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THE ADDITIVE STRUCTURE OF $G^*(L^n(p^k))$

SYUN-ICHI YANAGIDA

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Let $L^n(k)=L^n(k; 1, \cdots, 1)$ be the $(2n+1)$-dimensional standard lens space mod $k$ where $n$ and $k$ are positive integers and $k \geq 2$.


For the case $k=p^s$, $p$ a prime, there exist results by T. Kobayashi, M. Sugawara, and T. Kawaguchi [6], [7].

Let $p$ be a prime. By Adams [2], there is a cohomology theory $G^*$ which decomposes $K$-cohomology localized at $p$.

In this note we determine the additive structure of $G^*(L^n(r))$ where $r=p^k$, $2 \leq k < \infty$, which results to the determination of $K(L^n(p^k))$ for any prime $p$ and $KO(L^n(p^k))$ for an odd prime $p$.

After manuscript, Professor M. Sugawara kindly communicated to the author that recently N. Mahammed ([9]) has determined the additive structure of $K(L^n(p^k))$ and $KO(L^n(p^k))$ for $1 \leq k < \infty$. But the method of author's is not same as his. Our basic tool is the formal group of $G$-cohomology established by S. Araki [3].

In §1 we summarize the well known facts about $G$-cohomology of lens spaces. In §2 the coefficients of the formal power series $[p^k]_G(T)$ are partially discussed, and the order of the group $G^{2*}(L^n(p^k))$ is determined. In §3 we calculate the order of $e^i$ in $G^{2*}(L^n(p^k))$. In §4 we construct the elements $w_i$ which are in the form $w_i=e^i+\text{lower degree terms}$, and has a smaller order than $e^i$. In §5 it is proven that a part of the above $w_i$'s generate $G^{2*}(L^n(p^k))$ for $1 \leq \beta \leq p-1$, and in fact, they give a direct sum decomposition of $G^{2*}(L^n(p^k))$. In §6, the additive structure of $K(L^n(p^k))$ are determined by the preceding results.

The author owes to Professors S. Araki and Z. Yosimura their valuable discussions and criticisms. He wishes to express his hearty thanks to them.

1. Summary on $G$-cohomology groups of lens spaces

Following Adams [2], there is a generalized cohomology theory $G$ which gives a decomposition of $K$-cohomology localized at a prime $p$, i.e., for a finite $CW$-complex $X$
where \( \mathbb{Z}_{(\mathfrak{p})} \) is the ring of integers localized at \( \mathfrak{p} \).

The coefficient ring of \( G \)-cohomology is the following \([3]\); \[
\mathbb{G}^*(X) \otimes_{\mathbb{Z}_{(\mathfrak{p})}} \cong \mathbb{G}^*(X) \oplus \cdots \oplus \mathbb{G}^{i+1}(X) \oplus \cdots \oplus \mathbb{G}^{(p-1)}(X)
\]
Moreover, \( G \)-cohomology is complex oriented \([3]\) so is there defined the Euler class \( e(L) \) for a complex line bundle \( L \) over \( X \) such that \[
e(L) \in \mathbb{G}^2(X) \) and \( G^*(\mathbb{C}P^n) \cong G^*(\mathbb{C}P^n) \) \( e(\eta)](e(\eta)^{n+1} = 0) \).

where \( \eta \) is the canonical line bundle over \( \mathbb{C}P^n \).

The associated formal group \( F_G \) was investigated by Araki \([3]\).

The formal power series \( \mathcal{G}_G(T), k \in \mathbb{Z} \) is defined so that \( \mathcal{G}_G(e(\eta)) = e(\eta)^k \) in \( G^*(\mathbb{C}P^n) \) for all \( k \), where \( \eta^k \) is \( k \)-fold tensor power of \( \eta \).

Observing the Gysin sequence of the sphere bundle,
\[
S^i = S^i/\mathbb{Z}_{\mathfrak{p}^k} \to \mathbb{C}P^n \to \mathbb{C}P^n
\]
we have the following exact sequence \([8], [10]\):
\[
0 \to G^{i+1}(\mathbb{C}P^n) \to G^{i+1}(\mathbb{C}P^n) \to G^i(\mathbb{C}P^n) \to \psi
\]
where \( \psi \) is the Gysin homomorphism which is obtained by multiplying the element \( \mathcal{G}_G(e(\eta)) \).

Using (1.1) we have
\[
\text{Lemma 1.1.} \quad \mathcal{G}_G^*(L^n(p^k)) \cong G^*(\mathbb{C}P^n) \left[ e \right] / \left[ e^{n+1} \right], [p^k] \mathcal{G}_G(e(\eta))
\]
where \( e = \pi^*(e(\eta)) \) and \( G^*(\mathbb{C}P^n) \left[ e \right] \) means the subgroup of \( G^*(\mathbb{C}P^n) \left[ e \right] \) generated by \( G^*(\mathbb{C}P^n), e^i, i > 0 \).

The proof is straightforward by (1.1).

