ON W^*-AND C^*-DYNAMICAL SYSTEMS

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To my parents

Tomie and Kīchirō

ACKNOWLEDEMENT

It is my great pleasure to thank Professors Hirokichi Kudo, Seinosuke Kitagawa and Mr. Masayuki Fujita for their constant encouragements and valuable suggestions.

Special thanks go to Professors Yoshiomi Nakagami, Masamichi Takesaki and Doctor Hiroshi Takai for stimulating discussions on the present material and their warm encouragement.

I would like to express my deep gratitude to Professor Osamu Takenouchi, my advisor, for his guidance, instructive discussions and constant encouragement throughout the course of this reseach.

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Introduction.

This thesis is devoted to the study of W^*-dynamical systems and C^*-dynamical systems. When we have a W^*-dynamical system (M, G, α) or a C^*-dynamical system (A, G, α), we can construct from it a W^*-crossed product $G\times_\alpha M$ or a C^*-crossed product $G\times_\alpha A$. Many results in the theory of operator algebras are established in these terms. And the structure of these dynamical systems and crossed products are our main concern. There are many ways to pick up problems and I want to establish here three results.

This thesis consists of three chapters. Each chapter concerns with each one of these results. Now I will explain briefly the contents of each chapter.

In the first chapter, we will be concerned with the non-existence of conditional expectations on W^*-crossed products ([23]).

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{B}$ be a σ-subfield of $\mathcal{A}$. The conditional expectation E in statistics is a linear map of $L^1(X,\mathcal{A},\mu)$ onto $L^1(X,\mathcal{B},\mu)$ satisfying

$$\int_B f(x)\, d\mu(x) = \int_B (E(f))(x)\, d\mu(x)$$

for all $B\in\mathcal{B}$ and $f\in L^1(X,\mathcal{A},\mu)$. E is an $L^\infty(X,\mathcal{B},\mu)$-bimodule map, namely it satisfies $E(gfh) = gE(f)h$ for all $g,h\in L^\infty(X,\mathcal{B},\mu)$ and $f\in L^1(X,\mathcal{A},\mu)$. In the study of operator algebras, conditional expectations of $L^\infty(X,\mathcal{A},\mu)$ onto $L^\infty(X,\mathcal{B},\mu)$ are in question. The transposed operator E^* of E in the above is not the one to find because it is but the canonical inclusion map of $L^\infty(X,\mathcal{B},\mu)$ into $L^\infty(X,\mathcal{A},\mu)$. A desired conditional expectation

T of $L^{\infty}(X,\mathcal{A},\mu)$ onto $L^{\infty}(X,\mathcal{B},\mu)$ should be a map which makes correspond to each $f \in L^{\infty}(X,\mathcal{A},\mu)$ an element $T(f)$ in $L^{\infty}(X,\mathcal{B},\mu)$ such that

$$\int_X T(f)(x)g(x)\, d\mu(x) = \int_X f(x)g(x)\, d\mu(x)$$

for all $g \in L^1(X,\mathcal{B},\mu)$. Then T is shown to be an $L^{\infty}(X,\mathcal{B},\mu)$-bimodule map, namely it satisfies $T(gfh) = gT(f)h$ for all $g,h \in L^{\infty}(X,\mathcal{B},\mu)$ and $f \in L^{\infty}(X,\mathcal{A},\mu)$. This method may be applied in the context of general operator algebras, especially in the case of finite von Neumann algebras. Let M be a finite von Neumann algebra with a normal tracial state τ and let N be a von Neumann subalgebra of M. Then the conditional expectation T of M onto N will be defined as a map which satisfies for each $x \in M$

$$\tau(T(x)y) = \tau(xy)$$

for all $y \in N$. In this way, J. Dixmier [8] and H. Umegaki [50] have introduced the notion of the conditional expectation in operator algebras. And there are abundant studies concerning this ([8],[21],[36],[44],[46],[47],[50]--[53]). A map P of a C^*-algebra A onto a C^*-subalgebra B of A is called a conditional expectation if P is a B-bimodule linear map with $P(x) = x$ for all $x \in B$. J. Tomiyama [46] showed that every projection of norm one is a conditional expectation. And M. Takesaki [44] examined under what condition there exists a nomal conditional expectation of a von Neumann algebra M onto a von Neumann subalgebra N of M. It was shown that, if there is a normal conditional expectation of M onto N, then the type of M is greater than that of N (see[47]).

The study of W^*-crossed product with a discrete group was initiated by F. J. Murray and J. von Neumann, and many people investigated the structure of these crossed products. In their studies, the existence of a conditional expectation of $G\times_{\alpha}M$ onto M played an important role. If the group is not discrete, the situation is not quite favorable. Indeed, my result here is that

> there is not any normal conditional expectation of $G\times_{\alpha}M$ onto M if G is a locally compact connected group.

One of the reasons why the study of the crossed product with a continuous group is difficult will lie in this fact. As an application, we prove that $G\times_{\alpha}M$ is always properly infinite when G is a locally compact connected group. This is related to the von Neumann - Segal Theorem "the connected semisimple Lie group has not any non-trivial finite representation"([41] Corollary 1).

In the second chapter, we discuss about isomorphisms of Fourier algebras(see[12]) in crossed products. For two locally compact abelian groups G and H, Pontrjagin's theorem implies that $L^1(G)$ and $L^1(H)$ are isomorphic as Banach algebras if and only if G and H are isomorphic. Y. Kawada [26] and J. G. Wendel [49] proved the same statement for arbitrary locally compact groups. Now for an abelian group G, we set $A(G) \equiv \{\hat{f} ; f\in L^1(\hat{G})\}$ where $\hat{G}$ is the dual group of G and $\hat{f}$ is the Fourier transform of f (see [11] (3.6) and [40]) and we give a norm $\|\cdot\|$ on A(G) by $\|\hat{f}\| = L^1$-norm of f. Then A(G) turns out to be a Banach algebra with pointwise-multiplication. The above fact may be reformulated in this way; A(G) and A(H) are isomorphic as Banach algebras if and only if G and H are isomorphic. The algebra A(G) was later extended as the Fourier algebra of an arbitrary locally compact group G. The definition is due to P. Eymard [11] and it is shown to be isomorphic as Banach spaces to the predual $m(G)_*$ of the von Neumann algebra m(G) generated by the left regular representation of G. M. E. Walter [48] showed that A(G) and A(H) are isometrically isomorphic as Banach algebras if and only if G and H are isomorphic. Quite recently J. Cannière, M. Enoch and J. M. Schwartz [3, Theorème 2.9] established the same statement as the M. E. Walter's result in the category of Kac algebras. We have also the notion of Fourier algebra for W^*-dynamical system. It was defined by H. Takai [43] and M. Fujita [12]. Now, what we prove in this chapter is that

> two W^*-dynamical systems are equivalent or anti-equivalent if and only if their Fourier algebras are isomorphic as Banach algebras.

