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THE ADDITIVE STRUCTURE OF $G^*(L^n(p^k))$

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Let $L^n(k)=L^n(k; 1, \cdots, 1)$ be the $(2n+1)$-dimensional standard lens space mod k where n and k are positive integers and $k\geq 2$.

The structure of K-ring and KO-ring of $L^n(k)$ are determined by J.F. Adams [1] for $k=2$ and by T. Kambe [5] for k an odd prime.

For the case $k=p^2$, p a prime, there exist results by T. Kobayashi, M. Sugawara, and T. Kawaguchi [6], [7].

Let p be a prime. By Adams [2], there is a cohomology theory G^* which decomposes K-cohomology localized at p.

In this note we determine the additive structure of $\tilde{G}^*(L^n(r))$ where $r=p^k$, $2\leq k<\infty$, which results to the determination of $K(L^n(p^k))$ for any prime p and $KO(L^n(p^k))$ for an odd prime p.

After manuscript, Professor M. Sugawara kindly communicated to the author that recently N. Mahammed ([9]) has determined the additive structure of $K(L^n(p^k))$ and $KO(L^n(p^k))$ for $1\leq k<\infty$. But the method of author's is not same as his. Our basic tool is the formal group of G-cohomology established by S. Araki [3].

In §1 we summarize the well known facts about G-cohomology of lens spaces. In §2 the coefficients of the formal power series $[p^k]_G(T)$ are partially discussed, and the order of the group $\tilde{G}^{2*}(L^n(p^k))$ is determined. In §3 we calculate the order of e^i in $\tilde{G}^{2*}(L^n(p^k))$. In §4 we construct the elements w_i which are in the form $w_i=e^i+$*lower degree terms,* and has a smaller order than e^i. In §5 it is proven that a part of the above w_i*'s* generate $\tilde{G}^{2\beta}(L^n(p^k))$ for $1\leq\beta\leq p-1$, and in fact, they give a direct sum decomposition of $\tilde{G}^{2\beta}(L^n(p^k))$. In §6, the additive structure of $\tilde{K}(L^n(p^k))$ are determined by the preceding results.

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1. Summary on G-cohomology groups of lens spaces

Following Adams [2], there is a generalized cohomology theory G which gives a decomposition of K-cohomology localized at a prime p, i.e., for a finite CW-complex X

$$\tilde{K}^0(X)\otimes Z_{(p)}\cong\tilde{G}^2(X)\oplus\cdots\oplus\tilde{G}^{2i}(X)\oplus\cdots\oplus\tilde{G}^{2(p-1)}(X)$$
$$\tilde{K}^1(X)\otimes Z_{(p)}\cong\tilde{G}^3(X)\oplus\cdots\oplus\tilde{G}^{2(p-1)+1}(X)$$

where $Z_{(p)}$ is the ring of integers localized at p.

The coefficient ring of G-cohomology is the following [3];

$$G^*(pt)\cong Z_{(p)}[u_1, u_1^{-1}],\ u_1\in G^{-2(p-1)}(pt)\,.$$

Moreover, G-cohomology is complex oriented [3]; so is there defined the Euler class $e(L)$ for a complex line bundle L over X such that

$$e(L)\in\tilde{G}^2(X) \text{ and } G^*(CP^n)\cong G^*(pt)\,[e(\eta)]/(e(\eta)^{n+1}=0)\,.$$

where η is the canonical line bundle over CP^n.

The associated formal group F_G was investigated by Araki [3].

The formal power series $[k]_G(T)$, $k\in Z$ is defined so that $[k]_G(e(\eta))=e(\eta^k)$ in $G^*(CP^n)$ for all k, where η^k is k-fold tensor power of η.

Observing the Gysin sequence of the sphere bundle,

$$S^1=S^1/Z_{p^k}\to L^n(p^k)\xrightarrow{\pi} CP^n$$

we have the following exact sequence [8], [10]:

$$\begin{aligned}(1.1)\qquad & 0\cong\tilde{G}^{2i+1}(CP^n)\to\tilde{G}^{2i+1}(L^n(p^k))\to\tilde{G}^{2i}(CP^n)\\ & \xrightarrow{\Psi}\tilde{G}^{2i+2}(CP^n)\xrightarrow{\pi^*}\tilde{G}^{2i+2}(L^n(p^k))\to\tilde{G}^{2i+1}(CP^n)\cong 0\end{aligned}$$

where Ψ is the Gysin homomorphism which is obtained by multiplying the element $[p^k]_G(e(\eta))$. Using (1.1) we have

Lemma 1.1. $\tilde{G}^{2*}(L^n(p^k))\cong G^*(pt)\,[e]/(e^{n+1}, [p^k]_G(e))$

where $e=\pi^*(e(\eta))$ and $G^*(pt)\,[e]$ means the subgroup of $G^*(pt)\,[e]$ generated by $G^*(pt)$, e^i, $i>0$.

The proof is straightfowards *by* (1.1).

Let ι; $L^{n-1}(p^k)\to L^n(p^k)$ be the inclusion.

