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Author(s)	Yanagida, Syun-ichi
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THE ADDITIVE STRUCTURE OF $G^*(L^n(p^k))$

Syun-ichi YANAGIDA

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Let $L^{n}(k) = L^{n}(k; 1, \dots, 1)$ be the (2n+1)-dimensional standard lens space mod k where n and k are positive integers and $k \ge 2$.

The structure of K-ring and KO-ring of $L^{n}(k)$ are determined by J.F. Adams [1] for k=2 and by T. Kambe [5] for k an odd prime.

For the case $k=p^2$, p a prime, there exist results by T. Kobayashi, M. Sugawara, and T. Kawaguchi [6], [7].

Let p be a prime. By Adams [2], there is a cohomology theory G^* which decomposes K-cohomology localized at p.

In this note we determine the additive structure of $\tilde{G}^*(L^n(r))$ where $r=p^k$, $2 \le k < \infty$, which results to the determination of $K(L^n(p^k))$ for any prime p and $KO(L^n(p^k))$ for an odd prime p.

After manuscript, Professor M. Sugawara kindly communicated to the author that recently N. Mahammed ([9]) has determined the additive structure of $K(L^{n}(p^{k}))$ and $KO(L^{n}(p^{k}))$ for $1 \le k < \infty$. But the method of author's is not same as his. Our basic tool is the formal group of G-cohomology established by S. Araki [3].

In §1 we summarize the well known facts about G-cohomology of lens spaces. In §2 the coefficients of the formal power series $[p^k]_G(T)$ are partially discussed, and the order of the group $\tilde{G}^{2*}(L^n(p^k))$ is determined. In §3 we calculate the order of e^i in $\tilde{G}^{2*}(L^n(p^k))$. In §4 we construct the elements w_i which are in the form $w_i = e^i + lower$ degree terms, and has a smaller order than e^i . In §5 it is proven that a part of the above w_i 's generate $\tilde{G}^{2\beta}(L^n(p^k))$ for $1 \le \beta \le p-1$, and in fact, they give a direct sum decomposition of $\tilde{G}^{2\beta}(L^n(p^k))$. In §6, the additive structure of $\tilde{K}(L^n(p^k))$ are determined by the preceding results.

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1. Summary on G-cohomology groups of lens spaces

Following Adams [2], there is a generalized cohomology theory G which gives a decomposition of K-cohomology localized at a prime p, i.e., for a finite CW-complex X

$$\widetilde{K}^{0}(X) \otimes Z_{(p)} \cong \widetilde{G}^{2}(X) \oplus \cdots \oplus \widetilde{G}^{2i}(X) \oplus \cdots \oplus \widetilde{G}^{2(p-1)}(X)$$

 $\widetilde{K}^{1}(X) \otimes Z_{(p)} \cong \widetilde{G}^{3}(X) \oplus \cdots \oplus \widetilde{G}^{2(p-1)+1}(X)$

where $Z_{(p)}$ is the ring of integers localized at p.

The coefficient ring of G-cohomology is the following [3];

$$G^*(pt) \cong Z_{(p)}[u_1, u_1^{-1}], u_1 \in G^{-2(p-1)}(pt)$$

Moreover, G-cohomology is complex oriented [3]; so is there defined the Euler class e(L) for a complex line bundle L over X such that

$$e(L) \in \widetilde{G}^{2}(X)$$
 and $G^{*}(CP^{n}) \simeq G^{*}(pt) [e(\eta)]/(e(\eta)^{n+1} = 0)$.

where η is the canonical line bundle over CP^n .

The associated formal group F_G was investigated by Araki [3].

The formal power series $[k]_G(T)$, $k \in \mathbb{Z}$ is defined so that $[k]_G(e(\eta)) = e(\eta^k)$ in $G^*(\mathbb{CP}^n)$ for all k, where η^k is k-fold tensor power of η .

Observing the Gysin sequence of the sphere bundle,

$$S^{1} = S^{1}/Z_{p^{k}} \to L^{n}(p^{k}) \xrightarrow{\pi} CP'$$

we have the following exact sequence [8], [10]:

(1.1)
$$0 \cong \widetilde{G}^{2i+1}(CP^n) \to \widetilde{G}^{2i+1}(L^n(p^k)) \to \widetilde{G}^{2i}(CP^n)$$
$$\stackrel{\Psi}{\to} \widetilde{G}^{2i+2}(CP^n) \stackrel{\pi^*}{\to} \widetilde{G}^{2i+2}(L^n(p^k)) \to \widetilde{G}^{2i+1}(CP^n) \cong 0$$

where Ψ is the Gysin homomorphism which is obtained by multiplying the element $[p^k]_G(e(\eta))$. Using (1.1) we have

Lemma 1.1. $\tilde{G}^{2*}(L^{n}(p^{k})) \simeq G^{*}(pt) [e]/(e^{n+1}, [p^{k}]_{G}(e))$

where $e=\pi^*(e(\eta))$ and $G^*(pt)$ [e] means the subgroup of $G^*(pt)$ [e] generated by $G^*(pt)$, e^i , i>0.

The proof is straightfowards by (1.1).

Let ι ; $L^{n-1}(p^k) \rightarrow L^n(p^k)$ be the inclusion.