Let \( \iota; L^{n-1}(p^k) \to L^n(p^k) \) be the inclusion.

\[
\text{Lemma 1.2.} \quad \text{If } i-n \equiv 0 \mod p-1,
\]
\[
\iota^*; \mathcal{G}_G^i(L^n(p^k)) \cong \mathcal{G}_G^i(L^{n-1}(p^k));
\]
if \( i-n \equiv 0 \mod p-1 \), \( \iota^* \) is epimorphic and \( \text{Kernel } \iota^* \) is the cyclic subgroup generated by \( \alpha \iota \mathbb{C}P^n \), \( \alpha = (i-n)/(p-1) \).

Proof. By Lemma 1.1, we see immediately that \( \iota^* \) is epimorphic.

Next, take a truncated polynomial \( f(e(\eta)) \) of \( \mathcal{G}_G^i(\mathbb{C}P^n) \) such that \( \pi^*(f(e(\eta))) \subseteq \text{Kernel } \iota^* \). Since \( \pi^*f(e(\eta)) = 0 \), we have that
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\[ f(e) = [p^k]_G(e) \cdot x(e) \mod e^n. \]

That is, if \( y \in \tilde{G}^{2i}(L'(p^k)) \) belongs to Kernel \( \iota^* \), then \( y = xe^n, x \in G^{2(i-n)}(pt) \). Therefore the result is immediate.

2. The coefficients of \([p^k]_G(T)\) and the order of \(\tilde{G}^{2*}(L'(p^k))\)

For simplicity we put

\[ p_t = 1 + p + \cdots + p^{l-1}, \quad l \geq 0, \quad \text{i.e., } p_t = (p^l - 1)/(p - 1). \]

First we observe certain divisibilities of coefficients of \([p^k]_G(T)\) by powers of \(p\).

**Proposition 2.1.** Put \([p^k]_G(T) = \sum_{i=1}^{\infty} a_{i-1} T^i\), then

1. \( a_i = 0 \) if \( i \equiv 0 \mod p - 1 \),
2. \( a_0 = p^k \),
3. \( p^{k-1} \mid a_{p^l-1}, p^{k-l+1} \mid a_{p^l-1}, \) for \( 1 \leq l \leq k \),
4. \( p^{k-l+1} \mid a_i \) for \( p^l - 1 < i < p^{l+1} - 1, 1 \leq l \leq k - 1 \),
5. \( p^{k-l} \mid a_i \) for \( p^l - 1 \leq i < p^{l+1} - 1, 1 \leq l \leq k - 1 \).

Proof. (1) is trivial by the sparseness of \( G^*(pt) \), and (2) is well-known.

Let \( \log_G(T) \) be the logarithm of \( F_G \) (see [3]), we have

\[ \log_G(T) = T + (1/p) u_t T^p + (1/p^2) u_t p^2 T^{p^p} + \cdots + (1/p^l) u_t p^l T^{p^l} + \cdots, \]  

and

\[ \log_G([p^k]_G(T)) = p^k \cdot \log_G(T) \]

where \( \circ \) means the composition of formal power series.

We prove (3) and (4) by induction on \( i \). If \( i = p^l - 1 \), by substituting (2.2) into (2.3), we get

\[ a_{p-1} T^p + (1/p) u_t (a_0 T)^p = p^{k-1} u_t T^p \mod T^{p+1}. \]

As \( a_0 = p^k, p^{k-1} \mid a_{p-1} \) but \( p^k \nmid a_{p-1} \).

Next assume that the proposition holds for any \( i \) such that \( p^l - 1 \leq i < p^l - 1 < p^k - 1 \), and also assume that \( p^l - 1 \leq r < p^{l+1} - 1 \leq p^k - 1 \). Substituting (2.2) into (2.3), we obtain

\[ a_r T^{p^l} + (1/p) u_t (\sum_{j=1}^{r} a_{j-1} T^j)^p + \cdots + (1/p^l) u_t p^l (\sum_{j=1}^{r} a_{j-1} T^j) p^l \equiv p^{k-l} u_t p^l T^{p^l} \mod T^{p^l + 2}. \]

By the assumption of induction, we see
\[ p^{k-l+1} | a_{j-1} \text{ for } 1 \leq j \leq r \text{ and } j-1 \neq p^l-1, \text{ and } p^{k-l} | a_{p'-1}. \]

The coefficient of \( T^{r+1} \) of \( (\sum_{j=1}^{r} a_{j-1} T^j)^{p^s} \) is the sum of monomials of \( a_{j-1} \), differing from \( (a_{p'-1})^{p^s} \).

Therefore, \( (\sum_{j=1}^{r} a_{j-1} T^j)^{p^s} \) for \( s \leq l \) is divisible by

\[ p^{k-l+1+(p^l-1)(k-1)} \geq p^{k-l+1+2} \]

Therefore, if \( r = p^l - 1, p^{k-l} | a_r, p^{k-l+1} / a_r \)

and if \( r > p^l - 1, p^{k-l+1} | a_r. \)

The case \( r = p^k - 1 \) is also easily proven by the same argument with a little care to degrees.

Finally (5) follows from (3) and (4).

