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# ASYMPTOTIC ARC-SINE LAWS FOR FINITE-DIMENSIONAL INTERACTING DIFFUSIONS

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**Abstract**

We consider finite-dimensional interacting diffusions which are defined by adding a linear drift term to independent one dimensional diffusions. For these processes we prove that the distribution of the occupation time at the first quadrant converges to a generalized arc-sine law.

## 1. Introduction

Let $S$ be a finite set, and let $A = \{A_{ij}\}_{i \neq j \in S}$ be a matrix with non-negative elements. Let us consider the following stochastic differential equation (SDE):

$$dX_i(t) = \alpha(X_i(t))\, dB_i(t) + \sum_{j \in S} A_{ij}(X_j(t) - X_i(t))\, dt, \quad (i \in S), \tag{1.1}$$

where $\{B_i(t)\}_{i \in S}$ is an independent system of one-dimensional standard Brownian motions.

Assume that $\alpha\colon \mathbb{R} \to \mathbb{R}_+$ is a Borel measurable function satisfying the following conditions:
[A-1] For some positive constant $C > 0$,

$$\alpha(x) \leq C(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}. \tag{1.2}$$

[A-2] For each compact set $K$, there exists a positive constant $c_K$ such that $\alpha(x) \geq c_K$ $(x \in K)$,
one can see by standard arguments to use the Girsanov theorem that for any initial distribution on $\mathbb{R}^S$, the SDE (1.1) has a unique weak solution, which defines a diffusion process $(X(t), P_x)$ on $\mathbb{R}^S$. We call the diffusion process *a finite-dimensional interacting diffusion*.

In this paper we are concerned with limiting distribution as $t \to \infty$ of the occupation time of $X(t)$ at the first quadrant $\mathbb{R}_+^S = [0, \infty)^S$ of $\mathbb{R}^S$

$$\frac{1}{t} \int_0^t I_{\mathbb{R}_+^S}(X(s))\, ds. \tag{1.3}$$

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In non-interacting case where $A = \{A_{ij}\}$ is absent, each coordinate process is a diffusion process $(X(t), P_x)$ on $\mathbb{R}$ governed by the following SDE:

$$dX(t) = \alpha(X(t))\,dB(t). \tag{1.4}$$

For the one-dimensional diffusion process $(X(t), P_x)$ governed by (1.4) Watanabe [5] proved that the distribution of

$$\frac{1}{t}\int_0^t I_{\mathbb{R}_+}(X(s))\,ds$$

converges to a non-degenerate distribution as $t \to \infty$ if and only if

$$m_+(x) = \int_0^x \alpha(u)^{-2}\,du, \quad m_-(x) = \int_{-x}^0 \alpha(u)^{-2}\,du \quad (x \geq 0)$$

satisfy the following condition; for some $0 < p < 1$

$$m_\pm(x) = x^{1/p-1}K_\pm(x) \tag{1.5}$$

with slowly varying functions $K_+(x)$ and $K_-(x)$ at $x = \infty$ and

$$\lim_{x\nearrow\infty} \frac{K_+(x)}{K_-(x)} = b \in (0, \infty). \tag{1.6}$$

Then it holds that

$$\frac{1}{t}\int_0^t I_{\mathbb{R}_+}(X(s))\,ds \stackrel{(d)}{\Longrightarrow} Y_{p,q} \quad (t \to \infty),$$

where $q$ is given by

$$q = \frac{b^p}{1+b^p} \in (0, \infty),$$

and $\stackrel{(d)}{\Longrightarrow}$ denotes convergence in distribution and $Y_{p,q}$ is a $[0,1]$-valued random variable with the Stieltjes transform given by

$$E\left[\frac{1}{u+Y_{p,q}}\right] = \frac{q(u+1)^{p-1} + (1-q)u^{p-1}}{q(u+1)^p + (1-q)u^p}, \quad u > 0.$$

The family $Y_{p,q}$, $0 < p \leq 1$, $0 < q < 1$, was introduced by Lamperti [2], of which distribution is called *a generalized arc-sine law*. In particular, the distribution of $Y_{1/2,1/2}$ is the arc-sine law, of which density function is given by

$$\frac{1}{\pi\sqrt{x(1-x)}}.$$

For general $0 < p < 1$ and $0 < q < 1$, $Y_{p,q}$ has the density $f_{p,q}(x)$ on $[0, 1]$;

$$f_{p,q}(x) = \frac{\sin p\pi}{\pi} \frac{q(1-q)x^{p-1}(1-x)^{p-1}}{q^2(1-x)^{2p} + (1-q)^2x^{2p} + 2q(1-q)x^p(1-x)^p \cos p\pi}.$$

For the finite-dimensional interacting diffusion $(X(t), P_x)$ governed by (1.1) we investigate the limiting distribution of (1.3) under the following condition:
[B-1] $\alpha(x)$ is regularly varying both at $x \to \infty$ and $x \to -\infty$ with the common exponent $-\infty < \gamma < 1/2$, and

$$\lim_{x\to\infty} \frac{\alpha(-x)}{\alpha(x)} = c \in (0, \infty).$$

[B-2] An $S \times S$-matrix $A = \{A_{ij}\}_{i,j\in S}$, of which diagonal element is defined by

$$A_{ii} = -\sum_{j\in S,\, j\neq i} A_{ij} \quad (i \in S),$$

is irreducible.
We note that by [B-2]

$$Q_t = \exp tA$$

defines a transition matrix of an irreducible Markov process on $S$, so that there exists a probability vector $m = \{m_i\}_{i\in S}$ with $m_i > 0$ such that for some $\delta > 0$

$$|Q_t(i, j) - m_j| \leq e^{-\delta t} \quad (i, j \in S). \tag{1.7}$$

The main result of this paper is the following.