Lemma 1.2. If $i - n \equiv 0 \mod p - 1$,

$$\iota^*: \widetilde{G}^{2i}(L^n(p^k)) \cong \widetilde{G}^{2i}(L^{n-1}(p^k));$$

if $i-n\equiv 0 \mod p-1$, ι^* is epimorphic and Kernel ι^* is the cyclic subgroup generated by $u_1^{\alpha}e^n$, $\alpha = (i-n)/(p-1)$.

Proof. By Lemma 1.1, we see immediately that ι^* is epimorphic.

Next, take a truncated polynomial $f(e(\eta))$ of $\tilde{G}^{2i}(CP^n)$ such that $\pi^*(f(e(\eta))) \in Kernel \iota^*$. Since $\pi^*\iota^*(f(e(\eta)))=0$, we have that

$$f(e) = [p^k]_G(e) \cdot x(e) \mod e^n$$

That is, if $y \in \tilde{G}^{2i}(L^n(p^k))$ belongs to Kernel ι^* , then $y = xe^n$, $x \in G^{2(i-n)}(pt)$. Therefore the result is immediate.

2. The coefficients of $[p^k]_G(T)$ and the order of $\tilde{G}^{2*}(L^n(p^k))$

For simplicity we put

(2.1)
$$p_l = 1 + p + \dots + p^{l-1}, l \ge 0, \text{ i.e, } p_l = (p^l - 1)/(p-1).$$

First we observe certain divisibilities of coefficients of $[p^k]_G(T)$ by powers of p.

Proposition 2.1. Put
$$[p^k]_G(T) = \sum_{i=1}^{\infty} a_{i-1}T^i$$
, then

(1)
$$a_i = 0$$
 if $i \equiv 0 \mod p - 1$,
(2) $a_0 = p^k$,
(3) $p^{k^{-l}} |a_{p^{l-1}}, p^{k^{-l+1}} \not| a_{p^{l-1}}, \text{ for } 1 \leq l \leq k$,
(4) $p^{k^{-l+1}} |a_i \quad \text{ for } p^l - 1 < i < p^{l+1} - 1, 1 \leq l \leq k - 1$,
(5) $p^{k^{-l}} |a_i \quad \text{ for } p^l - 1 \leq i < p^{l+1} - 1, 1 \leq l \leq k - 1$.

Proof. (1) is trivial by the sparsness of $G^*(pt)$, and (2) is well-known. Let $\log_G(T)$ be the logarithm of F_G (see [3]), we have

(2.2)
$$\log_{G}(T) = T + (1/p)u_{1}T^{p} + (1/p^{2})u_{1}^{p_{2}}T^{p^{2}} + \dots + (1/p^{i})u_{1}^{p_{i}}T^{p^{i}} + \dots, \text{ and }$$

(2.3)
$$(\log_G) \circ [p^k]_G(T) = p^k \cdot \log_G(T)$$

where \circ means the composition of formal power series.

We prove (3) and (4) by induction on *i*. If i=p-1, by substituting (2.2) into (2.3), we get

$$a_{p-1}T^{p} + (1/p)u_{1}(a_{0}T)^{p} = p^{k-1}u_{1}T^{p} \mod T^{p+1}$$

As $a_0 = p^k$, $p^{k-1} | a_{p-1}$ but $p^k \not\mid a_{p-1}$.

Next assume that the proposition holds for any *i* such that $p-1 \le i \le r-1 < p^{k}-1$, and also assume that $p^{i}-1 \le r < p^{i+1}-1 \le p^{k}-1$. Substituting (2.2) into (2.3), we obtain

$$a_{r}T^{r+1} + (1/p)u_{1}(\sum_{j=1}^{r} a_{j-1}T^{j})^{p} + \dots + (1/p^{l})u_{1}^{p_{l}}(\sum_{j=1}^{r} a_{j-1}T^{j})^{p^{l}} \equiv p^{k-l}u_{1}^{p_{l}}T^{p^{l}} \mod T^{r+2}.$$

By the assumption of induction, we see

$$p^{k^{-l+1}}|a_{j-1}$$
 for $1 \le j \le r$ and $j-1 \ne p^{l}-1$, and $p^{k^{-l}}|a_{p^{l-1}}$.

The coefficient of T^{r+1} of $(\sum_{j=1}^{r} a_{j-1}T^{j})^{p^{s}}$ is the sum of monomials of a_{j-1} , differing from $(a_{p^{l}-1})^{p^{s}}$.

Therefore, $(\sum_{j=1}^{r} a_{j-1}T^{j})^{p^{s}}$ for $s \leq l$ is divisible by $p^{k^{-l+1+(p^{s}-1)(k^{-1})} \geq p^{k^{-l+1+s}}$

Therefore, if $r=p^{l}-1$, $p^{k^{-l}}|a_{r}, p^{k^{-l+1}} \not| a_{r}$ and if $r > p^{l}-1$, $p^{k^{-l+1}}|a_{r}$.

The case $r=p^k-1$ is also easily proven by the same s argument with a little care to degrees.

q.e.d.

Finally (5) follows from (3) and (4).

Proposition 2.2. In
$$\tilde{G}^{2n}(L^n(p^k))$$
, order $e^n = p^k$.