**q.e.d.**

**Proposition 2.2.** In \( \bar{G}^2_n(L^n(p^k)) \), order \( e^n = p^k. \)

**Proof.** By Proposition 2.1, (2), we see

\[ p^k e^n = [p^k]_{c(e)} \cdot e^{n-1} \text{ mod } e^{n+1}; \text{ i.e.,} \]

\[ p^k e^n = 0 \text{ in } \bar{G}^2_n(L^n(p^k)). \]

On the other hand, assume that \( p^l e^n = 0 \) for \( l < k. \)

Then, there exists an element \( x(e) = \sum_{i=1}^{n} x_i e^i \) of \( \bar{G}^2_n(L^n(p^k)), x_i \in G^{2^{k-2}}(pt), \) such that

\[ [p^k]_{c(e)} \cdot x(e) \equiv p^l e^n \text{ mod } e^{n+1}. \]

Comparing the coefficients of both sides, we see that

\[ x_1 = x_2 = \cdots = x_{n-2} = 0 \text{ and } p^k x_{n-1} = p^l, x_{n-1} \in G^2(pt) \cong Z(p). \]

This is a contradiction.

Next, we calculate the order of the group \( \bar{G}^2_n(L^n(p^k)). \)

**Proposition 2.3.** \( |\bar{G}^2_\beta(L^n(p^k))| = p^{k(1+[(n-\beta)/(p-1)])} \) for \( 1 \leq \beta \leq p-1. \)

**Proof.** The proof is by induction on \( n. \) In the case \( n = 1, \)

The proof is straightforward by Lemma 1.1.

Next, assume that the equality holds for \( n-1. \)

By Lemma 1.2 and Proposition 2.2 we get that if \( n - \beta \equiv 0 \mod p-1 \)

\[ |\bar{G}^2_\beta(L^n(p^k))| = p^{k(1+[(n-\beta)/(p-1)])} = p^{k(1+[(n-\beta)/(p-1)])}, \]

and if \( n - \beta \equiv 0 \mod p-1, \)

\[ |\bar{G}^2_\beta(L^n(p^k))| = p^{k(1+[(n-\beta)/(p-1)])} \cdot p^k = p^{k(1+[(n-\beta)/(p-1)])}. \]
3. The order of $e^i$ in $\tilde{G}^{2i}(L^n(p^k))$

We assume $k \geq 2$ from now until last section, and $n \geq p$ in this section and the next section.

**Proposition 3.1.** In $\tilde{G}^{2i}(L^n(p^k))$,

1. order $e^i = p^{k+\lceil (n-i)/(p-1) \rceil}$ for $1 \leq i \leq n$.
2. $bp^{k-1+\lceil (n-i)/(p-1) \rceil} e^i = p^{k+\lceil (n-i)/(p-1) \rceil} e^{(p-1)}$

for $p \leq i \leq n$, where $b$ is a unit element of $G^{2(p-1)}(pt)$.

Proof. The proof is by induction on descending order of $i$. For $i=n$, (1) follows from Proposition 2.2.

Next, multiplying $[p^k]_G(e)$ by $e^{n-p}$, we have that

$$p^k e^{n-(p-1)} + a_{p-1} e^n = 0.$$ 

If we put

$$b = -(a_{p-1})|p^{k-1},$$

then $b$ is a unit element by Proposition 2.1, (3), and we obtain (2).

Next, assume (1) and (2) holds for $i$ such that $p < j + 1 \leq i \leq n$. We prove (2) for $i=j$. Multiply $[p^k]_G(e)$ by $p^{\lceil (n-j)/(p-1) \rceil} e^{j-p}$, then we obtain,

$$0 = p^{k+\lceil (n-j)/(p-1) \rceil} e^{j-(p-1)} - b' p^{k-1+\lceil (n-j)/(p-1) \rceil} e^j + \sum_{i=2}^{\infty} a_{(i-1)} p^{\lceil (n-j)/(p-1) \rceil} e^{j+(i-1)(p-1)}$$

by Proposition 2.1, (1).

If $t > p_k$, then, $k + [(n-j+(t-1)(p-1))/(p-1)] \leq [(n-j)/(p-1)]$ because $p_k \geq k$.

Next, let $p_l \leq t < p_{l+1}$ for $l \geq 2$, then, by Proposition 2.1, (5),

$$p^{k-1+\lceil (n-j)/(p-1) \rceil} |a_{t(p-1)}| p^{\lceil (n-j)/(p-1) \rceil}$$

and

$$k + [(n-j+(t-1)(p-1))/(p-1)] \leq k - l + [(n-j)/(p-1)]$$

because $t-1 \geq p_l - 1 > l-1$ for $l \geq 2$.

Finally, if $p_{l+1} \leq t < p_{l+2}$,

$$p^{k+\lceil (n-j)/(p-1) \rceil} |a_{t(p-1)}| p^{\lceil (n-j)/(p-1) \rceil}$$

by Proposition 2.1, (4), but

$$k + [(n-j+(t-1)(p-1))/(p-1)] \leq k + [(n-j)/(p-1)]$$

Therefore, by the assumption of induction, we obtain from (3.1) that
\[ p^{k+\lceil (m-j)/(p-1) \rceil} e^{j-(p-1)} = b' p^{k-1+\lceil (m-j)/(p-1) \rceil} e^j. \]

Then, we have (2) for \( i=j \).

