi_m)\Phi(u)) - \mathcal{E}'^c(\Phi(\varphi_n), (\Phi(\varphi_m)\Phi(u))^2) \\ &= \int_{X'} \Phi(\varphi_n) d\mu_{\langle \Phi(\varphi_m)\Phi(u) \rangle}^c, \end{aligned}$$

where  $\mu_{\langle \Phi(u) \rangle}^c$  is the continuous part of the energy measure  $\mu'_{\langle \Phi(u) \rangle}$  related to a regular Dirichlet space  $(\mathcal{E}', \mathcal{F}')$  (see §3.2 in [7].) Since for  $n < m$ ,  $\Phi(\varphi_m) = 1$  on some neighbourhood of  $\text{supp}[\Phi(\varphi_n)]$  by Lemma 2.1 (ii), the right hand side of (10) is

equal to  $\int_{X'} \Phi(\varphi_n) d\mu_{\langle \Phi(u) \rangle}^c$  by Lemma 3.2.5 in [7] again.

Since  $\|\Phi(\varphi_n)\|_\infty = \|\varphi_n\|_\infty \leq 1$ ,

$$\int_{X'} \Phi(\varphi_n) d\mu_{\langle \Phi(u) \rangle}^c \leq \int_{X'} d\mu_{\langle \Phi(u) \rangle}^c \leq 2\mathcal{E}'(\Phi(u), \Phi(u)) = 2\tilde{\mathcal{E}}(u, u).$$

Hence, we can conclude that

$$(11) \quad \int_X \mu_{\langle u \rangle} \leq 2\tilde{\mathcal{E}}(u, u),$$

and  $u \in \mathcal{F}^{ref}$ . The inequality (11) is extended to any  $u \in \tilde{\mathcal{F}}$ , thereby completing the proof of Theorem 2.1.

**REMARK 1.** It was shown in Chen [3] that  $(\mathcal{E}^{ref}, \mathcal{F}^{ref})$  is a Dirichlet space. Hence, we see from Theorem 2.1 that  $(\mathcal{E}^{ref}, \mathcal{F}^{ref})$  is the maximum element in  $\mathcal{A}(\mathcal{E}, \mathcal{F})$  with respect to the semi-order  $<$  on  $\mathcal{A}(\mathcal{E}, \mathcal{F})$  defined by

$$(\mathcal{E}^1, \mathcal{F}^1) < (\mathcal{E}^2, \mathcal{F}^2) \quad \text{if} \quad \mathcal{F}^1 \subset \mathcal{F}^2 \quad \text{and} \quad \mathcal{E}^1(u, u) \geq \mathcal{E}^2(u, u) \quad \text{for} \quad u \in \mathcal{F}^1.$$

An important implication of Assumption A is the next lemma proved in Sturm [16].

**Lemma 2.2.** *Under Assumption A, the function  $\rho_p(x) = \rho(p, x)$  belongs to  $\mathcal{F}_{loc} \cap C(X)$  and  $\mu_{\langle \rho_p \rangle} \leq m$ . Moreover, every ball  $B_r(p) = \{x : \rho_p(x) < r\}$  is relatively compact. Here  $p \in X$  is a fixed point.*

**Theorem 2.2.** *Under Assumption A, the Silverstein extension of  $(\mathcal{E}, \mathcal{F})$  is unique,  $\#(\mathcal{A}(\mathcal{E}, \mathcal{F})) = 1$ .*

**Proof.** Set

$$\varphi_n(x) = \begin{cases} 1 & x \leq n \\ n+1-x & n \leq x \leq n+1 \\ 0 & x \geq n+1. \end{cases}$$