Proof. By Proposition 2.1, (2), we see

$$p^{k}e^{n} \equiv [p^{k}]_{G}(e) \cdot e^{n-1} \mod e^{n+1}; i.e.,$$

$$p^{k}e^{n} = 0 \text{ in } \tilde{G}^{2n}(L^{n}(p^{k})).$$

On the other hand, assume that $p^l e^n = 0$ for l < k. Then, there exists an element $x(e) = \sum_{i=1}^{n} x_i e^i$ of $\tilde{G}^{2*}(L^n(p^k))$, $x_i \in G^{2*-2i}(pt)$, such that

 $[p^k]_G(e) \cdot x(e) \equiv p^l e^n \mod e^{n+1}.$

Comparing the coefficients of both sides, we see that

$$x_1 = x_2 = \cdots = x_{n-2} = 0$$
 and $p^k x_{n-1} = p^l$, $x_{n-1} \in G^0(pt) \cong Z_{(p)}$.

This is a contradiction.

Next, we calculate the order of the group $\tilde{G}^{2*}(L^n(p^k))$.

Proposition 2.3. $|\tilde{G}^{2\beta}(L^{n}(p^{k}))| = p^{k^{(1+[(n-\beta)/(p-1)])}} \text{ for } 1 \le \beta \le p-1.$

Proof. The proof is by induction on n. In the case n=1, The proof is straightfoward by Lemma 1.1.

Next, assume that the equality holds for n-1. By Lemma 1.2 and Proposition 2.2 we get that if $n-\beta \equiv 0 \mod p-1$

$$|\tilde{G}^{2\beta}(L^{n}(p^{k}))| = p^{k(1+[(n-1-\beta)/(p-1)])} = p^{k(1+[(n-\beta)/(p-1)])}$$

and if $n - \beta \equiv 0 \mod p - 1$,

$$|\tilde{G}^{2\beta}(L^{n}(p^{k}))| = p^{k(1 + [(n-1-\beta)/(p-1)])} \cdot p^{k} = p^{k(1 + [(n-\beta)/(p-1)])}$$

3. The order of e^i in $\tilde{G}^{2i}(L^n(p^k))$

We assume $k \ge 2$ from now until last section, and $n \ge p$ in this section and the next section.

Proposition 3.1. In $\tilde{G}^{2*}(L^n(p^k))$,

- (1) order $e^i = p^{k + [(n-i)/(p-1)]}$ for $1 \le i \le n$.
- (2) $bp^{k^{-1}+[(n-i)/(p-1)]}e^i = p^{k+[(n-i)/(p-1)]}e^{i-(p-1)}$

for $p \le i \le n$, where b is a unit element of $G^{-2(p-1)}(pt)$.

Proof. The proof is by induction on decending order of *i*. For i=n, (1) follows from Proposition 2.2.

Next, multiplying $[p^k]_G(e)$ by e^{n-p} , we have that

$$p^{k}e^{n-(p-1)} + a_{p-1}e^{n} = 0$$

If we put

$$b = -(a_{p-1})/p^{k-1}$$

then b is a unit element by Proposition 2.1, (3), and we obtain (2).

Next, assume (1) and (2) holds for *i* such that $p < j+1 \le i \le n$. We prove (2) for i=j. Multiply $[p^k]_G(e)$ by $p^{[(n-j)/(p-1)]}e^{j-p}$, then we obtain,

(3.1)
$$0 = p^{k + [(n-j)/(p-1)]} e^{j - (p-1)} - b' p^{k-1 + [(n-j)/(p-1)]} e^{j} + \sum_{i=2}^{\infty} a_{t(p-1)} p^{[(n-j)/(p-1)]} e^{j + (t-1)(p-1)}$$
 by Proposition 2.1, (1).

If $t > p_k$, then, $k + [(n - \{j + (t-1)(p-1)\})/(p-1)] \le [(n-j)/(p-1)]$ because $p_k \ge k$.

Next, let $p_l \le t < p_{l+1}$ for $l \ge 2$, then, by Proposition 2.1, (5),

$$p^{k^{-l+\lceil (n-j)/(p-1)\rceil}} | a_{t(p-1)}p^{\lceil (n-j)/(p-1)\rceil}, \text{ and } k+\lceil (n-\{j+(t-1)(p-1)\})/(p-1)\rceil \le k-l+\lceil (n-j)/(p-1)\rceil$$

because $t-1 \ge p_l - 1 > l-1$ for $l \ge 2$.

Finally, if $p_1 + 1 \le t < p_2$,

$$p^{k+[(n-j)/(p-1)]} | a_{t(p-1)} p^{[(n-j)/(p-1)]}$$

by Propositon 2.1, (4), but

$$k + [(n - \{j + (t-1)(p-1)\})/(p-1)] \le k + [(n-j)/(p-1)]$$

Therefore, by the assumption of induction, we obtain from (3.1) that

 $p^{k+[(n-j)/(p-1)]}e^{j-(p-1)} = b'p^{k-1+[(n-j)/(p-1)]}e^j.$

Then, we have (2) for i=j.

Next, apply (2) for i=j+(p-1),

$$p^{k^{-1}+[(n-j)/(p-1)]}e^{j} = bp^{k^{-1}+[(n-j-(p-1))/(p-1)]}e^{j+(p-1)}.$$

Therefore, if we assume that order $e^{j} < p^{k + [(n-j)((p-1)]]}$, then,

order
$$e^{j+(p-1)} < p^{k+[(n-(j+(p-1)))/(p-1)]}$$

This cotradicts to (1) for i=j+(p-1). Finally we have

$$p^{k+[(n-j)/(p-1)]}e^{j} = bp^{k+[(n-(j+p-1))/(p-1)]}e^{j+p-1} = 0$$

So we obtain (1) for i=j.

