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ON UNIQUENESS PROBLEM FOR LOCAL DIRICHLET FORMS

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1. Introduction

Let X be a locally compact separable metric space and let m be a positive Radon measure on X with everywhere dense support. Let $(\mathscr{E},\mathscr{F})$ be a regular Dirichlet space satisfying the strong local property, i.e., $\mathscr{E}(u,v)=0$ if u is constant on a neighbourhood of the support of the measure $|v|\cdot m$. Then, the form $\mathscr{E}$ can be written as

$$\mathscr{E}(u,u)=\frac{1}{2}\int_X d\mu_{\langle u\rangle}, \qquad u\in\mathscr{F},$$

where $\mu_{\langle u\rangle}$ is the *energy measure* of $u\in\mathscr{F}$ (cf. §3.2 in [7]). We say that a function u is *locally in* $\mathscr{F}$ ($u\in\mathscr{F}_{loc}$ in notation) if, for any relatively compact open subset G of X, there exists a function $w\in\mathscr{F}$ such that $u=w$ m-a.e. on G. Because of the strong locality of $(\mathscr{E},\mathscr{F})$, the energy measure $\mu_{\langle u\rangle}$ can be defined for $u\in\mathscr{F}_{loc}$.

A pseudo metric ρ on X associated with $(\mathscr{E},\mathscr{F})$ is defined by

$$(1) \qquad \rho(x,y)=\sup\{u(x)-u(y): u\in\mathscr{F}_{loc}\cap C(X),\ \mu_{\langle u\rangle}\leq m\},$$

where $\mu_{\langle u\rangle}\leq m$ means that the energy measure $\mu_{\langle u\rangle}$ is absolutely continuous with respect to m with Radon-Nikodym derivative $\frac{d\mu_{\langle u\rangle}}{dm}\leq 1$ m-a.e. The pseudo metric ρ is called *intrinsic metric* and its properties has been investigated by Biroli and Mosco [1] and Sturm [17], [18]. Now, we make the following:

ASSUMPTION A. ρ is a metric on X and the topology induced by it coincides with the original one. Moreover, (X,ρ) is a complete metric space.

The objective of this paper is to show the uniqueness of the extensions of $(\mathscr{E},\mathscr{F})$ under Assumption A. In §2, we shall prove that if $(\mathscr{E},\mathscr{F})$ fulfills Assumption A, then it has a unique extension in Silverstein's sense (Theorem 2.2), which was introduced in [14] in order to classify the symmetric Markov semigroups dominating the semigroup associated with $(\mathcal{E},\mathcal{F})$.

Suppose that X is a smooth manifold and the domain of the self-adjoint operator A corresponding to $(\mathcal{E},\mathcal{F})$ contains the space $C_0^\infty(X)$, the set of infintely differentiable functions with compact support. We can then consider self-adjoint extensions of the symmetric operator $A\upharpoonright C_0^\infty(X)$, where $A\upharpoonright C_0^\infty(X)$ denotes the restriction of A to $C_0^\infty(X)$. In §3, we shall show that if A is hypoelliptic, Assumption A implies the essential self-adjoitness of $A\upharpoonright C_0^\infty(X)$ (Theorem 3.1).

Let (M,g) be a connected, smooth Riemannian manifold and Δ the Laplace-Beltrami operator, that is, the self-adjoint operator associated with the regular Dirichlet space

$$(2)\qquad \begin{cases} \mathcal{E}(u,v)=\displaystyle\int_M (\operatorname{grad} u, \operatorname{grad} v)dV_g \\ \mathcal{F}=\text{the closure of } C_0^\infty(M) \text{ with respect to } \mathcal{E}+(\,,\,)_{V_g}, \end{cases}$$

where V_g denotes the Riemannian volume. Then, the intrinsic metric associated with the regular Dirichlet form (2) is nothing but the Riemaniann distance, and Assumption A is equivalent to the completeness of the Riemannian manifold (M,g). Hence, Theorem 3.1 tells us that if (M,g) is complete, then the operator $\Delta\upharpoonright C_0^\infty(M)$ has a unique self-adjoint extension. This fact is well known (see Davies [5]) and thus Theorem 3.1 is regarded as an extension of it.

We emphasis that if a regular Dirichlet form is given, its extensions in Silverstein's sense always can be considered. Accordingly, Theorem 2.2 applies to singular Dirichlet forms as given in §4.