**Theorem 1.1.** *Assume the conditions* [B-1] *and* [B-2]. *Then*

$$\frac{1}{t}\int_0^t \delta_{X(s)}\, ds \overset{(d)}{\Longrightarrow} Y_{p,q}\delta_{+\underline{\infty}} + (1 - Y_{p,q})\delta_{-\underline{\infty}} \quad (t \to \infty), \tag{1.8}$$

*where* $+\underline{\infty} = \{x_i \equiv +\infty\}$, $-\underline{\infty} = \{x_i \equiv -\infty\}$, $\delta_{X(s)}$, $\delta_{+\underline{\infty}}$ *and* $\delta_{-\underline{\infty}}$ *stand for the one point mass at* $X(s)$, $+\underline{\infty}$ *and* $-\underline{\infty}$ *respectively, and* $\overset{(d)}{\Longrightarrow}$ *denotes the weak convergence as* $\mathcal{P}([-\infty, \infty]^S)$*-valued random variables, and here* $p$, $q$ *are given by*

$$p = \frac{1}{2(1-\gamma)}, \quad q = \frac{c^{2p}}{1 + c^{2p}}.$$

From Theorem 1.1 it follows immediately that

**Corollary 1.2.** *Assume the same assumptions as in* Theorem 1.1. *Then*

$$\frac{1}{t}\int_0^t I_{\mathbb{R}_+^S}(X(s))\,ds \overset{(d)}{\Longrightarrow} Y_{p,q} \quad (t \to \infty). \tag{1.9}$$

The result of Theorem 1.1 can be interpreted as follows. Since $S$ is a finite set, the effect of the interaction $A = \{A_{ij}\}$ is so strong that all component processes diverge to $\infty$ or $-\infty$ as $t \to \infty$ simultaneously. Hence the phenomena would be quite similar to the one-dimensional case. Nevertheless the one-dimensional analysis as in Watanabe [5] cannot be applied, so, in the next section, we will investigate a scaling limit for the finite-dimensional interacting diffusion $(X(t), P_x)$ on $\mathbb{R}^S$.

## 2. A scaling limit of $X(t)$

By the condition [B-1] $\alpha(x)$ has the following form;

$$\alpha(x) = |x|^\gamma L(x) \quad (|x| > 0),$$

where $L(x)$ is a slowly varying function both at $\infty$ and $-\infty$ satisfying that

$$\lim_{x\to\infty}\frac{L(-x)}{L(x)} = c \in (0,\infty).$$

Let

$$p = \frac{1}{2(1-\gamma)} \quad \text{and} \quad \theta_\lambda = \lambda L(\lambda^p)^{-2} \quad (\lambda > 0).$$

We introduce a rescaled process $(X^\lambda(t), B^\lambda(t))$ by

$$X_i^\lambda(t) = \lambda^{-p} X_i^\lambda(\theta_\lambda t), \quad B_i^\lambda(t) = \theta_\lambda^{-1/2} B_i(\theta_\lambda t), \quad i \in S.$$

Note that $\{B_i^\lambda(t)\}_{i\in S}$ are independent Brownians motion and the rescaled process $(X^\lambda(t), B^\lambda(t))$ satisfies the following SDE;

$$dX_i^\lambda(t) = \alpha_\lambda\bigl(X_i^\lambda(t)\bigr)\,dB_i^\lambda(t) + \theta_\lambda \sum_{j\in S} A_{ij}\bigl(X_j^\lambda(t) - X_i^\lambda(t)\bigr)\,dt,$$

where

$$\alpha_\lambda(x) = \lambda^{-p}\theta_\lambda^{1/2}\alpha(\lambda^p x).$$

Moreover it holds that

$$\lim_{\lambda\to\infty}\alpha_\lambda(x) = \begin{cases} x^\gamma & (0 < x), \\ c|x|^\gamma & (0 > x). \end{cases}$$

In order to describe the limiting processes of the $(X^\lambda(t))$ we introduce a class of skew Bessel processes on natural scale.

Let

$$\overline{\alpha}(x) = \begin{cases} \|m\|_2 x^\gamma & (0 \le x), \\ \|m\|_2 c|x|^\gamma & (0 > x). \end{cases}$$

where $\|m\|_2 = \sqrt{\sum_{i\in S} m_i^2}$, $\overline{\alpha}(0) = \infty$ if $\gamma < 0$, and $\overline{\alpha}(0) = \|m\|_2$ if $\gamma = 0$.