Next, apply (2) for \( i=j+(p-1) \),

\[ p^{k-1+\lceil (m-j)/(p-1) \rceil} e^j = b p^{k-1+\lceil (m-j-(p-1))/(p-1) \rceil} e^{j+(p-1)} . \]

Therefore, if we assume that \( \text{ord } e^j < p^{k+\lceil (m-j)/(p-1) \rceil} \), then,

\( \text{ord } e^{j+(p-1)} < p^{k+\lceil (m-(j+(p-1)))/(p-1) \rceil} \)

This contradicts to (1) for \( i=j+(p-1) \). Finally we have

\[ p^{k+\lceil (m-j)/(p-1) \rceil} e^j = b p^{k+\lceil (m-(j+(p-1)))/(p-1) \rceil} e^{j+(p-1)} = 0 \]

So we obtain (1) for \( i=j \). q.e.d.

4. The construction of the generators \( \{w_i\} \)

Put \( z_i(k) = \lceil k/(p^i+1-p^i) \rceil \) for \( l \geq 0 \). As is easily seen

\begin{align*}
(4.1) & \quad z_i(a)+z_i(b)+1 \geq z_i(a+b) \geq z_i(a) + z_i(b) . \\
(4.2) & \quad z_i(a-d)+z_i(b+d)+1 \geq z_i(a)+z_i(b) .
\end{align*}

Lemma 4.1. Fix an integer \( l \) such that \( 1 \leq l < k \).

For each integer \( t \) such that \( t \geq 1 \) and \( p^t+t(p-1) \leq n \), we have,

\[ p^{k-t+z_i(n-(p^t+t(p-1)))+z_i(t(p-1)-1)} = p^{k-t+z_i(n-(p^t+t(p-1)))+z_i(t(p-1))} . \]

Proof. (1) In case \( p^t+t(p-1) \geq p^k \):

\[ z_i(n-p^t) \geq z_i(n-(p^t+t(p-1)))+z_i(t(p-1)) \]

by (4.1). On the other hand, \( t(p-1) \geq p^k-p^t \) by the assumption, thus

\[ z_i(t(p-1)) \geq k-l . \]

Therefore, we get

\[ z_i(n-p^t) \geq k-l+z_i(n-(p^t+t(p-1))) . \]

(2) In case \( p^t+t(p-1) < p^k \): Fix an integer \( m \) such that

\[ p^m \leq p^t+t(p-1) < p^{m+1}, \ (so, \ m \geq l) . \]

Then, \( p^{k-m} a_{p^t+t(p-1)-1} \), by Proposition 2.1, (5). So, we have only to see that

\[ z_i(n-p^t)+k-m \geq k-l+z_i(n-(p^t+t(p-1))) . \]

But, \( n-p^m+(p^{m+1}-p^t)(m-l) \leq n-p^t \), because \( m \geq l \). Therefore,
and we obtain that

\[ z_i(n-p^m)+m-l \leq z_i(n-p^l), \]

and we obtain that

\[ z_i(n-p^l)+k-m \geq k-l+z_i(n-p^m) \geq k-l+z_i(n-\{p^l+t(p-1)\}). \]  

q.e.d.

Lemma 4.2. Fix an integer \( l \) such that \( 1 \leq l < k \). For integers \( t, j \), such that \( t \geq 1 \), \( p^l+t(p-1) \leq n \), and \( 2 \leq j \leq p^l \), we have that

\[ p^{k-l+z_i(n-p^l)+z_i(p^l-j)+1} \mid p^{*,(p^l+t(p-1)-j)+1} p^{*,(n-p^l)} a^{p^l+t(p-1)-1}. \]

Proof. In case \( p^l+t(p-1) \geq p^k \), we have to see that

\[ z_i(p^l+t(p-1)-j) \geq z_i(p^l-j)+k-l. \]

And in case \( p^l+t(p-1) < p^k \), let \( m \) be an integer such that \( p^m \leq p^l+t(p-1) < p^{m+1} \). Therefore, we have only to see that

\[ z_i(p^l+t(p-1)-j)+z_i(n-p^l)+k-m \geq k-l+z_i(n-p^l)+z_i(p^l-j). \]

But these results is easily obtained by similar argument in Lemma 4.1. q.e.d.

Now, we prove the following important result which is a generalization of Proposition 4 of [4].

Theorem 4.3. Fix an integer \( l \) such that \( p^l \leq n \) and \( 1 \leq l \leq k-1 \). For \( i \) such that \( p^l \leq p^l+i \leq n \), in \( G^{*(L^n(p^k))} \) we have the equality

\[ p^{k-l+z_i(n-(p^l+i))} e^{p^l+i} = p^{k-l+z_i(n-(p^l+i))} \sum_{j=1}^{p^l} \lambda_{i,j} e^j \]

and \( p_i^{*(p^l+i-j)} | \lambda_{i,j} \) where \( \lambda_{i,j} \in G^{*(p^l+i-j)(pt)} \).