The third chapter is an attempt to extend some results on C^*-dynamical systems with locally compact abelian groups to C^*-dynamical systems with non-abelian groups. W. Arveson [2] constructed the theory of spectral subspaces for W^*- and C^*-dynamical systems with locally compact abelian groups. Using this, A. Connes [6] defined the Connes spectrum $\Gamma(\alpha)$ which is a closed subgroup of the dual group $\hat{G}$ of G, and established a beautiful structure theory of factors of type III. A. Connes and M. Takesaki ([7],[45]) proved for a W^*-dynamical system (M, G,

α) with an abelian group G that $G\times_\alpha M$ is a factor if and only if $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M. There were two ways to generalize this statement, the one was toward C^*-dynamical systems ([27],[28],[37],[38],[39] etc.), the other one was toward W^*-dynamical systems with non-abelian groups ([29],[30],[32],[33],[34]).

Now, suppose that we have a C^*-dynamical system (A, G, α), when G is an abelian group, we have a dual action $\hat{\alpha}$ of α on the C^*-crossed product $G\times_\alpha A$ and we can consider the Connes spectrum $\Gamma(\hat{\alpha})$ for the C^*-dynamical system ($G\times_\alpha A$, $\hat{G}$, $\hat{\alpha}$)(see[37]). Unless the group G is abelian, we can not construct the dual action. Instead, a co-action δ on $G\times_\alpha A$ can be constructed, which plays the same role as the dual action in the case of abelian groups.

In this chapter,

for a C^*-dynamical system (A, G, α) we introduce the notion of δ-invariantness for C^*-subalgebras of $G\times_\alpha A$, and, using this, we define the essential spectrum $\Gamma(\delta)$, which is coincident with $\Gamma(\hat{\alpha})$ in the case of abelian groups. It is shown that A is prime if and only if A is G-prime and $\Gamma(\delta) = G$.

D. E. Evans and T. Sund [10] investigated C^*-dynamical systems with compact groups and mentioned that $\Gamma(\alpha)$ is not invariant under exterior equivalence. We prove that

$\Gamma(\hat{\hat{\alpha}})$ is invariant under exterior equivalence ($\hat{\hat{\alpha}}$ is the bidual action of α) by characterizing $\Gamma(\hat{\hat{\alpha}})$ in terms of the dual co-action δ of α.

Our final result is that

a von Neumann algebra should be hyperfinite when a compact group acts on it ergodically.

This was proposed as a problem in the preprint of [17] of R. Høegh-Krohn, M. B. Landstad and E. Størmer, and, when the paper appeared, it was poved independently with us.

Chapter I Expectation.

The conditional expectation in operator algebras played an important role from the outset in the theory of operator algebras. Thus J. von Neumann and F. J. Murray were able to show the existence of type II_1 factor by using it. The notion of conditional expectations in general was first introdued by J. Dixmier and H. Umegaki in a finite von Neumann algebra. There are abundant literatures on conditional expectations (See[8], [21], [36], [44], [46], [47], [50]-[53]).

Now let (M, G, α) be a W*-dynamical system and $G\times_\alpha M$ be the crossed product constructed from it (Precise definition will appear in section 1). An important problem is whether there is a conditional expectation of $N\equiv G\times_\alpha M$ onto M. When G is a discrete group, there exists a faithful and normal conditional expectation. But it was not known whether there exists a normal conditional expectation of N onto M in the case when G is non-discrete.

In this chapter we establish that there is no normal conditional expectation of the crossed product N with a locally compact connected group G onto M under certain conditions.

In spite of this result, a normal semi-finite operator valued weight from N into M can always be found. This was shown by U. Haagerup[13] prior to our result.

1. Non-Existence of Expectation. Let M be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and G be a locally compact group. The triple (M, G, α) is said a W*-dynamical system if the mapping α of G into the group Aut(M) of all automorphisms of M is a homomorphisms and the function $g \to \omega \circ \alpha_g(x)$ is continuous on G for any $x \in M$ and $\omega \in M_*$ (M_* is the predual of M).

The crossed product $G \times_\alpha M$ of M with G is the von Neumann algebra on $L^2(G, \mathfrak{H})$ generated by the family of the operators $\{\pi_\alpha(x), \lambda(g);\ x \in M,\ g \in G\}$;

$$\begin{cases} (\pi_\alpha(x)\zeta)(h) = \alpha_h^{-1}(x)\zeta(h), & \zeta \in L^2(G, \mathfrak{H}) \\ (\lambda(g)\zeta)(h) = \zeta(g^{-1}h) & , \quad \zeta \in L^2(G, \mathfrak{H}). \end{cases} \tag{1.1}$$

The mapping π_α is then a normal isomorphism of M onto $\pi_\alpha(M)$ such that $\lambda(g)\pi_\alpha(x)\lambda(g)^* = \pi_\alpha(\alpha_g(x))$ for all $g \in G$ and $x \in M$. We often identify the von Neumann algebra M with the von Neumann algebra $\pi_\alpha(M)$.

Let T be a linear mapping of a von Neumann algebra M onto a von Neumann subalgebra N of M.

Definition 1.1. T is called a conditional expectation of M onto N if T has the following properties.

(i) $T(1) = 1$, where 1 is the identity operator

(ii) $T(axb) = a(T(x))b$, for all $a, b \in N$, $x \in M$.

Moreover T is called normal if ${}^tT(N_*) \subset M_*$.

Let ϕ be an automorphism of a von Neumann algebra M.

Definition 1.2. ϕ is said freely acting if the element x of M with the property that $x\phi(y) = yx$ for any $y \in M$ is necessarily zero. For each automorphism ψ of M, there is a unique central projection q of M such that;

(i) $\psi(q) = q$

(ii) $\psi|M_q$ is an inner automorphism of M_q

(iii) $\psi|M_{(1-q)}$ is a freely acting automorphism of $M_{(1-q)}$.