Let  $u \in \mathcal{F}_b^{ref} (\subset \mathcal{F}_{loc})$ . Note that by Lemma 2.2  $\varphi_n(\rho_p)$  is an element of  $\mathcal{F}_b$  and  $\text{supp}[\varphi_n(\rho_p)]$  is a compact set according to Lemma 2.2. Hence, we have  $u \cdot \varphi_n(\rho_p) \in \mathcal{F}_b$  and

$$\begin{aligned} & \mathcal{E}(u\varphi_n(\rho_p) - u\varphi_m(\rho_p), u\varphi_n(\rho_p) - u\varphi_m(\rho_p)) = \frac{1}{2} \int_X d\mu_{\langle u(\varphi_n(\rho_p) - \varphi_m(\rho_p)) \rangle} \\ & \leq \int_X \tilde{u}^2 d\mu_{\langle \varphi_n(\rho_p) - \varphi_m(\rho_p) \rangle} + \int_X (\varphi_n(\rho_p) - \varphi_m(\rho_p))^2 d\mu_{\langle u \rangle} \\ & = \int_X \tilde{u}^2 (\varphi'_n(\rho_p) - \varphi'_m(\rho_p))^2 d\mu_{\langle \rho_p \rangle} + \int_X (\varphi_n(\rho_p) - \varphi_m(\rho_p))^2 d\mu_{\langle u \rangle}. \end{aligned}$$

Since the first term of the right hand side is dominated by  $\int_{\{n \leq \rho_p \leq n+1\} \cup \{m \leq \rho_p \leq m+1\}} \tilde{u}^2 dm$  on account of Lemma 2.2, it converges to 0 as  $n, m \rightarrow \infty$ . The second term also converges to 0 by the dominated convergence theorem. Noting

that  $u\varphi_n(\rho_p) \rightarrow u$  in  $L^2$ , we see that  $u$  belongs to  $\mathcal{F}$ , which implies the theorem on account of Theorem 2.1.  $\square$

REMARK 2. Let  $(\mathcal{E}^1, \mathcal{F})$  and  $(\mathcal{E}^2, \mathcal{F})$  be regular Dirichlet forms on  $L^2(X; m^1)$  and  $L^2(X; m^2)$ . Suppose that these Dirichlet forms are quasi-equivalent: there exist constants  $c_1, c_2 \geq 1$  such that

$$c_1^{-1} \mathcal{E}^1(u, u) \leq \mathcal{E}^2(u, u) \leq c_1 \mathcal{E}^1(u, u) \quad \text{for } u \in \mathcal{F}, \quad c_2^{-1} m^1 \leq m^2 \leq c_2 m^1.$$

Then, by the domination principle (cf. [10])

$$c_1^{-1} \mu_{\langle u \rangle}^1 \leq \mu_{\langle u \rangle}^2 \leq c_1 \mu_{\langle u \rangle}^1,$$

where  $\mu_{\langle u \rangle}^1$  (resp.  $\mu_{\langle u \rangle}^2$ ) is the energy measure of  $u$  associated with  $(\mathcal{E}^1, \mathcal{F})$  (resp.  $(\mathcal{E}^2, \mathcal{F})$ ). Thus, we have

$$\mathcal{F}^{1, ref} = \mathcal{F}^{2, ref}$$

by the definition of the reflected Dirichlet space. Here  $\mathcal{F}^{1, ref}$  and  $\mathcal{F}^{2, ref}$  are reflected Dirichlet spaces associated with  $(\mathcal{E}^1, \mathcal{F})$  and  $(\mathcal{E}^2, \mathcal{F})$ . Therefore, we can conclude that the uniqueness of Silverstein's extension is stable under quasi-equivalence.

REMARK 3. Let  $N \subset X$  be a closed set with  $\text{Cap}(N) = 0$ , where  $\text{Cap}$  denotes the 1-capacity associated with the Dirichlet form  $(\mathcal{E}, \mathcal{F})$ . Set  $D = X \setminus N$  and let  $(\mathcal{E}^D, \mathcal{F}^D)$  be the part of  $(\mathcal{E}, \mathcal{F})$  on  $D$ . Then, by the same argument as in Remark 4.3 in [11], Theorem 2.2 can be extended as follows: under Assumption A, the extension of  $(\mathcal{E}^D, \mathcal{F}^D)$  in Silverstein's sense is unique,  $\#(\mathcal{A}(\mathcal{E}^D, \mathcal{F}^D)) = 1$ .