Let us consider the following one-dimensional SDE:

$$\begin{aligned} dZ(t) &= \overline{\alpha}(Z(t))\, dB(t), \\ Z(0) &= x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

If $-\infty < \gamma \le 0$, the SDE (2.1) has a law unique solution, however, if $0 < \gamma < 1/2$, the law uniqueness for (2.1) fails. In this case, if we add the non-sticky condition to (2.1), i.e.

$$\int_0^t I_{\{0\}}(Z(s))\, ds = 0 \quad (\forall t > 0), \quad P\text{-a.s.,} \tag{2.2}$$

the law uniqueness holds. In fact, the solution can be constructed from a Brownian motion through the time change method. Thus we have a diffusion process $(Z(t), P_x)$ on $\mathbb{R}$, which is called *a skew Bessel process on natural scale*.

**Theorem 2.1.** *Assume the conditions* [B-1] *and* [B-2], *and* $X(0)$ *is a* $\mathbb{R}^S$*-valued random variable independent of* $B(t) = \{B_i(t)\}_{i\in S}$. *Then*

$$(X^\lambda(t) = \{X_i^\lambda(t)\}_{i\in S}) \overset{(\mathcal{L})}{\Longrightarrow} (X^\infty(t) = \{X_i^\infty(t)\}_{i\in S}) \quad (\lambda \to \infty), \tag{2.3}$$

*where* $\overset{(\mathcal{L})}{\Longrightarrow}$ *stands for the weak convergence of the probability laws on the path space induced by* $\{X^\lambda(t)\}$. *Moreover, all component processes of* $\{X_i^\infty(t)\}_{i\in S}$ *coincide with each other and the common process is equivalent to a skew Bessel diffusion on natural scale* $(Z(t))$ *governed by* (2.1) *with* $Z(0) = 0$ *being imposed the non-sticky condition whenever* $0 < \gamma < 1/2$;

$$\int_0^t I_{\{0\}}(Z(s))\, ds = 0 \quad (t > 0), \quad P\text{-a.s.} \tag{2.4}$$

From Theorem 2.1 it follows the following

**Corollary 2.2.** *Under the same assumption of in* Theorem 2.1,

$$X(t) \overset{(d)}{\Longrightarrow} q\delta_{+\underline{\infty}} + (1-q)\delta_{-\underline{\infty}} \quad (t \to \infty). \tag{2.5}$$

**Proof of Theorem 1.1.** Theorem 1.1 follows immediately from Theorem 2.1. In fact, since

$$\int_0^t I(Z(s) = 0)\, ds = 0,$$

by Theorem 2.1 we can see that for every bounded continuous function $f$ on $[-\infty, \infty]$ it holds that

$$\begin{aligned}\frac{1}{\theta_\lambda}\int_0^{\theta_\lambda} f(X(s))\, ds &= \int_0^1 f(\lambda^p X^\lambda(s))\, ds \\ &\overset{(d)}{\Longrightarrow} f(+\underline{\infty})\int_0^1 I(Z(s) > 0)\, ds + f(-\underline{\infty})\int_0^1 I(Z(s) < 0)\, ds \\ &= Y_{p,q} f(+\underline{\infty}) + (1 - Y_{p,q}) f(-\underline{\infty}),\end{aligned}$$

because of

$$\int_0^1 I(Z(s) > 0)\, ds \overset{(d)}{=} Y_{p,q}.$$

For the last relation see Watanabe [5]. □

## 3. Proof of Theorem 2.1

To avoid complication of arguments we prove Theorem 2.1 under the following condition [B-3] instead of [B-1], since the proof is essentially the same even under the condition [B-1].
[B-3] Let $-\infty < \gamma < 1/2$, and for some $\alpha_+ > 0$ and $\alpha_- > 0$

$$\lim_{x\to\infty} \frac{\alpha(x)}{x^\gamma} = \alpha_+, \quad \lim_{x\to-\infty} \frac{\alpha(x)}{|x|^\gamma} = \alpha_-. \tag{3.1}$$

In what follows we assume the conditions [A-1], [A-2], [B-2] and [B-3]. Let

$$p = \frac{1}{2(1-\gamma)}, \tag{3.2}$$

and for $\lambda > 0$ we set

$$\alpha_\lambda(x) = \lambda^{-p+1/2}\alpha(\lambda^p x), \tag{3.3}$$

and

$$\alpha_\infty(x) = \begin{cases} \alpha_+ x^\gamma & (0 \le x), \\ \alpha_- |x|^\gamma & (x < 0). \end{cases} \tag{3.4}$$

where

$$\alpha_\infty(0) = \begin{cases} \infty & (\gamma < 0), \\ \alpha_+ & (\gamma = 0). \end{cases}$$

Moreover we set

$$\overline{\alpha}(x) = \|m\|_2 \alpha_\infty(x), \tag{3.5}$$

where $\{m_i\}_{i\in S}$ is a probability vector in (1.7), and $\|m\|_2 = \sqrt{\sum_{i\in S} m_i^2}$.