4. The construction of the generators $\{w_i\}$

Put $z_l(k) = [k/(p^{l+1}-p^l)]$ for $l \ge 0$. As is easily seen

$$(4.1) z_{l}(a) + z_{l}(b) + 1 \ge z_{l}(a+b) \ge z_{l}(a) + z_{l}(b) .$$

(4.2)
$$z_{l}(a-d)+z_{l}(b+d)+1\geq z_{l}(a)+z_{l}(b)$$
.

Lemma 4.1. Fix an integer l such that $1 \le l < k$. For each integer t such that $t \ge 1$ and $p^{t}+t(p-1) \le n$, we have,

 $p^{k^{-l+z_l}(n-(p^l+t(p-1)))} | p^{z_l(n-p^l)} a_{p^l+t(p-1)-1}.$

Proof. (1) In case $p^{t}+t(p-1) \ge p^{k}$:

$$z_{l}(n-p^{l}) \ge z_{l}(n-\{p^{l}+t(p-1)\})+z_{l}(t(p-1))$$

by (4.1). On the other hand, $t(p-1) \ge p^k - p^l$ by the assumption, thus

$$z_l(t(p-1)) \ge k-l$$
.

Therefore, we get

$$z_l(n-p') \ge k-l+z_l(n-\{p'+t(p-1)\}).$$

(2) In case $p'+t(p-1) < p^k$: Fix an integer *m* such that

$$p^m \le p^l + t(p-1) < p^{m+1}$$
, (so, $m \ge l$).

Then, $p^{k^{-m}} | a_{p^{l+t(p-1)-1}}$, by Proposition 2.1, (5). So, we have only to see that

$$z_l(n-p')+k-m\geq k-l+z_l(n-\{p'+t(p-1)\})$$

But, $n-p^m+(p^{l+1}-p^l)(m-l) \le n-p^l$, because $m \ge l$. Therefore,

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q.e.d.

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$$z_{l}(n-p^{m})+m-l\leq z_{l}(n-p^{l}),$$

and we obtain that

$$z_l(n-p')+k-m \ge k-l+z_l(n-p^m) \ge k-l+z_l(n-\{p'+t(p-1)\})$$
. q.e.d.

Lemma 4.2. Fix an integer l such that $1 \le l < k$. For integers t, j, such that $t \ge 1, p'+t(p-1) \le n$, and $2 \le j \le p'$, we have that

$$p^{k-l+z_l(n-p^l)+z_l(p^l-j)+1} | p^{z_l(p^l+t(p-1)-j)+1} p^{z_l(n-p^l)} a_{p^l+t(p-1)-1}$$

Proof. In case $p^{l}+t(p-1) \ge p^{k}$, we have to see that

 $z_l(p'+t(p-1)-j) \ge z_l(p'-j)+k-l$. And in case $p'+t(p-1) < p^k$, let *m* be an integer such that $p^m \le p'+t(p-1) < p^{m+1}$, then $p^{k-m} |a_{p'+t(p-1)}|$. Therefore we have only to see that

$$z_l(p^l+t(p-1)-j)+z_l(n-p^l)+k-m\geq k-l+z_l(n-p^l)+z_l(p^l-j)$$
.

But these results is easily obtained by similar argument in Lemma 4.1. q.e.d.

Now, we prove the following important result which is a generalization of Proposition 4 of [4].

Therem 4.3. Fix an integer l such that $p^{l} \le n$ and $1 \le l \le k-1$. For i such that $p^{l} \le p^{l}+i \le n$, in $\tilde{G}^{2*}(L^{n}(p^{k}))$ we have the equality

$$p^{k^{-l+z_{l}(n-(p^{l}+i))}}e^{p^{l}+i} = p^{k^{-l+z_{l}(n-(p^{l}+i))}}\sum_{j=1}^{p^{l-1}}\lambda_{i,j}e^{j}$$

and $p^{z_i(p^l+i-j)+1}|\lambda_{i,j}$ where $\lambda_{i,j} \in G^{2(p^l+i-j)}(pt)$.

Proof. The proof is by induction on *n*. For n=p, we obtain that $p^{k-1}e^p = p^{k-1}\lambda e$ and $p|\lambda$ by Proposition 3.1, (2) for n=i=p. Thus the case n=p is valid.

Next assume that the statement holds in $\tilde{G}^{2*}(L^{n-1}(p^k))$, i.e., for fixed l such that $p^{l} \leq n-1$, $1 \leq l \leq k-1$, and for i such that $p^{l} \leq p^{l}+i \leq n-1$,

$$p^{k^{-l+z_l(n-1-(p^l+i))}}e^{p^l+i} = p^{k^{-l+z_l(n-1-(p^l+i))}}\sum_{j=1}^{p^{l-1}}\lambda_{i,j}e^{j^{l-1}}$$

and $p^{z_{l}(p^{l}+i-j)+1}|\lambda_{i,j}$ in $\tilde{G}^{2*}(L^{n-1}(p^{k}))$.