Proof. The proof is by induction on \( n \). For \( n=p \), we obtain that \( p^{k-1} e^p = p^{k-1} \lambda e \) and \( p | \lambda \) by Proposition 3.1, (2) for \( n=i=p \). Thus the case \( n=p \) is valid.

Next assume that the statement holds in \( G^{*(L^{n-1}(p^k))} \), i.e., for fixed \( l \) such that \( p^l \leq n-1 \), \( 1 \leq l \leq k-1 \), and for \( i \) such that \( p^l \leq p^l+i \leq n-1 \),

\[ p^{k-l+z_i(n-1-(p^l+i))} e^{p^l+i} = p^{k-l+z_i(n-1-(p^l+i))} \sum_{j=1}^{p^l} \lambda_{i,j} e^j \]

and \( p_i^{*(p^l+i-j)} | \lambda_{i,j} \) in \( G^{*(L^{n-1}(p^k))} \).

Applying the homorphism \( g: G^{*(L^{n-1}(p^k))} \rightarrow G^{*(L^n(p^k))} \) by defined \( g(x) = e \cdot x \) (which is well-defined by Lemma 1.1), we have the following lemma.

Lemma 4.4. Fix an integer \( l \) such that \( p^l \leq n-1 \). \( 1 \leq l \leq k-1 \) and for \( i' \) such that \( p^l+1 \leq p^l+i' \leq n \), then in \( G^{*(L^n(p^k))} \)

\[ p^{k-l+z_i(n-(p^l+i'))} e^{p^l+i'} \]
where we have put \( i' = i + 1 \), \( j' = j + 1 \), and \( \lambda_{i', j'} = \lambda_{i'-1, j'-1} \).

This Lemma stands close to the statement of Theorem 4.3., but the definitive obstruction to go ahead is the existence of \( e^{pl} \)-term at the right hand side. So we prepare the next Lemma which is a special case of Theorem 4.3.

**Lemma 4.5.** Fix an integer \( l \) such that \( p^l \leq n \), \( 1 \leq l \leq k - 1 \). Then, in 
\[ G^\otimes(L^n(p^k)) \]
\[ p^{k-1+l, (n-p^l)} e^{p^l} = p^{k-1+l, (n-p^l)} \sum_{j=1}^{k-1} \lambda_{0, j} e^{j}, \]
and
\[ p^{k-1+l, (n-p^l)} | \lambda_{0, j}. \]

**Proof.** Multiplying \([p^k]_o(e)\) by \( p^{l, (n-p^l)} \), we have that
\[ \sum_{j=1}^{k-1} p^{l, (n-p^l)} a_{j(p-1)} e^{(p-j)(p-1)} + p^{l, (n-p^l)} a_{j(p-1)} e^{p^l} \]
\[ + \sum_{j=1}^{k-1} p^{l, (n-p^l)} a_{j(p-1)} e^{p^l} = 0. \]

The above second term is equal to \( p^{k-1+l, (n-p^l)} b e^{p^l} \),
by Proposition 2.1, (3), where \( b \) is a unit.
Next,
\[ p^{k-1+l, (n-p^l)} | p^{l, (n-p^l)} a_{s(p-1)}, \text{ for } 0 \leq s \leq p_i - 1 \]
by Proposition 2.1, (5).

On the other hand, we obtain trivially that
\[ z_i(p^l - (s(p-1)+1)) = 0. \]
Therefore, the above first term is given in a form of
\[ p^{k-1+l, (n-p^l)} \sum_{j=1}^{k-1} \lambda_{j} e^{j}, \]
\[ p^{k-1+l, (n-p^l)} | \lambda_{j}. \]

Thus we obtain the equation
\[ \text{(4.3)} \]
\[ bp^{k-1+l, (n-p^l)} e^{p^l} = p^{k-1+l, (n-p^l)} \sum_{j=1}^{k-1} \lambda_{j} e^{j} \]
\[ + \sum_{j=1}^{k-1} p^{l, (n-p^l)} a_{j(p-1)} e^{p^l} + c_{p^l} \]
and \( p^{k-1+l, (n-p^l)} | \lambda_{j}. \)

Now we calculate the last term. In case \( n = p^l \)
\[ \sum_{t=1}^{\infty} p^t e^{t(p-1)+i} = 0, \text{ because } p^t + t(p-1) > n. \]

In case \( p' \leq n-1 \), we may apply Lemmas 4.1 and 4.4 to obtain

\[
\sum_{t=1}^{\infty} p^t e^{t(p-1)+i} = \sum_{t=1}^{\infty} p^t e^{t(p-1)+i} = p^t e^{t(p-1)+i} + \sum_{j=1}^{p'-1} p^{j+1} e^{j+1} |\lambda_j, \text{ and } p^t |\lambda_{p'}. \]

Then, applying Lemma 4.2 to each term of above sum and summing over \( j \), we have

\[
\sum_{t=1}^{\infty} p^t e^{t(p-1)+i} = p^{k-l-z_l(n-(p+i))} e^{k-l-z_l(n-p)+i} \sum_{j=1}^{k-l-z_l(n-p)} |\lambda_j, \text{ and } p^t |\lambda_{p'}. \]

Finally, in either case, we know by (4.3), that,

\[
(b+p\lambda)p^{k-l-z_l(n-(p+i))} e^{k-l-z_l(n-p)+i} = p^{k-l-z_l(n-(p+i))} \sum_{j=1}^{k-l-z_l(n-p)} |\lambda_j, \text{ and } p^t |\lambda_{p'}. \]

Then, we obtain Lemma 4.5.