Lemma 1.2. *If $i-n\not\equiv 0 \bmod p-1$,*

$$\iota^*\colon\tilde{G}^{2i}(L^n(p^k))\cong\tilde{G}^{2i}(L^{n-1}(p^k));$$

if $i-n\equiv 0 \bmod p-1$, ι^* is epimorphic and Kernel ι^* is the cyclic subgroup generated by $u_1^\alpha e^n$, $\alpha=(i-n)/(p-1)$.

Proof. By Lemma 1.1, we see immediately that ι^* is epimorphic.

Next, take a truncated polynomial $f(e(\eta))$ of $\tilde{G}^{2i}(CP^n)$ such that $\pi^*(f(e(\eta)))\in$ *Kernel* ι^*. Since $\pi^*\iota^*(f(e(\eta)))=0$, we have that

$$f(e) = [p^k]_G(e)\cdot x(e) \mod e^n .$$

That is, if $y \in \tilde{G}^{2i}(L^n(p^k))$ belongs to *Kernel* ι^*, then $y = xe^n$, $x \in G^{2(i-n)}(pt)$. Therefore the result is immediate.

2. The coefficients of $[p^k]_G(T)$ and the order of $\tilde{G}^{2*}(L^n(p^k))$

For simplicity we put

$$p_l = 1+p+\cdots+p^{l-1},\ l\geq 0,\ \text{.i.e,}\ p_l = (p^l-1)/(p-1) . \tag{2.1}$$

First we observe certain divisibilities of coefficients of $[p^k]_G(T)$ by powers of p.

Proposition 2.1. *Put* $[p^k]_G(T) = \sum_{i=1}^{\infty} a_{i-1}T^i$, *then*

(1) $a_i = 0$ *if* $i \not\equiv 0 \mod p-1$,

(2) $a_0 = p^k$,

(3) $p^{k-l} \mid a_{p^l-1},\ p^{k-l+1} \nmid a_{p^l-1},$ *for* $1 \leq l \leq k$,

(4) $p^{k-l+1} \mid a_i$ *for* $p^l-1 < i < p^{l+1}-1,\ 1 \leq l \leq k-1$,

(5) $p^{k-l} \mid a_i$ *for* $p^l-1 \leq i < p^{l+1}-1,\ 1 \leq l \leq k-1$.

Proof. (1) is trivial by the sparsness of $G^*(pt)$, and (2) is well-known. Let $\log_G(T)$ be the logarithm of F_G (see [3]), we have

$$\log_G(T) = T+(1/p)u_1T^p+(1/p^2)u_1^{p_2}T^{p^2}+\cdots+ +(1/p^i)u_1^{p_i}T^{p^i}+\cdots,\ \text{and} \tag{2.2}$$

$$(\log_G)\circ[p^k]_G(T) = p^k\cdot\log_G(T) \tag{2.3}$$

where $\circ$ means the composition of formal power series.

We prove (3) and (4) by induction on i. If $i = p-1$, by substituting (2.2) into (2.3), we get

$$a_{p-1}T^p+(1/p)u_1(a_0T)^p = p^{k-1}u_1T^p \mod T^{p+1} .$$

As $a_0 = p^k$, $p^{k-1} \mid a_{p-1}$ but $p^k \nmid a_{p-1}$.

Next assume that the proposition holds for any i such that $p-1 \leq i \leq r-1 < p^k-1$, and also assume that $p^l-1 \leq r < p^{l+1}-1 \leq p^k-1$. Substituting (2.2) into (2.3), we obtain

$$a_rT^{r+1}+(1/p)u_1(\sum_{j=1}^{r} a_{j-1}T^j)^p+\cdots+(1/p^l)u_1^{p_l}(\sum_{j=1}^{r} a_{j-1}T^j)^{p^l}$$
$$\equiv p^{k-l}u_1^{p_l}T^{p^l} \mod T^{r+2} .$$

By the assumption of induction, we see

$$p^{k-l+1} | a_{j-1} \text{ for } 1 \leq j \leq r \text{ and } j-1 \neq p^l-1, \text{ and } p^{k-l} | a_{p^l-1}\,.$$

The coefficient of T^{r+1} of $(\sum_{j=1}^{r} a_{j-1}T^j)^{p^s}$ is the sum of monomials of a_{j-1}, differing from $(a_{p^l-1})^{p^s}$.

Therefore, $(\sum_{j=1}^{r} a_{j-1}T^j)^{p^s}$ for $s \leq l$ is divisible by

$$p^{k-l+1+(p^s-1)(k-1)} \geq p^{k-l+1+s}$$

Therefore, if $r=p^l-1$, $p^{k-l} | a_r$, $p^{k-l+1} \not| a_r$
and if $r > p^l-1$, $p^{k-l+1} | a_r$.

The case $r=p^k-1$ is also easily proven by the same s argument with a little care to degrees.

Finally (5) follows from (3) and (4). q.e.d.

Proposition 2.2. *In $\tilde{G}^{2n}(L^n(p^k))$, order $e^n=p^k$.*

Proof. By Proposition 2.1, (2), we see

$$p^k e^n \equiv [p^k]_G(e) \cdot e^{n-1} \bmod e^{n+1}; \textit{ i.e.,}$$
$$p^k e^n = 0 \text{ in } \tilde{G}^{2n}(L^n(p^k))\,.$$

On the other hand, assume that $p^l e^n=0$ for $l<k$.