This central projection q will be denoted by $p(\psi)$ (cf. Kallmann [20]).

Let M be a von Neumann algebra. We also identify M_f with $fMf = \{fxf;\ x \in M\}$ where f is a projection of M or M'.

Theorem 1.1. Let (M, G, α) be a W^*-dynamical system and we suppose that $\sup\{p(\alpha_g);\ g \in G,\ g \neq e\} \neq 1$, where e is the identity of G. Then, the following statements are equivalent;

(i) G is a discrete group

(ii) there exists a normal conditional expectation of $G \times_\alpha M$ onto M.

Remark 1.2. That (i) implies (ii) is well known (cf. [6] Proposition 1, 4, 6, [35] § 4 and [15] § 2). In fact if G is a discrete group, the Hilbert space $L^2(G, \mathfrak{h})$ is identified with $\mathfrak{h} \otimes \ell^2(G)$. On the other hand, for each g in G, put

$$\varepsilon_g(h) = \delta_g^h = \begin{cases} 1 & (g = h) \\ 0 & (g \neq h), \end{cases}$$

then the Hilbert space $L^2(G, \mathfrak{h})$ is identifiable with the direct sum $\sum_{g \in G} \mathfrak{h} \otimes \varepsilon_g$ of subspaces $\mathfrak{h} \otimes \varepsilon_g\ (g \in G)$. For each g in G and η in $\mathfrak{h}$, put $J_g \eta = \eta \otimes \varepsilon_g$, then J_g is an isometry of $\mathfrak{h}$ onto $\mathfrak{h} \otimes \varepsilon_g$. Every x in $\mathcal{L}(L^2(G, \mathfrak{h}))$ has a matrix representation with an operator on $\mathfrak{h}$ as elements

$$(x)_{g,h} = J_g^* x J_h,$$

where $\mathcal{L}(R)$ is the algebra of all bounded linear operators on the Hilbert space R. Especially, we have

$$(\pi_\alpha(x))_{g,h} = \delta_g^h \alpha_{g^{-1}}(x) \qquad (x \in M,\ g,\ h, \in G)$$

$$(\lambda(k))_{g,h} = \delta_g^{kh} \qquad (g,\ h,\ k \in G).$$

Put $T(y) = (y)_{e,e}$ for $y \in G \times_\alpha M$. Then T is a faithful normal conditional expectation of $G \times_\alpha M$ onto M.

Before we prove (ii) ⇨ (i), we will give two lemmas. Lemma 1.3 will be used repeatedly in the whole of our study.

Lemma 1.3. Let T be a conditional expectation of $G\times_{\alpha}M$ onto M. We then have $T(\lambda(g))_{(1-p(\alpha_g))} = 0$ for any $g\in G$.

Proof. For each $y\in M_{(1-p(\alpha_G))}$, we have;

$$yT(\lambda(g)^*) = T(y\lambda(g)^*) = T(\lambda(g)^*\lambda(g)y\lambda(g)^*).$$

Since $\lambda(g)y\lambda(g)^* = \alpha_g(y)$ is an element of M,

$$yT(\lambda(g)^*) = T(\lambda(g)^*)\alpha_g(y).$$

Therefore $T(\lambda(g)^*)_{(1-p(\alpha_g))} = 0$ because α_g is a freely acting automorphism of $M_{(1-p(\alpha_g))}$.

Lemma 1.4. $\sup\{p(\alpha_g);\ g\in G,\ g\neq e\}$ is a G-invariant central projection of M.

Proof. For any $y\in M$, g, h, $\in G$ with $g\neq e$, we have

$$\alpha_{hgh^{-1}}(y\alpha_h(p(\alpha_g))) = \alpha_h(U)y\alpha_h(p(\alpha_g))\alpha_h(U)^*,$$

where U is an element of M such that $\alpha_g|M_{p(\alpha_g)} = \mathrm{Ad}U$, $U^*U = p(\alpha_g)$ and $UU^* = p(\alpha_g)$ $(\mathrm{Ad}U(x) = UxU^*$ for $x\in M_{p(\alpha_g)})$.

Therefore we get $\alpha_n(p(\alpha_g)) \leqq p(\alpha_{hgh^{-1}})$, so that

$$\alpha_h(\sup\{(\alpha_g);\ g\in G,\ g\neq e\}) \leqq \sup\{p(\alpha_g);\ g\in G,\ g\neq e\}.$$

Hence $\sup\{p(\alpha_g);\ g\in G,\ g\neq e\}$ is a G-invariant central projection of M.

[The proof of Theorem 1.1]. Lemma 1.4. implies that it is sufficient to prove the Theorem in the case when $p(\alpha_g) = 0$ for all $g\in G$ except the identity e. It follows that $T(\lambda(g)) = 0$ for all $g\in G$ except e by Lemma 1.3.

Suppose that T is a normal conditional expectation of $G\times_{\alpha}M$ onto M. Let K(G, M) be the family of M-valued, σ-weakly continuous functions on G with a compact support. By [14] Lemma 2.3, a *-representation μ of the involutive algebra $K(G, M)$ is defined,

$$\mu(\xi) = \int_G \lambda(g)\pi_\alpha(\xi(g))d\nu(g)$$

where $\xi \in K(G, M)$ and ν is a left Haar measure of G. Moreover the representation μ maps $K(G, M)$ onto a σ-weakly dense subalgebra of $G \times_\alpha M$. Since T is normal and $T(\lambda(g)) = 0$ for all $g \in G$ except e, we have

$$T(\mu(\xi)) = \int_G T(\lambda(g))\pi_\alpha(\xi(g))\, d\nu(g) = \pi_\alpha(\xi(e))\nu(\{e\}).$$

Therefore $\nu(\{e\})$ must be a positive number, so G must be a discrete group.

Remark 1.5 Let (M, G, α) be a W^*-dynamical system. Let V be a strongly continuous unitary representation of G into M such that $\alpha_g = \mathrm{Ad}V_g$ for any $g \in G$.