Applying the homorhpism $g: \tilde{G}^{2*}(L^{n-1}(p^k)) \to \tilde{G}^{2*}(L^n(p^k))$ by defined $g(x) = e \cdot x$ (which is well-defined by Lemma 1.1), we have the following lemma.

Lemma 4.4. Fix an integer l such that $p' \le n-1$. $1 \le l \le k-1$ and for i' such that $p'+1 \le p'+i' \le n$, then in $\tilde{G}^{2*}(L^n(p^k))$

$$p^{k-l+z} i^{(n-(p'+i'))} e^{p^l+i'}$$

$$= p^{k-l+z_{l}(n-(p^{l}+i'))} \sum_{j'=2}^{p^{l}-1} \tilde{\lambda}_{i,j} e^{j'} \\ + p^{k-l+z_{l}(n-(p^{l}+i'))} \tilde{\lambda}_{i',p^{l}} e^{p^{l}}, and \\ p^{z_{l}(p^{l}+i'-j')+1} |\tilde{\lambda}_{i',j'}, p^{z_{l}(i')+1}| \tilde{\lambda}_{i',p^{l}}$$

where we have put i'=i+1, j'=j+1, and $\tilde{\lambda}_{i',j'}=\lambda_{i'-1,j'-1}$.

This Lemma stands close to the statement of Theorem 4.3., but the definitive obstruction to go ahead is the existence of e^{p^l} -term at the right hand side. So we prepare the next Lemma which is a special case of Theorem 4.3.

Lemma 4.5. Fix an integer l such that $p^{l} \leq n$, $1 \leq l \leq k-1$. Then, in $\tilde{G}^{2*}(L^{n}(p^{k}))$,

$$p^{k^{-l+z_{l}(n-p^{l})}e^{p^{l}}} = p^{k^{-l+z_{l}(n-p^{l})}\sum_{j=1}^{p^{l}-1}}\lambda_{0,j}e^{j}$$

and $p^{z_{l}(p^{l}-j)+1}|\lambda_{0,j}$.

Proof. Multiplying $[p^k]_G(e)$ by $p^{z_l(n-p^l)}$, we have that

$$\sum_{s=0}^{p_{l-1}} p^{z_{l}(n-p^{l})} a_{s(p-1)} e^{s(p-1)+1} + p^{z_{l}(n-p^{l})} a_{p^{l}-1} e^{p^{l}} + \sum_{t=1}^{\infty} p^{z_{l}(n-p^{l})} a_{p^{l}+t(p-1)-1} e^{p^{l}+t(p-1)} = 0.$$

The above second term is equal to $p^{k^{-l+z_l}(n-p^l)}be^{p^l}$,

by Proposition 2.1, (3), where b is a unit. Next,

$$p^{k^{-l+z_{l}(n-p')+1}} | p^{z_{l}(n-p')} a_{s(p-1)}, \text{ for } 0 \le s \le p_{l} - 1$$

by Proposition 2.1, (5).

On the other hand, we obtain trivially that

$$z_i(p^i - (s(p-1)+1)) = 0$$
.

Therefore, the above first term is given in a form of

$$p^{k-l+z_l(n-p^l)}\sum_{j=1}^{p^l-1}\lambda_j e^j, p^{z_l(p^l-j)+1}|\lambda_j.$$

Thus we obtain the equation

(4.3)
$$bp^{k^{-l+z_{i}(n-p^{l})}e^{p^{l}}} = p^{k^{-l+z_{i}(n-p^{l})}\sum_{j=1}^{p^{l}-1}}\lambda_{j}e^{j} + \sum_{t=1}^{\infty} p^{z_{i}(n-p^{l})}a_{p^{l+t(p-1)-1}}e^{p^{l}+t(p-1)},$$

and $p^{z_{i}(p^{l}-j)+1}|\lambda_{j}.$

Now we calculate the last term. In case $n=p^{t}$

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$$\sum_{t=1}^{\infty} p^{z_t(n-p^t)} a_{p^t+t(p-1)-1} e^{p^t+t(p-1)} = 0, \text{ because } p^t+t(p-1) > n$$

In case $p' \le n-1$, we may apply Lemmas 4.1 and 4.4 to obtain

$$\sum_{i=1}^{\infty} p^{z_{i}(n-p^{l})} a_{p^{l}+t(p-1)-1} e^{p^{l}+t(p-1)}$$

$$= \sum_{i=1}^{\infty} p^{z_{i}(n-p^{l})} a_{p^{l}+t(p-1)-1} \sum_{j=2}^{p^{l}-1} \tilde{\lambda}_{t,j} e^{j}$$

$$+ \sum_{i=1}^{\infty} p^{z_{i}(n-p^{l})} a_{p^{l}+t(p-1)-1} \tilde{\lambda}_{t,p^{l}} e^{p^{l}}$$

$$p^{z_{i}(p^{l}+t(p-1)-j)+1} | \tilde{\lambda}_{t,j}, \text{ and}$$

$$p^{z_{i}(t(p-1))+1} | \tilde{\lambda}_{t,p^{l}}$$

where

Then, applying Lemma 4.2 to each term of above sum and summing over j, we have

$$\sum_{l=1}^{\infty} p^{z_{l}(n-p^{l})} a_{p^{l}+t(p-1)-1} e^{p^{l}+t(p-1)}$$

$$= p^{k^{-l+z_{l}(n-p^{l})}} \sum_{j=2}^{p^{l-1}} \overline{\lambda}_{j} e^{j} + p^{k^{-l+z_{l}(n-p^{l})}} \overline{\lambda}_{p^{l}} e^{p^{l}}$$

$$p^{z_{l}(p^{l}-j)+1} |\overline{\lambda}_{j}, p| \overline{\lambda}_{p^{l}}.$$

where

Finally, in either case, we know by (4.3), that,

$$(b+p\lambda)p^{k-l+z_l(n-p^l)}e^{p^l} = p^{k-l+z_l(n-p^l)}\sum_{j=1}^{p^{l-1}}\lambda_j e^j, \text{ and } p^{z_l(p^{l-j})+1}|\lambda_j.$$

Then, we obtain Lemma 4.5.