Next return to the proof of Theorem 4.3.

Then the coefficient of \( e^{p'-i} \)-term of the right hand side of the equation of Lemma 4.4 is divisible by \( p^{k-l-z_l(n-p)} \), and we may apply Lemma 4.5 to this, so that we obtain

\[
p^{k-l-z_l(n-(p'-i))} e^{k-l-z_l(n-p)+i} \sum_{j=1}^{k-l-z_l(n-p)} |\lambda_j, \text{ and } p^t |\lambda_{p'}. \]

By (4.1), the above sum can be written that,

\[
p^{k-l-z_l(n-(p'-i))} \sum_{j=1}^{k-l-z_l(n-p)} |\lambda_j, \text{ and } p^t |\lambda_{p'}. \]

Therefore we obtain Theorem 4.3 in case \( p' \leq n-1 \), and \( p' + 1 \leq p' + i \leq n \).

The statement of the theorem in another case has ever been proven by Lemma 4.5.

**Corollary 4.6.** For \( i \) such that \( p \leq i \leq \min(n, p^k-1) \), there exist the element
\( w_i \in \tilde{G}^\beta(L^n(p^k)), 1 \leq \beta \leq p - 1, \) which has the form \( e^i + \text{lower degree terms}, \) precisely,

\[
w_i = u_i^{\beta(i)}e_i + \sum_{j=1}^{\beta} \lambda_{ij}u_i^{\beta(i)-j}e_i^{(p^k-1)} \quad \text{where } i = \alpha(i)(p-1) + \beta, 1 \leq \beta \leq p - 1.
\]

Moreover, if \( p^l \leq i < p^{l+1}, \) order \( w_i \leq p^{k-l}e_i^{(n-1)}. \)

Proof. For \( i \) such that \( p \leq i \leq \min(n, p^k - 1) \) there exists unique \( l \) such that \( 1 \leq l \leq k - 1, \) \( p^l \leq i \leq n, \) and \( p^{l-1} \leq i < p^{l+1}. \)

Fix this \( l, \) then we obtain,

\[
p^{k-l}e_i^{(n-1)}e_i = p^{k-l}e_i^{(n-1)}\sum_{j=1}^{\beta} \lambda_{ij}e_i^j, \text{ by Theorem 4.3.}
\]

Putting \( w_i = u_i^{\alpha(i)}(e_i^{(p^k-1)} - \sum_{j=1}^{\beta} \lambda_{ij}e_i^j), \) by the sparseness of \( G^*(pt), \)

\[
\lambda_j = 0 \text{ unless } i \equiv j \text{ mod } (p-1).
\]

Therefore we obtain the desired elements \( w_i. \)

5. The additive structure of \( \tilde{G}^\beta(L^n(p^k)) \)

Proposition 5.1. \( \tilde{G}^\beta(L^n(p^k)) 1 \leq \beta \leq p - 1 \) is generated by

\[
\{u_i^{\beta(i)}e_i^j, j = 0, 1, \ldots, \min((n-\beta)/(p-1)), p_k-1\}.
\]

\( (p_k \text{ is defined by } (2.1)). \)

Proof. (1) If \( n < p^k, \) as \( [(n-\beta)/(p-1)] \leq p_k - 1, \) \( \min((n-\beta)/(p-1)), p_k-1) = [(n-\beta)/(p-1)]. \)

By Lemma 1.1, if we prove that

\[
(1 + [(n-\beta)/(p-1))] (p-1) + \beta > n
\]

then we obtain the result. But this statement is easily seen.

(2) Assume the statement is true for \( n-1 \) and we prove it for \( n \geq p^k. \) (It means that \( \min((n-\beta)/(p-1)), p_k-1) = p_k-1). \)

By Lemma 1.2, we obtain,

\[
\begin{align*}
0 & \rightarrow \text{Kernel } \iota^* \rightarrow \tilde{G}^\beta(L^n(p^k)) \xrightarrow{\iota^*} \tilde{G}^\beta(L^{n-1}(p^k)) \rightarrow 0 \\
\text{Kernel } \iota^* & = \begin{cases} u_i^{\alpha(i)}e_i^j & \text{if } \beta-n = \alpha(n)(p-1) \\
0 & \text{otherwise.} \end{cases}
\end{align*}
\]

Thus, we have only to see that \( u_i^{\alpha(i)}e_i^j \) is the linear combination of \( \{u_i^{\beta(i)}e_i^j, j = 0, 1, \ldots, p_k-1\}. \)