We define a unitary operator W on $L^2(G, \mathfrak{H}) = \mathfrak{H} \otimes L^2(G)$

$$(W\xi)(g) = V_g \xi(g)$$

for all $\xi \in L^2(G, \mathfrak{H})$. We get;

$$W\pi_\alpha(x)W^* = x \otimes 1 \qquad \text{for any } x \in M$$

$$W\lambda(g)W^* = V_g \otimes \rho(g) \quad \text{for any } g \in G.$$

where ρ is the left regular representation of G on $L^2(G)$. We therefore get

$$W(G \times_\alpha M)W^* = M \otimes \rho(G)'', \quad W\pi_\alpha(M)W^* = M \otimes 1.$$

Whence we know that there are many normal conditional expectations of $G \times_\alpha M$ onto M, according to the result of [47] Theorem 1.1.

We will have a decisive result about the existence of a normal conditional expectation in case of a connected group.

Theorem 1.6. Let G be a locally compact connected group and (M, G, α) be a W^*-dynamical system. If there is an element h in G such that α_h is an outer automorphism of M, then there does not exist any normal conditional expectation of $G \times_\alpha M$ onto M.

Proof. We suppose that there exists a normal conditional expectation T of $G \times_\alpha M$ onto M.

Assume first that there is an element g in G such that g is on a one-parameter subgroup x(t) at t = s and $\alpha_g = \alpha_{x(s)}$ is an outer automorphism of M. $p(\alpha_{x(s)})$ is then a central projection of M which is not the identity operator of M. For any $n \in \mathbb{N}$, we get,

$$p(\alpha_{x(\frac{s}{n})}) \leqq p(\alpha_{x(s)})$$

because $(\alpha_{x(\frac{s}{n})})^n = \alpha_{x(s)}$.

From Lemma 1.3, $T(\lambda(x(\frac{s}{n})))_{(1-p(\alpha_{x(\frac{s}{n})}))} = 0$, so we have

$$T(\lambda(x(\tfrac{s}{n})))_{(1-p(\alpha_{x(s)}))} = 0$$

for any $n \in \mathbb{N}$. Therefore we get,

$$T(\lambda(e))_{(1-p(\alpha_{x(s)}))} = \text{w-}\lim_{n\to\infty} T(\lambda(x(\tfrac{s}{n})))_{(1-p(\alpha_g))} = 0,$$

so we get $1 = p(\alpha_g)$, which is a contradiction. So the assumed situation does not take place.

When an element g in G is on a one-parameter subgroup of G, we write $e \sim g$. By the above argument, α_g must be an inner automorphism of M for any g in $\{g \in G;\ e \sim g\}$. Now G is equal to the closed subgroup K generated by $\{g \in G;\ e \sim g\}$. Indeed, suppose that there are an element g in G and an open neighborhood U of e in G such that the intersection of gU and K is empty. By [31] Theorem 4.6, there exists in U a compact normal subgroup H such that G/H is a Lie group. Then there is a neighborhood V of e in G such that V contains H and each point of V/H is on a one-parameter subgroup in G/H. Since G/H is also a connected group, G/H is the group generated by V/H, so that there are a finite subset

$\{g_iH;\ i = 1, 2, \ldots, n\}$ in G/H, and one-parameter subgroups $x_i(t)$ $(i = 1, 2, \ldots, n)$ in G/H such that $\prod_{i=1}^{n} g_iH = gH$, g_iH is on the one-parameter subgroup $x_i(t)$ of G/H at $t = s_i$ $(i = 1, 2, \ldots, n)$ and $g_i \in V$ $(i = 1, 2, \ldots, n)$. By [31] Theorem 4.15, there are one-parameter subgroups $y_i(t)$ of G $(i = 1, 2, \ldots, n)$ such that $y_i(t)H = x_i(t)$ for any $t \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. The element $g^{-1}\prod_{i=1}^{n} y_i(s_i)$ is contained in $H \subset U$ because $\prod_{i=1}^{n} y_i(s_i)H = gH$. Then the intersection of K and gU is not empty since $\prod_{i=1}^{n} y_i(s_i)$ is contained in both K and gU, which is a contradiction.

As the group generated by $\{g \in G;\ e \sim g\}$ was shown to be dense in G, there is a net $\{g_i\}_{i \in I}$ in this group such that it converges to h in G, h being the element in the statement of the Theorem. Since α_g are inner automorphisms of M for any g in $\{g \in G;\ e \sim g\}$, α_{g_i} are inner automorphisms of M for any $i \in I$. Then we get;

$$p(\alpha_{g_i^{-1}h}) = p(\alpha_h),$$

$$T(\lambda(g_i^{-1}h))_{(1-p(\alpha_{g_i^{-1}h}))} = T(\lambda(g_i^{-1}h))_{(1-p(\alpha_h))} = 0,$$

$$T(\lambda(e))_{(1-p(\alpha_h))} = \operatorname*{w-lim}_{i \in I} T(\lambda(g_i^{-1}h))_{(1-p(\alpha_h))} = 0,$$

so that $1-p(\alpha_h) = 0$, which is not the case. We get thus a contradiction and the proof is complete.

Remark 1.7. If the group is not supposed connected, there are W^*-dynamical systems with a non-discrete locally compact group such that there is an element h in G with the freely acting automorphism α_h of M and there is a normal conditional expectation of $G \times_\alpha M$ onto M. For instance, let G be a locally compact group $G_1 \times G_2$ where G_1 is a discrete group and G_2 is a non-discrete locally compact group. Then $(L^\infty(G_1), G_1\times G_2, \alpha)$ and $(L^\infty(G_1), G_1, \sigma)$ are W^*-dynamical systems where the actions $\alpha_{(g,h)} = \sigma_g$ are the translation of $L^\infty(G_1)$ for all $(g,h)\in G_1\times G_2$. Then $G\times_\alpha L^\infty(G_1)$ is isomorphic to $G_1\times_\sigma L^\infty(G_1)\otimes\rho(G_2)''$ where ρ is the left regular representation of G_2 on $L^2(G_2)$. Let ω be a normal state of $\rho(G_2)''$, p_ω be a slice mapping associated with ω (See [47]) of $G_1\times_\sigma L^\infty(G_1)\otimes\rho(G_2)''$ onto $G_1\times_\sigma L^\infty(G_1)$. Let T be a normal conditional expectation of $G_1\times_\sigma L^\infty(G_1)$ onto $L^\infty(G_1)$ (Remark 1.2.). Then $T\cdot p_\omega$ is a normal conditional expectation of $G\times_\alpha L^\infty(G_1)$ onto $L^\infty(G_1)$.