Next return to the proof of Theorem 4.3.

$$k-l+z_l(n-(p'+i'))+z_l(i')+1\geq k-l+z_l(n-p')$$
, by (4.1).

Thus the coefficient of e^{p^l} -term of the right hand side of the equation of Lemma 4.4 is divisible by $p^{k^{-l+z_l(n-p^l)}}$, and we may apply Lemma 4.5 to this, so that we obtain

$$p^{k^{-l+z_l(n-(p^l+i'))+z_l(i')+1}}\sum_{j=1}^{p^l-1}\lambda_j e^j$$
, and $p^{z_l(p^l-j)+1}|\lambda_j$.

By (4.1), the above sum can be written that,

$$p^{k-l+z_l(n-p^l+i'))} \sum_{j=1}^{p^l-1} \lambda_j' e^j$$
, and $p^{z_l(p^l+i'-j)+1} |\lambda_j'|$

Therefore we obtain Theorem 4.3 in case $p' \le n-1$, and $p'+1 \le p'+i \le n$.

The statement of the theorem in another case has ever been proven by Lemma 4.5. q.e.d.

Corollary 4.6. For i such that $p \le i \le \min(n, p^k - 1)$, there exist the element

$$w_i \in \tilde{G}^{2\beta}(L^n(p^k)), 1 \le \beta \le p-1$$
, which has the form e^i +lower degree terms, precisely,
 $w_i = u_1^{\alpha(i)}e^i + \sum_{j=1} \lambda_{i,j}u_1^{\alpha(j)-j}e^{i-j(p-1)}$ where $i = \alpha(i) (p-1) + \beta, 1 \le \beta \le p-1$.
Moreover, if $p^l \le i < p^{l+1}$, order $w_i \le p^{k-l+z_i(n-i)}$.

Proof. For *i* such that $p \le i \le \min(n, p^k - 1)$ there exists unique *l* such that $1 \le l \le k-1$, $p^l \le n$, and $p^l \le i < p^{l+1}$. Fix this *l*, then we obtain,

 $p^{k^{-l+z_l(n-i)}}e^i = p^{k^{-l+z_l(n-i)}}\sum_{j=1}^{p^{l-1}}\lambda_j e^j, \text{ by Theorem 4.3.}$ Putting $w_i = u_1^{\alpha(i)}(e^i - \sum_{j=1}^{p^{l-1}}\lambda_j e^j)$, by the sparsness of $G^*(pt)$, $\lambda_j = 0$ unless $i \equiv j \mod (p-1)$.

Therefore we obtain the desired elements w_i .

5. The additive structure of $\tilde{G}^{2*}(L^n(p^k))$

Proposition 5.1. $\tilde{G}^{2\beta}(L^{*}(p^{k})) \ 1 \le \beta \le p-1$ is generated by

$$\{u_1^{je^{j(p-1)+\beta}}\}, j = 0, 1, \dots, \min([(n-\beta)/(p-1)], p_k-1).$$

(p_k is defined by (2.1).).

Proof. (1) If $n < p^k$, as $[(n-\beta)/(p-1)] \le p_k-1$, $min([(n-\beta)/(p-1)]$, $p_k-1)=[(n-\beta)/(p-1)]$. By Lemma 1.1, if we prove that

$$(1+[(n-\beta)/(p-1)])(p-1)+\beta > n$$

then we obtain the result. But this statement is easily seen.

(2) Assume the statement is true for n-1 and we prove it for $n \ge p^k$. (It means that $min([(n-\beta)/(p-1)], p_k-1) = p_k-1)$.

By Lemma 1.2, we obtain,

(5.1)
$$0 \to \text{Kernel } \iota^* \to \widetilde{G}^{2\beta}(L^n(p^k)) \xrightarrow{\iota^*} \widetilde{G}^{2\beta}(L^{n-1}(p^k)) \to 0$$
$$\text{Kernel } \iota^* = \begin{cases} u_1^{\alpha(n)}e^n & \text{if } \beta - n = \alpha(n) (p-1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have only to see that $u_1^{\alpha(n)}e^n$ is the linear combination of $\{u_1^{j}e^{j(p-1)+\beta}\}$ $j=0, 1, \dots, p_k-1$.