2. Uniqueness of extension in Silverstein's sense

For any Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$, denote by $\mathcal{F}_b$ the set of essentially bounded functions in $\mathcal{F}$. Then the space $\mathcal{F}_b$ is an algebra over the real field R (cf. [6] or A.4 in [7]). The following class of extensions was introduced by M. Silverstein [14]:

$$\mathcal{A}\mathcal{s}(\mathcal{E},\mathcal{F})=\left\{(\tilde{\mathcal{E}},\tilde{\mathcal{F}}):\begin{array}{l} (\tilde{\mathcal{E}},\tilde{\mathcal{F}}) \text{ is a symmetric Dirichlet space on } L^2(X;m), \\ \tilde{\mathcal{F}}\supset\mathcal{F},\ \tilde{\mathcal{E}}(u,u)=\mathcal{E}(u,u) \text{ for } u\in\mathcal{F}, \text{ and} \\ u\cdot v\in\mathcal{F} \text{ for } \forall u\in\tilde{\mathcal{F}}_b,\ \forall v\in\mathcal{F}_b \text{ (ideal property).} \end{array}\right\}$$

We call an element of $\mathcal{A}\mathcal{s}(\mathcal{E},\mathcal{F})$ an *extension of* $(\mathcal{E},\mathcal{F})$ *in Silverstein's sense.* For the meanings of the extension in Silverstein's sense, see Theorem 20.1 in [14] or A.4.4 in [7].

Let $\mathcal{F}^{ref}$ be the function space defined by

$$\mathscr{F}^{ref}=\{u\in L^2(X;m);\ u^{(n)}=(-n\vee u)\wedge n\in\mathscr{F}_{loc} \text{ for } \forall n>0,\ \sup_n \mu_{\langle u^{(n)}\rangle}(X)<\infty\},$$

and set

$$\mathscr{E}^{ref}(u,u)=\lim_{n\to\infty}\frac{1}{2}\mu_{\langle u^{(n)}\rangle}(X) \quad \text{for} \quad u\in\mathscr{F}^{ref}.$$

The above form $(\mathscr{E}^{ref},\mathscr{F}^{ref})$ is said to be the *reflected Dirichlet space* and was introduced by Z. Q. Chen [3]. We then have

Theorem 2.1. *For any* $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})\in\mathscr{A}\mathscr{J}(\mathscr{E},\mathscr{F})$

$$\tilde{\mathscr{F}}\subset\mathscr{F}^{ref}, \quad \tilde{\mathscr{E}}(u,u)\geq\mathscr{E}^{ref}(\mathrm{u},\mathrm{u}) \quad u\in\tilde{\mathscr{F}}.$$

The above theorem was obtained and a short proof was given in [21] ; however, we give a full proof for the reader's convenience. In order to do so, we need the fact shown in [20]. Let $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})\in\mathscr{A}\mathscr{J}(\mathscr{E},\mathscr{F})$ and let $(X',m',\mathscr{E}',\mathscr{F}',\Phi)$ be its regular representaion, i.e., $(\mathscr{E}',\mathscr{F}')$ is a regular Dirichlet form on $L^2(X';m')$ and Φ is an isometrically isomorphic map between two Dirichlet rings $\tilde{\mathscr{F}}_b$ and $\mathscr{F}'_b$ (see A.4 in [7] for detail). The map Φ is constructed through the Gel'fand representation of a certain closed subalgebra L of $L^\infty(X;m)$ satisfying

(L.1) L is countably generated.
(L.2) $\tilde{\mathscr{F}}\cap L$ is dense both in $(\tilde{\mathscr{F}},\tilde{\mathscr{E}}_1)$ and in $(L,\|\ \|_\infty)$.
(L.3) $L^1(X;m)\cap L$ is dense in $(L,\|\ \|_\infty)$.

For the existence of such a subalgebra L, see Theorem A.4.1 in [7]. By considering $\mathscr{F}\cap C_0(X)$ if necessary, we can assume that

$$C_0(X)\subset L. \tag{3}$$

Lemma 2.1. *For* $u,v,w\in C_0(X)$ *such that* $\operatorname{supp}[u]\cap\operatorname{supp}[v]=\emptyset$ *and* $w=k$ (*constant*) *on a neighbourhood of* $\operatorname{supp}[u]$,

(i) $\operatorname{supp}[\Phi(u)]\cap\operatorname{supp}[\Phi(v)]=\emptyset$
(ii) $\Phi(w)=k$ *on a neighbourhood of* $\operatorname{supp}[\Phi(u)]$.

Proof. (i) Take $f,g\in C_0(X)$ such that $\operatorname{supp}[f]\cap\operatorname{supp}[g]=\emptyset$ and f and g are equal to 1 on $\operatorname{supp}[u]$ and $\operatorname{supp}[v]$, respectively. Then, since $\Phi(u)=\Phi(fu)=\Phi(f)\Phi(u)$, $\Phi(f)=1$ on $\{x\in X':\Phi(u)(x)\neq 0\}$. On account of (3), $\Phi(f)$ is a continuous function on X' (cf. Lemma A.4.3 in [7]). Hence, $\operatorname{supp}[\Phi(u)]$ is included in the open set $\{\Phi(f)>0\}$, and by the same reason, $\operatorname{supp}[\Phi(v)]$ is included in the open set $\{\Phi(g)>0\}$. $\{\Phi(f)>0\}\cap\{\Phi(g)>0\}=\emptyset$ because $\Phi(f)\Phi(g)=\Phi(fg)=\Phi(0)=0$, so $\operatorname{supp}[\Phi(u)]\cap\operatorname{supp}[\Phi(v)]=\emptyset$.