Then, there exists an element $x(e)=\sum_{i=1}^{n} x_i e^i$ of $\tilde{G}^{2*}(L^n(p^k))$, $x_i \in G^{2*-2i}(pt)$, such that

$$[p^k]_G(e) \cdot x(e) \equiv p^l e^n \bmod e^{n+1}\,.$$

Comparing the coefficients of both sides, we see that

$$x_1 = x_2 = \cdots = x_{n-2} = 0 \text{ and } p^k x_{n-1} = p^l,\; x_{n-1} \in G^0(pt) \cong Z_{(p)}\,.$$

This is a contradiction.
Next, we calculate the order of the group $\tilde{G}^{2*}(L^n(p^k))$.

Proposition 2.3. $|\tilde{G}^{2\beta}(L^n(p^k))| = p^{k(1+[(n-\beta)/(p-1)])}$ *for* $1 \leq \beta \leq p-1$.

Proof. The proof is by induction on n. In the case $n=1$,
The proof is straightfoward by Lemma 1.1.

Next, assume that the equality holds for $n-1$.
By Lemma 1.2 and Proposition 2.2 we get that if $n-\beta \not\equiv 0$ *mod* $p-1$

$$|\tilde{G}^{2\beta}(L^n(p^k))| = p^{k(1+[(n-1-\beta)/(p-1)])} = p^{k(1+[(n-\beta)/(p-1)])}\,,$$

and if $n-\beta \equiv 0$ *mod* $p-1$,

$$|\tilde{G}^{2\beta}(L^n(p^k))| = p^{k(1+[(n-1-\beta)/(p-1)])} \cdot p^k = p^{k(1+[(n=\beta)/(p-1)])}\,.$$

3. The order of e^i in $\tilde{G}^{2i}(L^n(p^k))$

We assume $k\geq 2$ from now until last section, and $n\geq p$ in this section and the next section.

Proposition 3.1. In $\tilde{G}^{2*}(L^n(p^k))$,

(1) *order* $e^i=p^{k+[(n-i)/(p-1)]}$ *for* $1\leq i\leq n$.

(2) $bp^{k-1+[(n-i)/(p-1)]}e^i=p^{k+[(n-i)/(p-1)]}e^{i-(p-1)}$

for $p\leq i\leq n$, *where* b *is a unit element* of $G^{-2(p-1)}(pt)$.

Proof. The proof is by induction on decending order of i. For $i=n$, (1) follows from Proposition 2.2.

Next, multiplying $[p^k]_G(e)$ by e^{n-p}, we have that

$$p^ke^{n-(p-1)}+a_{p-1}e^n=0\,.$$

If we put

$$b=-(a_{p-1})/p^{k-1}\,,$$

then b is a unit element by Proposition 2.1, (3), and we obtain (2).

Next, assume (1) and (2) holds for i such that $p<j+1\leq i\leq n$. We prove (2) for $i=j$. Multiply $[p^k]_G(e)$ by $p^{[(n-j)/(p-1)]}e^{j-p}$, then we obtain,

$$0=p^{k+[(n-j)/(p-1)]}e^{j-(p-1)}-b'p^{k-1+[(n-j)/(p-1)]}e^j+\sum_{t=2}^{\infty}a_{t(p-1)}p^{[(n-j)/(p-1)]}e^{j+(t-1)(p-1)} \tag{3.1}$$

by Proposition 2.1, (1).

If $t>p_k$, then, $k+[(n-\{j+(t-1)(p-1)\})/(p-1)]\leq[(n-j)/(p-1)]$ because $p_k\geq k$.

Next, let $p_l\leq t<p_{l+1}$ for $l\geq 2$, then, by Proposition 2.1, (5),

$$p^{k-l+[(n-j)/(p-1)]}\mid a_{t(p-1)}p^{[(n-j)/(p-1)]},\ \text{and}$$
$$k+[(n-\{j+(t-1)(p-1)\})/(p-1)]\leq k-l+[(n-j)/(p-1)]$$

because $t-1\geq p_l-1>l-1$ for $l\geq 2$.

Finally, if $p_1+1\leq t<p_2$,

$$p^{k+[(n-j)/(p-1)]}\mid a_{t(p-1)}p^{[(n-j)/(p-1)]}$$

by Propositon 2.1, (4), but

$$k+[(n-\{j+(t-1)(p-1)\})/(p-1)]\leq k+[(n-j)/(p-1)]$$

Therefore, by the assumption of induction, we obtain from (3.1) that

$$p^{k+[(n-j)/(p-1)]}e^{j-(p-1)} = b'p^{k-1+[(n-j)/(p-1)]}e^{j} .$$

Then, we have (2) for $i=j$.

Next, apply (2) for $i=j+(p-1)$,

$$p^{k-1+[(n-j)/(p-1)]}e^{j} = bp^{k-1+[(n-j-(p-1))/(p-1)]}e^{j+(p-1)} .$$

Therefore, if we assume that *order* $e^{j} < p^{k+[(n-j)((p-1)]}$, then,

$$\text{order } e^{j+(p-1)} < p^{k+[(n-(j+(p-1)))/(p-1)]}$$

This cotradicts to (1) for $i=j+(p-1)$. Finally we have

$$p^{k+[(n-j)/(p-1)]}e^{j} = bp^{k+[(n-(j+p-1))/(p-1)]}e^{j+p-1} = 0$$

So we obtain (1) for $i=j$. q.e.d.

