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GIRSANOV-TYPE TRANSFORMATIONS OF LOCAL DIRICHLET FORMS: AN ANALYTIC APPROACH

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0. Introduction

In recent years, several authors have investigated Girsanov transformations of symmetric diffusion processes, that preserve the strong Markov property as well as the symmetrizability, and the corresponding transformations of the associated analytic objects, in particular the Dirichlet forms, cf. [12], [15], [16], [25], [3], [26], [22], [11] and [13].

As an illustration, consider the following simple situation:

Let $\boldsymbol{M}=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in\boldsymbol{R}^d})$ be the canonical Brownian motion on $\boldsymbol{R}^d$, $d\in\boldsymbol{N}$, and

$$\mathscr{E}(u,v)=\int \nabla u\cdot\nabla v\,dx,\qquad u,v\in H^{1,2}(\boldsymbol{R}^d),$$

the associated Dirichlet form. Let φ be a function in $H^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^d)$ satisfying $\varphi>0$ a.e. and $\int|\nabla\varphi|^2dx<\infty$. Fix a quasi-continuous modification $\tilde{\varphi}$ of φ, and let

$$M_t^{[\log\varphi]}:=\int_0^t\frac{\nabla\tilde{\varphi}}{\tilde{\varphi}}(X_s)dX_s,$$

defined up to the stopping time $\tau:=\sup_{n\in\boldsymbol{N}}\inf\{t>0;\ \int_0^t\frac{|\nabla\tilde{\varphi}|^2}{\tilde{\varphi}^2}(X_s)ds\geq n\}$.

By a result of P.A. Meyer and W.H. Zheng (cf.[15]), the process $\boldsymbol{M}^\varphi=(\Omega,\mathscr{F}^\varphi,(X_t)_{t\geq 0},(P_z^\varphi)_{z\in\boldsymbol{R}^d})$ obtained by transforming $\boldsymbol{M}$ with the multiplicative functional

$$(0.1)\qquad L_t^{[\varphi]}=\exp\left(M_t^{[\log\varphi]}-\frac{1}{2}\langle M^{[\log\varphi]}\rangle_t\right)\cdot I_{\{t<\tau\}}$$

is a φ^2dx-symmetric conservative diffusion;($\boldsymbol{M}^\varphi$ is defined as the unique Markov process with life-time ζ satisfying $\int_{\{t<\zeta\}}f(X_t)dP_z^\varphi=\int f(X_t)L_t^{[\varphi]}dP_z$ for any $z\in E$, $t\geq 0$, and $f:E\to\boldsymbol{R}$ measurable). Let $(\mathscr{E}^\varphi,D(\mathscr{E}^\varphi))$ be the Dirichlet form of $\boldsymbol{M}^\varphi$. Recently,

M. Takeda showed (s.[26, Th. 4.1]) that:

(0.2) $(\mathscr{E}^{\varphi}, D(\mathscr{E}^{\varphi}))$ is the closure of the bilinear form

$$\mathscr{E}^{\varphi}(u,v)=\int \nabla u \cdot \nabla v \ \varphi^2 dx, \qquad u,v \in C_0^{\infty}(\boldsymbol{R}^d).$$

There have been several approaches to carry over these facts to more general symmetrizable diffusions $\boldsymbol{M}$ living on more general (even non-locally-compact) state spaces. The transformation can be done as above, if, roughly speaking, φ is locally in the Dirichlet space of $\boldsymbol{M}$, and $M^{[\log \varphi]}$ is the local martingale additive functional part in Fukushima's decomposition of the additive functional $\log \tilde{\varphi}(X_t) - \log \tilde{\varphi}(X_0)$ (cf. [13, Sect. 6.3] for details). While the construction of the transformed process, as well as the proof of its symmetrizability have already been carried out in [25](s.also [13, Th. 6.3.3]) in a very general situation, the identification of the associated Dirichlet form remained open in general. Partial results for concrete situations or under additional assumptions on φ have been obtained in [4], [22, §10], [26] and [11]. In particular, during the time of preparation of this paper, I received a preprint of P.J. Fitzsimmons, in which he gives a representation of the transformed form generalizing (0.2) under very weak assumptions on the initial diffusion, cf.[11, Th. (5.2)]. However, Fitzsimmons supposes abstract conditions on φ (s.[11, (5.3), (5.4)]), that are not always easy to check in concrete situations.

In this paper, an alternative, more analytic approach is used, provided the Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ of the initial diffusion admits a square field operator (cf. Section 1 below for the definition). We proceed as follows: First, we construct a transformed Dirichlet form $(\mathscr{E}^{\varphi}, D(\mathscr{E}^{\varphi}))$ in a purely analytic way and prove analytically (s. Theorems 1.1 and 1.4 below) that this form is quasi-regular and has the strong local property (cf. Section 1). By general results on Dirichlet forms (cf.[14, IV. 6.7 and V.1.11]), this implies that it is associated with a diffusion process $\boldsymbol{M}^{\varphi}$. Moreover, we give a necessary and sufficient condition for some general "test-function-subspace" of $D(\mathscr{E})$ (e.g. the smooth functions with compact support, if the state space is a subset of $\boldsymbol{R}^n$ or a Riemannian manifold, or the bounded smooth cylinder functions in infinite dimensional situations) to be dense in $D(\mathscr{E}^{\varphi})$ (s.Theorem 1.2). As a consequence of this density property, a representation of $(\mathscr{E}^{\varphi}, D(\mathscr{E}^{\varphi}))$ similar to (0.2) holds. Finally, we prove that the diffusion associated with $(\mathscr{E}^{\varphi}, D(\mathscr{E}^{\varphi}))$ is a generalized Girsanov-transform of the initial diffusion, and thus obtain an absolute continuity result (s. Theorem 1.5). The two main ingredients in the proof of this result are Proposition 4.1 (treating the case of φ bounded from above and below), and the analytic considerations done before (which enable us to pass to more general functions φ). Proposition 4.1 is also contained in [11] (cf. Thm. 4.9), but we give an alternative proof using a method of S. Song [22].

The main advantages of this approach are:

- It always leads to a precise description of the Dirichlet form of the transformed process (cf. Theorems 1.1 and 1.5).
- It produces a necessary and sufficient, and rather concrete, condition on φ for a representation of this Dirichlet form similar to (0.2) to hold (cf. Theorem 1.2 and Corollary 1.3).
- It includes the case of non-locally-compact state spaces.
- A special localization procedure (in terms of balls w.r.t. a metric generated by functions in the Dirichlet space) enables us to prove all the results for not necessarily finite symmetrizing measures, and functions φ which are only locally (in our sense) contained in the Dirichlet space. For regular Dirichlet forms on complete locally compact separable metric spaces, our local Dirichlet space coincides with the usual one, but it is also useful in infinite dimensional situations (cf. the examples in Section 1).

In Section 5, we apply the results to some concrete situations, and obtain sufficient conditions for local absolute continuity of diffusions w.r.t. Brownian motions on Riemannian manifolds, reflected Brownian motions on smooth Euclidean domains, and diffusions associated with gradient Dirichlet forms on Banach spaces. In particular, we obtain generalizations of the main result (Theorem 1.5) from [4](s. Section 5c)below), as well as the result by M.Takeda mentioned above (s. Remark (i) in Section 5 a)).

The results of this article can also be used to study the Markovian uniqueness of the generator of $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$. This will be done in detail in my PhD-thesis ([9]).

This work profited a lot from techniques developed in [4], [19] and [22].

1. Preparations and Main Results

Let E be a metrizable Lusin space (i.e. homeomorphic to a Borel subset of a Polish space), $\mathscr{B}(E)$ its Borel-σ-algebra, and m a σ-finite positive measure on $(E,\mathscr{B}(E))$.

Recall that a symmetric closed bilinear form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(E,m)$ is called a **Dirichlet form**, if $u^+\wedge 1\in D(\mathscr{E})$ and $\mathscr{E}(u^+\wedge 1,u^+\wedge 1)\leq\mathscr{E}(u,u)$ whenever $u\in D(\mathscr{E})$. $D(\mathscr{E})$ is a Hilbert space w.r.t. the inner product $\mathscr{E}_1(u,v):=\mathscr{E}(u,v)+\int uv\,dm$. The space of all measurable functions representing a class in $D(\mathscr{E})$ is denoted $D(\mathscr{E})$ as well. The **generator** of $(\mathscr{E},D(\mathscr{E}))$ is the unique negative-definite self-adjoint operator $(L, D(L))$ such that $D(\mathscr{E})=D(\sqrt{-L})$ and $\mathscr{E}(u,v)=\int\sqrt{-L}u\sqrt{-L}v\,dm$ for all $u,v\in D(\mathscr{E})$.

We fix a Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ with generator $(L,D(L))$, which satisfies the following assumptions:

(D1) **Strong local property:** $\mathscr{E}((f-f(0))\circ u, (g-g(0))\circ u)=0$ for all $u\in D(\mathscr{E})$ and $f,g\in C_0^{\infty}(\boldsymbol{R})$ with $\operatorname{supp}f\cap\operatorname{supp}g=\emptyset$,

(D2) **Existence of a square field operator:** There is a positive symmetric continuous bilinear operator $\Gamma: D(\mathscr{E})\times D(\mathscr{E})\to L^1(E,m)$, such that

$$\mathcal{E}(uv,u)-\tfrac{1}{2}\mathcal{E}(v,u^2)=\int v\Gamma(u,u)dm \text{ for all bounded } u,v\in D(\mathcal{E}).$$

We also write $\mathcal{E}(u)$ and $\Gamma(u)$ instead of $\mathcal{E}(u,u)$ resp. $\Gamma(u,u)$. Note that it is enough to check (D1) and (D2) with $D(\mathcal{E})$ replaced by a dense subspace of bounded functions (s.[7, I, 4.1.3 and 5.1.5]). The relation of (D1) to other forms of the strong local property is discussed in [7, Sect. I.5 and Notes]. An important consequence of (D1) is the following **"energy image density property"**: For a function $u\in D(\mathcal{E})$ the law of u under the measure $\Gamma(u)\cdot m$ is absolutely continuous. In particular, $\Gamma(u)$ (and thus $\Gamma(u,v)$, $v\in D(\mathcal{E})$) vanishes m-a.e. on $\{u=0\}$, cf. [7, I, 7.1.1].

For an increasing sequence $(F_k)_{k\in N}$ of Borel subsets of E we set

$$D_0(\mathcal{E},(F_k)):=\{u\in D(\mathcal{E});\ u=0\ m\text{-a.e. on } E\backslash F_k \text{ for some } k\in N\}.$$

Recall that $(F_k)_{k\in N}$ is called an **$\mathcal{E}$-nest**, if each F_k is closed and $D_0(\mathcal{E},(F_k))$ is dense in $D(\mathcal{E})$. A set $N\subset E$ is **$\mathcal{E}$-exceptional**, if it is contained in $\bigcap_{k\in N}(E\backslash F_k)$ for some $\mathcal{E}$-nest $(F_k)_{k\in N}$, and a property of points in E holds **$\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.)**, if it holds up to an $\mathcal{E}$-exceptional set. Finally, a function $f: E\to \boldsymbol{R}$ is called **$\mathcal{E}$-quasi-continuous** if there is an $\mathcal{E}$-nest $(F_k)_{k\in N}$ such that the restriction of f to F_k is continuous for all $k\in N$.

We are going to introduce a special kind of local Dirichlet spaces. We first need some preparations:

Let $\mathcal{C}$ be a set of $\mathcal{E}$-quasi-continuous functions in $D(\mathcal{E})$, such that

$$\Gamma(\xi)\leq 1 \qquad \forall \xi\in\mathcal{C}. \tag{1.1}$$

We assume that $\mathcal{C}$ is symmetric (i.e. $\mathcal{C}=-\mathcal{C}$), and that there is an $\mathcal{E}$-nest $(F_k)_{k\in N}$ such that all the restrictions of functions in $\mathcal{C}$ to sets $F_k, k\in N$, are continuous. This is e.g. the case if $\mathcal{C}$ is a subset of a countably generated linear space of quasicontinuous functions (cf.[14, Prop. I.3.3]) or consists of continuous functions. We define a pseudo-metric $\rho: E\times E\to[0,\infty]$ by

$$\rho(x,y):=\sup_{\xi\in\mathcal{C}}(\xi(x)-\xi(y)). \tag{1.2}$$

Fix $p\in E$ and let $\rho_p:=\rho(\cdot,p)$. We **assume**

$$(E,\rho) \text{ is separable} \qquad \text{and} \tag{1.3}$$

$$\rho_p<\infty \ m\text{-a.e.}\,. \tag{1.4}$$

Note that (1.3) holds whenever the ρ-topology is weaker than the original topology on E. For $k\in N$ let

$$E_k:=\{z\in E;\ \rho_p(z)\leq k\} \quad \text{and} \quad e_k(z):=(k-\rho_p(z))^+\wedge 1\ (z\in E). \tag{1.5}$$

Modifying similar considerations of M. Biroli / U. Mosco ([6]) and K.T. Sturm ([23]), we can show that the functions e_k are nice cut-off functions in the following sense:

(1.6) For all $k \in \boldsymbol{N}$, $e_k(z)=0 \forall z \in E \backslash E_k$, and $e_k(z)=1 \forall z \in E_{k-1}$

(1.7) $e_k \nearrow 1$ m-a.e.

(1.8) $u \cdot e_k \in D(\mathscr{E})$ and $\Gamma(u \cdot e_k) \leq 2 \cdot (u^2 + \Gamma(u))$ whenever $u \in D(\mathscr{E})$ and $k \in \boldsymbol{N}$.

(1.6) is obvious, (1.7) follows from (1.4), and the proof of (1.8) is given in the appendix (Proposition A1).

It is a consequence of (1.8) that $(E_k \cap F_k)_{k \in \boldsymbol{N}}$ is an $\mathscr{E}$-nest (s. Appendix, Corollary A2). In particular, $D_0(\mathscr{E},(E_k))$ is dense in $D(\mathscr{E})$. Let

$$D^{1,\infty}(\mathscr{E}) := \{u \in D(\mathscr{E});\ u, \Gamma(u) \in L^\infty(E,m)\} \quad \text{and}$$

$$D_0^{1,\infty}(\mathscr{E},(E_k)) := D_0(\mathscr{E},(E_k)) \cap D^{1,\infty}(\mathscr{E}).$$

In addition to (D1) and (D2) we assume:

(D3) $D(L) \cap D_0^{1,\infty}(\mathscr{E},(E_k))$ is dense in $D(\mathscr{E})$.

We will show in Section 5 below how to check (D3) in concrete situations.

Now we define the **local Dirichlet space** $D_{\text{loc}}(\mathscr{E},(E_k))$ as the set of all measurable functions $u: E \to \boldsymbol{R}$ satisfying

(1.9) $\forall k \in \boldsymbol{N} \exists u_k \in D(\mathscr{E})$: $u = u_k$ m-a.e. on E_k.

EXAMPLES

1) We may always choose $\mathscr{C} = \{0\}$. In this case $\rho = 0$, thus $e_k = 1$ and $E_k = E$ for all $k \in \boldsymbol{N}$, hence $D_0(\mathscr{E},(E_k)) = D_{\text{loc}}(\mathscr{E},(E_k)) = D(\mathscr{E})$.

2) Assume that E is a locally compact separable metric space and $(\mathscr{E}, D(\mathscr{E}))$ is regular (i.e. $C_0(E) \cap D(\mathscr{E})$ is dense both in $D(\mathscr{E})$ w.r.t. the $\mathscr{E}_1$-norm and in $C_0(E)$ w.r.t. the uniform norm). Let $\mathscr{C} := \{\xi \in C_0(E) \cap D(\mathscr{E});\ \Gamma(\xi) \leq 1\}$. The associated pseudo-metric ρ is called the **intrinsic pseudo-metric** of $(\mathscr{E}, D(\mathscr{E}))$ (s. [6], [24]). Suppose that ρ is a **metric** which gives back the original topology on E, and that the metric space (E,ρ) is complete. Then E_k is relatively compact for any $k \in \boldsymbol{N}$, cf. [24, Th. 2]. On the other hand, any compact subset of E is contained in some E_k. Thus $D_0(\mathscr{E},(E_k))$ consists of the elements of $D(\mathscr{E})$ with compact support, and $D_{\text{loc}}(\mathscr{E},(E_k))$ coincides with the local Dirichlet space as it is usually defined in the locally compact case (i.e. $u \in D_{\text{loc}}(\mathscr{E},(E_k))$ iff for any relatively compact open subset G of E there is a function $u' \in D(\mathscr{E})$ such that $u = u'$ m-a.e. on G).

3) For the choice of $\mathscr{C}$ and the definition of non-trivial local Dirichlet spaces in certain infinite dimensional situations cf. Example c), Choice B, in Section 5 below.

Because of the energy image density property, we can assign to any $u,v \in D_{\text{loc}}(\mathscr{E},(E_k))$ a (unique) m-class $\Gamma(u,v)$ such that $\Gamma(u,v) = \Gamma(u',v')$ m-a.e. on E_k, whenever

$k \in \boldsymbol{N}$ and $u', v' \in D(\mathscr{E})$ such that $u = u'$ and $v = v'$ m-a.e. on E_k (s.[7, I, 7.1.4]).