For the diffusion process $(X(t), P_x)$ governed by (1.1) we introduce a rescaled process $X^\lambda(t)$ $(\lambda > 0)$ by

$$X_i^\lambda(t) = \lambda^{-p} X_i(\lambda t) \quad (i \in S),$$

which satisfies the following SDE:

$$dX_i^\lambda(t) = \alpha_\lambda\big(X_i^\lambda(t)\big)\, dB_i^\lambda(t) + \lambda \sum_{j\in S} A_{ij}\big(X_j^\lambda(t) - X_i^\lambda(t)\big)\, dt. \tag{3.6}$$

For the proof of Theorem 2.1 we may assume that the initial condition $X(0)$ is non-random, i.e.

$$X(0) = \{x_i\}_{i\in S} \in \mathbb{R}^S.$$

We first prepare several moment estimates of the rescaled process $X_i^\lambda(t)$.

**Lemma 3.1.** *Let $-\infty < \gamma < 1/2$. For $a \geq 2$ there exists a constant $C = C(a, p) > 0$ such that*

$$\sum_{i\in S} m_i E\big[\big|X_i^\lambda(t)\big|^a\big] \leq C\left(\lambda^{-pn} + \lambda^{-pn} \sum_{i\in S} m_i |x_i|^a + t^{pa}\right) \quad (t \geq 0,\ \lambda > 0). \tag{3.7}$$

Proof. Using the Itô formula and taking expectations, we have

$$\begin{aligned} \frac{d}{dt} \sum_{i\in S} m_i E[|X_i(t)|^a] &= a \sum_{i\in S}\sum_{j\in S} m_i A_{ij} E\big[|X_i(t)|^{a-1} \operatorname{sgn}(X_i(t))(X_j(t) - X_i(t))\big] \\ &\quad + \frac{1}{2} a(a-1) \sum_{i\in S} m_i E\big[|X_i(t)|^{a-2} \alpha^2(X_i(t))\big]. \end{aligned} \tag{3.8}$$

Note that

$$\sum_{i\in S}\sum_{j\in S} m_i A_{ij} |x_i|^{a-1} \operatorname{sgn}(x_i)(x_j - x_i) \leq 0, \tag{3.9}$$

because, using $\sum_{j\in S} A_{ij} = 0$, $\sum_{i\in S} m_i A_{ij} = 0$ and a simple inequality

$$t^{a-1}s \le \frac{a-1}{a}t^a + \frac{1}{a}s^a \quad (t > 0,\ s > 0),$$

we see

$$\begin{aligned}
&\sum_{i\in S}\sum_{j\in S} m_i A_{ij}|x_i|^{a-1}\operatorname{sgn}(x_i)(x_j - x_i)\\
&\le \sum_{i\in S}\sum_{j\in S} m_i A_{ij}\big(|x_i|^{a-1}|x_j| - |x_i|^a\big)\\
&\le \frac{1}{a}\sum_{i\in S}\sum_{j\in S} m_i A_{ij}(|x_j|^a - |x_i|^a)\\
&= 0.
\end{aligned}$$

Note that by the conditions [A-1], [A-2] and [B-3] there exists constants $C_1 > 0$ and $C_2 > 0$ satisfying

$$C_1(1+|x|)^\gamma \le \alpha(x) \le C_2(1+|x|)^\gamma, \quad (x \in \mathbb{R}), \tag{3.10}$$

so that there exists a constant $C_3$ such that

$$\sum_{i\in S} m_i |x_i|^{a-2}\alpha^2(x_i) \le C_3\left(1 + \sum_{i\in S} m_i |x_i|^a\right)^{1-1/ap}, \tag{3.11}$$

Hence, by (3.9), (3.10) and (3.11) $F(t) = \sum_{i\in S} m_i E[|X_i(t)|^{2a}]$ satisfies

$$\frac{d}{dt}F(t) \le C_3(1+F(t))^{1-1/ap}.$$

Thus we obtain, for some $C_4 > 0$,

$$\sum_{i\in S} m_i E[|X_i(t)|^a] \le C_4\left(1 + \sum_{i\in S} m_i |x_i|^a + t^{ap}\right). \tag{3.12}$$

(3.7) follows immediately from (3.12). □

Let

$$U_{i,j}(t) = X_i(t) - X_j(t) \quad (i \ne j \in S),$$

and for $\lambda > 0$ let

$$U^\lambda_{i,j}(t) = X^\lambda_i(t) - X^\lambda_j(t) \quad (i \ne j \in S).$$

**Lemma 3.2.** (i) *For any $a \geq 2$ there exists a constant $C > 0$ such that*

$$E\left[\left|U^{\lambda}_{i,j}(t)\right|^{a}\right] \leq \begin{cases} C\lambda^{-a/2}(1+t^{ap\gamma}) & (0 \leq \gamma < 1/2), \\ C\lambda^{-ap} & (-\infty < \gamma < 0). \end{cases} \tag{3.13}$$