4. The construction of the generators $\{w_i\}$

Put $z_l(k)=[k/(p^{l+1}-p^l)]$ for $l\geq 0$. As is easily seen

$$(4.1) \qquad z_l(a)+z_l(b)+1\geq z_l(a+b)\geq z_l(a)+z_l(b) .$$

$$(4.2) \qquad z_l(a-d)+z_l(b+d)+1\geq z_l(a)+z_l(b) .$$

Lemma 4.1. *Fix an integer l such that $1\leq l<k$. For each integer t such that $t\geq 1$ and $p^l+t(p-1)\leq n$, we have,*

$$p^{k-l+z_l(n-(p^l+t(p-1)))} \mid p^{z_l(n-p^l)}a_{p^l+t(p-1)-1} .$$

Proof. (1) In case $p^l+t(p-1)\geq p^k$:

$$z_l(n-p^l)\geq z_l(n-\{p^l+t(p-1)\})+z_l(t(p-1))$$

by (4.1). On the other hand, $t(p-1)\geq p^k-p^l$ by the assumption, thus

$$z_l(t(p-1))\geq k-l .$$

Therefore, we get

$$z_l(n-p^l)\geq k-l+z_l(n-\{p^l+t(p-1)\}) .$$

(2) In case $p^l+t(p-1)<p^k$: Fix an integer m such that

$$p^m\leq p^l+t(p-1)<p^{m+1}, \text{ (so, } m\geq l) .$$

Then, $p^{k-m}\mid a_{p^l+t(p-1)-1}$, by Proposition 2.1, (5). So, we have only to see that

$$z_l(n-p^l)+k-m\geq k-l+z_l(n-\{p^l+t(p-1)\}) .$$

But, $n-p^m+(p^{l+1}-p^l)(m-l)\leq n-p^l$, because $m\geq l$. Therefore,

$$z_l(n-p^m)+m-l\leq z_l(n-p^l)\,,$$

and we obtain that

$$z_l(n-p^l)+k-m\geq k-l+z_l(n-p^m)\geq k-l+z_l(n-\{p^l+t(p-1)\})\,. \quad \text{q.e.d.}$$

Lemma 4.2. *Fix an integer l such that $1\leq l<k$. For integers t,j, such that $t\geq 1$, $p^l+t(p-1)\leq n$, and $2\leq j\leq p^l$, we have that*

$$p^{k-l+z_l(n-p^l)+z_l(p^l-j)+1}\mid p^{z_l(p^l+t(p-1)-j)+1}p^{z_l(n-p^l)}a_{p^l+t(p-1)-1}\,.$$

Proof. In case $p^l+t(p-1)\geq p^k$, we have to see that

$z_l(p^l+t(p-1)-j)\geq z_l(p^l-j)+k-l$. And in case $p^l+t(p-1)<p^k$, let m be an integer such that $p^m\leq p^l+t(p-1)<p^{m+1}$, then $p^{k-m}\mid a_{p^l+t(p-1)}$. Therefore we have only to see that

$$z_l(p^l+t(p-1)-j)+z_l(n-p^l)+k-m\geq k-l+z_l(n-p^l)+z_l(p^l-j)\,.$$

But these results is easily obtained by similar argument in Lemma 4.1. q.e.d.

Now, we prove the following important result which is a generalization of Proposition 4 of [4].

Therem 4.3. *Fix an integer l such that $p^l\leq n$ and $1\leq l\leq k-1$. For i such that $p^l\leq p^l+i\leq n$, in $\tilde{G}^{2*}(L^n(p^k))$ we have the equality*

$$p^{k-l+z_l(n-(p^l+i))}e^{p^l+i}=p^{k-l+z_l(n-(p^l+i))}\sum_{j-1}^{p^l-1}\lambda_{i,j}e^j$$

and $p^{z_l(p^l+i-j)+1}\mid\lambda_{i,j}$ where $\lambda_{i,j}\in G^{2(p^l+i-j)}(pt)$.

Proof. The proof is by induction on n. For $n=p$, we obtain that $p^{k-1}e^p=p^{k-1}\lambda e$ and $p\mid\lambda$ by Proposition 3.1, (2) for $n=i=p$. Thus the case $n=p$ is valid.

Next assume that the statement holds in $\tilde{G}^{2*}(L^{n-1}(p^k))$, i.e., for fixed l such that $p^l\leq n-1$, $1\leq l\leq k-1$, and for i such that $p^l\leq p^l+i\leq n-1$,

$$p^{k-l+z_l(n-1-(p^l+i))}e^{p^l+i}=p^{k-l+z_l(n-1-(p^l+i))}\sum_{j=1}^{p^l-1}\lambda_{i,j}e^j$$

and $p^{z_l(p^l+i-j)+1}\mid\lambda_{i,j}$ in $\tilde{G}^{2*}(L^{n-1}(p^k))$.

Applying the homorhpism g: $\tilde{G}^{2*}(L^{n-1}(p^k))\rightarrow\tilde{G}^{2*}(L^n(p^k))$ by defined $g(x)=e\cdot x$ (which is well-defined by Lemma 1.1), we have the following lemma.