We fix a function $\varphi \in D_{\mathrm{loc}}(\mathscr{E}, (E_k))$, $\varphi > 0$ m-a.e. Note that the m- and $\varphi^2 m$-classes of functions $E \to \boldsymbol{R}$ coincide. We define a symmetric bilinear form $(\mathscr{E}^\varphi, \mathscr{D}^\varphi)$ on $L^2(E, \varphi^2 m)$ by

$$\mathscr{D}^\varphi := \left\{ u \in D(\mathscr{E});\ \int (\Gamma(u) + u^2) \varphi^2 dm < \infty \right\}, \tag{1.10}$$

$$\mathscr{E}^\varphi(u, v) := \int \Gamma(u, v) \varphi^2 dm.$$

In particular, we have $\mathscr{D}^\varphi = D(\mathscr{E})$ whenever φ is bounded. In Section 2 below we prove:

Theorem 1.1. *$(\mathscr{E}^\varphi, \mathscr{D}^\varphi)$ is densely defined and closable on $L^2(E, \varphi^2 \cdot m)$. The closure $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ is a Dirichlet form with the strong local property.*

To state the next result, let $\mathscr{A}$ be a linear space consisting of functions in $D_0^{1,\infty}(\mathscr{E}, (E_k))$. Suppose that $\mathscr{A}$ is dense in $D(\mathscr{E})$ and satisfies

(1.11) $\mathscr{A}$ is closed under composition with Lipschitz-continuous functions $f \in C_b^\infty(\boldsymbol{R}^d)$, $d \in \boldsymbol{N}$, vanishing at the origin.

Here, $C_b^\infty(\boldsymbol{R}^d)$ is the set of all infinitely often differentiable functions such that all partial derivatives are bounded. Clearly, $\mathscr{A}$ is contained in $\mathscr{D}^\varphi$. $\mathscr{A}$ should be thought of as a space of "test-functions" — typical examples are the smooth functions with compact support on a subset of $\boldsymbol{R}^n$ or a manifold and the bounded smooth cylinder functions on a topological vector space, see also Remark 5.4. We denote the closure of $\mathscr{A}$ w.r.t. the $\mathscr{E}_1^\varphi$-norm as $\bar{\mathscr{A}}$, and fix functions $f_n \in C^1(\boldsymbol{R})$ such that $\chi_{[-n,n]} \leq f_n \leq \chi_{[-n-2,n+2]}$ and $|f_n'| \leq 1$.

Theorem 1.2. *$(\mathscr{E}^\varphi, \bar{\mathscr{A}})$ is a Dirichlet form, which coincides with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ if and only if*

$$f_n(\log \varphi) \cdot e_k \in \bar{\mathscr{A}} \quad \textit{for all } k, n \in \boldsymbol{N}. \tag{1.12}$$

Note that for $u \in \mathscr{A}$ we have $\Gamma(u) \in L^1(E, m) \cap L^\infty(E, m) = \bigcap_{1 \leq p \leq \infty} L^p(E, m)$.

Corollary 1.3. *Suppose $\mathscr{C} \subset \mathscr{A}$. Let $p, q \in [2, \infty]$ such that $p^{-1} + q^{-1} = 2^{-1}$, and assume that the following conditions hold for any $k, n \in \boldsymbol{N}$:*

$$\int_{E_k} |\varphi|^q dm < \infty. \tag{1.13}$$

(1.14) *There is a sequence $u_l \in \mathscr{A} (l \in \boldsymbol{N})$ such that $u_l \to f_n(\log \varphi)$ m-a.e. on E_k and*

$$\sup_{l\in N}\int_{E_k}\Gamma(u_l)^{p/2}dm<\infty.$$

Then $(\mathscr{E}^{\varphi},\bar{\mathscr{A}})=(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$.

The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 2 below.

REMARKS. (i) If φ is bounded on any $E_k, k\in N$, (1.13) and (1.14) always hold with $q=\infty$ and $p=2$. In fact, $f_n\circ\log:(0,\infty)\to \boldsymbol{R}$ can be extended to a C_b^1-map from $\boldsymbol{R}$ to $\boldsymbol{R}$ vanishing at 0, so we have $f_n(\log\varphi)\in D_{\mathrm{loc}}(\mathscr{E},(E_k))$. Since $\mathscr{A}$ is dense in $D(\mathscr{E})$, this implies (1.14).

(ii) For certain gradient-type Dirichlet forms on finite and infinite dimensional state spaces, it can be shown that (1.12) holds for any $\varphi\in D_{\mathrm{loc}}(\mathscr{E},(E_k))$; see in particular the remarks in Section 5 a) and c).

(iii) Suppose $\mathscr{A}\subset D(L)\cap D_0^{1,\infty}(\mathscr{E},(E_k))$ and $L(\mathscr{A})\subset L^2(E,\varphi^2\cdot m)$. Then it can be shown that $\mathscr{A}$ is contained in the domain $D(L^{\varphi})$ of the generator L^{φ} of $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ and $L^{\varphi}u=Lu+\varphi^{-1}\Gamma(\varphi,u)\ \forall u\in\mathscr{A}$. Moreover, Theorem 1.2 can be applied to obtain a criterion for $(L^{\varphi},\mathscr{A})$ to be **Markov-unique** (i.e. $(L^{\varphi},D(L^{\varphi}))$ is the only self-adjoint operator on $L^2(E,\varphi^2 m)$ extending $(L^{\varphi},\mathscr{A})$ that generates a Markovian semigroup). This criterion, which generalizes results of [19], will be proved in my PhD-thesis ([9]).

Recall that the Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ is called **quasi-regular**, if the following conditions hold:
(i) There is an $\mathscr{E}$-nest of compacts,
(ii) Every $u\in D(\mathscr{E})$ has an $\mathscr{E}$-quasi continuous modification,
(iii) There are an $\mathscr{E}$-exceptional set $N\subset E$ and $\mathscr{E}$-quasi continuous functions $u_n\in D(\mathscr{E})$ $(n\in N)$ that separate the points of $E\backslash N$.

The notion of quasi-regularity is important, because a symmetric Dirichlet form on $L^2(E,m)$ is quasi-regular if and only if there is an associated right process (s. [14, IV.6.7]).

In Section 3 we show:

Theorem 1.4. (i) *Every* $\mathscr{E}$*-nest is an* $\mathscr{E}^{\varphi}$*-nest.*
(ii) *If* $(\mathscr{E},D(\mathscr{E}))$ *is quasi-regular, then* $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ *is quasi-regular.*

Now suppose that $(\mathscr{E},D(\mathscr{E}))$ is quasi regular. The strong local property (D1) implies in particular, that $\mathscr{E}(u,v)=0$ whenever $u,v\in D(\mathscr{E})$ such that $u\cdot v=0$ m-a.e. (s. [7, I, Prop. 5.1.3]). Thus by [14, V.1.11 and IV.6.7], there is a diffusion process $\boldsymbol{M}=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in E_{\Delta}})$ **properly associated** with $(\mathscr{E},D(\mathscr{E}))$, i.e.

$$(p_tf)(z):=E_z[f(X_t)]\qquad (z\in E)$$

is an $\mathscr{E}$-q.c. m-version of $e^{tL}f$ for any square integrable bounded function $f:E\to\boldsymbol{R}$ and $t>0$. Here Δ is the cemetery point, E_z means expectation w.r.t. P_z, and we set $f(\Delta):=0$. M is m-**symmetric**, i.e. $\int p_t f g\, dm = \int f p_t g\, dm$ for all $t \geq 0$ and all bounded measurable $f, g: E \to \mathbf{R}$. We may assume, that M is **canonical**, i.e. Ω is the space of continuous E-valued paths $\omega:[0,\zeta(\omega)) \to E$ with life-time $\zeta \in [0,\infty]$, $X_t(\omega) = \omega(t)$ $\forall 0 \leq t < \zeta(\omega)$ and $X_t(\omega) = \Delta$ $\forall t \geq \zeta(\omega)$.

Because of Theorems 1.1 and 1.4, we also have a $\varphi^2 m$-symmetric canonical diffusion $M^\varphi = (\Omega, \mathscr{F}^\varphi, (X_t)_{t \geq 0}, (P_z^\varphi)_{z \in E_\Delta})$ properly associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$.

We will show in Lemma 3.1, that the functions in $D_{\mathrm{loc}}(\mathscr{E},(E_k))$ have $\mathscr{E}$-quasi-continuous modifications. Let $\tilde{\varphi}$ be such a modification of φ. Because of Corollary A2 in the appendix, we can find an $\mathscr{E}$-nest $(E_k')_{k \in \mathbf{N}}$ such that $E_k' \subset E_k$ and $\tilde{\varphi}|_{E_k'}$ is continuous for any $k \in \mathbf{N}$. Let

$$E_k^\varphi := E_k' \cap \{k^{-1} \leq \tilde{\varphi} \leq k\}. \tag{1.15}$$

We will show in Lemma 3.2 below, that $(E_k^\varphi)_{k \in \mathbf{N}}$ is an $\mathscr{E}^\varphi$-nest. Since φ is in $D_{\mathrm{loc}}(\mathscr{E},(E_k))$, we can find functions $\varphi^{(k)} \in D(\mathscr{E})$ $(k \in \mathbf{N})$ such that

$$\varphi = \varphi^{(k)} = \varphi_k \quad m\text{-a.e. on } E_k^\varphi. \tag{1.16}$$

where $\varphi_k := (\varphi^{(k)} \wedge k) \vee k^{-1}$. Let $\tilde{\varphi}_k$ be an $\mathscr{E}$-q.c. modification of φ_k, and set $\tilde{\varphi}_k(\Delta) := 0$. By [13, Thm. 5.5.1], we have the Fukushima decomposition (w.r.t. M)

$$\log \tilde{\varphi}_k(X_t) - \log \tilde{\varphi}_k(X_0) = M_t^{[\log \varphi_k]} + N_t^{[\log \varphi_k]}$$

of the additive functional on the left-hand side into a martingale additive functional locally of finite energy $M_t^{[\log \varphi_k]}$ and a continuous additive functional locally of zero energy $N_t^{[\log \varphi_k]}$. Because of the regularization method of S. Albeverio, Z.M. Ma and M. Röckner (s. [14, Ch. VI]), the decomposition exists, even if the state space is not locally compact and $(\mathscr{E}, D(\mathscr{E}))$ is not regular; cf. Section 4b) below for details.

For $t \geq 0$ and $k \in \mathbf{N}$ let $\mathscr{B}_t := \sigma(X_s; 0 \leq s \leq t)$ and

$$\sigma_k := \inf\{s > 0;\ X_s \in E \backslash E_k^\varphi\}.$$

For a positive measure μ on $\mathscr{B}(E)$ we set $P_\mu := \int P_z[\,\cdot\,]\mu(dz)$. M is called **conservative**, if $\zeta = \infty$ P_z-a.s. for $\mathscr{E}$-q.e. z. Our final theorem shows that M^φ is a generalized Girsanov transform of M, provided M is conservative:

Theorem 1.5. *Suppose that M is conservative. Then for $\mathscr{E}^\varphi$-q.e. $z \in E$, P_z^φ is locally absolutely continuous w.r.t. P_z up to the life-time ζ, i.e.,*

$$P_z^\varphi|_{\mathscr{B}_t \cap \{t < \zeta\}} \ll P_z|_{\mathscr{B}_t \cap \{t < \zeta\}} \qquad \forall t \geq 0.$$

In particular, $P_{\varphi^2 \cdot m}^\varphi$ is locally absolutely continuous w.r.t. P_m up to ζ.

More precisely, we have:

(i) $P_z^\varphi[\sup_{k \in \mathbf{N}} \sigma_k < \zeta] = 0$ *for $\mathscr{E}^\varphi$-q.e. $z \in E$.*

(ii) *Let $t \geq 0$ and $k \in \mathbf{N} \backslash \{1\}$. Then for $\mathscr{E}^\varphi-q.e.$ $z \in E$, P_z^φ and P_z are equivalent on the σ-algebra $\mathscr{B}_t \cap \{t < \sigma_{k-1}\}$ with density*

$$\left.\frac{dP_z^\varphi}{dP_z}\right|_{\mathscr{B}_t \cap \{t<\sigma_{k-1}\}} = \exp\left(M_t^{[\log \varphi_k]} - \frac{1}{2}\langle M^{[\log \varphi_k]}\rangle_t\right).$$

The proof is given in Section 4.

REMARK. Theorem 1.5 does not make a direct statement about the law of the life-time under $P^\varphi_{\varphi^2 \cdot m}$. In fact, the diffusion $\boldsymbol{M}^\varphi$ may explode with strictly positive probability, although $\boldsymbol{M}$ is conservative. Sufficient criteria for conservativeness of $\boldsymbol{M}^\varphi$ are given in [27] and [13, Th. 6.3.3], e.g. it is enough that $\int \Gamma(\varphi)\, dm < \infty$.

In Section 5 we show how to apply Theorems 1.1 - 1.5 to concrete examples. In particular, we discuss Girsanov type transformations of Brownian motion on manifolds, reflected Brownian motion on smooth Euclidean domains, and diffusions associated with gradient Dirichlet forms on Banach spaces.

2. Transformation of the Dirichlet Form

In this section we prove Theorem 1.1 and 1.2, and Corollary 1.3. The following lemma will be needed in the proof of Theorem 1.1:

Lemma 2.1. *Let ψ be a bounded function in $D_{\mathrm{loc}}(\mathscr{E},(E_k))$ satisfying $\psi > 0$ m-a.e. Then for all functions $u,v \in D_0^{1,\infty}(\mathscr{E},(E_k))$*

$$\int \Gamma(u,v)\psi^2\, dm = -\int u \cdot L^\psi v\, \psi^2\, dm,$$

where $L^\psi v = Lv + 2\psi^{-1}\Gamma(v,\psi) \in L^2(E,\psi^2 m)$.

Proof. Fix $u,v \in D(L) \cap D_0^{1,\infty}(\mathscr{E},(E_k))$. Since u vanishes m-a.e. on $E \backslash E_k$ for some $k \in \boldsymbol{N}$, we can find a bounded function $\hat{\psi} \in D(\mathscr{E})$ such that $\hat{\psi} = \psi$ m-a.e. on $\{u \neq 0\}$. $\hat{\psi}^2$ and $u\psi^2$ $(= u\hat{\psi}^2)$ are in $D(\mathscr{E})$. Using the functional calculus for Dirichlet forms with strong local property (cf.[7, Sect. I.6]) and the fact that $\Gamma(u,v)$ vanishes m-a.e. on $\{u = 0\}$, we obtain

$$\begin{aligned}
\int \Gamma(u,v)\psi^2\, dm &= \int \Gamma(u,v)\hat{\psi}^2\, dm = \int \Gamma(u\hat{\psi}^2, v)\, dm - \int u\Gamma(\hat{\psi}^2, v)\, dm \\
&= \mathscr{E}(u\hat{\psi}^2, v) - \int u 2\hat{\psi}\Gamma(\hat{\psi}, v) dm = \mathscr{E}(u\psi^2, v) - \int u 2\hat{\psi}^{-1}\Gamma(\hat{\psi}, v)\hat{\psi}^2\, dm \\
&= -\int u \cdot (Lv + 2\psi^{-1}\Gamma(\psi, v))\psi^2\, dm.
\end{aligned}$$

Now choose $\psi' \in D(\mathscr{E})$ such that $\psi = \psi'$ m-a.e. on $\{v \neq 0\}$. Then we have $|\Gamma(\psi,v)| = |\Gamma(\psi',v)| \le \Gamma(\psi')^{1/2} \cdot \|\Gamma(v)\|_\infty^{1/2}$ m-a.e. on $\{v \neq 0\}$, and $\Gamma(\psi,v) = 0$ m-a.e. on

$\{v=0\}$. We obtain $\Gamma(\psi,v)\in L^2(E,m)$, and thus $\psi^{-1}\Gamma(\psi,v)\in L^2(E,\psi^2 m)$. The claim follows from the fact that $Lv\in L^2(E,m)\subset L^2(E,\psi^2 m)$, because ψ is bounded. □

Proof of Theorem 1.1. If φ is bounded, $\mathscr{D}^\varphi=D(\mathscr{E})$, and by [14, Prop.I.3.3.], the closability stated is a direct consequence of Lemma 2.1 and (D3).

Now, we consider the case of not necessarily bounded φ. We want to aproximate $u\in L^2(E,\varphi^2 m)$ by elements of $\mathscr{D}^\varphi$ w.r.t. the $L^2(E,\varphi^2 m)$-norm. Without loss of generality we may assume that u vanishes m-a.e. outside $\{i^{-1}\le\varphi\le i\}$ for some $i\in\boldsymbol{N}$. Choose $u_n\in D_0(\mathscr{E},(E_k))$ such that $u_n\to u$ in $L^2(E,m)$, and $f\in C_0^\infty((0,\infty))$, $0\le f\le 1$, such that $f=1$ on $[i^{-1},i]$. Then $f(\varphi)\cdot u_n$ converges to u in $L^2(E,m)$ and thus in $L^2(E,\varphi^2 m)$. For $n\in\boldsymbol{N}$ we choose $\hat\varphi\in D(\mathscr{E})$ s.t. $\varphi=\hat\varphi$ m-a.e. on $\{u_n\neq 0\}$. Then we have $f(\varphi)\cdot u_n=f(\hat\varphi)\cdot u_n\in D(\mathscr{E})$ and

$$\begin{aligned}\int((f(\varphi)\cdot u_n)^2+\Gamma(f(\varphi)\cdot u_n))\varphi^2\,dm&=\int((f(\hat\varphi)\cdot u_n)^2+\Gamma(f(\hat\varphi)\cdot u_n))\varphi^2\,dm\\&\le c^2\cdot\mathscr{E}_1(f(\hat\varphi)\cdot u_n),\end{aligned}$$

where c is the supremum of the support of f. So $f(\varphi)\cdot u_n$ is in $\mathscr{D}^\varphi$ for all $n\in\boldsymbol{N}$, and hence $\mathscr{D}^\varphi$ is dense in $L^2(E,\varphi^2 m)$.