### 3. Uniqueness of self-adjoint extension

Let  $A$  be the self-adjoint operator associated with  $(\mathcal{E}, \mathcal{F})$ . Throughout this section, we suppose that  $X$  is a smooth manifold and the space  $C_0^\infty(X)$  is included in the domain of  $A$ . Let us denote by  $S$  the symmetric operator  $A \upharpoonright C_0^\infty(X)$ , the restriction of  $A$  to  $C_0^\infty(X)$ . Furthermore, we assume that  $S$  is a hypoelliptic differential operator in the sense that

$$\mathcal{N}_1 = \{u \in \mathcal{D}(S^*); (1 - S^*)u = 0\} \subset C^\infty(X),$$

where  $S^*$  is the adjoint operator of  $S$ .

Then, by following the proof of Theorem 5.2.3 in Davies [5], we obtain

**Theorem 3.1.** *Under Assumption A, the operator  $S$  is essentially self-adjoint.*

Proof. Take  $g \in \mathcal{N}_1$ . By the hypoellipticity of  $S$ ,  $g \in C^\infty(X)$  and  $Sg = g$ . Let  $\psi$  be the function defined on  $[0, \infty)$  by

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 1-x & 1 \leq x < 2 \\ 0 & 2 \leq x, \end{cases}$$

and put  $\phi_n(x) = \psi(\frac{\rho_p(x)}{n})$ . Noting that  $\phi_n \in \mathcal{F}_{loc} \cap L^2(X; m)$  and that

$$\begin{aligned} \mu_{\langle \phi_n \rangle}(X) &= \frac{1}{n^2} \int_X \psi' \left( \frac{\rho_p(x)}{n} \right)^2 d\mu_{\langle \rho_p \rangle} \\ &\leq \frac{1}{n^2} \int_{\{n \leq \rho_p(x) \leq 2n\}} dm < \infty, \end{aligned}$$

we see from Theorem 2.2 that  $\phi_n \in \mathcal{F}$ .

Let  $\varphi \in C_0^\infty(X)$  such that  $\varphi = 1$  on a neighbourhood of  $\{\rho_p(x) \leq 2n\}$ . Then,

$$0 \geq - \int_X \phi_n^2 g^2 dm = - \int_X \phi_n^2 g Sg dm = - \int_X \phi_n^2 g S(g\varphi) dm,$$

and since  $\phi_n^2 g, g\varphi \in \mathcal{F}$ , the right hand side equals

$$\begin{aligned} \frac{1}{2} \int_X d\mu_{\langle \phi_n^2 g, g\varphi \rangle} &= \frac{1}{2} \int_X \phi_n^2 d\mu_{\langle g, g\varphi \rangle} + \int_X g \phi_n d\mu_{\langle \phi_n, g\varphi \rangle} \\ &= \frac{1}{2} \int_X \phi_n^2 d\mu_{\langle g \rangle} + \int_X g \phi_n d\mu_{\langle \phi_n, g \rangle}. \end{aligned}$$

Hence, by virtue of Lemma 5.6.1 in [7],

$$\begin{aligned} \int_X \phi_n^2 d\mu_{\langle g \rangle} &\leq -2 \int_X g \phi_n d\mu_{\langle \phi_n, g \rangle} \\ &\leq 2 \left( \int_X g^2 d\mu_{\langle \phi_n \rangle} \right)^{1/2} \left( \int_X \phi_n^2 d\mu_{\langle g \rangle} \right)^{1/2}, \end{aligned}$$

and so

$$\int_X \phi_n^2 d\mu_{\langle g \rangle} \leq 4 \int_X g^2 d\mu_{\langle \phi_n \rangle}.$$