Lemma 4.4. *Fix an integer l such that $p^l\leq n-1$. $1\leq l\leq k-1$ and for i' such that $p^l+1\leq p^l+i'\leq n$, then in $\tilde{G}^{2*}(L^n(p^k))$*

$$p^{k-l+z_l(n-(p^l+i'))}e^{p^l+i'}$$

$$= p^{k-l+z_l(n-(p^l+i'))}\sum_{j'=2}^{p^l-1}\tilde{\lambda}_{i,j}e^{j'}$$
$$+p^{k-l+z_l(n-(p^l+i'))}\tilde{\lambda}_{i',p^l}e^{p^l}, \textit{ and}$$
$$p^{z_l(p^l+i'-j')+1}\,|\,\tilde{\lambda}_{i',j'},\ p^{z_l(i')+1}\,|\,\tilde{\lambda}_{i',p^l}$$

where we have put $i'=i+1$, $j'=j+1$, *and* $\tilde{\lambda}_{i',j'}=\lambda_{i'-1,j'-1}$.

This Lemma stands close to the statement of Theorem 4.3., but the definitive obstruction to go ahead is the existence of e^{p^l}*-term* at the right hand side. So we prepare the next Lemma which is a special case of Theorem 4.3.

Lemma 4.5. *Fix an integer* l *such that* $p^l\leq n$, $1\leq l\leq k-1$. *Then, in* $\tilde{G}^{2*}(L^n(p^k))$,

$$p^{k-l+z_l(n-p^l)}e^{p^l} = p^{k-l+z_l(n-p^l)}\sum_{j=1}^{p^l-1}\lambda_{0,j}e^j$$

and $p^{z_l(p^l-j)+1}\,|\,\lambda_{0,j}$.

Proof. Multiplying $[p^k]_G(e)$ by $p^{z_l(n-p^l)}$, we have that

$$\sum_{s=0}^{p_l-1}p^{z_l(n-p^l)}a_{s(p-1)}e^{s(p-1)+1}$$
$$+p^{z_l(n-p^l)}a_{p^l-1}e^{p^l}$$
$$+\sum_{t=1}^{\infty}p^{z_l(n-p^l)}a_{p^l+t(p-1)-1}e^{p^l+t(p-1)} = 0\,.$$

The above second term is equal to $p^{k-l+z_l(n-p^l)}be^{p^l}$,

by Proposition 2.1, (3), where b is a unit.

Next,

$$p^{k-l+z_l(n-p^l)+1}\,|\,p^{z_l(n-p^l)}a_{s(p-1)}\,,\quad \text{for } 0\leq s\leq p_l-1$$

by Proposition 2.1, (5).

On the other hand, we obtain trivially that

$$z_l(p^l-(s(p-1)+1)) = 0\,.$$

Therefore, the above first term is given in a form of

$$p^{k-l+z_l(n-p^l)}\sum_{j=1}^{p^l-1}\lambda_je^j,\ p^{z_l(p^l-j)+1}\,|\,\lambda_j\,.$$

Thus we obtain the equation

$$bp^{k-l+z_l(n-p^l)}e^{p^l} = p^{k-l+z_l(n-p^l)}\sum_{j=1}^{p^l-1}\lambda_je^j + \sum_{t=1}^{\infty}p^{z_l(n-p^l)}a_{p^l+t(p-1)-1}e^{p^l+t(p-1)}\,, \tag{4.3}$$

and $p^{z_l(p^l-j)+1}\,|\,\lambda_j$.

Now we calculate the last term. In case $n=p^l$

$$\sum_{t=1}^{\infty} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1} e^{p^l+t(p-1)} = 0\,, \quad \text{because} \quad p^l+t(p-1)>n\,.$$

In case $p^l \leq n-1$, we may apply Lemmas 4.1 and 4.4 to obtain

$$\begin{aligned}
&\sum_{t=1}^{\infty} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1} e^{p^l+t(p-1)} \\
&= \sum_{t=1}^{\infty} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1} \sum_{j=2}^{p^l-1} \tilde{\lambda}_{t,j} e^j \\
&\quad + \sum_{t=1}^{\infty} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1} \tilde{\lambda}_{t,p^l} e^{p^l}
\end{aligned}$$

where
$$p^{z_l(p^l+t(p-1)-j)+1} \mid \tilde{\lambda}_{t,j}, \quad \text{and}$$
$$p^{z_l(t(p-1))+1} \mid \tilde{\lambda}_{t,p^l}$$

Then, applying Lemma 4.2 to each term of above sum and summing over j, we have

$$\begin{aligned}
&\sum_{t=1}^{\infty} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1} e^{p^l+t(p-1)} \\
&= p^{k-l+z_l(n-p^l)} \sum_{j=2}^{p^l-1} \bar{\lambda}_j e^j + p^{k-l+z_l(n-p^l)} \bar{\lambda}_{p^l} e^{p^l}
\end{aligned}$$

where
$$p^{z_l(p^l-j)+1} \mid \bar{\lambda}_j, \ p \mid \bar{\lambda}_{p^l}\,.$$

Finally, in either case, we know by (4.3), that,

$$(b+p\lambda) p^{k-l+z_l(n-p^l)} e^{p^l} = p^{k-l+z_l(n-p^l)} \sum_{j=1}^{p^l-1} \lambda_j e^j, \quad \text{and}$$
$$p^{z_l(p^l-j)+1} \mid \lambda_j\,.$$

Then, we obtain Lemma 4.5.