(ii) Suppose that $w=k$ on an open set U $(\supset \mathrm{supp}[u])$ and take $f\in C_0(X)$ such that $f=1$ on $\mathrm{supp}[u]$ and $\mathrm{supp}[f]\subset U$. Then, $\Phi(w)=k$ on $\{\Phi(f)>0\}$ because $\Phi(w)\Phi(f)=\Phi(wf)=\Phi(kf)=k\Phi(f)$. □

According to the Beurling-Deny formula, the regular Dirichlet form $(\mathscr{E}',\mathscr{F}')$ can be decomposed as

$$\mathscr{E}'(u,v)=\mathscr{E}'^c(u,v)+\int_{X'\times X'-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))J'(dxdy)+\int_{X'}\tilde{u}(x)\tilde{v}(x)k'(dx),$$
$$\text{for } u,v\in\mathscr{F}'$$

where $\tilde{u}$ and $\tilde{v}$ mean quasi continuous versions of u and v. Let us define Radon measures J on $X\times X-d$ and k on X as follows: for $f,g\in C_0(X)$ with $\mathrm{supp}[f]\cap\mathrm{supp}[g]=\emptyset$

$$\int_{X\times X}f(x)g(y)J(dxdy)=\int_{X'\times X'}\Phi(f)(x)\Phi(g)(y)J'(dxdy) \tag{4}$$

and for $f\in C_0(X)$

$$\int_X f(x)k(dx)=\int_{X'}\Phi(f)(x)k'(dx). \tag{5}$$

Note that J and k are well defined in view of Lemma 2.1. Finally, define the form $\tilde{\mathscr{E}}^c(u,v)$ on $\mathscr{F}\cap C_0(X)$ by

$$\tilde{\mathscr{E}}^c(u,v)=\mathscr{E}'^c(\Phi(u),\Phi(v)). \tag{6}$$

By Lemma 2.1 (ii), $\tilde{\mathscr{E}}^c$ becomes a local form. We then see that the Dirichlet form $\tilde{\mathscr{E}}$ can be decomposed as, for $u,v\in\mathscr{F}\cap C_0(X)$

$$\begin{aligned}\tilde{\mathscr{E}}(u,v)&=\mathscr{E}'(\Phi(u),\Phi(v))\\&=\mathscr{E}'^c(\Phi(u),\Phi(v))+\int_{X'\times X'-d}(\Phi(u)(x)-\Phi(u)(y))(\Phi(v)(x)-\Phi(v)(y))J'(dxdy)\\&\quad+\int_{X'}\tilde{u}(x)\tilde{v}(x)k'(dx)\\&=\tilde{\mathscr{E}}^c(u,v)+\int_{X\times X-d}(u(x)-u(y))(v(x)-v(y))J(dxdy)+\int_X u(x)v(x)k(dx).\end{aligned} \tag{7}$$

On the other hand, $\tilde{\mathscr{E}}=\mathscr{E}$ on $\mathscr{F}\cap C_0(X)$. Hence, $J=0$ and $k=0$ on account of the regularlity and strong locality of $(\mathscr{E},\mathscr{F})$. As a result, we have $\mathscr{E}'(\Phi(f),\Phi(g))=\mathscr{E}'^c(\Phi(f),\Phi(g))$ for $f,g\in\mathscr{F}\cap C_0(X)$, and thus

$$\mathscr{E}'(\Phi(f),\Phi(g))=\mathscr{E}'^{c}(\Phi(f),\Phi(g)) \quad \text{for} \quad f,g\in\mathscr{F}, \tag{8}$$

by virtue of the regularity of $(\mathscr{E},\mathscr{F})$.

Proof of Theorem 2.1. Let $u\in\tilde{\mathscr{F}}_b$. Then the function u is an element of $\mathscr{F}_{loc}$ by the ideal property of $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$, so the energy measure $\mu_{\langle u\rangle}$ can be defined.

Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of relatively compact open sets such that $\Omega_1\subset\bar{\Omega}_1\subset\cdots\subset\Omega_n\subset\bar{\Omega}_n\subset\cdots$, $\Omega_n\uparrow X$. Let $\varphi_n\in\mathscr{F}\cap C_0(X)$ be functions satisfying

$$0\le\varphi_n\le 1, \qquad \varphi_n=\begin{cases}1 & \text{on} \quad \Omega_n\\ 0 & \text{on} \quad X\backslash\Omega_{n+1}.\end{cases}$$

We then have from the derivation property of $\mu_{\langle u\rangle}$ (see Lemma 3.2.5 in [7])

$$\mu_{\langle u\rangle}(X)=\lim_{n\to\infty}\int_X\varphi_n d\mu_{\langle u\rangle}=\lim_{n\to\infty}\lim_{m\to\infty}\int_X\varphi_n d\mu_{\langle\varphi_m u\rangle}.$$

Since $\varphi_m u$ belongs to $\mathscr{F}$,

$$\begin{aligned}\int_X\varphi_n d\mu_{\langle\varphi_m u\rangle}&=2\mathscr{E}(\varphi_n\varphi_m u,\varphi_m u)-\mathscr{E}(\varphi_n,(\varphi_m u)^2)\\&=2\tilde{\mathscr{E}}(\varphi_n\varphi_m u,\varphi_m u)-\tilde{\mathscr{E}}(\varphi_n,(\varphi_m u)^2).\end{aligned}\tag{9}$$

Let $(X',m',\mathscr{E}',\mathscr{F}',\Phi)$ be the regular representation of $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$ stated above. Then, by virtue of (8), the right hand side is equal to

$$\begin{aligned}&2\mathscr{E}'(\Phi(\varphi_n)\Phi(\varphi_m)\Phi(u),\Phi(\varphi_m)\Phi(u))-\mathscr{E}'(\Phi(\varphi_n),(\Phi(\varphi_m)\Phi(u))^2)\\&=2\mathscr{E}'^{c}(\Phi(\varphi_n)\Phi(\varphi_m)\Phi(u),\Phi(\varphi_m)\Phi(u))-\mathscr{E}'^{c}(\Phi(\varphi_n),(\Phi(\varphi_m)\Phi(u))^2)\\&=\int_{X'}\Phi(\varphi_n)d\mu'^{c}_{\langle\Phi(\varphi_m)\Phi(u)\rangle},\end{aligned}\tag{10}$$

where $\mu'^{c}_{\langle\Phi(u)\rangle}$ is the continuous part of the energy measure $\mu'_{\langle\Phi(u)\rangle}$ related to a regular Dirichlet space $(\mathscr{E}',\mathscr{F}')$ (see §3.2 in [7].) Since for $n<m$, $\Phi(\varphi_m)=1$ on some neighbourhood of $\mathrm{supp}[\Phi(\varphi_n)]$ by Lemma 2.1 (ii), the right hand side of (10) is equal to $\int_{X'}\Phi(\varphi_n)d\mu'^{c}_{\langle\Phi(u)\rangle}$ by Lemma 3.2.5 in [7] again.

Since $\|\Phi(\varphi_n)\|_\infty=\|\varphi_n\|_\infty\le 1$,

$$\int_{X'}\Phi(\varphi_n)d\mu'^{c}_{\langle\Phi(u)\rangle}\le\int_{X'}d\mu^{c}_{\langle\Phi(u)\rangle}\le 2\mathscr{E}'(\Phi(u),\Phi(u))=2\tilde{\mathscr{E}}(u,u).$$

Hence, we can conclude that

$$\int_X \mu_{\langle u \rangle} \leq 2\tilde{\mathscr{E}}(u,u), \tag{11}$$

and $u \in \mathscr{F}^{ref}$. The inequality (11) is extended to any $u \in \tilde{\mathscr{F}}$, thereby completing the proof of Theorem 2.1.

REMARK 1. It was shown in Chen [3] that $(\mathscr{E}^{ref}, \mathscr{F}^{ref})$ is a Dirichlet space. Hence, we see from Theorem 2.1 that $(\mathscr{E}^{ref}, \mathscr{F}^{ref})$ is the maximum element in $\mathscr{A}\mathscr{S}(\mathscr{E},\mathscr{F})$ with respect to the semi-order $<$ on $\mathscr{A}\mathscr{S}(\mathscr{E},\mathscr{F})$ defined by

$$(\mathscr{E}^1,\mathscr{F}^1) < (\mathscr{E}^2,\mathscr{F}^2) \quad \text{if} \quad \mathscr{F}^1 \subset \mathscr{F}^2 \quad \text{and} \quad \mathscr{E}^1(u,u) \geq \mathscr{E}^2(u,u) \quad \text{for} \quad u \in \mathscr{F}^1.$$

An important implication of Assumption A is the next lemma proved in Sturm [16].