Since the right hand side is dominated by  $\frac{4}{n^2} \int_{\{n \leq \rho_p(x) \leq 2n\}} g^2 dm$ ,

$$\mu_{\langle g \rangle}(X) = \lim_{n \rightarrow \infty} \int_X \phi_n^2 d\mu_{\langle g \rangle} = 0.$$

Hence,  $g \in \mathcal{F}$  and  $\mathcal{E}(g, g) = 0$  by Theorem 2.2. Noting that  $|\mathcal{E}(g, v)| \leq \mathcal{E}(g, g)^{1/2} \mathcal{E}(v, v)^{1/2} = 0$  for any  $v \in \mathcal{F}$ , we can conclude that  $g \in \mathcal{D}(A)$  and  $Ag = 0$ . Therefore,  $g = S^*g = Ag = 0$ , which leads us to the theorem.  $\square$

REMARK 4. Let  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$  be an element of  $\mathcal{A}(\mathcal{E}, \mathcal{F})$  and  $\tilde{A}$  the self-adjoint operator associated with  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ . Then, under the situation of this section,  $\tilde{A}$  is a self-adjoint extension of  $S$  and so Theorem 3.1 implies Theorem 2.2. In fact, take  $u \in \mathcal{D}(\tilde{A})$  and  $\varphi \in C_0^\infty(X)$ . Let  $(X', m', \mathcal{E}', \mathcal{F}', \Phi)$  be the regular representation of  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$  as in §2. Then

$$(-\tilde{A}u, \varphi) = \tilde{\mathcal{E}}(u, \varphi) = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}(u^n, \varphi) \quad (u^n = (-n \vee u) \wedge n)$$

and by (8)

$$\tilde{\mathcal{E}}(u^n, \varphi) = \mathcal{E}'(\Phi(u^n), \Phi(\varphi)) = \mathcal{E}'^c(\Phi(u^n), \Phi(\varphi)).$$

Take  $\psi \in \mathcal{F} \cap C_0(X)$  such that  $\psi = 1$  on a neighbourhood of  $\text{supp}[\varphi]$ . Then the right hand side is equal to

$$\mathcal{E}'^c(\Phi(u^n\psi), \Phi(\varphi)) = \mathcal{E}'(\Phi(u^n\psi), \Phi(\varphi)) = \tilde{\mathcal{E}}(u^n\psi, \varphi)$$

on account of Lemma 2.1 (ii). Noting that  $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \in \mathcal{A}(\mathcal{E}, \mathcal{F})$ , we get

$$\tilde{\mathcal{E}}(u^n\psi, \varphi) = \mathcal{E}(u^n\psi, \varphi) = (u^n, -S\varphi).$$

Therefore,  $(\tilde{A}u, \varphi) = (u, S\varphi)$  and so  $\tilde{A} \subset S^*$ .

#### 4. Examples

EXAMPLE 1. Let  $(M, g)$  be a complete smooth Riemannian manifold. For  $\psi \in L_{loc}^2(M; V_g)$  with  $\psi > 0$ ,  $V_g$ -a.e., consider the symmetric form on  $L^2(M; \psi^2 V_g)$ :

$$\mathcal{E}^\psi(u, v) = \frac{1}{2} \int_M (\text{grad } u, \text{grad } v) \psi^2 dV_g \quad u, v \in C_0^\infty(M).$$