Multiplying
$$[p^k]_G(e)$$
 by $u_1^{\omega(n)-p_k} \cdot e^{n-p^k}$, in $\widetilde{G}^{2\beta}(L^n(p^k))$,

(5.2)
$$u_1^{\omega(n)}e^n = p \sum_{j=0}^{\omega(n)-1} \mu_j u_1^{j} e^{j(p-1)+\beta} \mu_j \in Z_{(p)}, \text{ by Proposition 2.1, (3).}$$

On the other hand, by the assumption of the induction and by

(5.1), we know,

(5.3)
$$u_1^{j} e^{j(p-1)+\beta} = \sum_{t=0}^{p_k-1} \mu_t u_1^{t} e^{t(p-1)+\beta} + \mu u_1^{\omega(n)} e^n .$$

Substituting (5.3) into the right side of (5.2), we obtain,

$$(1+p\mu)u_1^{o(n)}e^n = \sum_{t=0}^{p_{k-1}} \mu_t u_1^{t} e^{t(p-1)+\beta} .$$
 g.e.d.

Corollary 5.2. Fix integers n and β such that $n \ge \beta$ and $1 \le \beta \le p-1$. Then $\tilde{G}^{2\beta}(L^n(p^k))$ is generated by $\{e^\beta\}$ and $\{w_{j(p-1)+\beta}\}$ where $j=1, 2, \cdots, \min([(n-\beta)/(p-1)], p_k-1)$.

REMARK. In case $[(n-\beta)/(p-1)]=0$, we observe that the only generater is e^{β} .

Proof. In case $[(n-\beta)/(p-1)]=0$, the proof is straightfowards by Lemma 1.1.

Thus we may assume that $n \ge p$. Then we have only to see that, there is $w_{j(p-1)+\beta}$ of Corollary 4.6. for $1 \le j \le \min([(n-\beta)/(p-1)], p_k-1)$. But we see easily that, for such j

$$j(p-1)+\beta \leq \min(n, p^k-1)$$
.

Therefore we obtain the result.

Next we put

$$\begin{split} V_n &= k + [(n-\beta)/(p-1)] + \sum_{j=1}^{p_{2-1}} \{(k-1) + z_1(n-j(p-1)-\beta)\} \\ &+ \sum_{j=p_2}^{p_{3-1}} \{(k-2) + z_2(n-j(p-1)-\beta)\} + \cdots \\ &+ \sum_{j=p_1}^{p_{j+1}-1} \{(k-l) + z(n-j(p-1)-\beta)\} + \cdots \\ &+ \sum_{j=p_{m(N)}}^{M(n)} \{(k-m(n) + z_{m(n)}(n-j(p-1)-\beta)\} \\ &= M(n) = \min([(n-\beta)/(p-1)], p_k - 1), \\ &= M(n) = \begin{cases} i(n) & \text{if } [(n-\beta)/(p-1)] \le p_k - 1, \\ k-1 & \text{if } [(n-\beta)/(p-1)] \ge p_k - 1, \end{cases} \end{split}$$

where

and
$$i(n)$$
 is a number such that

$$p_{i(n)} \ge [(n-\beta)/(p-1)] < p_{i(n)+1}, p_l = (p^l-1)/(p-1),$$

and we put $p_0 = 0$.

We note that $M(n) \ge M(n-1)$, $m(n) \ge m(n-1)$. And it is convenient to put $V_n = 0$ if $n < \beta$.

Theorem 5.3. Fix integers n, k, β such that $n \ge 1$, $k \ge 2$, and $1 \le \beta \le p-1$, then

$$\widetilde{G}^{2\beta}(L^n(p^k)) \cong \langle e^{\beta} \rangle \oplus \sum_j \langle w_{j(p-1)+\beta} \rangle \ j = 1, 2, \cdots$$

..., $\min([(n-\beta)/(p-1)], p_k-1)$

where $\langle x \rangle$ is the cyclic subgroup generated by x, and

order
$$e^{\beta} = p^{k+[(n-\beta)/(p-1)]}$$
,
order $w_{j(p-1)+\beta} = p^{k-l+z_l(n-j(p-1)-\beta)}$, if $p^l \le j(p-1) + \beta < p^{l+1}$.

Proof. The order of the group of right hand side is less or equal than $p^{\mathbf{v}}n$ by Proposition 3.1 and Corollary 4.6. If we prove $p^{\mathbf{v}}n=p^{k(1+\lfloor(n-\beta)/(p-1)\rfloor)}$ = $|\tilde{G}^{2\beta}(L^n(p^k))|$ then observing Corollary 5.2, we get the proof of all statements of Theorem 5.3 Therefore we prove the next lemma.

Lemma 5.4. For $n \ge 1$, $k \ge 2$, $1 \le \beta \le p-1$ ' we have

$$V_n = k(1 + [(n-\beta)/(p-1)])$$
.

Proof. We put $Y_n = k(1 + [(n - \beta)/(p - 1)])$. (1) In case $n \le \beta$, the proof is easy. (2) If $n - \beta \equiv 0 \mod p - 1$, as $[(n - \beta)/(p - 1)] = [(n - 1 - \beta)/(p - 1)]$, M(n - 1) = M(n) and m(n - 1) = m(n). Moreover $z_i(n - j(p - 1) - \beta) = z_i(n - 1 - j(p - 1) - \beta)$. Therefore $V_n = V_{n-1} = Y_n$. (3) If $n - \beta = d(p - 1)$, $d \ge 1$, then, $Y_n = Y_{n-1} + k$.