Lemma 2.2. *Under Assumption A, the function $\rho_p(x) = \rho(p,x)$ belongs to $\mathscr{F}_{loc} \cap C(X)$ and $\mu_{\langle \rho_p \rangle} \leq m$. Moreover, every ball $B_r(p) = \{x : \rho_p(x) < r\}$ is relatively compact. Here $p \in X$ is a fixed point.*

Theorem 2.2. *Under Assumption A, the Silverstein extension of $(\mathscr{E},\mathscr{F})$ is unique, $\#(\mathscr{A}\mathscr{S}(\mathscr{E},\mathscr{F})) = 1$.*

Proof. Set

$$\varphi_n(x) = \begin{cases} 1 & x \leq n \\ n+1-x & n \leq x \leq n+1 \\ 0 & x \geq n+1. \end{cases}$$

Let $u \in \mathscr{F}_b^{ref}$ $(\subset \mathscr{F}_{loc})$. Note that by Lemma 2.2 $\varphi_n(\rho_p)$ is an element of $\mathscr{F}_b$ and $\mathrm{supp}[\varphi_n(\rho_p)]$ is a compact set according to Lemma 2.2. Hence, we have $u \cdot \varphi_n(\rho_p) \in \mathscr{F}_b$ and

$$\mathscr{E}(u\varphi_n(\rho_p)) - u\varphi_m(\rho_p), u\varphi_n(\rho_p) - u\varphi_m(\rho_p)) = \frac{1}{2}\int_X d\mu_{\langle u(\varphi_n(\rho_p) - \varphi_m(\rho_p))\rangle}$$

$$\leq \int_X \tilde{u}^2 d\mu_{\langle \varphi_n(\rho_p) - \varphi_m(\rho_p)\rangle} + \int_X (\varphi_n(\rho_p)) - \varphi_m(\rho_p))^2 d\mu_{\langle u \rangle}$$

$$= \int_X \tilde{u}^2(\varphi'_n(\rho_p) - \varphi'_m(\rho_p))^2 d\mu_{\langle \rho_p \rangle} + \int_X (\varphi_n(\rho_p) - \varphi_m(\rho_p))^2 d\mu_{\langle u \rangle}.$$

Since the first term of the right hand side is dominated by $\int_{\{n \leq \rho_p \leq n+1\} \cup \{m \leq \rho_p \leq m+1\}} \tilde{u}^2 dm$ on account of Lemma 2.2, it converges to 0 as $n,m \to \infty$. The second term also converges to 0 by the dominated convergence theorem. Noting that $u\varphi_n(\rho_p) \to u$ in L^2, we see that u belongs to $\mathscr{F}$, which implies the theorem on account of Theorem 2.1. $\square$

REMARK 2. Let $(\mathscr{E}^1,\mathscr{F})$ and $(\mathscr{E}^2,\mathscr{F})$ be regular Dirichlet forms on $L^2(X;m^1)$ and $L^2(X;m^2)$. Suppose that these Dirichlet forms are quasi-equivalent: there exist constants $c_1,c_2 \geq 1$ such that

$$c_1^{-1}\mathscr{E}^1(u,u) \leq \mathscr{E}^2(u,u) \leq c_1\mathscr{E}^1(u,u) \quad \text{for} \quad u \in \mathscr{F}, \quad c_2^{-1}m^1 \leq m^2 \leq c_2 m^1.$$

Then, by the domination principle (cf. [10])

$$c_1^{-1}\mu^1_{\langle u\rangle} \leq \mu^2_{\langle u\rangle} \leq c_1 \mu^1{}_{\langle u\rangle},$$

where $\mu^1{}_{\langle u\rangle}$ (resp. $\mu^2{}_{\langle u\rangle}$) is the energy measure of u associated with $(\mathscr{E}^1,\mathscr{F})$ (resp. $(\mathscr{E}^2,\mathscr{F})$). Thus, we have

$$\mathscr{F}^{1,ref} = \mathscr{F}^{2,ref}$$

by the definition of the reflected Dirichlet space. Here $\mathscr{F}^{1,ref}$ and $\mathscr{F}^{2,ref}$ are reflected Dirichlet spaces associated with $(\mathscr{E}^1,\mathscr{F})$ and $(\mathscr{E}^2,\mathscr{F})$. Therefore, we can conclude that the uniqueness of Silverstein's extension is stable under quasi-equivalence.

REMARK 3. Let $N \subset X$ be a closed set with $\mathrm{Cap}(N)=0$, where Cap denotes the 1-capacity associated with the Dirichlet form $(\mathscr{E},\mathscr{F})$. Set $D=X\backslash N$ and let $(\mathscr{E}^D,\mathscr{F}^D)$ be the part of $(\mathscr{E},\mathscr{F})$ on D. Then, by the same argument as in Remark 4.3 in [11], Theorem 2.2 can be extended as follows: under Assumption A, the extension of $(\mathscr{E}^D,\mathscr{F}^D)$ in Silverstein's sense is unique, $\sharp(\mathscr{A}\mathscr{d}(\mathscr{E}^D,\mathscr{F}^D))=1$.