Next return to the proof of Theorem 4.3.

$$k-l+z_l(n-(p^l+i'))+z_l(i')+1 \geq k-l+z_l(n-p^l), \quad \text{by (4.1)}.$$

Thus the coefficient of e^{p^l}*-term* of the right hand side of the equation of Lemma 4.4 is divisible by $p^{k-l+z_l(n-p^l)}$, and we may apply Lemma 4.5 to this, so that we obtain

$$p^{k-l+z_l(n-(p^l+i'))+z_l(i')+1} \sum_{j=1}^{p^l-1} \lambda_j e^j, \quad \text{and} \quad p^{z_l(p^l-j)+1} \mid \lambda_j\,.$$

By (4.1), the above sum can be written that,

$$p^{k-l+z_l(n-p^l+i')} \sum_{j=1}^{p^l-1} \lambda_j' e^j, \quad \text{and} \quad p^{z_l(p^l+i'-j)+1} \mid \lambda_j'$$

Therefore we obtain Theorem 4.3 in case $p^l \leq n-1$, and $p^l+1 \leq p^l+i \leq n$.

The statement of the theorem in another case has ever been proven by Lemma 4.5. q.e.d.

Corollary 4.6. *For i such that $p \leq i \leq \min(n, p^k-1)$, there exist the element*

$w_i \in \tilde{G}^{2\beta}(L^n(p^k))$, $1 \leq \beta \leq p-1$, *which has the form* e^i+*lower degree terms, precisely,*

$$w_i = u_1^{\alpha(i)}e^i + \sum_{j=1} \lambda_{i,j} u_1^{\alpha(i)-j} e^{i-j(p-1)} \text{ where } i = \alpha(i)(p-1)+\beta,\ 1 \leq \beta \leq p-1 .$$

Moreover, if $p^l \leq i < p^{l+1}$, *order* $w_i \leq p^{k-l+z_l(n-i)}$.

Proof. For i such that $p \leq i \leq \min(n, p^k-1)$ there exists unique l such that $1 \leq l \leq k-1$, $p^l \leq n$, and $p^l \leq i < p^{l+1}$.
Fix this l, then we obtain,

$$p^{k-l+z_l(n-i)} e^i = p^{k-l+z_l(n-i)} \sum_{j=1}^{p^l-1} \lambda_j e^j, \text{ by Theorem 4.3.}$$

Putting $w_i = u_1^{\alpha(i)}(e^i - \sum_{j=1}^{p^l-1} \lambda_j e^j)$, by the sparsness of $G^*(pt)$,

$$\lambda_j = 0 \ \text{ unless } \ i \equiv j \bmod (p-1) .$$

Therefore we obtain the desired elements w_i.

5. The additive structure of $\tilde{G}^{2*}(L^n(p^k))$

Proposition 5.1. $\tilde{G}^{2\beta}(L^n(p^k))$ $1 \leq \beta \leq p-1$ *is generated by*

$\{u_1{}^j e^{j(p-1)+\beta}\}$, $j = 0, 1, \cdots, \min([(n-\beta)/(p-1)], p_k-1)$.
(p_k *is defined by* (2.1).).

Proof. (1) If $n < p^k$, as $[(n-\beta)/(p-1)] \leq p_k-1$, $min([(n-\beta)/(p-1)], p_k-1) = [(n-\beta)/(p-1)]$.
By Lemma 1.1, if we prove that

$$(1+[(n-\beta)/(p-1)])(p-1)+\beta > n$$

then we obtain the result. But this statement is easily seen.

(2) Assume the statement is true for $n-1$ and we prove it for $n \geq p^k$. (It means that $min([(n-\beta)/(p-1)], p_k-1) = p_k-1$).

By Lemma 1.2, we obtain,

$$(5.1) \qquad 0 \to \text{Kernel } \iota^* \to \tilde{G}^{2\beta}(L^n(p^k)) \xrightarrow{\iota^*} \tilde{G}^{2\beta}(L^{n-1}(p^k)) \to 0$$

$$\text{Kernel } \iota^* = \begin{cases} u_1{}^{\alpha(n)} e^n & \text{if } \beta - n = \alpha(n)(p-1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have only to see that $u_1{}^{\alpha(n)} e^n$ is the linear conbination of $\{u_1{}^j e^{j(p-1)+\beta}\}$ $j=0, 1, \cdots, p_k-1$.

Multiplying $[p^k]_G(e)$ by $u_1{}^{\alpha(n)-p_k} \cdot e^{n-p^k}$, in $\tilde{G}^{2\beta}(L^n(p^k))$,

$$(5.2) \qquad u_1{}^{\alpha(n)} e^n = p \sum_{j=0}^{\alpha(n)-1} \mu_j u_1{}^j e^{j(p-1)+\beta} \ \mu_j \in Z_{(p)}, \text{ by Proposition 2.1, (3).}$$

On the other hand, by the assumption of the induction and by

(5.1), we know,

$$(5.3) \qquad u_1{}^j e^{j(p-1)+\beta} = \sum_{t=0}^{p_k-1} \mu_t u_1{}^t e^{t(p-1)+\beta} + \mu u_1{}^{\alpha(n)} e^n .$$

Substituting (5.3) into the right side of (5.2), we obtain,

$$(1+p\mu)u_1{}^{\alpha(n)} e^n = \sum_{t=0}^{p_k-1} \mu_t u_1{}^t e^{t(p-1)+\beta} .$$

q.e.d.