Proposition 1.8. Let (M, G, α) be a W^*-dynamical system, Γ be an open subgroup of G and ω be a faithful normal semi-finite weight on M. Then there is a faithful normal conditional expectation T of $G\times_\alpha M$ onto $W^*(M, \Gamma, \alpha) \equiv \{\pi_\alpha(M), \lambda(\Gamma)\}''$ such that $\tilde{\omega}\circ T = \tilde{\omega}$ and $T(\lambda(g)) = 0$ if $g\notin\Gamma$ where $\tilde{\omega}$ is the dual weight associated with ω.

Proof. By [14] Theorem 3.2 we get,

$$\sigma_t^{\tilde{\omega}}(\pi_\alpha(x)) = \pi_\alpha(\sigma_t^\omega(x)) \qquad \text{for all } x\in M$$

$$\sigma_t^{\tilde{\omega}}(\lambda(g)) = \Delta(g)^{it}\lambda(g)\pi_\alpha((D\omega\circ\alpha_g : D\omega)_t) \qquad \text{for all } g\in G.$$

Therefore $W^*(M, \Gamma, \alpha)$ is $\sigma_t^{\tilde{\omega}}$-invariant for all $t\in\mathbb{R}$. Let $K(\Gamma, A_\omega)$ be the family of all A_ω-valued continuous functions on Γ with a compact support where A_ω is the left Hilbert algebra associated with ω. We regard $K(\Gamma, A_\omega)$ as $\{f\in K(G, A_\omega);\ f = 0 \text{ outside } \Gamma\}$. Then by [14] Theorem 3.2, $\omega|_{W^*(M, \Gamma, \alpha)}$ is semi-finite. Then by [44] Theorem, there is a unique faithful normal conditional expectation T of $G\times_\alpha M$ onto $W^*(M, \Gamma, \alpha)$ such that $\tilde{\omega}\circ T = \tilde{\omega}$. Moreover we find, by the construction of T in [44], that

$T(x) = \phi(E\ x\ E)$ for all $x \in G \times_\alpha M$ where E is the projection of $L^2(G, \mathfrak{h})$ onto $L^2(\Gamma, \mathfrak{h})$ and Φ is the canonical automorphism of $\Gamma \times_\beta M$ onto $W^*(M, \Gamma, \alpha)$;

$$\Phi(\pi_\beta(x)) = \pi_\alpha(x) \qquad \text{for all } x \in M$$
$$\Phi(\lambda(g)) = \lambda(g) \qquad \text{for all } g \in \Gamma$$

(where the action β is the restriction of α on Γ). For all $x(g) \in K(G, M)$, we obtain,

$$T(\int_G \pi_\alpha(x(g))\lambda(g)dg)$$
$$= \Phi(E \int_G \pi_\alpha(x(g))\lambda(g)dg\ E)$$
$$= \int_\Gamma \pi_\alpha(x(g))\lambda(g)dg.$$

Then we get $T(\lambda(g)) = 0$ for all $g \notin \Gamma$ since $E\lambda(g)E = 0$ for $g \notin \Gamma$.

Remark 1.9. The above proposition was proved by H. Choda in case of a discrete group ([4] Proposition 2).

2. Applications.

Corollary 2.1. Let G be a locally compact connected group and (M, G, α) be a W^*-dynamical system. If there is an element $g \in G$ such that $p(\alpha_g) = 0$, then the crossed product $G \times_\alpha M$ is properly infinite.

Corollary 2.2. Let (M, G, α) be a W^*-dynamical system. If the group G is not discrete and $p(\alpha_g) = 0$ for all $g \in G$ except $g = e$, then the crossed product $G \times_\alpha M$ is properly infinite.

We prove the Corollary 2.1 only, as we can prove the Corollary 2.2 in the same way.

[Proof of Corollary 2.1]. We suppose $G \times_\alpha M$ is not properly infinite and let $(G \times_\alpha M)_p$ be the greatest finite portion of $G \times_\alpha M$.

Since p is a central projection of $G\times_\alpha M$, p is a projection of $\pi_\alpha(M)'$. Let q be the central support of p in $\pi_\alpha(M)'$. Then we get that q is a G-invariant projection of $\pi_\alpha(M)\cap\pi_\alpha(M)'$ because p is $Ad\lambda(g)$-invariant for all $g\in G$. The von Neumann algebra M_p is a von Neumann subalgebra of a finite von Neumann algebra $(G\times_\alpha M)_p$, so that there is a normal conditional expectation T_1 of $(G\times_\alpha M)_p$ onto M_p (See [8] Théorème 8 or [50] Theorem 1). We define a new normal conditional expectation T of $(G\times_\alpha M)_q$ onto M_q;

$$T(x) = \Phi(T_1(pxp))$$

for all $x\in(G\times_\alpha M)_q$ where Φ is the canonical isomorphism of M_p onto M_q.

For $a,b\in M_q$, $x\in(G\times_\alpha M)_q$, we have

$$\begin{aligned} T(axb) &= \Phi(T_1(paxbp)) \\ &= \Phi\{T_1((pap)(pxp)(pbp))\} \\ &= \Phi\{pap\ T_1(pxp)\ pbp\} \\ &= a\{\Phi\circ T_1(pxp)\}b = a(T(x))b, \end{aligned}$$

and

$$T(q) = \Phi\, T_1(pqp) = \Phi(p) = q.$$

Therefore T is a conditional expectation of $(G\times_\alpha M)_q$ onto M_q. The normality of T is clear. On the other hand, $(G\times_\alpha M)_q$ is the crossed product with the W^*-dynamical system $(M_q,\ G,\ \alpha|_{M_q})$. This contradicts Theorem 1.6.

Chapter II Isomorphism of Fourier algebra.

For locally compact abelian groups G and H, Pontrjagin's duality theorem mentions that $L^1(G)$ is isomorphic to $L^1(H)$ as Banach algebras if and only if G is isomorphic to H as locally compact abelian groups. Y. Kawada [26] and J. G. Wendel [49] proved the same statement for arbitrary locally compact groups.