Multiplying \( [p^k]e_i^j \) by \( u_i^{\alpha(i)}e_i^j \), \( e_i^j \), in \( \tilde{G}^\beta(L^n(p^k)), \)

\[
u_i^{\alpha(i)}e^j = \mu_j \sum_{j=0}^{\beta} \mu_{ij}u_i^{\beta(i)-j}e_i^{(p^k-1)+j}, \mu_j \in \mathbb{Z}_{p^j}, \text{ by Proposition 2.1, } (3).
\]

On the other hand, by the assumption of the induction and by
(5.1), we know,

\[(5.3) \quad u_t e^{j(p-1)\beta} = \sum_{l=0}^{p-1} \mu_l u_l e^{(p-1)\beta} + \mu u_0 e^\beta.\]

Substituting (5.3) into the right side of (5.2), we obtain,

\[(1+\rho)u_t e^{\alpha(j)} = \sum_{l=0}^{p-1} \mu_l u_l e^{(p-1)\beta}.\]

q.e.d.

**Corollary 5.2.** Fix integers \(n\) and \(\beta\) such that \(n \geq \beta\) and \(1 \leq \beta \leq p-1\). Then \(\mathcal{G}_{\beta\beta}(L^n(p^k))\) is generated by \(\{e^\beta\}\) and \(\{w_{j\beta(p-1)+\beta}\}\) where \(j=1, 2, \ldots, \min([(n-\beta)/(p-1)], p_k-1)\).

**Remark.** In case \([(n-\beta)/(p-1)] = 0\), we observe that the only generator is \(e^\beta\).

**Proof.** In case \([(n-\beta)/(p-1)] = 0\), the proof is straightforward by Lemma 1.1.

Thus we may assume that \(n \geq p\). Then we have only to see that, there is \(w_{j\beta(p-1)+\beta}\) of Corollary 4.6. for \(1 \leq j \leq \min([(n-\beta)/(p-1)], p_k-1)\). But we see easily that, for such \(j\)

\[j(p-1)+\beta \leq \min(n, p^k-1) .\]

Therefore we obtain the result.

Next we put

\[V_n = k + [(n-\beta)/(p-1)] + \sum_{j=1}^{p_k-1} \{(k-1)+z(n-j(p-1)-\beta)\}
+ \sum_{j=1}^{p_k-1} \{(k-2)+z(n-j(p-1)-\beta)\} + \cdots
+ \sum_{j=1}^{p_k-1} \{(k-l)+z(n-j(p-1)-\beta)\} + \cdots
+ \sum_{j=1}^{M(n)} \{(k-m(n))+z_{m(n)}(n-j(p-1)-\beta)\}
\]

where

\[M(n) = \min([(n-\beta)/(p-1)], p_k-1),\]

\[m(n) = \begin{cases} i(n) & \text{if } [(n-\beta)/(p-1)] \leq p_k-1, \\ k-1 & \text{if } [(n-\beta)/(p-1)] \geq p_k-1, \end{cases}\]

and \(i(n)\) is a number such that

\[p_{i(n)} \geq [(n-\beta)/(p-1)] < p_{i(n)+1}, \quad p_t = (p^t-1)/(p-1),\]

and we put \(p_0 = 0\).

We note that \(M(n) \geq M(n-1), m(n) \geq m(n-1)\). And it is convenient to put \(V_n=0\) if \(n < \beta\).
Theorem 5.3. Fix integers $n, k, \beta$ such that $n \geq 1$, $k \geq 2$, and $1 \leq \beta \leq p - 1$, then

$$\mathcal{C}^\beta(L^n(p^k)) = \langle e^\beta \rangle \oplus \sum_j \langle w_j(p-1)+\beta \rangle \quad j = 1, 2, \ldots$$

where $\langle x \rangle$ is the cyclic subgroup generated by $x$, and

order $e^\beta = p^{k + \lceil (n-\beta)/(p-1) \rceil}$,

order $w_j(p-1)+\beta = p^{k-1+l_j(n-\beta(p-1)-\beta)}$, if $p^l \leq j(p-1)+\beta < p^{l+1}$.

Proof. The order of the group of right hand side is less or equal than $p^n$ by Proposition 3.1 and Corollary 4.6. If we prove $p^n = p^{k + \lceil (n-\beta)/(p-1) \rceil}$, then observing Corollary 5.2, we get the proof of all statements of Theorem 5.3 Therefore we prove the next lemma.

Lemma 5.4. For $n \geq 1$, $k \geq 2$, $1 \leq \beta \leq p - 1$ we have

$$V_n = k(1 + [(n-\beta)/(p-1)])$$

Proof. We put $Y_n = k(1 + [(n-\beta)/(p-1)])$.

(1) In case $n \leq \beta$, the proof is easy.

(2) If $n-\beta \equiv 0 \mod p-1$, as $[(n-\beta)/(p-1)] = [(n-1-\beta)/(p-1)]$, $M(n-1) = M(n)$ and $m(n-1) = m(n)$.

Moreover $z_l(n-j(p-1)-\beta) = z_l(n-1-j(p-1)-\beta)$.