Next, we define a symmetric bilinear form $(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$ on $L^2(E,\varphi^2 m)$ by

$$(2.1)\qquad \begin{aligned}D(\mathscr{E}^{\varphi,+})&=\left\{u\in\textstyle\bigcap_{n\in\boldsymbol{N}}D(\mathscr{E}^{\varphi\wedge n});\ \sup_{n\in\boldsymbol{N}}\mathscr{E}_1^{\varphi\wedge n}(u)<\infty\right\},\\ \mathscr{E}^{\varphi,+}(u)&=\sup\nolimits_{n\in\boldsymbol{N}}\mathscr{E}^{\varphi\wedge n}(u)\end{aligned}$$

(-actually all the spaces $D(\mathscr{E}^{\varphi\wedge n})$, $n\in\boldsymbol{N}$, coincide, but this is not important here). As a supremum of closed forms, $(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$ is closed. This can be shown similarly to [14, Prop. I. 3.7], cf. [10, Lemma A5]. Moreover, $(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$ extends $(\mathscr{E}^\varphi,\mathscr{D}^\varphi)$, which is hence closable. $(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$ is an extension of the closure $(\mathscr{E}^\varphi,D(\mathscr{E}^\varphi))$ as well. We will show later, that actually both forms coincide (s. Corollary 2.4 below).

For $u\in\mathscr{D}^\varphi$, $\varepsilon>0$, and a smooth function $g_\varepsilon:\boldsymbol{R}\to[-\varepsilon,1+\varepsilon]$ such that $g_\varepsilon=\mathrm{id}$ on $[0,1]$ and $|g_\varepsilon'|\le 1$, we have $g_\varepsilon\circ u\in D(\mathscr{E})$ and

$$\int\Gamma(g_\varepsilon\circ u)\varphi^2\,dm=\int(g_\varepsilon'\circ u)^2\Gamma(u)\varphi^2\,dm\le\mathscr{E}^\varphi(u).$$

Thus $g_\varepsilon\circ u$ is in $D(\mathscr{E}^\varphi)$ and

$$(2.2)\qquad \mathscr{E}^\varphi(g_\varepsilon\circ u)\le\mathscr{E}^\varphi(u);$$

so by [14, I.4.10], $(\mathscr{E}^\varphi,D(\mathscr{E}^\varphi))$ is a Dirichlet form.

Finally, by the chain rule for Γ, we have

$$\mathscr{E}^{\varphi}(f\circ u-f(0),g\circ u-g(0))=\int f'\circ u\ g'\circ u\ \Gamma(u)\ \varphi^2\,dm=0$$

whenever $u\in\mathscr{D}^{\varphi}$ and $f,g\in C_0^{\infty}(\boldsymbol{R})$ with $\operatorname{supp}f\cap\operatorname{supp}g=\emptyset$. Since $\mathscr{D}^{\varphi}$ is dense in $D(\mathscr{E}^{\varphi})$, this implies the strong local property for $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$. □

We now fix functions $f_n, n\in\boldsymbol{N}$, as in Theorem 1.2, and define $(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$ as in (2.1). The following proposition is crucial for our approach. It is the essential step not only in the proof of Theorem 1.2, but also in that of Theorem 1.4 and Lemma 3.2 in the next section.

Proposition 2.2. *Let $\mathscr{G}$ be a dense subset in $D(\mathscr{E})$ satisfying*:

(2.3) *For any $u\in\mathscr{G}$ and $c\geq 0$ there is a map $T:\boldsymbol{R}\to[-c-1,c+1]$ such that $T(s)=s\ \forall s\in[-c,c]$, $|T(s)-T(t)|\leq|s-t|\ \forall s,t\in\boldsymbol{R}$, and $T\circ u\in\mathscr{G}$.*

Then $\mathscr{G}_{\varphi}:=\operatorname{span}\{f_n(\log\varphi)\cdot e_k\cdot u;\,k,n\in\boldsymbol{N},\,u\in\mathscr{G}\cap L^{\infty}(E,m)\}$ is a subspace of $D(\mathscr{E}^{\varphi})$, which is dense in $D(\mathscr{E}^{\varphi,+})$.

Since we can always apply the proposition with $\mathscr{G}=D(\mathscr{E})$, we obtain in particular that $D(\mathscr{E}^{\varphi})$ is dense in, and hence equal to, $D(\mathscr{E}^{\varphi,+})$. Thus we have shown:

Corollary 2.3. $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))=(\mathscr{E}^{\varphi,+},D(\mathscr{E}^{\varphi,+}))$.

For the proof of Proposition 2.2 we need a preparatory lemma:

Lemma 2.4. *Let $c>0$.*
(i) *Let u be a function in $D(\mathscr{E})$ vanishing m-a.e. on $\{\varphi>c\}$. Then u is in $D(\mathscr{E}^{\varphi})$ and $\mathscr{E}^{\varphi}(u)=\mathscr{E}^{\varphi\wedge c}(u)$.*
(ii) *The form $(\mathscr{E}^{\varphi\wedge c},D(\mathscr{E}^{\varphi\wedge c}))$ admits a square field operator $\Gamma^{\varphi\wedge c}$, and $\Gamma^{\varphi\wedge c}(u,v)=\Gamma(u,v)$ for all $u,v\in D(\mathscr{E})$.*

Proof. (i) Since $\Gamma(u)$ vanishes m-a.e. on $\{u=0\}$, we have

$$\int(u^2+\Gamma(u))\varphi^2\,dm=\int_{\{\varphi\leq c\}}(u^2+\Gamma(u))\varphi^2\,dm\leq c^2\cdot\mathscr{E}_1(u)<\infty,$$

hence u is in $D(\mathscr{E}^{\varphi})$. Moreover,

$$\mathscr{E}^{\varphi}(u)=\int_{\{\varphi\leq c\}}\Gamma(u)\varphi^2\,dm=\int_{\{\varphi\leq c\}}\Gamma(u)(\varphi\wedge c)^2\,dm=\mathscr{E}^{\varphi\wedge c}(u).$$

For $u,v\in D(\mathscr{E})\cap L^{\infty}(E,m)$ we have

$$\mathcal{E}^{\varphi\wedge c}(uv,u)-\tfrac{1}{2}\mathcal{E}^{\varphi\wedge c}(v,u^2)=\int(\Gamma(uv,u)-\tfrac{1}{2}\Gamma(v,u^2))(\varphi\wedge c)^2\,dm$$
$$=\int v\cdot\Gamma(u,u)(\varphi\wedge c)^2\,dm$$

by the product rule for Γ. Since $D(\mathcal{E})\cap L^\infty(E,m)$ is dense in $D(\mathcal{E}^{\varphi\wedge c})$, the claim follows by [7, Sect. I. 4.1]. □

Proof of Proposition 2.2. We denote the closure of a subset $\mathcal{F}\subset D(\mathcal{E}^{\varphi,+})$ w.r.t. the $\mathcal{E}_1^{\varphi,+}$-norm by $\bar{\mathcal{F}}$.

Step 1. $\mathcal{G}_\varphi\subset D(\mathcal{E}^\varphi)$.
Fix $l,n\in\boldsymbol{N}$ and $u\in\mathcal{G}\cap L^\infty(E,m)$. The map $f_n\circ\log:(0,\infty)\to\boldsymbol{R}$ can be extended to a C_b^1-map from $\boldsymbol{R}$ to $\boldsymbol{R}$ vanishing at 0. Thus we have $f_n(\log\varphi)\in D_{\text{loc}}(\mathcal{E},(E_k))\cap L^\infty(E,m)$ and (by (1.8)) $f_n(\log\varphi)\cdot e_l\cdot u\in D(\mathcal{E}^\varphi)$. The claim follows, since Lemma 2.4(i) implies $f_n(\log\varphi)\cdot e_l\cdot u\in D(\mathcal{E}^\varphi)$.

Step 2. $D(\mathcal{E}^{\varphi,+})=\overline{D_0(\mathcal{E}^{\varphi,+},(E_k))}$.
Fix $u\in D(\mathcal{E}^{\varphi,+})$ and $k\in\boldsymbol{N}$. Because of Lemma 2.4(ii), we may apply Proposition A1 from the appendix to each of the forms $(\mathcal{E}^{\varphi\wedge i},D(\mathcal{E}^{\varphi\wedge i}))$, $i\in\boldsymbol{N}$. We obtain $u\cdot e_k\in D(\mathcal{E}^{\varphi\wedge i})$ and

$$\mathcal{E}^{\varphi\wedge i}(u\cdot e_k)\leq 2\cdot\mathcal{E}_1^{\varphi\wedge i}(u)\leq 2\cdot\mathcal{E}_1^{\varphi,+}(u).$$

Thus $u\cdot e_k$ is in $D(\mathcal{E}^{\varphi,+})$, and $\mathcal{E}^{\varphi,+}(u\cdot e_k)\leq 2\cdot\mathcal{E}_1^{\varphi,+}(u)$. But $(u\cdot e_k)_{k\in\boldsymbol{N}}$ converges to u in $L^2(E,\varphi^2m)$ by Lebesgue's theorem and (1.7), so the theorems of Banach / Alaoglu and Banach / Saks (cf. [14, Appendix 2]) imply, that the Césaro means of a subsequence of $(u\cdot e_k)_{k\in\boldsymbol{N}}$ converge to u in $D(\mathcal{E}^{\varphi,+})$. This proves the claim of Step 2, since the functions $u\cdot e_k$ $(k\in\boldsymbol{N})$, and thus their Césaro means, are in $D_0(\mathcal{E}^{\varphi,+},(E_k))$.

Step 3. $D_0(\mathcal{E}^{\varphi,+},(E_k))\subset\bar{\mathcal{H}}$,
where $\mathcal{H}:=D_0(\mathcal{E}^{\varphi,+},(E_k\cap\{\varphi\in[k^{-1},k]\}))\cap L^\infty(E,m)$.

We fix $u\in D_0(\mathcal{E}^{\varphi,+},(E_k))\cap L^\infty(E,m)$ and $k\in\boldsymbol{N}$ such that u vanishes m-a.e. outside E_k. Let $\psi\in D(\mathcal{E})$ such that $\varphi=\psi$ m-a.e. on E_k. By (1.8), we have for all $n\in\boldsymbol{N}$

$$\text{(2.4)}\qquad f_n(\log\varphi)\cdot e_k=f_n(\log\psi)\cdot e_k\in D(\mathcal{E})\qquad\text{and}$$
$$\Gamma(f_n(\log\varphi)\cdot e_k)\leq 2\cdot(1+\Gamma(f_n(\log\psi)))$$
$$\leq 2\cdot(1+\|f_n'\|_\infty^2\cdot\psi^{-2}\Gamma(\psi))\leq 2\cdot(1+\psi^{-2}\Gamma(\psi)).$$

Because of the strong local property we obtain

$$\int\Gamma(f_n(\log\varphi)\cdot e_k)\varphi^2\,dm=\int_{E_k}\Gamma(f_n(\log\varphi)\cdot e_k)\varphi^2\,dm$$

$$\leq 2\int_{E_k}(1+\psi^{-2}\Gamma(\psi))\varphi^2\,dm = 2\int_{E_k}(\psi^2+\Gamma(\psi))dm \leq 2\cdot\mathscr{E}_1(\psi).$$

Thus $f_n(\log\varphi)\cdot e_k$ is in $D(\mathscr{E}^{\varphi}) \subset D(\mathscr{E}^{\varphi,+})$ for all $n \in \boldsymbol{N}$ and

$$\sup_{n\in\boldsymbol{N}}\mathscr{E}^{\varphi,+}(f_n(\log\varphi)\cdot e_k) \leq 2\cdot\mathscr{E}_1(\psi) < \infty.$$

Since all the forms $(\mathscr{E}^{\varphi\wedge i}, D(\mathscr{E}^{\varphi\wedge i}))$, $i \in \boldsymbol{N}$, are Dirichlet forms, $(\mathscr{E}^{\varphi,+}, D(\mathscr{E}^{\varphi,+}))$ is a Dirichlet form, too. Therefore we obtain

$$f_n(\log\varphi)\cdot u = f_n(\log\varphi)\cdot e_k\cdot u \in D(\mathscr{E}^{\varphi,+}) \qquad \text{and}$$

$$\sup_{n\in\boldsymbol{N}}\mathscr{E}^{\varphi,+}(f_n(\log\varphi)\cdot u)^{\frac{1}{2}} \leq \|u\|_\infty\cdot(2\cdot\mathscr{E}_1(\psi))^{\frac{1}{2}} + \mathscr{E}^{\varphi,+}(u)^{\frac{1}{2}} < \infty.$$

But the sequence $(f_n(\log\varphi)\cdot u)_{n\in\boldsymbol{N}}$ converges to u m-a.e. and thus in $L^2(E,\varphi^2 m)$, so the Césaro means of a properly chosen subsequence are elements of $\mathscr{H}$ that converge to u in $D(\mathscr{E}^{\varphi,+})$.

Hence we have shown $D_0(\mathscr{E}^{\varphi,+},(E_k)) \cap L^\infty(E,m) \subset \bar{\mathscr{H}}$, which implies the claim of Step 3.

Step 4. $\mathscr{H} \subset \bar{\mathscr{G}}_\varphi$.

Fix $u \in \mathscr{H}$. We choose $k, n \in \boldsymbol{N}$ such that $u = f_n(\log\varphi)\cdot e_k\cdot u$, and $i \in \boldsymbol{N}$ such that $f_n \circ \log = 0$ on (i,∞). In particular, $u = 0$ m-a.e. on $\{\varphi > i\}$. By assumption, $\mathscr{G}$ is dense in $D(\mathscr{E})$ w.r.t. the $\mathscr{E}_1$-norm and thus in $D(\mathscr{E}^{\varphi\wedge i})$ w.r.t. the $\mathscr{E}_1^{\varphi\wedge i}$-norm. Since $u \in D(\mathscr{E}^{\varphi,+}) \subset D(\mathscr{E}^{\varphi\wedge i})$, we can find a sequence $u_l \in \mathscr{G}$ $(l \in \boldsymbol{N})$ such that $\lim_{l\to\infty}\mathscr{E}_1^{\varphi\wedge i}(u_l - u) = 0$. For any $l \in \boldsymbol{N}$, we choose a contraction $T_l : \boldsymbol{R} \to [-\|u\|_\infty - 1, \|u\|_\infty + 1]$ such that $T_l(s) = s$ $\forall s \in [-\|u\|_\infty, \|u\|_\infty]$ and $T_l(u_l) \in \mathscr{G}$.

I claim that the Césaro means w_l of a subsequence of

$$v_l := f_n(\log\varphi)\cdot e_k\cdot T_l(u_l) \ (l \in \boldsymbol{N})$$

converge to u w.r.t. the $\mathscr{E}_1^{\varphi\wedge i}$-norm. In fact, we have $f_n(\log\varphi)\cdot e_k \in D(\mathscr{E}^{\varphi\wedge i})$, and thus $v_l \in D(\mathscr{E}^{\varphi\wedge i})$ for all $l \in \boldsymbol{N}$,

$$\text{(2.5)} \quad |v_l - u| = |f_n(\log\varphi)\cdot e_k\cdot T_l(u_l) - f_n(\log\varphi)\cdot e_k\cdot T_l(u)| \leq |u_l - u|, \quad \text{and}$$

$$\text{(2.6)} \quad \sup_{l\in\boldsymbol{N}}\mathscr{E}^{\varphi\wedge i}(v_l)^{\frac{1}{2}} \leq \sup_{l\in\boldsymbol{N}}\mathscr{E}^{\varphi\wedge i}(u_l)^{\frac{1}{2}} + (\|u\|_\infty + 1)\cdot\mathscr{E}^{\varphi\wedge i}(f_n(\log\varphi)e_k)^{\frac{1}{2}} < \infty.$$

By (2.5), $(v_l)_{l\in\boldsymbol{N}}$ converges to u in $L^\infty(E,(\varphi\wedge i)^2 m)$, so the claim follows from (2.6) by the usual arguments.

Lemma 2.4 (i) and the fact that the w_l vanish on $\{\varphi > i\}$ now imply $w_l \in D(\mathscr{E}^{\varphi})$

$\forall l \in \boldsymbol{N}$, and

$$\mathscr{E}_1^\varphi(w_l - w_m) = \mathscr{E}_1^{\varphi\,\wedge\,i}(w_l - w_m) \to 0 \ (l,m \to \infty).$$

So $(w_l)_{l\in\boldsymbol{N}}$ converges in $D(\mathscr{E}^\varphi)$, the limit has to be u. This completes the proof of Step 4, since the w_l are in $\mathscr{G}_\varphi$.

Step 2, 3 and 4 now imply $\bar{\mathscr{G}}_\varphi = D(\mathscr{E}^{\varphi,+})$. □

Proof of Theorem 1.2. For $u \in \mathscr{A}$, $\varepsilon > 0$, and a C_b^∞-function $g_\varepsilon : \boldsymbol{R} \to [-\varepsilon, 1+\varepsilon]$ such that $g_\varepsilon(s) = s \ \forall s \in [0,1]$ and $|g_\varepsilon'| \le 1$, we have $g_\varepsilon \circ u \in \mathscr{A}$ by (1.11), and $\mathscr{E}^\varphi(g_\varepsilon \circ u) \le \mathscr{E}^\varphi(u)$ by (2.2). Thus the closed form $(\mathscr{E}^\varphi, \bar{\mathscr{A}})$ is a Dirichlet form.