For an abelian group G, we set $A(G) \equiv \{\hat{f};\ f \in L^1(\hat{G})\}$ where $\hat{G}$ is the dual group of G and $\hat{f}$ is the Fourier transform of f (see[11] (3.6) and [40]) and we give a norm $\|\cdot\|$ on A(G) by $\|\hat{f}\| \equiv L^1$-norm of f. Then A(G) turns out to be a Banach algebra with pointwise-multiplication. The above fact may be reformulated in this way; A(G) and A(H) are isomorhpic as Banach algebras if and only if G and H are isomorhpic as locally compact abelian groups. The algebra A(G) was later extended as the Fourier algebra of an arbitrary locally compact group G. The definition is due to P. Eymard [11] and it is shown to be isomorphic as Banach spaces to the predual $m(G)_*$ of the von Neumann algebra m(G) generated by the left regular representation of G. M. E. Walter [48] showed that A(G) and A(H) are isometrically isomorphic as Banach algebras if and only if G and H are isomorphic. For a W^*-dynamical system (M, G, α), the Fourier space $F_\alpha(G;M_*)$ was defined by H. Takai in [40] such that $F_\alpha(G;M_*)$ is isometrically isomorphic to the predual of the crossed product $G\times_\alpha M$ as Banach spaces. M. Fujita [12] defined a Banach algebra structure in the Fourier space $F_\alpha(G;M_*)$. Then he showed that the group of all characters $\widetilde{F_\alpha(G;M_*)}$ of $F_\alpha(G;M_*)$ is isomorphic to G and studied the support of the operators in $G\times_\alpha M$.

In this chapter we generalize the Walter's result for W^*-dynamical system (M, G, α).

Let (M, G, α) be a W^*-dynamical system. The covariant representation (π_α, λ) defined in (1.1) will be denoted by (π_α, λ_G) in this chapter. Each element ω in the predual $(G\times_\alpha M)_*$ of $G\times_\alpha M$ may be regarded as an element u_ω of $C^b(G;M_*)$;

$$u_\omega[g](x) = \langle \pi_\alpha(x)\lambda_G(g) , \omega \rangle \qquad (2.1)$$

for all $x\in M$, $g\in G$ where $C^b(G;M_*)$ is the space of all bounded continuous M_*-valued functions on G. We denote $F_\alpha(G;M_*) = \{ u_\omega ;\omega\in(G\times_\alpha M)_*\}\subset C^b(G;M_*)$. A norm $\|\cdot\|$ is defined on $F_\alpha(G;M_*)$ by $\| u_\omega\| = \|\omega\|$. Then $\|u\|_\infty \leqq \|u\|$ for all $u\in F_\alpha(G;M_*)$ where $\| \|_\infty$ is the sup-norm on $C^b(G;M_*)$. We define a product on $F_\alpha(G;M_*)$ by

$$(u * v)[g](x) = u[g](x)v[g](1) \qquad (2.2)$$

for all $u, v\in F_\alpha(G;M_*)$, $x\in M$ and $g\in G$. Then $F_\alpha(G;M_*)$ is a Banach algebra ([12] Theorem 3.5). So we can define products between $G\times_\alpha M$ and $F_\alpha(G;M_*)$;

$$\langle uT , v \rangle = \langle T, v*u \rangle$$
$$\langle Tu , v \rangle = \langle T, u*v \rangle$$

for $T\in G\times_\alpha M$, $u, v\in F_\alpha(G;M_*)$ ((3.7),(3.9) in [12]). Let T be an operator in $G\times_\alpha M$. Then the supp(T) of T the set of all $g\in G$ satisfying the condition that $\lambda_G(g)$ belongs to the σ-weak closure of $TF_\alpha(G;M_*)$ (see [12] Proposition 4.1).

Theorem 1. Let (M, G, α), (N, H, β) be W^*-dynamical systems and $F_\alpha(G;M_*)$, $F_\beta(H;N_*)$ their associated Fourier algebras. Let ϕ be an isometric isomorphism of $F_\alpha(G;M_*)$ onto $F_\beta(H;N_*)$ as Banach algebras.

Then we have five elements (k, p, q, I, θ) with the following properties:

(1) $k \in G$ such that $\lambda_G(k) = {}^t\phi(\lambda_H(e))$ where ${}^t\phi$ is the transposed map of ϕ and e is the identity of H,

(2) I is an isomorphism or anti-isomorphism of H onto G,

(3) p(resp. q) is a projection of $Z_M \cap M^G$ (resp. $Z_N \cap N^H$) where Z_M (resp. Z_N) is the center of M (resp. N) and $M^G = \{x \in M;\ \alpha_g(x) = x \text{ for all } g \in G\}$, $N^H = \{x \in N;\ \beta_h(x) = x \text{ for } h \in H\}$,

(4) θ is an isometric linear map of N onto M such that θ is an isomorphism of N_q onto M_p and θ is an anti-isomorphism of N_{1-q} onto M_{1-p},

(5) $\phi(u)[h](y) = ({}_ku)[I(h)](\theta(y)p) + ({}_ku)[I(h)](\alpha_{I(h)}(\theta(y)(1-p)))$

for all $y \in N$, $h \in H$ and $u \in F_\alpha(G;M_*)$, where $({}_ku)[g](y)=u[kg](\alpha_k(y))$,

(6) $\theta[\beta_k(y)]=[\alpha_{I(h)}\theta(y)]p + [\alpha^{-1}_{I(h)}\theta(y)](1-p)$ for all $y \in N$, $h \in H$.