Corollary 5.2. *Fix integers n and* β *such that* $n \geq \beta$ *and* $1 \leq \beta \leq p-1$. *Then* $\tilde{G}^{2\beta}(L^n(p^k))$ *is generated by* $\{e^\beta\}$ *and* $\{w_{j(p-1)+\beta}\}$ *where* $j=1, 2, \cdots, \min([(n-\beta)/(p-1)], p_k-1)$.

REMARK. In case $[(n-\beta)/(p-1)]=0$, we observe that the only generater is e^β.

Proof. In case $[(n-\beta)/(p-1)]=0$, the proof is straightfowards by Lemma 1.1.

Thus we may assume that $n \geq p$. Then we have only to see that, there is $w_{j(p-1)+\beta}$ of Corollary 4.6. for $1 \leq j \leq min([(n-\beta)/(p-1)], p_k-1)$. But we see easily that, for such j

$$j(p-1)+\beta \leq \min(n, p^k-1) .$$

Therefore we obtain the result.

Next we put

$$\begin{aligned} V_n = k &+ [(n-\beta)/(p-1)] + \sum_{j=1}^{p_2-1} \{(k-1)+z_1(n-j(p-1)-\beta)\} \\ &+ \sum_{j=p_2}^{p_3-1} \{(k-2)+z_2(n-j(p-1)-\beta)\} + \cdots \\ &+ \sum_{j=p_l}^{p_{l+1}-1} \{(k-l)+z(n-j(p-1)-\beta)\} + \cdots \\ &+ \sum_{j=p_{m(n)}}^{M(n)} \{(k-m(n)+z_{m(n)}(n-j(p-1)-\beta)\} \end{aligned}$$

where

$$M(n) = \min([(n-\beta)/(p-1)], p_k-1) ,$$

$$m(n) = \begin{cases} i(n) & \text{if } [(n-\beta)/(p-1)] \leq p_k-1 , \\ k-1 & \text{if } [(n-\beta)/(p-1)] \geq p_k-1 , \end{cases}$$

and $i(n)$ is a number such that

$$p_{i(n)} \geq [(n-\beta)/(p-1)] < p_{i(n)+1}, \; p_l = (p^l-1)/(p-1) ,$$

and we put $p_0=0$.

We note that $M(n) \geq M(n-1)$, $m(n) \geq m(n-1)$. And it is convenient to put $V_n=0$ if $n<\beta$.

Theorem 5.3. *Fix integers n, k, β such that $n\geq 1$, $k\geq 2$, and $1\leq\beta\leq p-1$, then*

$$\tilde{G}^{2\beta}(L^n(p^k))\cong\langle e^\beta\rangle\oplus\sum_j\langle w_{j(p-1)+\beta}\rangle\ j=1,2,\cdots$$

$\cdots$, $\min([(n-\beta)/(p-1)], p_k-1)$

where $\langle x\rangle$ is the cyclic subgroup generated by x, and

$$\text{order } e^\beta=p^{k+[(n-\beta)/(p-1)]},$$
$$\text{order } w_{j(p-1)+\beta}=p^{k-l+z_l(n-j(p-1)-\beta)}, \quad \text{if } p^l\leq j(p-1)+\beta<p^{l+1}.$$

Proof. The order of the group of right hand side is less or equal than $p^V n$ by Proposition 3.1 and Corollary 4.6. If we prove $p^V n=p^{k(1+[(n-\beta)/(p-1)])}$ $=|\tilde{G}^{2\beta}(L^n(p^k))|$ then observing Corollary 5.2, we get the proof of all statements of Theorem 5.3 Therefore we prove the next lemma.

Lemma 5.4. *For $n\geq 1$, $k\geq 2$, $1\leq\beta\leq p-1$' we have*

$$V_n=k(1+[(n-\beta)/(p-1)]).$$

Proof. We put $Y_n=k(1+[(n-\beta)/(p-1)])$.

(1) In case $n\leq\beta$, the proof is easy.

(2) If $n-\beta\not\equiv 0 \mod p-1$, as $[(n-\beta)/(p-1)]=[(n-1-\beta)/(p-1)]$, $M(n-1)=M(n)$ and $m(n-1)=m(n)$.

Moreover $z_l(n-j(p-1)-\beta)=z_l(n-1-j(p-1)-\beta)$.

Therefore $V_n=V_{n-1}=Y_{n-1}=Y_n$.