Moreover, for any $c > 0$ we can find a C_b^∞-map $T: \boldsymbol{R} \to [-c-1, c+1]$ such that $T(s) = s \ \forall s \in [-c,c]$ and $|T'| \le 1$. By (1.11), $T \circ u$ is in $\mathscr{A}$ whenever $u \in \mathscr{A}$, so we may apply Proposition 2.3 with $\mathscr{G} = \mathscr{A}$ to conclude that $\mathscr{A}_\varphi$ (as defined in the proposition) is dense in $D(\mathscr{E}^\varphi)$.

Now suppose (1.12) holds. Then $\bar{\mathscr{A}}$ contains $\mathscr{A}_\varphi$, since it is a Dirichlet space, and thus we obtain $\bar{\mathscr{A}} = D(\mathscr{E}^\varphi)$. On the other hand, $\bar{\mathscr{A}} = D(\mathscr{E}^\varphi)$ implies (1.12) because of Lemma 2.5 (i). □

REMARK. For $u,v \in \mathscr{A}$ we have

$$\mathscr{E}^\varphi(uv,u) - \tfrac{1}{2}\mathscr{E}^\varphi(v,u^2) = \int (\Gamma(uv,u^2) - \tfrac{1}{2}\Gamma(v,u^2))\varphi^2\,dm = \int v \cdot \Gamma(u,u)\varphi^2\,dm.$$

Thus $(\mathscr{E}^\varphi, \bar{\mathscr{A}})$ admits a square field operator, which coincides with Γ on $\mathscr{A} \times \mathscr{A}$.

Proof of Corollary 1.3. Fix $k,n \in \boldsymbol{N}$. For $u \in \mathscr{A}$, Hölder's inequality implies

$$\int_{E_k} \Gamma(u)\varphi^2\,dm \le \left(\int_{E_k} \Gamma(u)^{p/2} dm\right)^{2/p} \cdot \left(\int_{E_k} \varphi^q\,dm\right)^{2/q}.$$

Thus by the assumptions, we can find a sequence $u_l \in \mathscr{A}$ ($l \in \boldsymbol{N}$) such that $u_l \to f_n(\log\varphi)$ m-a.e. on E_k and

$$\sup_{l\in\boldsymbol{N}} \int_{E_k} \Gamma(u_l)\varphi^2\,dm < \infty. \tag{2.7}$$

Since $f_n(\log\varphi)$ is bounded and $\mathscr{A}$ satisfies (1.11), we may assume that $(u_l)_{l\in\boldsymbol{N}}$ is uniformly bounded. For $l \in \boldsymbol{N}$ let $v_l := u_l \cdot e_k$. Then $(v_l)_{l\in\boldsymbol{N}}$ converges to $f_n(\log\varphi) \cdot e_k$ m-a.e., and $(|v_l|)_{l\in\boldsymbol{N}}$ is dominated by $\sup_{l\in\boldsymbol{N}} \|u_l\|_\infty \cdot \chi_{E_k}$ for all $l \in \boldsymbol{N}$. But $\int_{E_k} \varphi^2\,dm < \infty$, so $v_l \to f_n(\log\varphi) \cdot e_k$ in $L^2(E, \varphi^2 m)$ by Lebesgue's theorem. Since $\mathscr{C}$ is a subset of $\mathscr{A}$, the square field operators of $(\mathscr{E}, D(\mathscr{E}))$ and $(\mathscr{E}^\varphi, \bar{\mathscr{A}})$ coincide on $\mathscr{C}$ (s. the remark above). Hence we may apply Proposition A 1 from the appendix to $(\mathscr{E}^\varphi, \bar{\mathscr{A}})$. We obtain $v_l \in \bar{\mathscr{A}}$ and

$$\mathscr{E}^{\varphi}(v_l)=\int_{E_k}\Gamma(v_l)\varphi^2\,dm\le 2\int_{E_k}(\Gamma(u_l)+u_l^2)\varphi^2\,dm$$

$$\le 2\cdot\left(\sup_{l\in N}\int_{E_k}\Gamma(u_l)\varphi^2\,dm+\sup_{l\in N}\|u_l\|_\infty\cdot\int_{E_k}\varphi^2\,dm\right)<\infty\quad\forall l\in N.$$

Thus the Césaro means of a subsequence of $(v_l)_{l\in N}$ converge to $f_n(\log\varphi)\cdot e_k$ w.r.t. the $\mathscr{E}_1^{\varphi}$-norm. Hence $f_n(\log\varphi)\cdot e_k$ is in $\bar{\mathscr{A}}$ for all $k,n\in N$. The claim follows from Theorem 1.2. □

3. Potential Theory of the Transformaed Form

In this section we prove some potential theoretic properties of $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ which will be needed to show the existence of an associated diffusion, and to identify it as the Girsanov-transform of the initial diffusion. We first need a preparatory lemma:

Lemma 3.1. *If $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular, then any $u\in D_{\mathrm{loc}}(\mathscr{E},(E_k))$ has an $\mathscr{E}$-quasi-continuous modification $\tilde{u}$.*

Proof. For any $k\in N$ there is a function $u_k\in D(\mathscr{E})$ such that $u=u_k$ m-a.e. on $\{\rho_p<k\}$. Because of the quasi-regularity, we may choose an $\mathscr{E}$-q.c. modification $\tilde{u}_k$. But by Corollary A 2 (ii) in the appendix, ρ_p is $\mathscr{E}$-q.c., so $\{\rho_p<k\}$ is $\mathscr{E}$-quasi-open (i.e. there is an $\mathscr{E}$-nest $(F_l')_{l\in N}$ such that $\{\rho_p<k\}\cap F_l'$ is relatively open in F_l' for any $l\in N$. Thus for $k,l\in N$, $\tilde{u}_k$ and $\tilde{u}_l$ coincide $\mathscr{E}$-q.e. on $\{\rho_p<k\wedge l\}$ (cf. [13, Lemma 2.1.5]), and hence there is a modification $\tilde{u}$ of u such that $\tilde{u}=\tilde{u}_k$ $\mathscr{E}$-q.e. on $\{\rho_p<k\}$ for all $k\in N$. Because of Corollary A 2 (i), we can find an $\mathscr{E}$-nest $(F_k'')_{k\in N}$, such that for all $k\in N$ $F_k''\subset\{\rho_p<k\}$, $\tilde{u}|_{F_k''}=\tilde{u}_k|_{F_k''}$, and $\tilde{u}_k|_{F_k''}$ is continuous. Hence $\tilde{u}$ is $\mathscr{E}$-q.c. □

Proof of Theorem 1.4

(i) Suppose $(F_k)_{k\in N}$ is an $\mathscr{E}$-nest; so $D_0(\mathscr{E},(F_k))$ is dense in $D(\mathscr{E})$. For $c>0$ and $u\in D_0(\mathscr{E},(F_k))$ we have $(u\wedge c)\vee(-c)\in D_0(\mathscr{E},(F_k))$. Thus, by Proposition 2.2,

$$\mathscr{H}:=\mathrm{span}\{f_n(\log\varphi)\cdot e_k\cdot u;\ k,n\in N,\ u\in D_0(\mathscr{E},(F_k))\ \text{bounded}\}$$

is a dense subspace of $D(\mathscr{E}^{\varphi})$. Since $(F_k)_{k\in N}$ is increasing, $\mathscr{H}$ consisits of functions that vanish outside some F_k, $k\in N$. Thus, $D_0(\mathscr{E}^{\varphi},(F_k))$ is dense in $D(\mathscr{E}^{\varphi})$, and $(F_k)_{k\in N}$ is an $\mathscr{E}^{\varphi}$-nest.

(ii) Assume $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular. Then there is an $\mathscr{E}$-nest $(F_k)_{k\in N}$ of compacts, and by (i), $(F_k)_{k\in N}$ is an $\mathscr{E}^{\varphi}$-nest, too.

Moreover, the elements of $\mathscr{D}^{\varphi}$ have $\mathscr{E}$-q.c. modifications. But every $\mathscr{E}$-nest is an $\mathscr{E}^{\varphi}$-nest, so these modifications are $\mathscr{E}^{\varphi}$-q.c., too. Since $\mathscr{D}^{\varphi}$ is dense in $D(\mathscr{E}^{\varphi})$,every $u\in D(\mathscr{E}^{\varphi})$ has an $\mathscr{E}^{\varphi}$-q.c. modification (cf. [14, III. 3.5]).

Finally, fix $\mathscr{E}$-q.c. functions $u_l \in D(\mathscr{E})$ $(l \in N)$ that separate the points of E up to an $\mathscr{E}$-exceptional set N. We may assume, that the u_l are bounded. By Corollary A 2 in the appendix, the functions $e_k, k \in N$, are $\mathscr{E}$-q.c., and $e_k \nearrow 1$ $\mathscr{E}$-q.e. Moreover, we can choose an $\mathscr{E}$-q.c. modification $\tilde{\varphi}$ of φ by Lemma 3.1. We fix smooth functions $h_i : [0, \infty) \to [0,1]$ $(i \in N)$ such that $h_i(0) = 1$, $h_i(s) = 0$ for all $s \geq i$, $i \in N$, and $\sup_{i \in N} h_i(s) = 1$ for all $s \geq 0$. Then the functions

$$u_{l,k,i} := u_l \cdot e_k \cdot h_i(\tilde{\varphi}) \; (= u_l \cdot e_k + u_l \cdot e_k \cdot (h_i - 1)(\tilde{\varphi})) \; (l,k,i \in N)$$

are $\mathscr{E}$-q.c. functions in $D(\mathscr{E})$ (since $u_l \cdot e_k$, $(h_i - 1)(\tilde{\varphi}) \in D(\mathscr{E}) \cap L^\infty(E,m)$). Because of (i) and Lemma 2.4 (i), they are also $\mathscr{E}^\varphi$-q.c. elements of $D(\mathscr{E}^\varphi)$. But

$$u_l = \sup_{k,i} u_{l,k,i} \quad \mathscr{E}\text{-q.e. } \forall l \in N,$$

so $\{u_{l,k,i};\ l,k,i \in N\}$ separates the points of E up to an $\mathscr{E}$-exceptional, and hence $\mathscr{E}^\varphi$-exceptional, set. This proves the quasi-regularity of $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$. □

Now assume that $(\mathscr{E}, D(\mathscr{E}))$ is quasi-regular, and let $\tilde{\varphi}$ be an $\mathscr{E}$-q.c. modification of φ. By Theorem 1.4 (i), $\tilde{\varphi}$ is $\mathscr{E}^\varphi$-q.c., too. The set $\{\tilde{\varphi} = 0\}$ may not be $\mathscr{E}$-exceptional, but by the following lemma it is always $\mathscr{E}^\varphi$-exceptional:

Lemma 3.2. *Let $(F_k)_{k \in N}$ be an $\mathscr{E}$-nest such that $\tilde{\varphi}|_{F_k}$ is continuous for any $k \in N$, and let $F'_k := F_k \cap \{k^{-1} \leq \tilde{\varphi} \leq k\}$ $(k \in N)$. Then $(F'_k)_{k \in N}$ is an $\mathscr{E}^\varphi$-nest.*

Proof. The sets F_k, $k \in N$, are closed. Applying Proposition 2.2 with $\mathscr{G} := D_0(\mathscr{E}, (F_k))$ shows that $D_0(\mathscr{E}, (F'_k))$ is dense in $D(\mathscr{E}^\varphi)$. □

4. The Associated Diffusion

In this section we identify the diffusion process associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ as the (generalized) Girsanov transform of the diffusion associated with $(\mathscr{E}, D(\mathscr{E}))$, provided the latter is quasi-regular and conservative.

a) Identification under restrictions on $(\mathscr{E}, D(\mathscr{E}))$ and φ

In this subsection, we assume that E is a locally compact separable metric space, m is a positive Radon measure on $\mathscr{B}(E)$, $(\mathscr{E}, D(\mathscr{E}))$ is a **regular** (i.e. $C_0(E) \cap D(\mathscr{E})$ is both dense in $D(\mathscr{E})$ w.r.t. the $\mathscr{E}_1$-norm and in $C_0(E)$ w.r.t. the uniform norm) Dirichlet form on $L^2(E,m)$ satisfying (D 1) and (D 2), and

$$\varphi := (\psi \wedge n) \vee n^{-1}$$

for some $\psi \in D(\mathscr{E})$) and $n \in N$. In this case, we can define the transformed form on $L^2(E, \varphi^2 m)$ by

$$D(\mathscr{E}^\varphi) := D(\mathscr{E}), \qquad \mathscr{E}^\varphi(u,v) := \int \Gamma(u,v) \varphi^2 \, dm. \tag{4.1}$$

Since the $\mathscr{E}_1$ and $\mathscr{E}_1^\varphi$-norm are equivalent, $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ is a regular Dirichlet form as well. Let $\boldsymbol{M}=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in E_\Delta})$ and $\boldsymbol{M}^\varphi=(\Omega,\mathscr{F}^\varphi,(X_t)_{t\geq 0},(P_z^\varphi)_{z\in E_\Delta})$ be m-resp. $\varphi^2 m$-symmetric canonical diffusions properly associated with $(\mathscr{E},D(\mathscr{E}))$ resp. $(\mathscr{E}^\varphi,D(\mathscr{E}^\varphi))$. Since φ is bounded from below, $\log\varphi$ is locally in $D(\mathscr{E})$ (i.e.for any relatively compact open subset G of E there is a function $u\in D(\mathscr{E})$ such that $u=\log\varphi$ m-a.e. on G). Thus the Fukushima decomposition w.r.t. $\boldsymbol{M}$ of the additive functional $\log\tilde{\varphi}(X_t)-\log\tilde{\varphi}(X_0)$, where $\tilde{\varphi}$ is an $\mathscr{E}$-q.c. modification of φ, exists. Let $M^{[\log\varphi]}$ be the local martingale part.

The following proposition was first proved by S. Song (s. [22, §8, Lemma 2 and 3]) in the special situation of classical Dirichlet forms on topological vector spaces. Song partially extended a previous result by S. Albeverio, M. Röckner and T.S. Zhang ([4]). His proof carries over to our situation with minor changes. For the reader's convenience, we nevertheless give a detailed proof.

Proposition 4.1. *Suppose that $\boldsymbol{M}$ is conservative. Then P_z^φ and P_z are locally equivalent for $\mathscr{E}$-q.e. $z\in E$ with density*

$$\left.\frac{dP_z^\varphi}{dP_z}\right|_{\mathscr{B}_t}=\exp(M_t^{[\log\varphi]}-\tfrac{1}{2}\langle M^{[\log\varphi]}\rangle_t)\quad\forall t\geq 0. \tag{4.2}$$

Proof. For $t\geq 0$ let $L_t^{[\varphi]}:=\exp(M_t^{[\log\varphi]}-\frac{1}{2}\langle M^{[\log\varphi]}\rangle_t)$ and let $\mathscr{F}_t$ be the natural filtration of $\boldsymbol{M}$, i.e. $\Gamma\in\mathscr{F}_t$ if and only if Γ is in the P_μ-completion of $\sigma(X_s;0\leq s\leq t)$ for any probability measure μ on E_Δ. It is shown in [13, Lemma 6.3.5], that there is a $\varphi^2 m$-symmetric diffusion $\tilde{\boldsymbol{M}}^\varphi=(\Omega,\tilde{\mathscr{F}}^\varphi,(X_t)_{t\geq 0},(\tilde{P}_z^\varphi)_{z\in E_\Delta})$, such that

$$\tilde{E}_z^\varphi[F\cdot\chi_{\{T<\zeta\}}]=E_z[L_T^{[\varphi]}\cdot F]\quad\forall z\in E \tag{4.3}$$

for any bounded $(\mathscr{F}_t)$-stopping time T and any $\mathscr{F}_T$-measurable bounded function $F:\Omega\to\boldsymbol{R}$. Moreover, $\Gamma(\varphi)$ is well-defined, because φ is locally in $D(\mathscr{E})$, and

$$\int\Gamma(\varphi)\,dm\leq\int\Gamma(\psi)\,dm<\infty.$$

Thus, by [13, Lemma 6.3.7], $\tilde{\boldsymbol{M}}^\varphi$ is conservative. In particular, we have for $\mathscr{E}$-q.e. $z\in E$:

$$\tilde{E}_z^\varphi[F]=E_z[L_T^{[\varphi]}\cdot F]\quad\text{for any } T \text{ and } F \text{ as in (4.3).} \tag{4.4}$$

Since $M_t^{[\log\varphi]}$ is a local $(\mathscr{F}_t)$-martingale, $L_t^{[\varphi]}$ is a non-negative local $(\mathscr{F}_t)$-martingale, and hence a supermartingale under P_z for $\mathscr{E}$-q.e. z. But by (4.4), $E_z[L_t^{[\varphi]}]=1$ for all $t\geq 0$ and $\mathscr{E}$-q.e. z, so $L_t^{[\varphi]}$ is even an $(\mathscr{F}_t)$-martingale under P_z for $\mathscr{E}$-q.e. z.