(ii) *For each $T > 0$ there exists a constant $C_T > 0$ such that for every $\lambda \geq 1$*

$$E\left[\left|U^{\lambda}_{i,j}(t) - U^{\lambda}_{i,j}(s)\right|^{6}\right] \leq C_T\lambda^{-1}|t-s|^2 \quad (0 \leq s \leq t \leq T). \tag{3.14}$$

Proof. First, note that $X(t)$ satisfies

$$X_i(t) = \sum_{k\in S}\int_s^t Q_{t-u}(i,k)\alpha(X_k(u))\,dB_k(u) + \sum_{k\in S} Q_{t-s}(i,j)X_k(s) \quad (i \in S), \tag{3.15}$$

so that

$$\begin{aligned} U_{i,j}(t) - U_{i,j}(s) = {} & \sum_{k\in S}\int_s^t (Q_{t-u}(i,k) - Q_{t-u}(j,k))\alpha(X_k(u))\,dB_k(u) \\ & + \sum_{k\neq i} Q_{t-s}(i,k)U_{i,k}(s) + \sum_{k\neq j} Q_{t-s}(j,k)U_{j,k}(s). \end{aligned}$$

Using this and the Burkholder inequality, we have

$$\begin{aligned} & E[|U_{i,j}(t) - U_{i,j}(s)|^a] \\ & \leq C_1\sum_{k\in S} E\left[\left(\int_s^t (Q_{t-u}(i,k) - Q_{t-u}(j,k))^2\alpha^2(X_k(u))\,du\right)^{a/2}\right] \\ & \quad + C_1 E\left[\left(\sum_{k\in S} Q_{t-s}(i,k)U_{i,k}(s)\right)^{a}\right] \\ & \quad + C_1 E\left[\left(\sum_{k\in S} Q_{t-s}(j,k)U_{j,k}(s)\right)^{a}\right]. \end{aligned} \tag{3.16}$$

When $0 \leq \gamma < 1/2$, using this with $s = 0$, (1.7) and Lemma 3.1 we have a constant $C_2 > 0$ satisfying that

$$E[|U_{i,j}(t)|^a] \leq C_2(1+t^{a\gamma p}),$$

which yields (3.13). Using (1.7), (3.10), and Lemma 3.1, we see that the first term of the r.h.s. of (3.16) with $a = 6$ is dominated by

$$C_3\sum_{k\in S} E\left[\left(\int_s^t e^{-2\delta(t-u)}\alpha^2(X_k(u))\,du\right)^3\right] \leq C_4((t-s)\wedge 1)^3(1+t^{6p\gamma}).$$

Furthermore, by (3.13) the last two terms of (3.16) are dominated by

$$C_5((t-s)\wedge 1)^6(1+t^{6p\gamma}),$$

thus we have

$$E[|U_{i,j}(t)-U_{i,j}(s)|^6] \le C_6((t-s)\wedge 1)^3(1+t^{6p\gamma}). \tag{3.17}$$

From this it follows that

$$\begin{aligned} E\left[\left|U^{\lambda}_{i,j}(t)-U^{\lambda}_{i,j}(s)\right|^6\right] &\le C_7(\lambda(t-s)\wedge 1)^3\lambda^{-6p}(1+(\lambda t)^{6p\gamma}) \\ &\le C_T\lambda^{-1}(t-s)^2, \end{aligned}$$

which concludes (3.14). In the case $-\infty<\gamma<0$, since $\alpha(x)$ is bounded, $E[U_{i,j}(t)^6]$ is also bounded in $t\ge 0$. Hence it is easy to obtain (3.14). □

**Lemma 3.3.** *Suppose that $X(t)$ is a continuous martingale with $X(0)=0$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ with filtration $\{\mathcal{F}_t\}$, of which quadratic variation process satisfies*

$$\langle X\rangle(t)=\int_0^t \overline{\alpha}^2(X(s))\,ds,$$

*where $\overline{\alpha}(x)$ is of* (3.4). *If* $0<\gamma<1$, *we further assume the non-sticky condition*;

$$\int_0^t I_{\{0\}}(X(s))\,ds=0 \quad (t>0) \quad P\text{-a.s.}$$

*Then the probability law on the path space $W=C([0,\infty),\mathbb{R})$ induced by $(X(t))$ coincides with that of the skew Bessel process on natural scale $Z(t)$ starting at* 0 *governed by the SDE* (2.1) *with* (2.2).

Proof. Proof is to verify that $X(t)$ satisfies the SDE (2.1) for some Brownian motion $\overline{B}(t)$ using the time-change method, that is quite standard, so we omit it. □

**Proof of Theorem 2.1 in case 0 $\gamma<1/2$.** In this case the proof is rather standard, that is, first to verify the tightness of the probability laws $P^{\lambda}$ on $W$ induced by $\{X^{\lambda}(t)\}$ and next to identify the limit of $\{P^{\lambda}\}$ as $\lambda\to\infty$.