3. Uniqueness of self-adjoint extension

Let A be the self-adjoint aperator associated with $(\mathscr{E},\mathscr{F})$. Throughout this section, we suppose that X is a smooth manifold and the space $C_0^\infty(X)$ is included in the domain of A. Let us denote by S the symmetric operator $A \upharpoonright C_0^\infty(X)$, the restriction of A to $C_0^\infty(X)$. Furthermore, we assume that S is a hypoelliptic differential operator in the sense that

$$\mathscr{N}_1 = \{u \in \mathscr{D}(S^*); (1-S^*)u=0\} \subset C^\infty(X),$$

where S^* is the adjoint operator of S.

Then, by following the proof of Theorem 5.2.3 in Davies [5], we obtain

Theorem 3.1. *Under Assumption A, the operator S is essentially self-adjoint.*

Proof. Take $g \in \mathcal{N}_1$. By the hypoellipticity of S, $g \in C^\infty(X)$ and $Sg = g$. Let ψ be the function defined on $[0, \infty)$ by

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 1 - x & 1 \leq x < 2 \\ 0 & 2 \leq x, \end{cases}$$

and put $\phi_n(x) = \psi(\frac{\rho_p(x)}{n})$. Noting that $\phi_n \in \mathcal{F}_{loc} \cap L^2(X; m)$ and that

$$\begin{aligned} \mu_{\langle \phi_n \rangle}(X) &= \frac{1}{n^2} \int_X \psi'\left(\frac{\rho_p(x)}{n}\right)^2 d\mu_{\langle \rho_p \rangle} \\ &\leq \frac{1}{n^2} \int_{\{n \leq \rho_p(x) \leq 2n\}} dm < \infty, \end{aligned}$$

we see from Theorem 2.2 that $\phi_n \in \mathcal{F}$.

Let $\varphi \in C_0^\infty(X)$ such that $\varphi = 1$ on a neighbourhood of $\{\rho_p(x) \leq 2n\}$. Then,

$$0 \geq -\int_X \phi_n^2 g^2 dm = -\int_X \phi_n^2 g S g dm = -\int_X \phi_n^2 g S(g\varphi) dm,$$

and since $\phi_n^2 g$, $g\varphi \in \mathcal{F}$, the right hand side equals

$$\begin{aligned} \frac{1}{2} \int_X d\mu_{\langle \phi_n^2 g, g\varphi \rangle} &= \frac{1}{2} \int_X \phi_n^2 d\mu_{\langle g, g\varphi \rangle} + \int_X g \phi_n d\mu_{\langle \phi_n, g\varphi \rangle} \\ &= \frac{1}{2} \int_X \phi_n^2 d\mu_{\langle g \rangle} + \int_X g \phi_n d\mu_{\langle \phi_n, g \rangle}. \end{aligned}$$

Hence, by virtue of Lemma 5.6.1 in [7],

$$\begin{aligned} \int_X \phi_n^2 d\mu_{\langle g \rangle} &\leq -2 \int_X g \phi_n d\mu_{\langle \phi_n, g \rangle} \\ &\leq 2 \left(\int_X g^2 d\mu_{\langle \phi_n \rangle} \right)^{1/2} \left(\int_X \phi_n^2 d\mu_{\langle g \rangle} \right)^{1/2}, \end{aligned}$$

and so

$$\int_X \phi_n^2 d\mu_{\langle g \rangle} \leq 4 \int_X g^2 d\mu_{\langle \phi_n \rangle}.$$

Since the right hand side is dominated by $\frac{4}{n^2} \int_{\{n \leq \rho_p(x) \leq 2n\}} g^2 dm$,

$$\mu_{\langle g \rangle}(X) = \lim_{n \to \infty} \int_X \phi_n^2 d\mu_{\langle g \rangle} = 0.$$

Hence, $g\in\mathscr{F}$ and $\mathscr{E}(g,g)=0$ by Theorem 2.2. Noting that $|\mathscr{E}(g,v)|\leq\mathscr{E}(g,g)^{1/2}\mathscr{E}(v,v)^{1/2}=0$ for any $v\in\mathscr{F}$, we can conclude that $g\in\mathscr{D}(A)$ and $Ag=0$. Therefore, $g=S^*g=Ag=0$, which leads us to the theorem. $\square$

REMARK 4. Let $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$ be an element of $\mathscr{A}\mathscr{d}(\mathscr{E},\mathscr{F})$ and $\tilde{A}$ the self-adjoint operator associated with $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$. Then, under the situation of this section, $\tilde{A}$ is a self-adjoint extension of S and so Theorem 3.1 implies Theorem 2.2. In fact, take $u\in\mathscr{D}(\tilde{A})$ and $\varphi\in C_0^\infty(X)$. Let $(X',m',\mathscr{E}',\mathscr{F}',\Phi)$ be the regular representation of $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$ as in §2. Then

$$(-\tilde{A}u,\varphi)=\tilde{\mathscr{E}}(u,\varphi)=\lim_{n\to\infty}\tilde{\mathscr{E}}(u^n,\varphi)\quad (u^n=(-n\vee u)\wedge n))$$

and by (8)

$$\tilde{\mathscr{E}}(u^n,\varphi)=\mathscr{E}'(\Phi(u^n),\Phi(\varphi))=\mathscr{E}'^c(\Phi(u^n),\Phi(\varphi)).$$