Therefore $V_n = V_{n-1} = Y_{n-1} = Y_n$.

(3) If $n-\beta = d(p-1)$, $d \geq 1$, then

$$Y_n = Y_{n-1} + k.$$

On the other hand, for $j$ such that $p_j \leq j \leq p_{j+1}-1$, there exists only one $j$ such that

$$z_l(n-j(p-1)-\beta) = z_l(n-1-j(p-1)-\beta)+1,$$

and for other $j$,

$$z_l(n-j(p-1)-\beta) = z_l(n-1-j(p-1)-\beta).$$

Therefore

$$\sum_{j=n}^{k-1} \{k-l+z_l(n-j(p-1)-\beta)\} = \sum_{j=n}^{k-1} \{k-l+z_l(n-1-j(p-1)-\beta)\} + 1.$$
Thus we have only to see that
\[ V_n - V_{n-1} = k = Y_n - Y_{n-1}. \]

If \([(n-1-\beta)/(p-1)] \geq p_k - 1, M(n-1) = M(n) = p_k - 1, m(n-1) = m(n) = k - 1.\]
Therefore \[ V_n - V_{n-1} = k. \]

If \([(n-1-\beta)/(p-1)] < p_k - 1, then, M(n-1) = [(n-1-\beta)/(p-1)], M(n) = [(n-\beta)/(p-1)] = M(n-1) + 1, m(n-1) = i(n-1), m(n) = i(n).\]
In this case
\[ \sum_{j=M(n-1)+1}^{M(n)} \{k - m(n) + z_m(n) + j(p-1) - \beta\} = k - i(n). \]
Therefore, if we put
\[ W = \sum_{j=M(n-1)+1}^{M(n)} \{k - i(n) + z_i(n) + j(p-1) - \beta\} - \sum_{j=M(n-1)+1}^{M(n)} \{k - i(n-1) + z_i(n-1) + j(p-1) - \beta\}, \]
we have only to see that \( W = i(n) - i(n-1). \)

(3,a) If \( d < p_{i(n-1)+1}, \) then \( i(n) = i(n-1). \)
For \( j \) such that \( p_{i(n-1)} \leq j \leq M(n-1), \) we have,
\begin{align*}
  n - j(p-1) - \beta &< (p_{i(n-1)+1} - j)(p-1) \leq (p_{i(n-1)+1} - p_{i(n-1)})(p-1) \\
                    &= p^{(n-1)+1} - p^{(n-1)}.
\end{align*}
Therefore,
\[ z_i(n) + j(p-1) - \beta = 0, \] and also
\[ z_i(n-1) + j(p-1) - \beta = 0. \]
Hence \[ W = 0 = i(n) - i(n-1). \]

(3,b) If \( d = p_{i(n-1)+1}, \) then \( i(n) = i(n-1) + 1. \)
As same as above,
\[ \sum_{j=M(n-1)+1}^{M(n)} z_i(n) + j(p-1) - \beta = 0. \]
But, \( n - p_{i(n-1)}(p-1) - \beta = p^{(n-1)+1} - p^{(n-1)}, \) and for \( j \) such that \( j > p_{i(n-1)}, \) we have that, \( n - j(p-1) - \beta < p^{(n-1)+1} - p^{(n-1)} \).
Therefore,
\[ \sum_{j=M(n-1)+1}^{M(n)} \{z_i(n) + j(p-1) - \beta\} = 1. \]
Consequently \( W = 1 = i(n) - i(n-1). \) Thus we have completed the proof of this lemma.

\[ \text{q.e.d.} \]
6. The additive structure of $\tilde{K}(L^n(p^k))$ and $\tilde{KO}(L^n(p^k))$

By Theorem 5.3 we obtain,

**Theorem 6.1.** $\tilde{K}^q(L^n(p^k)) \cong \bigoplus_{i=1}^{M} \langle w_i' \rangle$ where $M = \min(n, p^k - 1)$, and order $w_i' = p^{-1+\frac{q}{2}(n-1)}$, if $p^i \leq t < p^{i+1}$.

Proof. If $1 \leq t \leq p - 1$, put $w_i' = e^t$, and if $p \leq t$, put $w_i' = w_i$.

As is well-known, for a finite CW-complex $X$ and for any odd prime $p$,

$$\tilde{KO}(X) \otimes \mathbb{Z}_p \cong \bigoplus_{i=1}^{(q-1)/2} G^i(X).$$

Observing this facts and Proposition 2.11 of [7], we obtain the next theorem.

**Theorem 6.2.** For any odd prime, $p$, and for any integer $k \geq 2$,

$$\tilde{KO}(L^n(p^k)) \cong \begin{cases} \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \langle w_{2i}' \rangle & \text{for } n \equiv 0 \mod 4 \\ \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \langle w_{2i}' \rangle \oplus \mathbb{Z}_2 & \text{for } n \equiv 0 \mod 4 \end{cases}$$

where $M = \min(n, p^k - 1)$ and order $w_{2i}' = p^{-1+\frac{q}{2}(n-1)}$, if $p^i \leq 2t < p^{i+1}$.

**Bibliography**