For the stationary probability vector $\{m_i\}$ of $Q_t$ we set

$$Y^{\lambda}(t)=\sum_{i\in S} m_i X^{\lambda}_i(t),$$

which satisfies the following equation;

$$dY^{\lambda}(t)=\sum_{i\in S} m_i\alpha_{\lambda}\left(X^{\lambda}_i(t)\right)dB^{\lambda}_i(t). \tag{3.18}$$

**Lemma 3.4.** *Let* $0 \le \gamma < 1/2$. *For each* $T > 0$ *there exists constant* $C_T > 0$ *such that for every* $\lambda > 0$,

$$E\left[\left|Y^\lambda(t) - Y^\lambda(s)\right|^4\right] \le C_T(t-s)^2, \quad (0 \le s, t \le T). \tag{3.19}$$

Proof. It is immediate from (3.18) and Lemma 3.1. □

**Lemma 3.5.** *Let* $0 \le \gamma < 1/2$.

$$\lim_{\varepsilon\to 0+} \limsup_{\lambda\to\infty} \int_0^t P\left(\left|X_i^\lambda(s)\right| \le \varepsilon\right) ds = 0. \quad (i \in S,\ t > 0). \tag{3.20}$$

Proof. For each $\varepsilon > 0$ define a function $\varphi_\varepsilon$ by

$$\varphi_\varepsilon''(x) = |x|^{-2\gamma} I(|x| \le \varepsilon),$$
$$\varphi_\varepsilon(x) = \int_0^{|x|} \int_0^y \varphi_\varepsilon''(u)\, du\, dy.$$

Applying Itô formula we obtain

$$E[\varphi_\varepsilon(Y^\lambda(t))] = \varphi_\varepsilon\left(\lambda^{-p} \sum_{i\in S} m_i x_i\right) + \sum_{i\in S} \int_0^t m_i^2 E\left[\alpha_\lambda^2(X_i^\lambda(s))\varphi_\varepsilon''(Y^\lambda(s))\, ds\right]. \tag{3.21}$$

Since

$$|\varphi_\varepsilon(x)| \le \frac{\varepsilon^{1-2\gamma}}{(1-2\gamma)}|x|,$$

using Lemma 3.1 we have

$$\lim_{\varepsilon\to 0+} \limsup_{\lambda\to\infty} \sum_{i\in S} \int_0^t m_j^2 E\left[\alpha_\lambda^2(X_i^\lambda(s))\varphi_\varepsilon''(Y^\lambda(s))\, ds\right] = 0. \tag{3.22}$$

Note that for some $C_1 > 0$

$$\alpha_\lambda^2(x) \ge C_1(\lambda^{-p} + |x|)^\gamma \quad (x \in \mathbb{R},\ \lambda > 0),$$

and for $y = \sum_i m_i x_i$

$$\begin{aligned}\sum_{i\in S} m_i \alpha_\lambda^2(x_i)\varphi_\varepsilon''(y) &\ge C_1 \sum_{i\in S} m_i(\lambda^{-p} + |x_i|)^{2\gamma}|y|^{-2\gamma} I(|y| < \varepsilon)\\ &\ge C_2(\lambda^{-p} + |y|)^{2\gamma}|y|^{-2\gamma} I(|y| < \varepsilon)\\ &\ge C_2 I(|y| < \varepsilon).\end{aligned}$$

Hence from this and (3.22) it follows that

$$\lim_{\varepsilon\to 0+}\limsup_{\lambda\to\infty}\int_0^t P(|Y^\lambda(s)|\le\varepsilon)\,du = 0. \tag{3.23}$$

Here we notice that

$$P(|X_i^\lambda(s)|\le\varepsilon)\le P(|Y^\lambda(s)|\le 2\varepsilon)+P\left(\left|X_i^\lambda(s)-Y^\lambda(s)\right|>\varepsilon\right),$$

and that for each $\varepsilon>0$ the second term vanishes as $\lambda\to\infty$. Hence (3.20) follows from (3.23). $\square$

Now we proceed to the proof of Theorem 2.1 in the case $0\le\gamma<1/2$. Let $P^\lambda$ be the probability measure on $W=C([0,\infty),\mathbb{R}^S)$ induced by $X^\lambda(t)$. We use the notation $E^{P^\lambda}$ for the expectation by $P^\lambda$. Then by Lemma 3.4 and Lemma 3.2 $\{P^\lambda\}$ is tight. Suppose that for some $\{\lambda_n\}$ tending to $\infty$, $P^{\lambda_n}$ converges weakly to $P^\infty$. Let

$$\overline{w}(t)=\sum_{i\in S}m_i w_i(t).$$

Since by (3.18) $\overline{w}(t)$ is a $P^\lambda$-martingale with quadratic variation process

$$\langle\overline{w}\rangle(t)=\sum_{i\in S}m_i^2\int_0^t\alpha_\lambda^2(w_i(s))\,ds\quad P^\lambda\text{-a.s.}, \tag{3.24}$$

using Lemma 3.1 we see easily that $\overline{w}(t)$ is a $P^\infty$-martingale with $\overline{w}(0)=0$. Moreover, it follows from Lemma 3.2 that

$$P^\infty(w_i(t)=w_j(t)\ (\forall t\ge 0))=1. \tag{3.25}$$

(3.24) implies that for every $0\le s<t$ and a $\mathcal{F}_s$-measurable and bounded continuous function $\Phi_s(w)$ on $W$

$$E^{P^\lambda}\left[\left(\overline{w}^2(t)-\overline{w}^2(s)-\sum_{i\in S}m_i^2\int_s^t\alpha_\lambda^2(w_i(u))\,du\right)\Phi_s(w)\right]=0. \tag{3.26}$$