Let $(p_t)_{t>0}$ and $(\tilde{p}_t^\varphi)_{t>0}$ be the transition semigroups of $\boldsymbol{M}$ resp. $\tilde{\boldsymbol{M}}^\varphi$. We denote by $(R_\alpha)_{\alpha>0}$ resp. $(\tilde{R}_\alpha^\varphi)_{\alpha>0}$ the resolvents of $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}^\varphi$, i.e.

$$(4.5)\qquad (R_\alpha f)(z) := E_z\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right] = \int_0^\infty e^{-\alpha t}(p_t f)(z)\,dt$$

$$(4.6)\qquad (\tilde{R}_\alpha^\varphi f)(z) := \tilde{E}_z^\varphi\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right] = \int_0^\infty e^{-\alpha t}(\tilde{p}_t^\varphi f)(z)\,dt$$

for all $z \in E$ and all bounded measurable $f: E \to \boldsymbol{R}$. Note that because of the m-resp. $\varphi^2 m$-symmetry of $\boldsymbol{M}$ resp. $\tilde{\boldsymbol{M}}^\varphi$, p_t, αR_α, $\tilde{p}_t^\varphi$ and $\alpha \tilde{R}_\alpha^\varphi$ are contractions w.r.t. the $L^1(E,m)$- resp. $L^1(E,\varphi^2 m)$-norm. Thus (4.5) and (4.6) make sense for any integrable $f: E \to \boldsymbol{R}$.

Now, we proceed in two steps: In Step 1 we show that

$$(4.7)\qquad \tilde{R}_\alpha^\varphi f = R_\alpha f + 2\tilde{R}_\alpha^\varphi \Gamma(R_\alpha f, \log\varphi) \qquad m\text{-a.e.}$$

for any bounded, m-square integrable $f: E \to \boldsymbol{R}$ and $\alpha > 0$. In Step 2 we conclude that $\tilde{\boldsymbol{M}}^\varphi$ is properly associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$. This completes the proof of the lemma: In fact, $\boldsymbol{M}^\varphi$ is properly associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ as well, and thus it follows from [14, IV. 6.4 and IV.6.2(ii)] and monotone class arguments, that

$$(4.8)\qquad P_z^\varphi = \tilde{P}_z^\varphi \quad \text{on } \mathscr{B}_\infty \ (:= \sigma(X_t; t \geq 0)) \quad \text{for } \mathscr{E}^\varphi\text{-q.e. } z \in E.$$

But the $\mathscr{E}_1$-and $\mathscr{E}_1^\varphi$-norm are equivalent, and thus (4.8) holds for $\mathscr{E}$-q.e. $z \in E$, too. (4.4) implies the claim of the lemma.

Step 1. First note, that for any u locally in $D(\mathscr{E})$

$$(4.9)\qquad \Gamma(u, \log\varphi) = \Gamma(u, \log\psi)\cdot \chi_{\{n^{-1} \leq \psi \leq n\}}.$$

Thus $\Gamma(R_\alpha f, \log\varphi)$ is integrable w.r.t. m and $\varphi^2 m$, and $\tilde{R}_\alpha^\varphi \Gamma(R_\alpha f, \log\varphi)$ exists m-a.e.

Because of the strong local property of $(\mathscr{E}, D(\mathscr{E}))$, $2\cdot\Gamma(R_\alpha f, \log\varphi)\cdot m$ is the Revuz measure of the additive functional $\langle M^{[R_\alpha f]}, M^{[\log\varphi]}\rangle$ (s. [13, Lemma 5.3.3 and (3.2.20)]). Thus

$$\langle M^{[R_\alpha f]}, M^{[\log\varphi]}\rangle = 2\cdot \int_0^\cdot \Gamma(R_\alpha f, \log\varphi)(X_t)dt \qquad P_z\text{-a.s. for } \mathscr{E}\text{-q.e. } z,$$

independent of the version of $\Gamma(R_\alpha f, \log\varphi)$ chosen (s. [10, Beispiel A72] for details). Moreover, by Itô's formula, we have P_z-a.s. for $\mathscr{E}$-q.e. z

$$L_s^{[\varphi]} = 1 + \int_0^s L_t^{[\varphi]} dM_t^{[\log\varphi]}, \qquad \text{and thus}$$

$$\langle M^{[R_\alpha f]}, L^{[\varphi]}\rangle_s = \int_0^s L_t^{[\varphi]} d\langle M^{[R_\alpha f]}, M^{[\log\varphi]}\rangle_t$$

$$=2\cdot\int_0^s L_t^{[\varphi]}\Gamma(R_\alpha f,\log\varphi)(X_t)\,dt\qquad \forall s\geq 0.$$

For $n\in\boldsymbol{N}$ let $T_n:=\inf\{t\geq 0;\ L_t^{[\varphi]}>n\}$. Then $T_n\uparrow\infty$ P_z-a.s. for $\mathscr{E}$-q.e. z. By Fubini's Theorem and by dominated convergence, we obtain for m-a.e. $z\in E$:

$$\begin{aligned}
(4.10)\qquad 2\tilde{R}_\alpha^\varphi\Gamma(R_\alpha f,\log\varphi)(z)&=2\cdot\int_0^\infty e^{-\alpha t}\tilde{E}_z^\varphi[\Gamma(R_\alpha f,\log\varphi)(X_t)]dt\\
&=2\cdot\int_0^\infty E_z\left[\left(\int_t^\infty \alpha e^{-\alpha s}ds\right)L_t^{[\varphi]}\cdot\Gamma(R_\alpha f,\log\varphi)(X_t)\right]dt\\
&=E_z\left[\int_0^\infty \alpha e^{-\alpha s}\left(\int_0^s 2L_t^{[\varphi]}\cdot\Gamma(R_\alpha f,\log\varphi)(X_t)\,dt\right)ds\right]\\
&=\lim_{n\to\infty}E_z\left[\int_0^\infty \alpha e^{-\alpha s}\left(\int_0^{s\wedge T_n} 2L_t^{[\varphi]}\cdot\Gamma(R_\alpha f,\log\varphi)(X_t)\,dt\right)ds\right]\\
&=\lim_{n\to\infty}E_z\left[\int_0^\infty \alpha e^{-\alpha s}\langle M^{[R_\alpha f]},L^{[\varphi]}\rangle_{s\wedge T_n}ds\right]\\
&=\lim_{n\to\infty}\int_0^\infty \alpha e^{-\alpha s}E_z[\langle M^{[R_\alpha f]},L^{[\varphi]}\rangle_{s\wedge T_n}]\,ds\\
&=\lim_{n\to\infty}\int_0^\infty \alpha e^{-\alpha s}E_z[M_{s\wedge T_n}^{[R_\alpha f]}L_{s\wedge T_n}^{[\varphi]}]\,ds.
\end{aligned}$$

Note that our use of Fubini's and Lebesgue's Theorem is justified, since for any $(\mathscr{F}_t)$-stopping time T and m-a.e. $z\in E$:

$$\begin{aligned}
(4.11)\qquad &\int_0^\infty \alpha e^{-\alpha s}E_z[|\langle M^{[R_\alpha f]},L^{[\varphi]}\rangle_{s\wedge T_n}|]\,ds\\
&=E_z\left[\int_0^\infty \alpha e^{-\alpha s}\cdot\left|\int_0^{s\wedge T} L_t^{[\varphi]}\Gamma(R_\alpha f,\log\varphi)(X_t)\,dt\right|ds\right]\\
&\leq E_z\left[\int_0^\infty \alpha e^{-\alpha s}\int_0^s L_t^{[\varphi]}\cdot|\Gamma(R_\alpha f,\log\varphi)(X_t)|\,dt\,ds\right]\\
&=\tilde{R}_\alpha^\varphi|\Gamma(R_\alpha f,\log\varphi)|(z)<\infty
\end{aligned}$$

by the same calculation as above. Thus all the integrands appearing above are integrable on the respective product spaces, and the integrand in the third line of (4.11) can be used as an integrable majorant.

Now, we will calculate the right hand side of (4.10). Since $\boldsymbol{M}$ is properly associated with $(\mathscr{E},D(\mathscr{E}))$, $R_\alpha f$ is $\mathscr{E}$-quasi continuous. Hence $t\mapsto R_\alpha f(X_t)$ is continuous P_z-a.s. for $\mathscr{E}$-q.e. z, because $\boldsymbol{M}$ is a conservative diffusion. Moreover, the Fukushima decomposition of $R_\alpha f(X_t) - R_\alpha f(X_0)$ w.r.t. $\boldsymbol{M}$ reduces to the usual semi-martingale decomposition and

$$M_s^{[R_\alpha f]} = R_\alpha f(X_s) - R_\alpha f(X_0) - \int_0^s (\alpha R_\alpha f - f)(X_t)\,dt \qquad \forall s \geq 0$$

P_z-a.s. for $\mathscr{E}$-q.e. z. Since f is bounded, we obtain by (4.4), dominated convergence and Fubini's Theorem for m-a.e. z:

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^\infty \alpha e^{-\alpha s} E_z[M_{s\wedge T_n}^{[R_\alpha f]}, L_{s\wedge T_n}^{[\varphi]}]\,ds \\
&= \lim_{n\to\infty} \int_0^\infty \alpha e^{-\alpha s} \tilde{E}_z^\varphi\left[R_\alpha f(X_{s\wedge T_n}) - R_\alpha f(X_0) - \int_0^{s\wedge T_n} (\alpha R_\alpha f - f)(X_t)\,dt \right] ds \\
&= \int_0^\infty \alpha e^{-\alpha s} \tilde{E}_z^\varphi\left[R_\alpha f(X_s) - R_\alpha f(X_0) - \int_0^s (\alpha R_\alpha f - f)(X_t)\,dt \right] ds \\
&= \int_0^\infty \alpha e^{-\alpha s} \left(\tilde{p}_s^\varphi R_\alpha f - R_\alpha f - \int_0^s \tilde{p}_t^\varphi (\alpha R_\alpha f - f)\,dt \right) ds \\
&= \alpha \tilde{R}_\alpha^\varphi R_\alpha f - R_\alpha f - \int_0^\infty \left(\int_t^\infty \alpha e^{-\alpha s} ds \right) \tilde{p}_t^\varphi (\alpha R_\alpha f - f)\,dt \\
&= \alpha \tilde{R}_\alpha^\varphi R_\alpha f - R_\alpha f - \tilde{R}_\alpha^\varphi (\alpha R_\alpha f - f) \qquad = \tilde{R}_\alpha^\varphi f - R_\alpha f
\end{aligned}$$

Together with (4.10), we obtain (4.7).

Step 2. Since $\tilde{\boldsymbol{M}}^\varphi$ is $\varphi^2 m$-symmetric, it is properly associated with a (symmetric) Dirichlet form $(\tilde{\mathscr{E}}^\varphi, D(\tilde{\mathscr{E}}^\varphi))$ on $L^2(E, \varphi^2 m)$. The **proper** association, i.e. the $\tilde{\mathscr{E}}^\varphi$-quasi-continuity of $\tilde{p}_t^\varphi f$ for any bounded square-integrable $f: E \to \boldsymbol{R}$, follows from [14, IV.6.7].

We are going to deduce from (4.7) that $(\tilde{\mathscr{E}}^\varphi, D(\tilde{\mathscr{E}}^\varphi))$ and $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ coincide. First note, that

$$\int \alpha \tilde{R}_\alpha^\varphi g f \varphi^2\,dm \to \int g f \varphi^2\,dm \qquad (\alpha \to \infty) \tag{4.12}$$

for any bounded $\varphi^2 m$-integrable $f: E \to \boldsymbol{R}$ and any $\varphi^2 m$-integrable $g: E \to \boldsymbol{R}$. In fact: Since $\alpha \tilde{R}_\alpha^\varphi$ ($\alpha > 0$) are contractions on $L^1(E, \varphi^2 m)$, it is enough to prove (4.12) for continuous bounded g. But in this case, $\lim_{t\downarrow 0} \tilde{p}_t^\varphi g = g$ m-a.e., hence $\lim_{\alpha\to\infty} \tilde{R}_\alpha^\varphi g = g$ m-a.e., and (4.12) follows by dominated convergence. Now fix $f \in C_0(E) \cap D(\mathscr{E}^\varphi)$ $(= C_0(E) \cap D(\mathscr{E}))$. Then (4.7) implies for $\alpha > 0$

$$(4.13)\qquad \int \alpha(f-\alpha\tilde{R}_\alpha^\varphi f) f\varphi^2\,dm = \int \alpha(f-\alpha R_\alpha f) f\varphi^2\,dm - 2\int \alpha\tilde{R}_\alpha^\varphi \Gamma(\alpha R_\alpha f, \log\varphi) f\varphi^2\,dm.$$

But $f\varphi^2$ is in $D(\mathscr{E})$, since the map $T: \boldsymbol{R}\to\boldsymbol{R}$, $T(s):=(((s\wedge n)\vee n^{-1})^2-n^{-2}$ is Lipschitz-continuous, bounded and vanishes at 0, and $f\varphi^2=f\cdot T(\psi)+n^{-2}f$. Hence the first term on the right-hand side of (4.13) converges to $\mathscr{E}(f,f\varphi^2)$ when $\alpha\to\infty$ (s. [13, Lemma 1.3.4]). Moreover,

$$\begin{aligned}&|\textstyle\int \alpha\tilde{R}_\alpha^\varphi \Gamma(\alpha R_\alpha f,\log\varphi) f\varphi^2\,dm - \int\Gamma(f,\log\varphi) f\varphi^2\,dm|\\ &\le |\textstyle\int \alpha\tilde{R}_\alpha^\varphi \Gamma(\alpha R_\alpha f - f,\log\varphi) f\varphi^2\,dm|\\ &\quad + |\textstyle\int \alpha\tilde{R}_\alpha^\varphi \Gamma(f,\log\varphi) f\varphi^2\,dm - \int\Gamma(\mathrm{f},\log\varphi) f\varphi^2\,dm|,\end{aligned}$$

which converges to 0 by (4.12) and the facts that $\alpha R_\alpha f\to f$ in $D(\mathscr{E})$, $\Gamma(\cdot,\log\varphi): D(\mathscr{E})\to L^1(E,\varphi^2 m)$ is continuous (by (4.9)), and $\alpha\tilde{R}_\alpha^\varphi$ is a contraction on $L^1(E,\varphi^2 m)$ for any $\alpha>0$. Thus

$$\begin{aligned}\lim_{\alpha\to\infty}\int \alpha(f-\alpha\tilde{R}_\alpha^\varphi f)\, f\varphi^2\,dm &= \mathscr{E}(f,f\varphi^2) - 2\int\Gamma(f,\log\varphi) f\varphi^2\,dm\\ &= \int\Gamma(f,f\varphi^2)dm - \int\Gamma(f,\varphi^2) f\,dm = \int\Gamma(f,f)\,\varphi^2\,dm = \mathscr{E}^\varphi(f,f).\end{aligned}$$

Hence we have $f\in D(\tilde{\mathscr{E}}^\varphi)$ and $\tilde{\mathscr{E}}^\varphi(f,f)=\mathscr{E}^\varphi(f,f)$ by [13, Lemma 1.3.4]. Since $C_0(E)\cap D(\mathscr{E}^\varphi)$ is dense in $D(\mathscr{E}^\varphi)$, we have shown that $(\tilde{\mathscr{E}}^\varphi, D(\tilde{\mathscr{E}}^\varphi))$ extends $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$.

On the other hand, for any bounded function $f\in D(\tilde{\mathscr{E}}^\varphi)$ we have

$$\begin{aligned}\left|\int \alpha\tilde{R}_\alpha^\varphi \Gamma(\alpha R_\alpha f,\log\varphi) f\,dm\right| &\le \|f\|_\infty\cdot n^2\cdot \int |\alpha\tilde{R}_\alpha^\varphi \Gamma(\alpha R_\alpha f,\log\varphi)|\varphi^2\,dm\\ \le \|f\|_\infty\cdot n^4\cdot\sup_{\alpha>0}\left(\int\Gamma(\alpha R_\alpha f)\,dm\right)^{\frac12}&\cdot\left(\int\Gamma(\log\varphi)\,dm\right)^{\frac12} =: c<\infty.\end{aligned}$$

Here we used that $\alpha\tilde{R}_\alpha^\varphi$ is a contraction on $L^1(E,\varphi^2 m)$, $\Gamma(\log\varphi)$ is in $L^1(E,m)$ by (4.9), and

$$\int\Gamma(\alpha R_\alpha f)\,dm = \mathscr{E}(\alpha R_\alpha f)\le\mathscr{E}(f)\qquad \forall\alpha>0$$

(cf. [14, I.2.22(ii) and I.2.22]). Thus by (4.7) and [13, Lemma 1.3.4] we obtain

$$(4.14)\qquad \limsup_{\alpha\to\infty}\int\alpha(f-\alpha R_\alpha f) f\,dm$$

$$\leq \limsup_{\alpha\to\infty} \int \alpha(f-\alpha\tilde{R}^{\varphi}_{\alpha}f)f\varphi^{-2}\varphi^{2}\,dm + 2\sup_{\alpha>0}\int \alpha\tilde{R}^{\varphi}_{\alpha}\Gamma(\alpha R_{\alpha}f,\log\varphi)f\,dm$$
$$\leq \tilde{\mathscr{E}}^{\varphi}(f,f\varphi^{-2})+c<\infty.$$

Note that, in fact, $f\varphi^{-2}$ is in $D(\tilde{\mathscr{E}}^{\varphi})$, since ψ is in $D(\mathscr{E})=D(\mathscr{E}^{\varphi})\subset D(\tilde{\mathscr{E}}^{\varphi})$, the map $T:\boldsymbol{R}\to\boldsymbol{R}$, $T(s):=((s\wedge n)\vee n^{-1})^{-2}-n^{2}$ is Lipschitz continuous, bounded and vanishes at 0, and $f\varphi^{-2}=f\cdot T(\psi)+n^{2}f$.