Take $\psi\in\mathscr{F}\cap C_0(X)$ such that $\psi=1$ on a neighbourhood of $\operatorname{supp}[\varphi]$. Then the right hand side is equal to

$$\mathscr{E}'^c(\Phi(u^n\psi),\Phi(\varphi))=\mathscr{E}'(\Phi(u^n\psi),\Phi(\varphi))=\tilde{\mathscr{E}}(u^n\psi,\varphi)$$

on account of Lemma 2.1 (ii). Noting that $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})\in\mathscr{A}\mathscr{d}(\mathscr{E},\mathscr{F})$, we get

$$\tilde{\mathscr{E}}(u^n\psi,\varphi)=\mathscr{E}(u^n\psi,\varphi)=(u^n,-S\varphi).$$

Therefore, $(\tilde{A}u,\varphi)=(u,S\varphi)$ and so $\tilde{A}\subset S^*$.

4. Examples

EXAMPLE 1. Let (M,g) be a complete smooth Riemannian manifold. For $\psi\in L^2_{loc}(M;V_g)$ with $\psi>0$, V_g-a.e., consider the symmetric form on $L^2(M;\psi^2V_g)$:

$$\mathscr{E}^\psi(u,v)=\frac{1}{2}\int_M(\operatorname{grad}u,\operatorname{grad}v)\psi^2dV_g\qquad u,v\in C_0^\infty(M).$$

If the above form is closable, we say that ψ is admissible. For conditions for ψ being admissible, see [6] and [8]. For an admissible function ψ, denote by $\mathscr{F}^\psi$ the closure of $C_0^\infty(M)$ with respect to $\mathscr{E}^\psi+(\,,)_{\psi^2V_g}$. Then, $(\mathscr{E}^\psi,\mathscr{F}^\psi)$ becomes a regular Dirichlet form and, independently of each admissible function ψ, the intrinsic metric associated with $(\mathscr{E}^\psi,\mathscr{F}^\psi)$ is identical to the Riemannian distance. Hence, Theorem 2.2 implies that $\sharp(\mathscr{A}\mathscr{d}(\mathscr{E}^\psi,\mathscr{F}^\psi))=1$ for any admissble function ψ, in particular, for any $\psi\in H^1_{loc}(M;V_g)$ with $\psi>0$, V_g-a.e. Here $H^1_{loc}(M;V_g)$ is the set of functions belonging locally to the Sobolev space of order 1.

On the other hand, the uniqueness of Markovian extensions is known only in the case where $M=R^d$, $\psi\in H^1_{loc}(R^d)$, and $\psi>0$ dx-a.e. (see [13], [15]). As a corollary of this result, we proved in [21] that $\sharp(\mathscr{A}\mathscr{I}(\mathscr{E}^\psi,\mathscr{F}^\psi))=1$ for $\psi\in H^1_{loc}(R^d)$ with $\psi>0$ dx-a.e.

EXAMPLE 2. Let $X=R^d$ and m the Lebesgue measure. Consider the following regular Dirichlet form:

$$(12)\qquad \begin{cases} \mathscr{E}(u,v)=\sum_{i,j=1}^{d}\int_{R^d} a_{i,j}(x)\dfrac{\partial u}{\partial x_i}\dfrac{\partial v}{\partial x_j}dx \\ \mathscr{F}=\text{the closure of } C_0^\infty(R^d) \text{ with respect to } \mathscr{E}_1. \end{cases}$$

Here, the coefficients $a_{i,j}$ are locally uniform elliptic and satisfy

$$(13)\qquad \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j\le k(|x|+2)^2(\log(|x|+2))^2|\xi|^2 \quad \text{for} \quad \xi\in R^d.$$

Denote by ρ the metric associated with $(\mathscr{E},\mathscr{F})$. Then, the local uniform ellipticity and (13) imply that the topology induced from ρ is equivalent with the usual topology on R^d.

Set

$$\psi(x)=\frac{1}{\sqrt{k}}\int_0^{|x|}\frac{ds}{(s+2)\log(s+2)}.$$

We then easily see that $\psi\in\mathscr{F}_{loc}\cap C(R^d)$ and on account of (13)

$$a_{i,j}(x)\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\le 1.$$

Hence, for $\forall r>0$

$$\{x\in R^d:\rho(0,x)\le r\}\subset\{x\in R^d:\psi(x)\le r\}$$
$$=\left\{x\in R^d:\int_0^{|x|}\frac{ds}{(s+2)\log(s+2)}\le\sqrt{k}r\right\}.$$

Hence, ρ fulfills Assumption A, and which implies that $\sharp(\mathscr{A}\mathscr{I}(\mathscr{E},\mathscr{F}))=1$. If a_{ij} are smooth, the essential self-adjointness is known (see [4]).