We claim that

$$\lim_{\lambda\to\infty}E^{P^\lambda}\left[\left(\int_s^t\alpha_\lambda^2(w_i(u))\,du\right)\Phi_s(w)\right]=E^{P^\infty}\left[\left(\int_s^t\overline{\alpha}^2(w_i(u))\,du\right)\Phi_s(w)\right]. \tag{3.27}$$

For $\varepsilon>0$ let $\varphi_\varepsilon$ be a smooth function on $\mathbb{R}$ satisfying

$$I_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(x)\le\varphi_\varepsilon(x)\le I_{\mathbb{R}\setminus[-\varepsilon/2,\varepsilon/2]}(x).$$

Since $\alpha_\lambda(x)$ converges to $\alpha_\infty(x)$ as $\lambda \to \infty$ compact uniformly in $\mathbb{R} \setminus \{0\}$ and

$$\alpha_\lambda(x) \le C_3(1 + |x|^\gamma) \quad (x \in \mathbb{R}),$$

using Lemma 3.1 we see that for every $\varepsilon > 0$

$$\begin{aligned}
&\lim_{\lambda\to\infty} E^{P^\lambda}\left[\left(\int_s^t \alpha_\lambda^2(w_i(u)\varphi_\varepsilon(w_i(u))\,du\right)\Phi_s(w)\right] \\
&= E^{P^\infty}\left[\left(\int_s^t \alpha_\infty^2(w_i(u)\varphi_\varepsilon(w_i(u))\,du\right)\Phi_s(w)\right].
\end{aligned} \tag{3.28}$$

On the other hand by Lemma 3.5

$$\begin{aligned}
&\lim_{\varepsilon\to+0}\limsup_{\lambda\to\infty} E^{P^\lambda}\left[\left(\int_s^t \alpha_\lambda^2(w_i(u))(1-\varphi_\varepsilon)(w_i(u))\,du\right)\Phi_s(w)\right] \\
&\le C_4 \lim_{\varepsilon\to+0}\limsup_{\lambda\to\infty}\int_0^t P(|X_i^\lambda(u)| \le \varepsilon)\,du = 0.
\end{aligned}$$

(3.27) follows from this and (3.28). Thus, $\overline{w}(t)$ is a $P^\infty$-martingale with quadratic variation provess

$$\langle w\rangle(t) = \sum_{i\in S} m_i^2 \int_0^t \alpha_\infty^2(w_i(u))\,du = \int_0^t \overline{\alpha}^2(\overline{w}(u))\,du.$$

Therefore by Lemma 3.3 $P^\infty$ coincides with the probability law of the skew Bessel process on natural scale, which completes the proof of Theorem 2.1 in the case $0 \le \gamma < 1/2$. □

**Proof of Theorem 2.1 in case $< \gamma < 0$.** In this case it seems hard to obtain the moment estimate for $Y^\lambda(t)$ as in Lemma 3.4 due to difficulty of negative power moment estimates, so we consider a spatial transformation by an asymptotic scale function $S(x)$;

$$S(x) = \begin{cases} x^{2(1-\gamma)} & (\gamma \ge 0), \\ |x|^{2(1-\gamma)} & (\gamma < 0). \end{cases}$$

**Lemma 3.6.** *Let $-\infty < \gamma < 0$. For each $T > 0$ there exists a constant $C_T > 0$ such that for every $\lambda \ge 1$*

$$E\left[\left|S(Y^\lambda(t)) - S(Y^\lambda(s))\right|^4\right] \le C_T|t-s|^2, \quad (0 \le s, t \le T). \tag{3.29}$$

Proof. Recall that $Y^\lambda(t)$ satisfies

$$dY^\lambda(t) = \alpha_\lambda(Y^\lambda(t))\,dV^\lambda(t), \tag{3.30}$$

where $V^\lambda(t)$ is a continuous martingale with quadratic variation process

$$\langle V^\lambda\rangle(t) = \sum_{i\in S} m_i^2 \int_0^t \frac{\alpha_\lambda^2(X_i^\lambda(u))}{\alpha_\lambda^2(Y^\lambda(u))}\, du. \tag{3.31}$$

