# ON THE ANALYTICITY OF SOLUTION OF EVOLUTION EQUATIONS

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Consider the abstract evolution equation

(E) $$du/dt + A(t)u = f(t), \quad 0 \leq t \leq T,$$

in a Banach space $X$. We are here concerned with sufficient conditions for the solutions $u(t)$ of $(E)$ to be analytic in $t$. Such conditions have been given, for example, in [3], [4].

The purpose of this note is to point out that there is a redundancy in the assumptions of some of the theorems of [4] and that, on eliminating it, we obtain theorems that seem to be definitive in this direction. As an application, we shall prove the analyticity of solutions of partial differential equations of a certain general type.

## 1. The main theorems

For the definition of the evolution operator $U(t, s)$ used below, see, for example, [4].

**Theorem 1.** *Let $\{A(t)\}$ be a family of densely defined, closed linear operators in a complex Banach space $X$, where $t$ varies over a convex complex neighborhood $\Delta$ of the closed real interval $[0, T]$. Assume that the resolvent set of $-A(t)$ contains a closed sector $\Sigma$: $|\arg z| \leq \pi/2 + \theta$, where $0 < \theta < \pi/2$, and that*

( 1 ) $$\|(\lambda I + A(t))^{-1}\| \leq M/|\lambda|, \quad \lambda \in \Sigma, \quad t \in \Delta.$$

*Assume further that $A(t)^{-1}$ (which is a bounded operator by the above assumption) is analytic for $t \in \Delta$. Then the evolution operator $U(t, s)$ exists for $0 \leq s \leq t \leq T$ and has a continuation to complex values of $s$ and $t$ such that it is analytic for $s, t \in \Delta$, $s \neq t$, $|\arg(t - s)| < \theta$ and strongly continuous up to $s = t$, with $U(s, s) = I$. Furthermore, it satisfies the following equations for such complex $s, t$: $U(t, r) = U(t, s) U(s, r)$, $(\partial/\partial t) U(t, s) = -A(t) U(t, s)$, $(\partial/\partial s) U(t, s) \phi = U(t, s) A(s) \phi$ for $\phi \in D(A(s))$.*

**Theorem 2.** *Let the assumptions of Theorem* 1 *be satisfied. Let f(t) be an X-valued function continuous in some real neighborhood of* $t_0 \in (0, T)$, *and let u(t) be a similar function satisfying* (E) *in some real neighborhood of* $t_0$. *If f(t) has an analytic continuation to some complex neighborhood of* $t_0$, *then u(t) also has an analytic continuation to some complex neighborhood of* $t_0$ *and satisfies* (E) *there.*

Proofs.  Theorem 1 has been proved in [4] as Theorem 5.1 under an additional condition called (A.4).  But this condition is redundant, being a consequence of other assumptions.  In fact, we shall show that the following strong form of (A.4) is true:

$$(2) \qquad \|(\partial/\partial t)(\lambda I + A(t))^{-1}\| \leq N'/|\lambda|, \quad \lambda \in \Sigma, \quad t \in \Delta',$$

where $\Delta'$ is any compact subset of $\Delta$ and $N'$ is a constant depending on $\Delta'$. To prove (2), we apply the Cauchy integral formula to $(\lambda I + A(t))^{-1}$ which is analytic in $t$ as shown in [4], obtaining

$$(3) \qquad \frac{\partial}{\partial t}(\lambda I + A(t))^{-1} = \frac{1}{2\pi i}\int_C (\lambda I + A(t'))^{-1}(t'-t)^{-2}dt',$$

where $C$ is the circle $|t'-t| = a$ with a small $a > 0$.  (2) follows immediately from (3) in virtue of (1).

Theorem 2 follows easily from the formula

$$u(t) = U(t, r)u(r) + \int_r^t U(t, s)f(s)ds, \quad 0 \leq r < t \leq T;$$

see [4].

Remarks.  1.  Theorem II of [3], another theorem on the analyticity, is now superseded by the above theorems.

2.  Theorem 5.2 of [4] is incorrectly stated and should be revised.  In any case, however, it is superseded by Theorem 2 above.

3.  The conclusions of the above theorems are not necessarily true if the assumptions of Theorem 1 are satisfied only for real $t$, even if $A(t)^{-1}$ is real-analytic.  As a counter-example, let $X = L^p(a, b)$, $1 < p < \infty$, $-\infty < a < b < \infty$, and assume that $(0, T) \cap (a, b)$ is not empty.  Set $(A(t)u)(x) = (x-t)^{-2}u(x)$. $-A(t)$ generates an analytic semigroup for all complex $t$, and it is even a bounded operator for non-real $t$.  $A(t)^{-1}$ is evidently analytic in $t$.  But the first assumption of Theorem 1 is not satisfied, for the spectrum of $A(t)$ is not contained in any fixed sector when $t$ varies over a complex neighborhood of $[0, T]$.  The solution of the homogeneous equation (E) (with $f = 0$) is given by

$$u(x, t) = \begin{cases} \exp((t-x)^{-1} - (s-x)^{-1})u(x, s) & \text{if } x > t \text{ or } x < s, \\ 0 & \text{if } s \leq x \leq t. \end{cases}$$

This shows that the conclusions of Theorems 1 and 2 do not hold in this example. (There is an incorrect statement in [3] about this example.)