By (4.14), f is in $D(\mathscr{E})$ and thus in $D(\mathscr{E}^{\varphi})$. Hence $(\tilde{\mathscr{E}}^{\varphi},D(\tilde{\mathscr{E}}^{\varphi}))=(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$, i.e. $\tilde{\boldsymbol{M}}^{\varphi}$ is properly associated with $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$. □

b) Regularization

From now on, we again assume only that E is a metrizable Lusin space, m is a σ-finite positive measure on $\mathscr{B}(E)$, and $(\mathscr{E},D(\mathscr{E}))$ is a quasi-regular Dirichlet form on $L^2(E,m)$ satisfying (D1) and (D2). In this subsection, we keep the assumptions on φ from a), and we define $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ as in (4.1). Since φ is bounded from above and below, it is obvious that every $\mathscr{E}$-nest is an $\mathscr{E}^{\varphi}$-nest, and $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ is quasi-regular. By the regularization method of Z. M. Ma and M. Röckner (s. [14, Ch.VI]), we can transfer the result of Proposition 4.1 to the more general situation considered here. A detailed description of the application of the regularization method in a similar situation is given in [10, p.56, "Reduktion von 43 auf 45"]. Here we just note (s. [10, Appendix, Section 6] for proofs resp. references to proofs) that, since φ is bounded from above and below, we can find an $\mathscr{E}$- (and hence $\mathscr{E}^{\varphi}$-) nest $(E_k)_{k\in\boldsymbol{N}}$ consisting of compact metrizable subsets in E, and a locally compact separable metric space $\hat{E}$ containing $Y:=\bigcup E_k$ as a dense subset, such that:

- $Y\in\mathscr{B}(\hat{E})$ and $\mathscr{B}(Y)=\mathscr{B}(\hat{E})\cap Y$,
- The trace topologies on E_k, $k\in\boldsymbol{N}$, induced by E, $\hat{E}$ respectively, coincide,
- The measure $\hat{m}$ obtained by restricting m to $\mathscr{B}(Y)$ and trivially extending to $\mathscr{B}(\hat{E})$ is a positive Radon measure,
- The Dirichlet forms $(\hat{\mathscr{E}},D(\hat{\mathscr{E}}))$ and $(\hat{\mathscr{E}}^{\varphi},D(\hat{\mathscr{E}}^{\varphi}))$ on $L^2(\hat{E},\hat{m})$ resp. $L^2(\hat{E},\hat{\varphi}^2\hat{m})$ defined by

$$D(\hat{\mathscr{E}})=\mathscr{T}(D(\mathscr{E})),\qquad \hat{\mathscr{E}}(\mathscr{T}u,\mathscr{T}v):=\mathscr{E}(u,v),$$
$$D(\hat{\mathscr{E}}^{\varphi})=\mathscr{T}^{\varphi}(D(\mathscr{E}^{\varphi})),\qquad \hat{\mathscr{E}}^{\varphi}(\mathscr{T}^{\varphi}u,\mathscr{T}^{\varphi}v):=\mathscr{E}^{\varphi}(u,v),$$

are regular. Here $\mathscr{T}:L^2(E,m)\to L^2(\hat{E},\hat{m})$ is the canonical isometry ($m(E\setminus Y)=\hat{m}(\hat{E}\setminus Y)=0$ and $m|_{\mathscr{B}(Y)}=\hat{m}|_{\mathscr{B}(Y)}$ imply $L^2(E,m)\cong L^2(Y,m|_Y)\cong L^2(\hat{E},\hat{m})$ canonically, $\hat{\varphi}:=\varphi\circ\mathscr{T}$, and $\mathscr{T}^{\varphi}:L^2(E,\varphi^2m)\to L^2(\hat{E},\hat{\varphi}^2\cdot\hat{m})$ is again the canonical isometry.

Clearly, $(\hat{\mathscr{E}}^{\varphi},D(\hat{\mathscr{E}}^{\varphi}))$ is the $\hat{\varphi}$-transform of $(\hat{\mathscr{E}},D(\hat{\mathscr{E}}))$ in the sense of (4.1). Moreover, if $\boldsymbol{M}=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in E_\Delta})$ and $\boldsymbol{M}^{\varphi}=(\Omega,\mathscr{F}^{\varphi},(X_t)_{t\geq 0},(P^{\varphi}_z)_{z\in E_\Delta})$ are canonical diffusions properly associated with $(\mathscr{E},D(\mathscr{E}))$ resp. $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$, we can find canonical diffusions $\hat{\boldsymbol{M}}=(\hat{\Omega},\hat{\mathscr{F}},(\hat{X}_t)_{t\geq 0},(\hat{P}_z)_{z\in\hat{E}_\Delta})$ and $\hat{\boldsymbol{M}}^{\varphi}=(\hat{\Omega},\hat{\mathscr{F}}^{\varphi},(\hat{X}_t)_{t\geq 0},(\hat{P}^{\varphi}_z)_{z\in\hat{E}_\Delta})$

properly associated with $(\hat{\mathscr{E}},D(\hat{\mathscr{E}}))$ resp. $(\hat{\mathscr{E}}^{\varphi},D(\hat{\mathscr{E}}^{\varphi}))$ such that

$$(4.15)\qquad P_z[\Omega\cap\hat{\Omega}]=\hat{P}_z[\Omega\cap\hat{\Omega}]=P_z^{\varphi}[\Omega\cap\hat{\Omega}]=\hat{P}_z^{\varphi}[\Omega\cap\hat{\Omega}]=1,$$
$$P_z|_{\Omega\cap\hat{\Omega}}=\hat{P}_z|_{\Omega\cap\hat{\Omega}}\quad\text{and}\quad P_z^{\varphi}|_{\Omega\cap\hat{\Omega}}=\hat{P}_z^{\varphi}|_{\Omega\cap\hat{\Omega}}$$

for $\mathscr{E}$-resp. $\mathscr{E}^{\varphi}$-q.e. $z\in Y$ and thus $\mathscr{E}$-/$\mathscr{E}^{\varphi}$-q.e. $z\in E$. Here $\Omega\cap\hat{\Omega}$ is the space of Y-valued paths with possibly finite life-time, which are continuous both w.r.t. the E-and $\hat{E}$-topology.

If $\boldsymbol{M}$ is conservative, $\hat{\boldsymbol{M}}$ is conservative, too. Hence we may apply Proposition 4.1 to conclude that for $\hat{\mathscr{E}}$-q.e. $z\in E$, $\hat{P}_z^{\varphi}$ and $\hat{P}_z$ are locally equivalent with density $\exp\,(\hat{M}_t^{[\log\hat{\varphi}]}-\frac{1}{2}\langle\hat{M}^{[\log\hat{\varphi}]}\rangle_t)$. Here $\hat{M}_t^{[\log\hat{\varphi}]}$ is the local martingale part in the Fukushima decomposition w.r.t. $\hat{\boldsymbol{M}}$. By restricting $\hat{M}_t^{[\log\hat{\varphi}]}$ and the corresponding locally-zero-energy-part $\hat{N}_t^{[\log\hat{\varphi}]}$ to $\Omega\cap\hat{\Omega}$, and then extending trivially to Ω, we obtain the Fukushima decomposition of $\log\varphi$ w.r.t. $\boldsymbol{M}$ (cp. [14, proof of VI.2.5.]). If $M_t^{[\log\hat{\varphi}]}$ denotes the local martingale part of this decomposition, then Proposition 4.1 holds because of (4.15).

c) The general case

Finally, we are going to prove Theorem 1.5 by localizing to the sets E_k^{φ}, $k\in\boldsymbol{N}$. The proofs in this subsection are modifications of those in [4]. The application of the techniques from this article to our more general situation becomes possible because of the results of Sections 2 and 3 above.

So consider again the setting from the introduction. In particular, we assume only $\varphi\in D_{\mathrm{loc}}(\mathscr{E},(E_k))$, $\varphi>0$ m-a.e., and we define E_k^{φ}, $\varphi^{(k)}$, φ^k and σ_k for $k\in\boldsymbol{N}$ as in (1.15) and below. Let $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi}))$ be defined as in the introduction and $(\mathscr{E}^{\varphi_k},D(\mathscr{E}^{\varphi_k}))$ for $k\in\boldsymbol{N}$ as in (4.1). For a space $\mathscr{F}$ of functions $E\to\boldsymbol{R}$ and $A\in\mathscr{B}(E)$ we set $\mathscr{F}_A:=\{u\in\mathscr{F};\ u=0\ m\text{-a.e. on } E\backslash A\}$.

Lemma 4.2. $(\mathscr{E}^{\varphi},D(\mathscr{E}^{\varphi})_{E_{k-1}^{\varphi}})=(\mathscr{E}^{\varphi_k},D(\mathscr{E}^{\varphi_k})_{E_{k-1}^{\varphi}})$ *for all* $k\in\boldsymbol{N}\backslash\{1\}$.

Proof. Fix $k\in\boldsymbol{N}$, and let $u\in D(\mathscr{E}^{\varphi_k})_{E_{k-1}^{\varphi}}$ $(=D(\mathscr{E})_{E_{k-1}^{\varphi}})$. Since $\varphi=\varphi_k$ m-a.e. on E_{k-1}^{φ}, we have

$$\int(\Gamma(u)+u^2)\,\varphi^2\,dm=\int(\Gamma(u)+u^2)\,\varphi_k^2\,dm=\mathscr{E}_1^{\varphi_k}(u)<\infty.$$

Hence u is in $D(\mathscr{E}^{\varphi})_{E_{k-1}^{\varphi}}$ and

$$\mathscr{E}^{\varphi}(u)=\int\Gamma(u)\,\varphi^2\,dm=\int\Gamma(u)\,\varphi_k^2\,dm=\mathscr{E}^{\varphi_k}(u).$$

On the other hand, let u be a bounded element of $D(\mathscr{E}^{\varphi})_{E_{k-1}^{\varphi}}$. Then there is a sequence $u_n\in\mathscr{D}^{\varphi}$ $(n\in\boldsymbol{N})$ that converges to u w.r.t. the $\mathscr{E}_1^{\varphi}$-norm. Now fix

$g_k \in C^1([0,\infty) \to [0,1])$ such that $g_k = 1$ on $[(k-1)^{-1}, k-1]$ and $g_k = 0$ outside (k^{-1}, k), and let $v_n := (u_n \wedge \|u\|_\infty) \cdot e_k \cdot g_k(\varphi)$. We can find a subsequence of $(u_n)_{n\in N}$ that converges to u m-a.e. But $v_n = u_n \wedge \|u\|_\infty$ m-a.e. on E_{k-1}^φ, and $|v_n - u| = |v_n| \le |u_n| = |u_n - u|$ m-a.e. on $E \setminus E_{k-1}^\varphi$ for any $n \in N$. Hence there is also a subsequence of $(v_n)_{n\in N}$ converging m-a.e. to u. Moreover, $e_k \cdot g_k(\varphi)$ is in $D(\mathscr{E})$ by the same argument as in (2.4), and in $D(\mathscr{E}^\varphi)$ by Lemma 2.4. So for any $n \in N$, v_n is in $D(\mathscr{E})$ $(= D(\mathscr{E}^{\varphi_k}))$, and

$$\mathscr{E}_1^{\varphi_k}(v_n) = \int (\Gamma(v_n) + v_n^2)\, \varphi_k^2\, dm = \int (\Gamma(v_n) + v_n^2)\, \varphi^2\, dm = \mathscr{E}_1^\varphi(v_n)$$

$$\le (\|u\|_\infty \cdot \mathscr{E}^\varphi(e_k \cdot g_k(\varphi))^{\frac{1}{2}} + \mathscr{E}^\varphi(u_n)^{\frac{1}{2}})^2 + \int u_n^2\, \varphi^2\, dm.$$

Since $(u_n)_{n\in N}$ is bounded w.r.t. the $\mathscr{E}_1^\varphi$-norm, $(v_n)_{n\in N}$ is bounded w.r.t. the $\mathscr{E}_1^{\varphi_k}$-norm. Therefore the Césaro moeans of a subsequence of $(v_n)_{n\in N}$ converge to u w.r.t. the $\mathscr{E}_1^{\varphi_k}$-norm, in particular u is in $D(\mathscr{E}^{\varphi_k})$. Thus $D(\mathscr{E}^\varphi)_{E_{k-1}^\varphi} \subset D(\mathscr{E}^{\varphi_k})_{E_{k-1}^\varphi}$, which completes the proof of the lemma. □

Let $M^\varphi = (\Omega, \mathscr{F}^\varphi, (X_t)_{t\ge 0}, (P_z^\varphi)_{z\in E_\Delta})$ and $M^{\varphi_k} = (\Omega, \mathscr{F}^{\varphi_k}, (X_t)_{t\ge 0}, (P_z^{\varphi_k})_{z\in E_\Delta})$, $k \in N$, be canonical diffusions, which are properly associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$, $(\mathscr{E}^{\varphi_k}, D(\mathscr{E}^{\varphi_k}))$ respectively.

Corollary 4.3. *Let $k \in N \setminus \{1\}$. Then for $\mathscr{E}^\varphi$-q.e. $z \in E$:*

$P_z^\varphi|_{\mathscr{B}_t \cap \{t < \sigma_{k-1}\}} = P_z^{\varphi_k}|_{\mathscr{B}_t \cap \{t < \sigma_{k-1}\}}$ *for all $t \ge 0$.*

Proof. For $t \ge 0$, $\alpha > 0$, $z \in E$, and any positive or bounded measurable $f: E \to \boldsymbol{R}$ let

$$p_t^k f(z) := E_z^{\varphi_k}[f(X_t), t < \sigma_{k-1}], \quad p_t^\infty f(z) := E_z^\varphi[f(X_t), t < \sigma_{k-1}],$$

$$R_\alpha^k f(z) := \int_0^\infty e^{-\alpha t} p_t^k f(z)\, dt = E_z^{\varphi_k}\left[\int_0^{\sigma_{k-1}} e^{-\alpha t} f(X_t)\, dt\right],$$

$$R_\alpha^\infty f(z) := \int_0^\infty e^{-\alpha t} p_t^\infty f(z)\, dt = E_z^\varphi\left[\int_0^{\sigma_{k-1}} e^{-\alpha t} f(X_t)\, dt\right].$$

We fix $\alpha > 0$ and a non-negative function $f: E \to \boldsymbol{R}$ which is square-integrable both w.r.t. m and $\varphi^2 m$. By [14, IV.5.6], and since $\sigma_{k-1} = 0$ $P_z^{\varphi_k}$-a.s. for m-a.e. $z \in E \setminus E_{k-1}^\varphi$, we have $R_\alpha^k f \in D(\mathscr{E}^{\varphi_k})_{E_{k-1}^\varphi}$ and

$$\mathscr{E}_\alpha^{\varphi_k}(R_\alpha^k f, v) = \int f v\, \varphi_k^2\, dm \qquad \forall v \in D(\mathscr{E}^{\varphi_k})_{E_{k-1}^\varphi}.$$

Similarly we obtain, using Lemma 4.2, $R_\alpha^\infty f \in D(\mathscr{E}^\varphi)_{E_{k-1}^\varphi} = D(\mathscr{E}^{\varphi_k})_{E_{k-1}^\varphi}$ and

$$\mathscr{E}_\alpha^{\varphi_k}(R_\alpha^\infty f,v)=\mathscr{E}_\alpha^\varphi(R_\alpha^\infty f,v)=\int f v\,\varphi^2\,dm=\int f v\,\varphi_k^2\,dm \qquad \forall v\in D(\mathscr{E}^{\varphi_k})_{E_{k-1}^\varphi}.$$

Hence $\mathscr{E}_\alpha^{\varphi_k}(R_\alpha^\infty f-R_\alpha^k f, R_\alpha^\infty f-R_\alpha^k f)=0$, and $R_\alpha^\infty f=R_\alpha^k f$ m-a.e.

Moreover, $R_\alpha^k f$ is $\mathscr{E}^{\varphi_k}$-quasi-continuous (cp. [14, proof of IV.5.25 (ii)]) and thus $\mathscr{E}$-, and by Theorem 1.4 (i), $\mathscr{E}^\varphi$-quasi-continuous. Since $R_\alpha^\infty f$ is $\mathscr{E}^\varphi$-quasi-continuous as well, we have

(4.16) $$R_\alpha^\infty f=R_\alpha^k f \;\; \mathscr{E}^\varphi\text{-q.e.}$$

By dominated convergence, we see that (4.16) holds for any bounded measurable $f\colon E\to \boldsymbol{R}$. Let $\mathscr{T}$ be a countable subset of $C_b(E)$, that separates the points of E, is closed under multiplication, and contains the constant function 1. The $\mathscr{E}^\varphi$-exceptional set in (4.16) can be chosen independent of $\alpha\in\boldsymbol{Q}^+$ and $f\in\mathscr{T}$. Since $t\mapsto p_t^\infty f(z)=E_z^\varphi[f(X_t),\, t<\sigma_{k-1}]$ and $t\mapsto p_t^k f(z)$ are right-continuous for any $f\in\mathscr{T}$ and $z\in E$, we obtain by the uniqueness of the Laplace-tramsform:

(4.17) $$p_t^\infty f=p_t^k f \quad \text{for all } t\geq 0 \text{ and } f\in\mathscr{T},\; \mathscr{E}^\varphi\text{-q.e.}$$

$\mathscr{T}$ generates $\mathscr{B}(E)$, because E is a metrizable Lusin space (s. [21, p. 102, Cor. 1, and p. 108, Lem. 18]). Thus, by monotone class arguments, (4.17) remains true if $\mathscr{T}$ is replaced by the set of all bounded $\mathscr{B}(E)$-measurable functions $f\colon E\to\boldsymbol{R}$. Moreover, by [14, IV.6.5], we obtain for $\mathscr{E}^\varphi$-q.e. $z\in E$:

$$P_z^\varphi[p_t^\infty f(X_s)=p_t^k f(X_s)]=1 \quad \text{for all } t,s\geq 0 \text{ and all bounded meas. } f\colon E\to\boldsymbol{R}.$$

The claim now follows by standard arguments, cp. [4, Cor. 4.5] or [10, Folgerung 67]. □

Proof of Theorem 1.5. By Lemma 3.2, $(E_k^\varphi)_{k\in\boldsymbol{N}}$ is an $\mathscr{E}^\varphi$-nest. Hence Claim (i) follows by [14, IV.5.30.(i)]. Claim (ii) is a direct consequence of Corollary 4.3 and Proposition 4.1, which can be applied to $(\mathscr{E}^{\varphi_k},D(\mathscr{E}^{\varphi_k}))$, $k\in\boldsymbol{N}$, because of the considerations in subsection b). For $\mathscr{E}^\varphi$-q.e. $z\in E$, P_z^φ is locally absolutely continuous w.r.t. P_z up to $\sup_{k\in\boldsymbol{N}}\sigma_k$ by (ii), and hence by (i) up to ζ. □

5. Examples

a) Brownian motion on a complete Riemannian manifold

Let $E:=M$ be a complete connected Riemannian manifold with metric g and corresponding volume element $d\mathfrak{v}$. For $u,v\in C_0^\infty(M)$ we set

$$\mathscr{E}(u,v):=\tfrac{1}{2}\int g(\nabla u,\nabla v)\,d\mathfrak{v}\;\left(=-\tfrac{1}{2}\int u\Delta v\,d\mathfrak{v}\right).$$

Here ∇ denotes the gradient and Δ the Laplace Beltrami operator on M.