Proof. The transposed map ${}^t\phi$ of ϕ is an isometric linear map of $H \times_\beta N$ onto $G \times_\alpha M$. Using [19] Theorem 7.10, we get;

$$ {}^t\phi = {}^t\phi(\lambda_H(e))(\gamma_I + \gamma_A) $$

where γ_I is an isomorphism of $(H \times_\beta N)_{z'}$ onto $(G \times_\alpha M)_z$, γ_A is an anti-isomorphism of $(H\times_\beta N)_{(1-z')}$ onto $(G\times_\alpha M)_{(1-z)}$, z (resp. z') being a central projection of $G\times_\alpha M$ (resp. $H\times_\beta N$). (2.3)

It follows from (2.2) that for all u, $v\in F_\alpha(G;M_*)$,

$$\begin{aligned} <{}^t\phi(\lambda_H(h)) , u*v> &= <\lambda_H(h) , \phi(u*v)> \\ &= <\lambda_H(h) , \phi(u)*\phi(v)> \\ &= <\lambda_H(h)\otimes\lambda_H(h) , \phi(u)\otimes\phi(v)> \\ &= <{}^t\phi(\lambda_H(h)), u> <{}^t\phi(\lambda_H(h)), v> \end{aligned}$$

Therefore ${}^t\phi(\lambda_H(h))$ is a character of $F_\alpha(G;M_*)$ for all $h\in H$, which implies that ${}^t\phi(\lambda_H(H))\subset\lambda_G(G)$ because the group of all characters $\widehat{F_\alpha(G;M_*)}$ is isomorphic to G ([12] Theorem 3.14), moreover since ϕ is an isomorphism,

$${}^t\phi(\lambda_H(H)) = \lambda_G(G).$$

We denote $\lambda_G(k) = {}^t\phi(\lambda_H(e))$. By the same argument in [48] Theorem 2, we get that

$$\gamma \equiv {}^t\phi(\lambda_H(e))^{-1}\, {}^t\phi = \gamma_I + \gamma_A \tag{2.4}$$

is a C^*-isomorphism in Kadison's sense [19] and $\gamma(\lambda_H(h_1)\lambda_H(h_2))$ is either $\gamma(\lambda_H(h_1))\gamma(\lambda_H(h_2))$ or $\gamma(\lambda_H(h_2))\gamma(\lambda_H(h_1))$, moreover if we put $\lambda_G(I(h)) = \gamma(\lambda_H(h))$, then I is either an isomorphism or an anti-isomorphism of H onto G. (2.5)

The transposed map ψ of γ is also an isometric isomorphism of $F_\alpha(G;M_*)$ onto $F_\beta(H;N_*)$. Then we get

$$\begin{aligned} <\gamma(\pi_\beta(y)) , u*v> &= <\pi_\alpha(y) , \psi(u*v)> \\ &= <\pi_\beta(y) , \psi(u)*\psi(v)> \\ &= <\pi_\beta(y)\otimes 1, \psi(u)\otimes\psi(v)> \\ &= <\gamma(\pi_\beta(y)) , u*v> \end{aligned}$$

for all $y\in N$, u, $v\in F_\alpha(G;M_*)$. By [29] Proposition 2.3, we obtain

$\gamma(\pi_\beta(y))$ is an element of $\pi_\alpha(M)$, so we can define an isometric surjective linear map θ of N onto M by $\theta = \pi_\alpha^{-1}\circ\gamma\circ\pi_\beta$. Since γ is a Jordan isomorphism,

$$\gamma(T)\gamma(z') + \gamma(z')\gamma(T) = \gamma([T,z']) = 2\gamma(Tz')$$

for all $T \in H\times_\beta N$, therefore we get $\gamma(Tz') = \gamma(T)z$. Hence

$$\gamma(\pi_\beta(xy))z = \gamma(\pi_\beta(x))\gamma(\pi_\beta(y))z$$

for all $x, y \in N$. Since z is a central projection of $G\times_\alpha M$, z is also a projection of $\pi_\alpha(M)'$, then we get

$$\gamma(\pi_\beta(xy))p = \gamma(\pi_\beta(x))\gamma(\pi_\beta(y))p \tag{2.6}$$

for all $x, y \in N$ where p is the central support of z in $\pi_\alpha(M)'$. We denote by q the central support of z' in $\pi_\beta(N)'$, then the equations $\gamma(q)z = \gamma(qz') = \gamma(z') = z$ imply that $\gamma(q)p = p$, similarly we obtain $\gamma^{-1}(p)q = q$ so that

$$\gamma(q) = \gamma(\gamma^{-1}(p)q) = \gamma(\gamma^{-1}(p))\gamma(q)p = p\gamma(q)p = p.$$

Hence θ is an isomorphism of N_q onto M_p and θ is an anti-isomorphism of $N_{(1-q)}$ onto $M_{(1-p)}$. The projection p (resp. q) is G-invariant (resp. H-invariant) since $\pi_\alpha(M)' = \lambda_G(g)\pi_\alpha(M)'\lambda_G(g)^*$ and $\lambda_G(g)z\lambda_G(g)^* = z$.
Now we have already proved (1) ∿ (4) and the statements (5),(6) still remain to prove. For all $y \in N$, $h \in H$ we get,

$$\begin{aligned}\{\pi_\alpha\circ\theta(\beta_h(y))\}z &= \gamma(\lambda_H(h)\pi_\beta(y)\lambda_H(h)^*z') \\ &= \lambda_G(I(h))\pi_\alpha\circ\theta(y)\lambda_G(I(h)^{-1})z \\ &= \{\pi_\alpha\circ\alpha_{I(h)}\theta\}(y)z,\end{aligned}$$

hence

$$\theta\circ\beta_h = \alpha_{I(h)}\circ\theta \quad \text{on } N_q,$$

and similarly

$$\theta\circ\beta_h = \alpha_{I(h^{-1})}\circ\theta \quad \text{on } N_{(1-q)}.$$

Therefore

$$\theta\circ\beta_h(y) = \alpha_{I(h)}\circ\theta(y)p + \alpha_{I(h^{-1})}\circ\theta(y)(1-p)$$

for all $y\in N$ and $h\in H$. To prove the statement (5), we shall show,

$$\operatorname{supp}\gamma(\pi_\beta(y)\lambda_H(h)) = \{I(h)\}.$$

For since $\gamma(\pi_\beta(y)\lambda_H(h))u = \gamma(\pi_\beta(y)\lambda_H(h)\psi(u))$ for all $u\in F_\alpha(G;M_*)$ and ψ is surjective,

$$[\gamma(\pi_\beta(y)\lambda_H(h))F_\alpha(G;M_*)]^{-\sigma-w} = \gamma[\pi_\beta(y)\lambda_H(h)F_\beta(H;N_*)]^{-\sigma-w}$$

where $[\ \dots\ ]^{-\sigma-w}$ means a σ-weak closure, on the other hand,

$$[\pi_\beta(y)\lambda_H(h)F_\beta(H;N_*)]^{-\sigma-w} \cap \lambda_H(H) = \mathbb{C}\lambda_H(h)$$

because of $\operatorname{supp}\pi_\beta(y)\lambda_H(h) = \{h\}$, so we obtain

$$[\gamma(\pi_\beta(y)\lambda_H(h))F_\alpha(G;M_*)]^{-\sigma-w} \cap \lambda_G(G) = \mathbb{C}\lambda_G(I(h))$$
$$\operatorname{supp}\gamma(\pi_\beta(y)\lambda_H(h)) = \{I(h)\}.$$