## 2. An application

In this section we consider an application of Theorem 2 of the preceding section making use of S. Agmon's result on elliptic boundary value problems [1]. For the notations we follow those of [1]. For each $t \in [0, T]$, let $A(x, t, D_x)$ be an elliptic operator of order $2m$ operating on functions defined in a bounded domain $G$ in $n$-space: $x = (x_1, \cdots, x_n)$, and let $\{B_j(x, t, D_x)\}_{j=1}^{m}$ be a system of boundary operators defined on $\partial G \times [0, T]$. We suppose that the coefficients of $B_j$, $j = 1, \cdots, m$, are defined in the whole of $\bar{G} \times [0, T]$. By $A^{\sharp}(x, t, D_x)$, $B_j^{\sharp}(x, t, D_x)$ we denote the principal parts of $A(x, t, D_x)$, $B_j(x, t, D_x)$ respectively. We make the following assumptions.

(i)  $G$ is a bounded domain whose boundary $\partial G$ is of class $C^{2m}$.

(ii)  For each $t \in [0, T]$ $A(x, t, D_x)$ is a strongly elliptic operator of order $2m$ in $G$, i.e. $\mathrm{Re}\, A^{\sharp}(x, t, i\xi) > 0$ when $(x, t) \in \bar{G} \times [0, T]$ and $\xi \neq 0$ is real.

Let $(x, t)$ be an arbitrary point on $\partial G \times [0, T]$ and $\nu$ be the outward normal vector to $\partial G$ at $x$. Let $\xi$ be a real vector parallel to $\partial G$ at $x$ and $\lambda$ be a complex number such that $\mathrm{Im}\, \lambda = 0$ and $(\xi, \lambda) \neq 0$. Then it follows from the assumption above that the polynomial $(-1)^m A^{\sharp}(x, t, \xi + \tau\nu) - \lambda$ in the variable $\tau$ has exactly $m$ roots $\tau_k^+(\xi, \lambda)$, $k = 1, \cdots, m$, with positive imaginary parts.

(iii)  $B_j(x, t, D_x)$ is of order $m_j < 2m$, $j = 1, \cdots, m$. $\{B_j(x, t, D_x)\}_{j=1}^{m}$ is normal, i.e. $m_j \neq m_k$ if $j \neq k$ and $\partial G$ is nowhere characteristic with respect to $B_j(x, t, D_x)$ for $j = 1, \cdots, m$.

(iv)  For any $(x, t) \in \partial G \times [0, T]$ and $(\xi, \lambda)$ as was stated just after the assumption (ii) the polynomials in $\tau$ $B_j^{\sharp}(x, t, \xi + \tau\nu)$, $j = 1, \cdots, m$, are linearly independent modulo the polynomial $\Pi_{k=1}^{m}(\tau - \tau_k^+(\xi, \lambda))$.

This is the condition of S. Agmon [1] in order that all rays $\{re^{i\theta}: r > 0, \pi/2 \leq \theta \leq 3\pi/2$ be of minimal growth with respect to the system $(A(x, t, D_x), \{B_j(x, t, D_x)\}, G)$.

(v)  The coefficients of $A(x, t, D_x)$, $B_j(x, t, D_x)$, $j = 1, \cdots, m$, are all extended to $\bar{G} \times \Delta$ such that they are analytic in $t \in \Delta$ for any fixed $x \in \bar{G}$ and they have derivatives in $t$ of all orders which are continuous in $\bar{G} \times \Delta$.

We shall denote by $A(t)$ the operator in $L^p(G)$, $1 < p < \infty$, defined as follows:

(a)  the domain of $A(t)$ is

$$\{u \in H_{2m, L^p}(G): B_j(x, t, D_x)u(x) = 0 \text{ on } \partial G, j = 1, \cdots, m\};$$

(b)  $(A(t)u)(x) = A(x, t, D_x)u(x)$ for $u \in D(A(t))$.

From the assumption (v) it follows that (ii), (iii), (iv) are all satisfied in

$\bar{G} \times \Delta$ if we replace $\Delta$ by another complex neighbourhood of $[0, T]$ if necessary. Hence in view of Theorem 2.1 of [1] the first assumption of Theorem 1 is satisfied if we replace $A(t)$ by $A(t)+kI$ with some real number $k$. In what follows we shall suppose that these replacement are made. Then the following *a priori* estimate holds (cf. [2])

$$(4) \qquad ||u||_{2m} \leqq C\{||A(x, t, D_x)u|| + \sum_{j=1}^{m} ||B_j(x, t, D_x)u||_{2m-m_j-p-1}\}.$$

We now prove that the second assumption of Theorem 1 is satisfied. For any $f \in L^p(G)$, $v(t)=A(t)^{-1}f$ is the solusion of the following boundary value problem

$$A(x, t, D_x)v(x, t)=f(x) \qquad\qquad x \in G,$$
$$B_j(x, t, D_x)v(x, t)=0, \quad j=1,\cdots, m, \qquad x \in \partial G.$$

Let $w$ be the solution of the inhomogeneous boundary value problem

$$A(x, t, D_x)w(x, t)=-\dot{A}(x, t, D_x)v(x, t) \qquad\qquad x \in G,$$
$$B_j(x, t, D_x)w(x, t)=-\dot{B}_j(x, t, D_x)v(x, t), \quad j=1,\cdots, m, \quad x \in \partial G,$$

where $\dot{A}(x, t, D_x)$, $\dot{B}_j(x, t, D_x)$, $j=1,\cdots, m$, are differential operators obtained by differentiating in $t$ the corresponding coefficients of $A(x, t, D_x)$, $B_j(x, t, D_x)$, $j=1,\cdots, m$, respectively. According to the normality of $\{B_j\}$, the above problem has a unique solution in $H_{2m, L^p}(G)$. If we apply (4) to $(v(t')-v(t))(t'-t)^{-1}-w(t)$, we can show without difficulty that $A(t)^{-1}$ is an analytic function of $t$ with values in $H_{2m, L^p}(G)$ and hence *a fortiori* in $L^p(G)$. Hence we can apply Theorem 2 and prove the analyticity in $t$ of the solution without estimating the derivatives of the solution directly.

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