Applying Itô formula to $S(x)$ together with Burkholder's inequality we see that

$$\begin{aligned}
&E[|S(Y^\lambda(t) - S(Y^\lambda(s)))|^4]\\
&\le C_1 E\left[\left(\int_s^t |S'(Y^\lambda(u))|^2\alpha_\lambda^2(Y^\lambda(u))\, d\langle V^\lambda\rangle(u)\right)^2\right]\\
&\quad + C_1 E\left[\left(\int_s^t S''(Y^\lambda(u))\alpha_\lambda^2(Y(\lambda u))\, d\langle V^\lambda\rangle(u)\right)^4\right]\\
&\le C_1 \int_s^t E[(S'\alpha_\lambda)^4(Y^\lambda(u))]\, du \int_s^t E[(\langle V^\lambda\rangle'(u))^4]\, du\\
&\quad + C_1\|S''\alpha_\lambda^2\|_\infty^4 (t-s)^3 \int_s^t E[(\langle V^\lambda\rangle'(u))^4]\, du,
\end{aligned} \tag{3.32}$$

where

$$\langle V^\lambda\rangle'(u) = \sum_{i\in S} m_i^2 \frac{\alpha_\lambda^2(X_i^\lambda(u))}{\alpha_\lambda^2(Y^\lambda(u))},$$

and we notice that $S''\alpha_\lambda^2(x)$ is bounded in $x\in\mathbb{R}$ and $\lambda\ge 1$. Note that

$$C_2\lambda^{2p-1}(1+\lambda^p|x|)^{2\gamma} \le \alpha_\lambda^2(x) \le C_3\lambda^{2p-1}(1+\lambda^p|x|)^{2\gamma},$$

then

$$\frac{\alpha_\lambda^2(x)}{\alpha_\lambda^2(y)} \le C_4\left(\frac{1+\lambda^p|y|}{1+\lambda^p|x|}\right)^{2|\gamma|} \le C_4(1+\lambda^p|x-y|)^{2|\gamma|}.$$

Hence,

$$E[(\langle V^\lambda\rangle'(u))^4] \le C_5\left(1+\lambda^{8p|\gamma|}\sum_{j\ne k} E\left[\left|U_{j,k}^\lambda(u)\right|^{8|\gamma|}\right]\right),$$

which is bounded in $u\ge 0$ by Lemma 3.2. Accordingly, it follows from this and (3.32) that

$$E[(S(Y^\lambda(t) - S(Y^\lambda(s)))^4] \le C_7(|t-s|^2 + |t-s|^4),$$

which completes the proof of Lemma 3.6. $\square$

**Lemma 3.7.** *Let* $-\infty < \gamma < 0$. *Then*

$$\lim_{\varepsilon\to 0}\limsup_{\lambda\to\infty}\int_0^t E\left[\alpha_\lambda^2(X_i^\lambda(s))I(|X_i^\lambda(s)| \le \varepsilon)\right]ds = 0 \quad (i \in S). \tag{3.33}$$

Proof. In the proof of Lemma 3.5, replacing $\varphi_\varepsilon(x)$ by $\varphi_\varepsilon''(x) = I_{[-\varepsilon,\varepsilon]}(x)$ we have

$$\lim_{\varepsilon\to 0+}\limsup_{\lambda\to\infty}\sum_{i\in S}\int_0^t m_i^2 E\left[\alpha_\lambda^2(X_i^\lambda(s))I_{[-\varepsilon,\varepsilon]}(Y^\lambda(s))\,ds\right] = 0. \tag{3.34}$$

Noting that

$$I_{[-\varepsilon,\varepsilon]}(X_i^\lambda(s)) \le I_{[-2\varepsilon,2\varepsilon]}(Y^\lambda(s)) + \sum_{j\in S} I_{[-\varepsilon,\varepsilon]}\left(X_j^\lambda(s) - X_i^\lambda(s)\right),$$

and by Lemma 3.2 we can see

$$\begin{aligned}&\lim_{\lambda\to\infty}\int_0^t E\left[\alpha_\lambda^2(X_i(s))I_{[-\varepsilon,\varepsilon]}\left(X_j^\lambda(s) - X_i^\lambda(s)\right)\right]ds\\ &\le \lim_{\lambda\to\infty}\lambda^{-2p+1}\|\alpha\|_\infty^2\int_0^t P\left(\left|U_{i,j}^\lambda(s)\right| > \varepsilon\right) = 0.\end{aligned}$$

Thus (3.33) follows from this and (3.34). □

Now we are in position to complete the proof of Theorem 2.1 in the case $-\infty < \gamma < 0$, but one can proceed the proof as in the case of $0 \le \gamma < 1/2$, so we shall only sketch the proof. By virture of Lemma 3.2 and Lemma 3.6, we may assume that $P^{\lambda_n}$ converges weakly to $P^\infty$ as $n \to \infty$ for some $\lambda_n \nearrow \infty$. Then, $\overline{w}(t)$ is $P^\infty$-martingale with $\overline{w}(0) = 0$ and

$$w_i(t) = w_j(t) = \overline{w}(t) \quad P^\infty\text{-a.s.} \quad (i, j \in S),$$

in the same way as $0 \le \gamma < 1/2$. Using the Lemma 3.7 instead of Lemma 3.5, we also have (3.27) which implies

$$E^{P^\infty}\left[\left(\overline{w}^2(t) - \overline{w}^2(s) - \int_s^t \overline{\alpha}^2(\overline{w}(u))\,du\right)\Phi_s(w)\right] = 0,$$

and then by Lemma 3.3, the probability law $(\overline{w}(t), P^\infty)$ coincides with that of the desired skew Bessel process on natural scale. Therefore Theorem 2.1 has been proved completely. □

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