On the other hand, for j such that $p_i \le j \le p_{i+1}-1$, there exists only one j such that

$$z_i(n-j(p-1)-\beta) = z_i(n-1-j(p-1)-\beta)+1$$
,

and for other j,

$$z_i(n-j(p-1)-\beta) = z_i(n-1-j(p-1)-\beta)$$

Therefore

$$\sum_{j=p_{l}}^{p_{l}+1-1} \{k-l+z_{l}(n-j(p-1)-\beta)\} = \sum_{j=p_{l}}^{p_{l}+1-1} \{k-l+z_{l}(n-1-j(p-1)-\beta)\} + 1.$$

Therefore,

$$V_n - V_{n-1} = m(n-1) + \sum_{j=b_{m(n-1)}}^{M(n-1)} \{k - m(n-1) + z_{m(n-1)}(n-j(p-1)-\beta)\}$$

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$$-\sum_{j=p}^{M(n-1)} \{k-m(n-1)+z_{m(n-1)}(n-1-j(p-1)-\beta)\} + \sum_{j=M(n-1)+1}^{M(n)} \{k-m(n)+z_{m(n)}(n-j(p-1)-\beta)\}$$

Thus we have only to see that

$$V_n - V_{n-1} = k = Y_n - Y_{n-1}$$
.

If $[(n-1-\beta)/(p-1)] \ge p_k-1$, $M(n-1)=M(n)=p_k-1$, m(n-1)=m(n)=k-1. Therefore $V_n - V_{n-1}=k$. If $[(n-1-\beta)/(p-1)] < p_k-1$, then, $M(n-1)=[(n-1-\beta)/(p-1)]$, $M(n)=[(n-\beta)/(p-1)]=M(n-1)+1$, m(n-1)=i(n-1), m(n)=i(n). In this case

$$\sum_{j=M(n-1)+1}^{M(n)} \{k-m(n)+z_{m(n)}(n-j(p-1)-\beta)\} = k-i(n)$$

Therefore, if we put

$$W = \sum_{j=p}^{M(n-1)} \{k - i(n-1) + z_{i(n-1)}(n-j(p-1)-\beta)\} - \sum_{j=p}^{M(n-1)} \{k - i(n-1) + z_{i(n-1)}(n-1-j(p-1)-\beta)\},\$$

we have only to see that W = i(n) - i(n-1).

(3,a) If $d < p_{i(n-1)+1}$, then i(n) = i(n-1). For *j* such that $p_{i(n-1)} \le j \le M(n-1)$, we have,

$$n - j(p-1) - \beta < (p_{i(n-1)+1} - j) (p-1) \le (p_{i(n-1)+1} - p_{i(n-1)}) (p-1)$$

= $p^{i(n-1)+1} - p^{i(n-1)}$.

Therefore,

 $z_{i(n-1)}(n-j(p-1)-\beta)=0$, and also

$$z_{i(n-1)}(n-1-j(p-1)-\beta)=0$$
.

Hence , W=0=i(n)-i(n-1). (3,b) If $d=p_{i(n-1)+1}$, then i(n)=i(n-1)+1. As same as above,

$$\sum_{i=p}^{M(n-1)} z_{i(n-1)}(n-1-j(p-1)-\beta) = 0.$$

But, $n-p_{i(n-1)}(p-1)-\beta=p^{i(n-1)+1}-p^{i(n-1)}$, and for j such that $j>p_{i(n-1)}$, we have that, $n-j(p-1)-\beta< p^{i(n-1)+1}-p^{i(n-1)}$.

Therefore,
$$\sum_{j=p_{i(n-1)}}^{M(n-1)} \{z_{i(n-1)}(n-j(p-1)-\beta)\} = 1.$$

Consequently W=1=i(n)-i(n-1). Thus we have completed the proof of this lemma. q.e.d.

6. The additive structure of $\tilde{K}(L^{n}(p^{k}))$ and $\tilde{KO}(L^{n}(p^{k}))$

By Theorem 5.3 we obtain,

Theorem 6.1. $\tilde{K}^{o}(L^{n}(p^{k})) \cong \bigoplus_{t=1}^{M} \langle w_{t}' \rangle$ where $M = \min(n, p^{k}-1)$, and order $w_{t}' = p^{k^{-l+2}t^{(n-t)}}$, if $p^{l} \le t < p^{l+1}$.

Proof. If $1 \le t \le p-1$, put $w_t' = e^t$, and if $p \le t$, put $w_t' = w_t$.

As is well-known, for a finite CW-complex X and for any odd prime p,

$$\widetilde{KO}(X) \otimes Z_{(p)} \cong \sum_{i=1}^{(p-1)/2} G^{4i}(X)$$

Observing this facts and Proposition 2.11 of [7], we obtain the next theorem.

Theorem 6.2. For any odd prime, p, and for any integer $k \ge 2$,

$$\widetilde{KO}(L^{n}(p^{k})) \approx \begin{cases} \sum_{t=1}^{\lfloor M/2 \rfloor} \langle w_{2t}' \rangle & \text{for } n \equiv 0 \mod 4 \\ \sum_{t=1}^{\lfloor M/2 \rfloor} \langle w_{2t}' \rangle \oplus Z_{2} & \text{for } n \equiv 0 \mod 4 \end{cases}$$

where $M = \min(n, p^{k}-1)$ and order $w_{2t}' = p^{k^{-l+z_{l}(n-2t)}}$, if $p^{i} \le 2t < p^{l+1}$.

OSAKA CITY UNIVERSITY

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