If the above form is closable, we say that  $\psi$  is admissible. For conditions for  $\psi$  being admissible, see [6] and [8]. For an admissible function  $\psi$ , denote by  $\mathcal{F}^\psi$  the closure of  $C_0^\infty(M)$  with respect to  $\mathcal{E}^\psi + (\cdot, \cdot)_{\psi^2 V_g}$ . Then,  $(\mathcal{E}^\psi, \mathcal{F}^\psi)$  becomes a regular Dirichlet form and, independently of each admissible function  $\psi$ , the intrinsic metric associated with  $(\mathcal{E}^\psi, \mathcal{F}^\psi)$  is identical to the Riemannian distance. Hence, Theorem 2.2 implies that  $\#(\mathcal{A}(\mathcal{E}^\psi, \mathcal{F}^\psi)) = 1$  for any admissible function  $\psi$ , in particular, for any  $\psi \in H_{loc}^1(M; V_g)$  with  $\psi > 0$ ,  $V_g$ -a.e. Here  $H_{loc}^1(M; V_g)$  is the set of functions belonging locally to the Sobolev space of order 1.

On the other hand, the uniqueness of Markovian extensions is known only in the case where  $M = R^d$ ,  $\psi \in H_{loc}^1(R^d)$ , and  $\psi > 0$   $dx$ -a.e. (see [13], [15]). As a corollary of this result, we proved in [21] that  $\#(\mathcal{A}(\mathcal{E}^\psi, \mathcal{F}^\psi)) = 1$  for  $\psi \in H_{loc}^1(R^d)$  with  $\psi > 0$   $dx$ -a.e.

EXAMPLE 2. Let  $X = R^d$  and  $m$  the Lebesgue measure. Consider the following regular Dirichlet form:

$$(12) \quad \begin{cases} \mathcal{E}(u, v) = \sum_{i,j=1}^d \int_{R^d} a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \\ \mathcal{F} = \text{the closure of } C_0^\infty(R^d) \text{ with respect to } \mathcal{E}_1. \end{cases}$$

Here, the coefficients  $a_{i,j}$  are locally uniform elliptic and satisfy

$$(13) \quad \sum_{i,j=1}^d a_{i,j}(x) \xi_i \xi_j \leq k(|x| + 2)^2 (\log(|x| + 2))^2 |\xi|^2 \quad \text{for } \xi \in R^d.$$

Denote by  $\rho$  the metric associated with  $(\mathcal{E}, \mathcal{F})$ . Then, the local uniform ellipticity and (13) imply that the topology induced from  $\rho$  is equivalent with the usual topology on  $R^d$ .

Set

$$\psi(x) = \frac{1}{\sqrt{k}} \int_0^{|x|} \frac{ds}{(s+2) \log(s+2)}.$$

We then easily see that  $\psi \in \mathcal{F}_{loc} \cap C(R^d)$  and on account of (13)

$$a_{i,j}(x) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \leq 1.$$

Hence, for  $\forall r > 0$

$$\begin{aligned} \{x \in R^d : \rho(0, x) \leq r\} &\subset \{x \in R^d : \psi(x) \leq r\} \\ &= \left\{ x \in R^d : \int_0^{|x|} \frac{ds}{(s+2) \log(s+2)} \leq \sqrt{kr} \right\}. \end{aligned}$$

Hence,  $\rho$  fulfills Assumption A, and which implies that  $\#(\mathcal{A}(\mathcal{E}, \mathcal{F})) = 1$ . If  $a_{ij}$  are smooth, the essential self-adjointness is known (see [4]).

EXAMPLE 3. Let  $X = R^d$  and  $m$  the Lebesgue measure. Suppose that the form  $(\mathcal{E}, C_0^\infty(R^d))$  is uniformly subelliptic, i.e., there exist constants  $\varepsilon, \lambda > 0$  and  $C$  such that

$$(14) \quad \frac{1}{\lambda} \|u\|_1^2 \geq \mathcal{E}(u, u) \geq \lambda \|u\|_\varepsilon^2 - C \|u\|_0^2, \quad \forall u \in C_0^\infty(R^d),$$