$(\mathscr{E},C_0^\infty(M))$ is closable on $L^2(M,\mathfrak{v})$ and the closure $(\mathscr{E},D(\mathscr{E}))$ is a regular Dirichlet form satisfying (D 1) and (D 2). $D(\mathcal{E})$ is the usual Sobolev space $H^{1,2}(M)$, and

$$\Gamma(u,v)=\tfrac{1}{2}g(\nabla u,\nabla v) \qquad (u,v\in H^{1,2}(M)).$$

We may choose

$$\mathscr{C}:=\{\xi\in C_0^\infty(M);\ g(\nabla\xi,\nabla\xi)\leq 1\}$$

$$\text{or} \qquad \mathscr{C}:=\{\xi\in C_0^\infty(M)\cap H^{1,2}(M);\ g(\nabla\xi,\nabla\xi)\leq 1 \text{ a.e.}\}.$$

In both cases the metric ρ generated by $\mathscr{C}$ coincides with the distance-function of (M,g). In particular, ρ is finite and (M,ρ) is separable.

By Example 2 in Section 1, $D_{\text{loc}}(\mathcal{E},(E_k))$ is the local Sobolev space $H^{1,2}_{\text{loc}}(M)$, and $D_0(\mathcal{E},(E_k))$ consists of all functions in $H^{1,2}(M)$ with compact support. Since $C_0^\infty(M)$ is dense in $H^{1,2}(M)$, (D3) is satisfied. Hense we may apply Theorems 1.1 and 1.4 to conclude that for any $\varphi\in H^{1,2}_{\text{loc}}(M)$, $(\mathcal{E}^\varphi,D(\mathcal{E}^\varphi))$ is a quasi-regular Dirichlet form with strong local property, that extends

$$\mathcal{E}^\varphi(u,v)=\frac{1}{2}\int g(\nabla u,\nabla v)\,\varphi^2\,d\mathfrak{v} \qquad (u,v\in C_0^\infty(M)). \tag{5.1}$$

$(\mathcal{E}^\varphi,D(\mathcal{E}^\varphi))$ is the closure of (5.1) if and only if (1.12) holds for $\mathscr{A}=C_0^\infty(M)$; e.g. the conditions in Corollary 1.3 are sufficient. The diffusion $\boldsymbol{M}$ properly associated with $(\mathcal{E},D(\mathcal{E}))$ is Brownian motion on (M,g). We assume that $\boldsymbol{M}$ is conservative. This is the case (s. e.g.[23, Th. 4]), if the volume growth condition

$$\int_1^\infty \frac{r}{\log V(r)}dr=\infty$$

holds. Here $V(r)$ is the volume of the ball of radius r around a fixed point $p\in M$. By Theorem 1.5, the law of the diffusion $\boldsymbol{M}^\varphi$ properly associated with $(\mathcal{E}^\varphi,D(\mathcal{E}^\varphi))$ is locally absolutely continuous, with density as in 1.5, w.r.t. Wiener measure on the path space of M.

REMARKS. (i) In the special case $M=\boldsymbol{R}^d$ with Euclidean metric, (1.12) holds for any $\varphi\in H^{1,2}_{\text{loc}}(\boldsymbol{R}^d)$. In fact, it is shown in [20, Proof of 3.1] by probabilistic arguments, that $f_n(\log\varphi)\cdot v$ is in $\overline{C_0^\infty(\boldsymbol{R}^d)}$ (i.e. in the closure of $C_0^\infty(\boldsymbol{R}^d)$ w.r.t. the $\mathcal{E}_1^\varphi$-norm) for any $n\in\boldsymbol{N}$ and $v\in C_0^\infty(\boldsymbol{R}^d)$. An application of Proposition A 1 in the appendix and of the subsequent remark to the form $(\mathcal{E}^\varphi,\overline{C_0^\infty(\boldsymbol{R}^d)})$ shows that e_k is in $\overline{C_0^\infty(\boldsymbol{R}^d)}$ for any $k\in\boldsymbol{N}$. Thus we obtain $f_n(\log\varphi)\cdot e_k\in\overline{C_0^\infty(\boldsymbol{R}^d)}$ $\forall k,n\in\boldsymbol{N}$ by approximating e_k with $C_0^\infty(\boldsymbol{R}^d)$-functions.

Consequently, the Dirichlet form of the Girsanov-transform of Brownian motion in $\boldsymbol{R}^d$ w.r.t. φ is the closure of (5.1) for any $\varphi\in H^{1,2}_{\text{loc}}(\boldsymbol{R}^d)$, i.e. we obtain M. Takeda's result mentioned in the introduction.

(ii) Similarly, sufficient criteria for absolute continuity w.r.t. diffusion processes generated by more general elliptic differential operators on manifolds (with not necessarily continuous coefficients) may be derived.

b) Reflected Brownian motion

Let $U \subset \boldsymbol{R}^d$, $d \in \boldsymbol{N}$, be a bounded Euclidean domain with smooth boundary ∂U, and $dm := dx$ Lebesgue-measure on U. The form given by

$$D(\mathscr{E}) := H^{1,2}(U), \qquad \mathscr{E}(u,v) := \frac{1}{2}\int \nabla u \cdot \nabla v \, dx$$

is a symmetric Dirichlet form on $L^2(U,dx)$ with strong local property and square-field-operator $\Gamma(u,v) = \nabla u \cdot \nabla v$. $(\mathscr{E}, D(\mathscr{E}))$ is not quasi-regular, but by identifying $L^2(U,dx) \cong L^2(\bar{U},dx)$ we may consider $(\mathscr{E}, D(\mathscr{E}))$ a regular Dirichlet form on $\bar{U}$. For $\mathscr{C} := \{0\}$, ρ is the Euclidean metric, and $D_{\mathrm{loc}}(\mathscr{E},(E_k)) = D_0(\mathscr{E},(E_k)) = D(\mathscr{E})$. (D 3) holds, because the image of $C^\infty(\bar{U})$ under the 1-resolvent of $(\mathscr{E}, D(\mathscr{E}))$ is dense in $D(\mathscr{E})$, and by elliptic regularity (cf. [5, 10.17, 10.18 and 8.8]), contained in $C^1(\bar{U}) \cap D(L) \subset D^{1,\infty}(\mathscr{E}) \cap D(L)$. By [13, Thm. 1.6.6], the corresponding diffusion process is conservative. It conincides with reflected Brownian motion on $\bar{U}$. For $\varphi \in H^{1,2}(U)$, Theorems 1.1 - 1.5 apply as above.

c) Infinite-dimensional gradient Dirichlet forms

Let E be a (real) separable Banach space and m a finite measure on $\mathscr{B}(E)$ which charges every weakly open set. For $K \subset E'$ let

$$\mathscr{F}C_b^\infty(K) := \{f(l_1, \cdots, l_n);\ n \in \boldsymbol{N},\ f \in C_b^\infty(\boldsymbol{R}^n),\ l_1, \cdots, l_n \in K\}$$

be the smooth cylinder functions based on K, and let $\mathscr{F}C_b^\infty := \mathscr{F}C_b^\infty(E')$. Here $C_b^\infty(\boldsymbol{R}^n)$ is the space of all smooth bounded functions on $\boldsymbol{R}^n$ with bounded partial derivatives. By the Hahn-Banach theorem, E' separates the points of E. If K is a dense subspace of E', then K and $\mathscr{F}C_b^\infty(K)$ separate the points, too. The support condition on m implies that we can regard $\mathscr{F}C_b^\infty(K)$ as a subspace of $L^2(E,m)$, and a monotone class argument shows that it is dense in $L^2(E,m)$. For $u = f(l_1, \cdots, l_n) \in \mathscr{F}C_b^\infty$ let $du : E \to E'$,

$$du(z) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(l_1(z), \cdots, l_n(z)) \cdot l_i,$$

be the differential of u.

We now assume in addition that we are given a separable real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ densely and continuously embedded into E. H should be thought of as a "tangent space"to E at each fixed point $z \in E$. Identifying H with its dual H', we have that

$$E' \subset H \subset E \text{ densely and continuously,} \tag{5.2}$$

and the dualization between E' and E restricted to $E' \times H$ coincides with $\langle \cdot, \cdot \rangle$. We define the gradient $\nabla u: E \to H$ as the function obtained from du via the imbedding (5.2). In particular, for $h \in H$ and $z \in E$ we have

$$\langle \nabla u(z), h \rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(l_1(z), \cdots, l_n(z)) \cdot l_i(h) = \frac{d}{ds} u(z+sh)|_{s=0} = \frac{\partial u}{\partial h}(z).$$

For $u, v \in \mathscr{F}C_b^\infty$ let

$$\mathscr{E}(u,v) := \int \langle \nabla u(z), \nabla v(z) \rangle m(dz). \tag{5.3}$$

We assume that there are a dense subspace $K \subset E'$ and functions $\beta_k \in L^2(E,m)$ ($k \in K$), such that the integration by parts formula

$$\int \frac{\partial u}{\partial k} v \, dm = - \int u \frac{\partial v}{\partial k} \, dm - \int u \, v \, \beta_k \, dm \tag{5.4}$$

holds for any $u, v \in \mathscr{F}C_b^\infty$ and $k \in K$. In this case, the bilinear form $(\mathscr{E}, \mathscr{F}C_b^\infty)$ is closable on $L^2(E,m)$, and the closure $(\mathscr{E}, D(\mathscr{E}))$ is a Dirichlet form with $\mathscr{F}C_b^\infty(K)$ in the domain of its generator (s. [19, Sect. 1]). Moreover, it follows from the chain rule, that $(\mathscr{E}, D(\mathscr{E}))$ satisfies (D 1) and (D 2). The square field operator is the unique continuous bilinear extension of $\Gamma(u,v) := \langle \nabla u, \nabla v \rangle$, $u, v \in \mathscr{F}C_b^\infty$, to $D(\mathscr{E})$. By [14, Sect. IV 4 b)], $(\mathscr{E}, D(\mathscr{E}))$ is quasi-regular. Let $\boldsymbol{M} = (\Omega, \mathscr{F}, (X_t)_{t \geq 0}, (P_z)_{z \in E})$ be the properly associated canonical diffusion.

There are two canonical choices for the set $\mathscr{C}$ generating the metric:

Choice A. If we just want to consider transformations with $\varphi \in D(\mathscr{E})$, $\varphi > 0$ m-a.e., we may set $\mathscr{C} := \{0\}$ (s. Example 1 in Section 1). Then $D_0(\mathscr{E}, (E_k)) = D_{\mathrm{loc}}(\mathscr{E}, (E_k))$ $= D(\mathscr{E})$. (D 3) holds, because $\mathscr{F}C_b^\infty(K)$ is dense in $D(\mathscr{E})$ by [2, Prop. 2.10]. Hence we may apply the theorems from Section 1 to conclude that $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ is a quasi-regular Dirichlet form with strong local property, that extends

$$\mathscr{E}^\varphi(u,v) = \int \langle \nabla u, \nabla v \rangle \, \varphi^2 \, dm \qquad (u, v \in \mathscr{F}C_b^\infty). \tag{5.5}$$

It is the closure of (5.5) if and only if $f_n(\log \varphi)$ is in the domain of this closure for any $n \in \boldsymbol{N}$, e.g. it is sufficient that φ is in this domain.

Let $\boldsymbol{M}^\varphi$ be the canonical diffusion properly associated with $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$. Since the constant function 1 is both in $D(\mathscr{E})$ and $D(\mathscr{E}^\varphi)$, and $\mathscr{E}(1) = \mathscr{E}^\varphi(1) = 0$, $\boldsymbol{M}$ and

$\boldsymbol{M}^{\varphi}$ are conservative (cf. [13, Th. 1.6.5]). Hence by Theorem 1.5, the path-space law of $\boldsymbol{M}^{\varphi}$ is locally absolutely continuous (up to ∞!) w.r.t. that of $\boldsymbol{M}$ for $\mathscr{E}^{\varphi}$-q.e. starting-point. This generalizes the main result of [4](Thm. 1.5).

REMARK. If E is large enough and

$$\int k^2(z)m(dz) < \infty \qquad \forall k \in K, \tag{5.6}$$

then it can be shown (s. [3, Thm. 6.6 and Ex. 6.4 (i)], that there is an E-valued stochastic process $(W_t)_{t\geq 0}$ on $(\Omega, \mathscr{F})$, which is a Wiener process starting at 0 with covariation

$$[l(W), l'(W)]_t = t \cdot \langle l, l' \rangle \qquad \forall l, l' \in E'$$

under P_z for $\mathscr{E}$-q.e. z, such that the stochastic equation

$$k(X_t) = k(X_0) + k(W_t) + \tfrac{1}{2}\int_0^t \beta_k(X_s)ds \qquad \forall t \geq 0 \ P_z\text{-a.s. for } \mathscr{E}\text{-q.e. } z \tag{5.7}$$

holds for any $k \in K$. In particular, if (E, H, m) is an abstract Wiener space, then $\boldsymbol{M}$ is an **infinite-dimensional Ornstein-Uhlenbeck process** (i.e. (5.7) holds with $\beta_k = k$ for any $k \in E'$), and if E, H, K and m are chosen as in [3, Sect. 7, II], then $\boldsymbol{M}$ is the stochastic quantization process of the **space-time resp. time-zero free quantum field** (-note that the space of tempered distributions in [3] may be replaced by a subspace which is a Banach space). If, moreover,

$$\int (k^2(z) + \beta_k^2(z))\varphi^2(z)m(dz) < \infty \qquad \forall k \in K, \tag{5.8}$$

then $\boldsymbol{M}^{\varphi}$ satisfies a stochastic equation of type (5.7) with β_k replaced by $\beta_k + k(\varphi^{-1}\nabla\varphi)$, cf. [10, Part I, (51)]. This is the starting-point for an alternative proof of our absolute continuity result based on an infinite-dimensional version of Girsanov's Theorem. This proof, which is carried out in [10, Satz 30], works in many concrete situations, including the infinite-dimensional Ornstein-Uhlenbeck processes and the processes from quantum-field-theory mentioned above. We also remark that in these situations, by [10, Proof of Satz 35 and Satz 22], $f_n(\log\varphi)$ is in the domain of the closure of (5.5) for any $n \in \boldsymbol{N}$, hence $\mathscr{F}C_b^{\infty}$ is dense in $D(\mathscr{E}^{\varphi})$.

Choice B. To introduce a suitable local Dirichlet space let

$$\mathscr{C} := \{u \in \mathscr{F}C_b^{\infty};\ \|du(z)\|_{E'} \leq M^{-1} \forall z \in E\},$$

where M denotes the operator norm of the imbedding $E' \hookrightarrow H$. Clearly, $\mathscr{C}$ consisits of continuous functions u satisfying $\Gamma(u) = \langle \nabla u, \nabla u \rangle \leq 1$. The metric ρ generated by $\mathscr{C}$ is given by

$$\rho(x,y) = M^{-1} \cdot \|x-y\|_E \qquad \forall x,y \in E.$$

In fact, "$\leq$" holds because

$$u(x) - u(y) \leq \sup_{z \in E} \|du(z)\|_{E'} \cdot \|x-y\|_E \leq M^{-1} \cdot \|x-y\|_E \qquad \forall u \in \mathscr{C},$$

and "$\geq$" follows from

$$\|x-y\|_E = \sup\{f(l(x)) - f(l(y));\ l \in E',\ \|l\|_{E'} = 1,\ f \in C_b^\infty(\boldsymbol{R}),\ |f'| \leq 1\}$$
$$\leq \sup\{u(x) - u(y);\ u \in \mathscr{F}C_b^\infty,\ \|du(z)\|_{E'} \leq 1\ \forall z \in E\} = M \cdot \rho(x,y)$$

In particular, ρ is finite, (E,ρ) is separable, and the sets E_k $(k \in \boldsymbol{N})$ are balls in E. If (D 3) holds, the theorems from Section 1 apply to any $\varphi \in D_{\text{loc}}(\mathscr{E},(E_k))$, i.e. to any measurable $\varphi: E \to \boldsymbol{R}$ that coincides with some function from $D(\mathscr{E})$ m-a.e. on every fixed ball.