EXAMPLE 3. Let $X=R^d$ and m the Lebesgue measure. Suppose that the form $(\mathscr{E},C_0^\infty(R^d))$ is uniformly subelliptic, i.e., there exist constants $\varepsilon,\lambda>0$ and C such that

$$(14) \qquad \frac{1}{\lambda}\|u\|_1^2 \geq \mathscr{E}(u,u) \geq \lambda\|u\|_\varepsilon^2 - C\|u\|_0^2, \qquad \forall u \in C_0^\infty(R^d),$$

where $\|u\|_\varepsilon^2 = \int_{R^d}|\hat{u}(\xi)|^2(1+|\xi|^2)^\varepsilon d\xi$ with $\hat{u}$ being the Fourier transformation of u. Then it is known that $(\mathscr{E}, C_0^\infty(R^d))$ is closable and its closure $(\mathscr{E},\mathscr{F})$ is a strongly local Dirichlet form. Moreover, the uniform subellipticity condition holds if and only if there exist constants $r_0>0$ and $C_0>0$ such that

$$C_0|x-y| \leq \rho(x,y) \leq \frac{1}{C_0}|x-y|^\varepsilon$$

for all $x,y \in R^d$ with $|x-y|<r_0$ (cf. [19]). Therefore, we can conclude that the intrinsic metric ρ fulfills Assumption A and $(\mathscr{E},\mathscr{F})$ has a unique Silverstein extension.

EXAMPLE 4. Let D be a bounded domain in R^d with smooth boundary ∂D, and $m(dx)=\sigma^b(x)dx$. Here σ is supposed to satisfy

$$\lambda d(x,\partial D) \leq \sigma(x) \leq \Lambda d(x,\partial D).$$

Let us consider the Dirichlet form defined by

$$(15) \qquad \begin{cases} \mathscr{E}(u,v) = \displaystyle\int_D (\operatorname{grad} u, \operatorname{grad} v)\sigma^a dx \\ \mathscr{F} = \text{the closure of} \quad C_0^\infty(D) \quad \text{with respect to } \mathscr{E}_1. \end{cases}$$

Suppose $a-b>2$ and set

$$\psi(x) = \frac{1}{\alpha\Lambda^{(a-b)/2}} d(x,\partial D)^\alpha \quad \left(\alpha = \frac{2+b-a}{2} < 0\right).$$

Then, we see that $(\operatorname{grad}\psi \cdot \operatorname{grad}\psi)\sigma^{(a-b)} \leq 1$ and thus for a fixed point $p \in D$

$$\{x \in D : \rho(p,x) \leq r\} \subset \{x \in D : \psi(x) \leq r + \psi(p)\}.$$

Since $\lim_{x \to \partial D}\psi(x) = \infty$ if $a-b>2$, Assumption A is satisfied. For the essential self-adjointness, see [9].

The final example tells us that the completeness is destroyed by some time change.

EXAMPLE 5. Let X be R^d $(d \geq 3)$ and m a smooth positive Radon measure in the sense of [7]. Let us consider the Dirichlet form on $L^2(R^d;m)$ defined by

$$(16)\qquad \begin{cases} \mathscr{E}(u,v)=\dfrac{1}{2}\displaystyle\sum_{i=1}^{d}\int_{R^d}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx \\ \mathscr{F}=\text{the closure of } \ C_0^\infty(R^d) \ \text{ with respect to } \mathscr{E}_1(=\mathscr{E}+(\,,\,)_m). \end{cases}$$

Denote by $\mathscr{C}$ the set of $C^\infty(R^d)$-functions f satisfying $\sum_{i=1}^{d}\int_{R^d}\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}dx<\infty$ and denote by $\tilde{\mathscr{F}}$ the closure of $\mathscr{C}$ with respect to $\mathscr{E}_1$. Then, it is shown in [3] that $\mathscr{F}=\tilde{\mathscr{F}}$ if and only if the measure m satisfies

$$(*)\qquad m(R^d\backslash A)=\infty \quad \text{for} \quad \forall A\in\mathscr{B}(R^d) \quad \text{with} \quad \mathrm{Cap}(A)<\infty,$$

where Cap means the 1-capacity associated with the classical Dirichlet form $(\frac{1}{2}\boldsymbol{D},H^1(R^d))$. Hence, if m does not satisfy the condition $(*)$, in particular, if m is a finite measure, then the extension of $(\mathscr{E},\mathscr{F})$ in Silverstein's sense is not unique. Accordingly, the pseudo metric corresponding to $(\mathscr{E},\mathscr{F})$ is not complete on account of Theorem 2.2.

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