(3) If $n-\beta=d(p-1)$, $d\geq 1$, then,

$$Y_n=Y_{n-1}+k.$$

On the other hand, for j such that $p_l\leq j\leq p_{l+1}-1$, therre exists only one j such that

$$z_l(n-j(p-1)-\beta)=z_l(n-1-j(p-1)-\beta)+1,$$

and for other j,

$$z_l(n-j(p-1)-\beta)=z_l(n-1-j(p-1)-\beta).$$

Therefore

$$\sum_{j=p_l}^{p_{l+1}-1}\{k-l+z_l(n-j(p-1)-\beta)\}=\sum_{j=p_l}^{p_{l+1}-1}\{k-l+z_l(n-1-j(p-1)-\beta)\}+1.$$

Therefore,

$$V_n-V_{n-1}=m(n-1)+\sum_{j=p_{m(n-1)}}^{M(n-1)}\{k-m(n-1)+z_{m(n-1)}(n-j(p-1)-\beta)\}$$

$$-\sum_{j=p_{m(n-1)}}^{M(n-1)} \{k-m(n-1)+z_{m(n-1)}(n-1-j(p-1)-\beta)\}$$
$$+\sum_{j=M(n-1)+1}^{M(n)} \{k-m(n)+z_{m(n)}(n-j(p-1)-\beta)\}$$

Thus we have only to see that

$$V_n - V_{n-1} = k = Y_n - Y_{n-1}\,.$$

If $[(n-1-\beta)/(p-1)] \geq p_k - 1$, $M(n-1)=M(n)=p_k-1$, $m(n-1)=m(n)=k-1$. Therefore $V_n - V_{n-1}=k$.

If $[(n-1-\beta)/(p-1)] < p_k - 1$, then, $M(n-1)=[(n-1-\beta)/(p-1)]$, $M(n)=[(n-\beta)/(p-1)]=M(n-1)+1$, $m(n-1)=i(n-1)$, $m(n)=i(n)$.

In this case

$$\sum_{j=M(n-1)+1}^{M(n)} \{k-m(n)+z_{m(n)}(n-j(p-1)-\beta)\} = k-i(n)\,.$$

Therefore, if we put

$$W = \sum_{j=p_{i(n-1)}}^{M(n-1)} \{k-i(n-1)+z_{i(n-1)}(n-j(p-1)-\beta)\}$$
$$-\sum_{j=p_{i(n+1)}}^{M(n-1)} \{k-i(n-1)+z_{i(n-1)}(n-1-j(p-1)-\beta)\}\,,$$

we have only to see that $W=i(n)-i(n-1)$.

(3,a) If $d < p_{i(n-1)+1}$, then $i(n)=i(n-1)$.

For j such that $p_{i(n-1)} \leq j \leq M(n-1)$, we have,

$$n-j(p-1)-\beta < (p_{i(n-1)+1}-j)\,(p-1) \leq (p_{i(n-1)+1}-p_{i(n-1)})\,(p-1)$$
$$= p^{i(n-1)+1} - p^{i(n-1)}\,.$$

Therefore,

$z_{i(n-1)}(n-j(p-1)-\beta)=0$, and also

$$z_{i(n-1)}(n-1-j(p-1)-\beta) = 0\,.$$

Hence , $W=0=i(n)-i(n-1)$.

(3,b) If $d=p_{i(n-1)+1}$, then $i(n)=i(n-1)+1$.

As same as above,

$$\sum_{j=p_{i(n-1)}}^{M(n-1)} z_{i(n-1)}(n-1-j(p-1)-\beta) = 0\,.$$

But, $n-p_{i(n-1)}(p-1)-\beta = p^{i(n-1)+1}-p^{i(n-1)}$, and for j such that $j > p_{i(n-1)}$, we have that, $n-j(p-1)-\beta < p^{i(n-1)+1}-p^{i(n-1)}$.

Therefore, $\displaystyle\sum_{j=p_{i(n-1)}}^{M(n-1)} \{z_{i(n-1)}(n-j(p-1)-\beta)\} = 1.$

Consequently $W=1=i(n)-i(n-1)$. Thus we have completed the proof of this lemma. q.e.d.

6. The additive structure of $\tilde{K}(L^n(p^k))$ and $\widetilde{KO}(L^n(p^k))$

By Theorem 5.3 we obtain,

Theorem 6.1. *$\tilde{K}^0(L^n(p^k))\cong\bigoplus_{t=1}^{M}\langle w_t'\rangle$ where $M=\min(n, p^k-1)$, and order $w_t'=p^{k-l+z_l(n-t)}$, if $p^l\leq t<p^{l+1}$.*

Proof. If $1\leq t\leq p-1$, put $w_t'=e^t$, and if $p\leq t$, put $w_t'=w_t$.

As is well-known, for a finite CW-complex X and for any odd prime p,

$$\widetilde{KO}(X)\otimes Z_{(p)}\cong\sum_{i=1}^{(p-1)/2}G^{4i}(X).$$

Observing this facts and Proposition 2.11 of [7], we obtain the next theorem.

Theorem 6.2. *For any odd prime, p, and for any integer $k\geq 2$,*

$$\widetilde{KO}(L^n(p^k))\cong\begin{cases}\sum_{t=1}^{[M/2]}\langle w_{2t}'\rangle & \text{for } n\not\equiv 0 \mod 4\\ \sum_{t=1}^{[M/2]}\langle w_{2t}'\rangle\oplus Z_2 & \text{for } n\equiv 0 \mod 4\end{cases}$$

where $M=\min(n, p^k-1)$ and order $w_{2t}'=p^{k-l+z_l(n-2t)}$, if $p^l\leq 2t<p^{l+1}$.

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