Claim. (D 3) holds, if the following two additional assumptions are satisfied:

(5.9) E is a Hilbert space

(5.10) m has a square-integrable logarithmic derivative, i.e. there is a function $\beta \in L^\infty(E \to E, m)$, such that (5.4) holds with $\beta_k = k(\beta)$ for any $k \in E'$.

Proof. We fix decreasing functions $g_k \in C_b^\infty([0,\infty))$, $k \in \boldsymbol{N}$, such that $g_k = 1$ on $[0,k]$, $g_k = 0$ on $[k+2,\infty)$, and $|g_k'| \leq 1$. Because of (5.9), the functions $\tilde{e}_k(z) := g_k(\|z\|_E)$ are in $C_b^\infty(E)$, the space of all infinitely often Fréchet-differentiable functions on E with all derivatives bounded. Hence $\mathscr{F}_0 := \{u \cdot \tilde{e}_k;\ u \in \mathscr{F}C_b^\infty,\ k \in \boldsymbol{N}\}$ is a subset of $C_b^\infty(E)$ as well. It is shown in [1, Lemma 6], that (5.10) implies that $C_b^\infty(E)$ is contained in the domain $D(L)$ of the generator of $(\mathscr{E}, D(\mathscr{E}))$. Moreover, every function from $\mathscr{F}_0$ vanishes outside some E_k, and

$$\Gamma(u) = \langle \nabla u, \nabla u \rangle_H \leq M^2 \cdot \|du\|_{E'}^2 \in L^\infty(E,m) \qquad \forall u \in \mathscr{F}_0.$$

Thus $\mathscr{F}_0 \subset D(L) \cap D_0^{1,\infty}(\mathscr{E},(E_k))$, and it only remains to show that span $\mathscr{F}_0$ is dense in $D(\mathscr{E})$. This is the case, since for any $v \in \mathscr{F}C_b^\infty$, $(v \cdot \tilde{e}_k)_{k \in \boldsymbol{N}}$ converges to v m-a.e., and $\sup_{k \in \boldsymbol{N}} \mathscr{E}_1(v \cdot \tilde{e}_k) \leq 3\mathscr{E}_1(v) < \infty$ by Prop. A 1 in the appendix. □

REMARK. If (5.9) and (5.10) hold, then the "test-function" space

$$\mathscr{A} := \{u \in C_b^\infty(E);\ \text{supp}\, u \subset E_k \quad \text{for some } k \in \boldsymbol{N}\}$$

contatins $\mathscr{F}_0$ and hence satisfies the conditions imposed above Theorem 1.2.

d) Other examples

Other examples may be treated similarly to those described above. In particular, we refer to [18], [17] and [8] for the setup needed to apply out theorems to Fleming-Viot processes and processes on path and loop spaces of Riemannian manifolds.

Appendix: Metrics and Cut-off-functions related to Dirichlet Forms

Let E be a separable Hausdorff space, m a σ-finite positive measure on its Borel-σ-algebra, and $(\mathscr{E},D(\mathscr{E}))$ a Dirichlet form on $L^2(E,m)$ satisfying (D 1) and (D 2). Suppose $\mathscr{C}$ is a symmetric (i.e. $\mathscr{C}=-\mathscr{C}$) set of functions $\xi\in D(\mathscr{E})$ satisfying

$$\Gamma(\xi)\leq 1 \qquad m\text{-a.e.}$$

Let $\rho:E\times E\to[0,\infty]$,

$$\rho(x,y):=\sup_{\xi\in\mathscr{C}}(\xi(x)-\xi(y))$$

be the pseudo-metric generated by $\mathscr{C}$. Assume that (E,ρ) is **separable**. We fix $p\in E$ and set $\rho_p:=\rho(\cdot,p)$.

Proposition A 1. *Let u be a function in $D(\mathscr{E})$, and let $g:\boldsymbol{R}\cup\{\infty\}\to[0,1]$ be a decreasing and Lipschitz-continuous function such that $g=1$ on $(-\infty,0]$ and $g=0$ on $[b,\infty]$ for some $b>0$. Then $u\cdot g(\rho_p)$ is in $D(\mathscr{E})$ and*

$$\Gamma(u\cdot g(\rho_p))\leq 2(\Gamma(u)+\|g'\|_\infty^2\cdot u^2) \quad m\text{-a.e.}$$

Moreover, $u\cdot g(\rho_p)$ is $\mathscr{E}$-quasi-continuous, if both u and the functions in $\mathscr{C}$ are $\mathscr{E}$-quasi-continuous.

REMARK. Under additional assumptions, it follows from the results of [6] and [23] that $g(\rho_p)$ is in $D(\mathscr{E})$ and $\Gamma(g(\rho_p))\leq\|g'\|_\infty^2$. If there is a function $u\in D(\mathscr{E})$ such that $u=1$ m-a.e. on $\{\rho_p\leq b\}$, then $g(\rho_p)\in D(\mathscr{E})$ is a consequence of the proposition.

Corollary A 2. *Suppose there is an $\mathscr{E}$-nest $(F_k)_{k\in\boldsymbol{N}}$ such that the restrictions of the functions from $\mathscr{C}$ to each F_k, $k\in\boldsymbol{N}$, are continuous. Let $\tilde{E}_k:=F_k\cap\{\rho_p\leq k\}$. Assume further $\rho_p<\infty$ m-a.e. Then:*

(i) *$(\tilde{E}_k)_{k\in\boldsymbol{N}}$ is an $\mathscr{E}$-nest.*
(ii) *If the functions in $D(\mathscr{E})$ have $\mathscr{E}$-q.c. modifications, then ρ_p is $\mathscr{E}$-q.c.*

Proof of the corollary. (i) Let $k\in\boldsymbol{N}$. Since ρ_p is a supremum of functions with continuous restrictions to F_k, $\rho_p|_{F_k}$ is lower-semi-continuous. Thus $\tilde{E}_k$ is relatively closed in F_k and therefore closed in E. Now fix $u\in D(\mathscr{E})$ and $\varepsilon>0$. We can find

$l \in N$ and $v \in D(\mathscr{E})$, such that v vanishes m-a.e. on $E \backslash F_l$, and $\mathscr{E}_1(u-v)^{\frac{1}{2}} \leq \varepsilon/2$. For $k \in N$ let $e_k := (k-\rho_p)^+ \wedge 1$. Since ρ_p is finite m-a.e., $(v \cdot e_k)_{k \in N}$ converges to v m-a.e. Morever, by the proposition, $v \cdot e_k$ is in $D(\mathscr{E})$ and

$$\sup_k \mathscr{E}_1(v \cdot e_k) = \sup_k \int (|v \cdot e_k|^2 + \Gamma(v \cdot e_k))\, dm$$
$$\leq 3 \cdot \int (v^2 + \Gamma(v))\, dm = 3 \cdot \mathscr{E}_1(v) < \infty.$$

Thus the Césaro means of a subsequence of $(v \cdot e_k)_{k \in N}$ converge to v in $D(\mathscr{E})$. But $v \cdot e_k$ vanishes m-a.e. on $E \backslash \tilde{E}_{k \vee l}$, and thus the Césaro means are in $D_0(\mathscr{E}, (\tilde{E}_k))$. Therefore we can find $w \in D_0(\mathscr{E}, (\tilde{E}_k))$, such that $\mathscr{E}_1(v-w)^{\frac{1}{2}} \leq \varepsilon/2$ and hence $\mathscr{E}_1(u-w)^{\frac{1}{2}} \leq \varepsilon$. Since u and ε are arbitrary, $D_0(\mathscr{E}, (\tilde{E}_k))$ is dense in $D(\mathscr{E})$, and thus $(\tilde{E}_k)_{k \in N}$ is an $\mathscr{E}$-nest.

(ii) By [14, III 3.6], there is an $\mathscr{E}$-q.c. function $u \in D(\mathscr{E})$ such that $u > 0$ $\mathscr{E}$-q.e. Let $n \in N$. The proposition implies that $u \cdot (n-\rho_p)^+$ is $\mathscr{E}$-q.c., hence $\rho_p \wedge n$ $(= n - u^{-1} \cdot u \cdot (n-\rho_p)^+ \, \mathscr{E}\text{-q.e.})$ is $\mathscr{E}$-q.c., too. By (i), we can find an $\mathscr{E}$-nest $(\tilde{F}_k)_{k \in N}$ such that on $\tilde{F}_k$, ρ_p is bounded by k, and $\rho_p|_{\tilde{F}_k} = \rho_p \wedge k|_{\tilde{F}_k}$ is continuous. Hence ρ_p is $\mathscr{E}$-q.c. □

Proof of the proposition

Step 1. Fix $y \in E$ and $\varepsilon > 0$. We show that there is a function $\xi = \xi_{y,\varepsilon} \in \mathscr{C}$ such that

(A 1) $\xi(x) - \xi(p) \leq \rho_p(x) \quad \forall x \in E,$ and

(A 2) $\xi(x) - \xi(p) \geq \rho_p(x) \wedge b - 3\varepsilon \quad \forall x \in B_\varepsilon(y),$

where $B_\varepsilon(y)$ denotes the ρ-ball of radius ε around y.

In fact, by definition of ρ, we can find $\xi \in \mathscr{C}$ such that

$$\xi(y) - \xi(p) \geq \rho_p(y) \wedge b - \varepsilon.$$

Using the triangle inequality, we obtain for $x \in B_\varepsilon(y)$

$$\xi(x) - \xi(p) = \xi(x) - \xi(y) + \xi(y) - \xi(p)$$
$$\geq -\rho(x,y) + \rho_p(y) \wedge b - \varepsilon \geq \rho_p(x) \wedge b - 3\varepsilon.$$

Thus (A 2) holds. (A 1) follows from the definition of ρ.

Step 2. We fix a bounded, non-negative function $u \in D(\mathscr{E})$. For y, ε and ξ as in Step 1 let

$$v_{y,\varepsilon}(x) := u(x) \cdot g(\xi(x) - \xi(p)), \qquad x \in E.$$

I claim that $v_{y,\varepsilon}$ has the following properties:

(A 3) $v_{y,\varepsilon}(x) \geq u(x) \cdot g(\rho_p(x)) \quad \forall x \in E$

(A 4) $v_{y,\varepsilon}(x) \leq u(x) \cdot g(\rho_p(x)) + 3\varepsilon \cdot \|u\|_\infty \cdot \|g'\|_\infty \quad \forall x \in B_\varepsilon(y)$

(A 5) $v_{y,\varepsilon} \in D(\mathscr{E})$ and $\Gamma(v_{y,\varepsilon}) \leq 2(\Gamma(u) + \|g'\|_\infty^2 \cdot u^2)$ m-a.e.

In fact, (A 3) follows from (A 1), since g is decreasing. Moreover, g vanishes on $[b, \infty]$, and thus

$$v_{y,\varepsilon} - u \cdot g(\rho_p) = u \cdot g(\xi - \xi(p)) - u \cdot g(\rho_p \wedge b)$$
$$\leq \|u\|_\infty \cdot \|g'\|_\infty \cdot (\rho_p \wedge b - (\xi - \xi(p)))^+ \leq \|u\|_\infty \cdot \|g'\|_\infty \cdot 3\varepsilon$$

on $B_\varepsilon(y)$ by (A 2). Finally, the map

$$\hat{g}: \boldsymbol{R} \to \boldsymbol{R}, \qquad \hat{g}(s) := g(s - \xi(p)) - g(-\xi(p)),$$

is Lipschitz-continuous with constant $\|g'\|_\infty$, bounded, and vanishes at 0. Thus we have

$$v_{y,\varepsilon} = u \cdot \hat{g}(\xi) + u \cdot g(-\xi(p)) \in D(\mathscr{E}) \qquad \text{and}$$

$$\Gamma(v_{y,\varepsilon}) = u \cdot \Gamma(\hat{g}(\xi), v_{y,\varepsilon}) + \hat{g}(\xi) \cdot \Gamma(u, v_{y,\varepsilon}) + g(-\xi(p)) \cdot \Gamma(u, v_{y,\varepsilon})$$
$$= u \cdot \Gamma(\hat{g}(\xi), v_{y,\varepsilon}) + g(\xi - \xi(p)) \cdot \Gamma(u, v_{y,\varepsilon})$$
$$= u^2 \cdot \Gamma(\hat{g}(\xi)) + 2u \cdot g(\xi - \xi(p)) \cdot \Gamma(\hat{g}(\xi), u) + g(\xi - \xi(p))^2 \cdot \Gamma(u)$$
$$\leq 2 \cdot (u^2 \cdot \Gamma(\hat{g}(\xi)) + g(\xi - \xi(p))^2 \cdot \Gamma(u)) \leq 2 \cdot (u^2 \cdot \|g'\|_\infty^2 + \Gamma(u)) \; m\text{-a.e.}$$

Step 3. We prove the assertion of the lemma for bounded, non-negative functions $u \in D(\mathscr{E})$. By (A 3) and (A 4), we obtain for u and $v_{y,\varepsilon}$ as in Step 2

(A 6) $$u(x) \cdot g(\rho_p(x)) = \inf_{n \in \boldsymbol{N}} \inf_{y \in M} v_{y,n^{-1}}(x) \qquad \forall x \in E,$$

where M is a countable dense subset of (E, ρ). On the other hand, the strong local property and (A 5) imply that for any finite subset $K \subset \boldsymbol{N} \times M$ we have $\inf_{(n,y) \in K} v_{y,n^{-1}} \in D(\mathscr{E})$,

(A 7) $$\Gamma\left(\inf_{(n,y) \in K} v_{y,n^{-1}}\right) \leq \sup_{(n,y) \in K} \Gamma(v_{y,n^{-1}}) \leq 2(\Gamma(u) + \|g'\|_\infty^2 \cdot u^2) \; m\text{-a.e.}$$

(s. [7, I, Ex. 7.2]), and thus

$$\sup_K \mathscr{E}_1\left(\inf_{(n,y) \in K} v_{y,n^{-1}}\right) \leq 2 \cdot (1 + \|g'\|_\infty^2) \cdot \mathscr{E}_1(u) < \infty,$$

where the supremum is taken over all finite subsets K of $\boldsymbol{N} \times M$.

Using the Theorems of Banach-Saks and Banach-Alaoglu, we conclude that there is an increasing sequence $(K_l)_{l \in \boldsymbol{N}}$ of finite subsets of $\boldsymbol{N} \times M$, such that $\bigcup K_l = \boldsymbol{N} \times M$, and the Césaro-means w_l of $v_l := \inf_{(n,y) \in K_l} v_{y,n^{-1}}$ $(l \in \boldsymbol{N})$ converge in

$D(\mathscr{E})$. Because of (A 6) the limit is $u \cdot g(\rho_p)$, which is hence in $D(\mathscr{E})$. The first part of the assertion now follows from (A 7) and the continuity of $\Gamma: D(\mathscr{E}) \times D(\mathscr{E}) \to L^1(E,m)$.

Moreover, if u and the elements of $\mathscr{C}$ are $\mathscr{E}$-quasi-continuous, the functions $v_{y,n^{-1}}$ ($n \in N$, $y \in M$), and hence v_l and w_l ($l \in N$), are $\mathscr{E}$-q.c., too. Since $w_l \to u \cdot g(\rho_p)$ pointwise (s. (A 6)) and in $D(\mathscr{E})$, $u \cdot g(\rho_p)$ is $\mathscr{E}$-q.c. as well (s. [14, III 3.5]).

Step 4. (general $u \in D(\mathscr{E})$):
For any non-negative function $u \in D(\mathscr{E})$ we have $(u \wedge n) \cdot g(\rho_p) \in D(\mathscr{E})$ and

$$\text{(A 8)} \qquad \Gamma((u \wedge n) \cdot g(\rho_p)) \leq 2(\Gamma(u \wedge n) + \|g'\|_\infty^2 \cdot (u \wedge n)^2) \leq 2(\Gamma(u) + \|g'\|_\infty^2 \cdot u^2) \ m\text{-a.e.} \quad \text{for all } n \in N.$$

Thus

$$\sup_{n \in N} \mathscr{E}_1((u \wedge n) \cdot g(\rho_p)) \leq 2 \cdot (1 + \|g'\|_\infty^2) \cdot \mathscr{E}_1(u) < \infty,$$

and the assertion for u follows by the same kind of argument as used in Step 3. Finally, for arbitrary functions $u \in D(\mathscr{E})$ we have

$$u \cdot g(\rho_p) = u^+ \cdot g(\rho_p) - u^- \cdot g(\rho_p) \in D(\mathscr{E}) \qquad \text{and}$$

$$\Gamma(u \cdot g(\rho_p)) = \Gamma(u^+ \cdot g(\rho_p)) + \Gamma(u^- \cdot g(\rho_p)) \leq 2(\Gamma(u^+) + \Gamma(u^-) + \|g'\|_\infty^2 \cdot ((u^+)^2 + (u^-)^2)) = 2(\Gamma(u) + \|g'\|_\infty^2 \cdot u^2) \ m\text{-a.e.}$$

Here we used that $\Gamma(u^+ \cdot g(\rho_p), u^- \cdot g(\rho_p)) = 0$ m-a.e., and $\Gamma(u^+)$ resp. $\Gamma(u^-)$ vanishes m-a.e. on $\{u \leq 0\}$ (resp. $\{u \geq 0\}$) and coincides with $\Gamma(u)$ m-a.e. on $\{u \geq 0\}$ (resp. $\{u \leq 0\}$).

This completes the proof of the proposition. □

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