where  $\|u\|_\varepsilon^2 = \int_{R^d} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^\varepsilon d\xi$  with  $\hat{u}$  being the Fourier transformation of  $u$ . Then it is known that  $(\mathcal{E}, C_0^\infty(R^d))$  is closable and its closure  $(\mathcal{E}, \mathcal{F})$  is a strongly local Dirichlet form. Moreover, the uniform subellipticity condition holds if and only if there exist constants  $r_0 > 0$  and  $C_0 > 0$  such that

$$C_0 |x - y| \leq \rho(x, y) \leq \frac{1}{C_0} |x - y|^\varepsilon$$

for all  $x, y \in R^d$  with  $|x - y| < r_0$  (cf. [19]). Therefore, we can conclude that the intrinsic metric  $\rho$  fulfills Assumption A and  $(\mathcal{E}, \mathcal{F})$  has a unique Silverstein extension.

EXAMPLE 4. Let  $D$  be a bounded domain in  $R^d$  with smooth boundary  $\partial D$ , and  $m(dx) = \sigma^b(x) dx$ . Here  $\sigma$  is supposed to satisfy

$$\lambda d(x, \partial D) \leq \sigma(x) \leq \Lambda d(x, \partial D).$$

Let us consider the Dirichlet form defined by

$$(15) \quad \begin{cases} \mathcal{E}(u, v) = \int_D (\text{grad } u, \text{grad } v) \sigma^a dx \\ \mathcal{F} = \text{the closure of } C_0^\infty(D) \text{ with respect to } \mathcal{E}_1. \end{cases}$$

Suppose  $a - b > 2$  and set

$$\psi(x) = \frac{1}{\alpha \Lambda^{(a-b)/2}} d(x, \partial D)^\alpha \quad (\alpha = \frac{2+b-a}{2} < 0).$$

Then, we see that  $(\text{grad } \psi \cdot \text{grad } \psi) \sigma^{(a-b)} \leq 1$  and thus for a fixed point  $p \in D$

$$\{x \in D : \rho(p, x) \leq r\} \subset \{x \in D : \psi(x) \leq r + \psi(p)\}.$$

Since  $\lim_{x \rightarrow \partial D} \psi(x) = \infty$  if  $a - b > 2$ , Assumption A is satisfied. For the essential self-adjointness, see [9].

The final example tells us that the completeness is destroyed by some time change.

EXAMPLE 5. Let  $X$  be  $R^d$  ( $d \geq 3$ ) and  $m$  a smooth positive Radon measure in the sense of [7]. Let us consider the Dirichlet form on  $L^2(R^d; m)$  defined by

$$(16) \quad \begin{cases} \mathcal{E}(u, v) = \frac{1}{2} \sum_{i=1}^d \int_{R^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \\ \mathcal{F} = \text{the closure of } C_0^\infty(R^d) \text{ with respect to } \mathcal{E}_1 (= \mathcal{E} + (\cdot, \cdot)_m). \end{cases}$$

Denote by  $\mathcal{C}$  the set of  $C^\infty(R^d)$ -functions  $f$  satisfying  $\sum_{i=1}^d \int_{R^d} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} dx < \infty$  and denote by  $\tilde{\mathcal{F}}$  the closure of  $\mathcal{C}$  with respect to  $\mathcal{E}_1$ . Then, it is shown in [3] that  $\mathcal{F} = \tilde{\mathcal{F}}$  if and only if the measure  $m$  satisfies

$$(*) \quad m(R^d \setminus A) = \infty \quad \text{for } \forall A \in \mathcal{B}(R^d) \text{ with } \text{Cap}(A) < \infty,$$

where  $\text{Cap}$  means the 1-capacity associated with the classical Dirichlet form  $(\frac{1}{2}D, H^1(R^d))$ . Hence, if  $m$  does not satisfy the condition (\*), in particular, if  $m$  is a finite measure, then the extension of  $(\mathcal{E}, \mathcal{F})$  in Silverstein's sense is not unique. Accordingly, the pseudo metric corresponding to  $(\mathcal{E}, \mathcal{F})$  is not complete on account of Theorem 2.2.

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