By [12] Theorem 4.4 or [32] Proposition 6.1, there exists an element x of M such that $\gamma(\pi_\beta(y)\lambda_H(h)) = \pi_\alpha(x)\lambda_G(I(h))$.

$$\pi_\alpha(x)\lambda_G(I(h))z = \gamma(\pi_\beta(y)\lambda_H(h))z$$
$$= \gamma(\pi_\beta(y))\gamma(\lambda_H(h))z = \pi_\alpha(\theta(y))\lambda_G(I(h))z,$$

then $xp = \theta(y)p$, and similarly $x(1-p) = \alpha_{I(h)}\theta(y)(1-p)$. we get

$$x = \theta(y)p + \alpha_{I(h)}\theta(y)(1-p),$$

$$\gamma(\pi_\beta(y)\lambda_H(h)) = \pi_\alpha(\theta(y)p)\lambda_G(I(h)) + \pi_\alpha(\alpha_{I(h)}\theta(y)(1-p))\lambda_G(I(h)).$$

By (2.1), $\phi(u) = \psi({}_k u)$ for $u \in F_\alpha(G;M_*)$ and the above equation, we can get the statement (5).

Remark 2. Theorem 1 is a generalization of [48] Theorem 2.

Corollary 3. Let (M, G, α), (N, H, β) be W^*-dynamical systems and the two actions α and β are ergodic on their centers (that is $Z_M \cap M^G = Z_N \cap N^H = \mathbb{C}$).

The following statements are equivalent,

(1) $F_\alpha(G;M_*)$ is isomorphic to $F_\beta(H;N_*)$ in the sense of Banach algebras

(2) there exists either an isomorphism I of H onto G, an isomorphism θ of N onto M such that $\theta \circ \beta_h = \alpha_{I(h)} \circ \theta$ for all $h \in H$ or an anti-isomorphism I of H onto G, an anti-isomorphism θ of N onto M such that $\theta \circ \beta_h = \alpha_{I(h^{-1})} \circ \theta$ for all $h \in H$.

Proof. Suppose ϕ is an isometric isomorphism of $F_\alpha(G;M_*)$ onto $F_\beta(H;N_*)$ and we use the same notations in Theorem 1. The projection p in (3) of Theorem 1 must be zero or 1 by the ergodicity of the action α, then θ is either an isomorphism or an anti-isomorphism of N onto M. When G is a locally compact abelian group (it follows from (2.5) that H is a locally compact abelian group), I in (2.5) can be regarded as both an isomorphism and an anti-isomorphism, therefore the statement (2) follows from Theorem 1 when G is abelian. Hence we may assume that G is non-abelian. When I is an anti-isomorphism of H onto G, the projection (1-z) in (2.3) must be non-zero. For if the projection z is the identity in $G \times_\alpha M$, then γ in (2.4) is an isomorphism of $H \times_\beta N$ onto $G \times_\alpha M$, so I is an isomorphism, which is a contradiction. Taking the central support of (1-z) in $\pi_\alpha(M)'$ as (2.6), θ is an anti-isomorphism of H onto G such that $\alpha_{I(h^{-1})} \circ \theta = \theta \circ \beta_h$ for all $h \in H$. If I is an isomorphism, θ is an isomorphism such that $\alpha_{I(h)} \circ \theta = \theta \circ \beta_h$.

Conversely suppose I is an isomorphism of H onto G such that $\theta \circ \beta_h = \alpha_{I(h)} \circ \theta$ for all $h \in H$. Then there exists an isomorphism Γ of $H \times_\beta N$ onto $G \times_\alpha M$ such that $\Gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in N$ and $\Gamma(\lambda_H(h)) = \lambda_G(I(h))$ for all $h \in H$ (cf. [45] Proposition 3.4). Then the transposed map ϕ of Γ is an isometric isomorphism of $F_\alpha(G;M_*)$ onto $F_\beta(H;N_*)$. Suppose I is an anti-isomorphism H onto G such that $\theta \circ \beta_h = \alpha_{I(h^{-1})} \circ \theta$ for all $h \in H$. Considering the opposite von Neumann algebra M° of M and the isomorphism J of H onto G by $J(h) = I(h^{-1})$ for all $h \in H$. There exists an isomorphism Γ of $H \times_\beta N$ onto $G \times_\alpha (M^\circ)$ such that $\Gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in N$, $\Gamma(\lambda_H(h)) = \lambda_G(J(h))$ for all $h \in H$. On the other hand, $G \times_\alpha (M^\circ)$ is isometrically isomorphic to $G \times_\alpha M$ as Banach spaces, therefore Γ is a σ-weakly continuous isometric linear map of $H \times_\beta N$ onto $G \times_\alpha M$. Then the transposed map ϕ of Γ is an isometric isomorphism of $F_\alpha(G;M_*)$ onto $F_\beta(H;N_*)$.

Chapter III C^*-dynamical system.

This chapter is an attempt to extend to some results on C^*-dynamical systems with locally compact abelian groups to C^*-dynamical systems with non-abelian groups. W. Arveson [2] constructed the theory of spectral subspaces for W^*-and C^*-dynamical systems. Using this , A. Connes [6] defined the Connes spectrum $\Gamma(\alpha)$ which is a closed subgroup of the dual group $\hat{G}$ of the group G, and established a beautiful structure theory of factors of type III. A. Connes and M. Takesaki ([7],[45]) proved for a W^*-dynamical system (M, G, α) with an abelian group G that $G\times_\alpha M$ is a factor if and only if $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M.

Now, suppose that we have a C^*-dynamical system (A, G, α). When G is an abelian group, Connes-Takesaki's statement was replaced by the following, (1) the C^*-crossed product $G\times_\alpha A$ is prime if and only if $\Gamma(\alpha) = \hat{G}$ and A is G-prime ([27],[38], [39]), (2) $G\times_\alpha A$ is simple if and only if for the strong Connes spectrum $\tilde{\Gamma}(\alpha)$, $\tilde{\Gamma}(\alpha) = \hat{G}$ and A is G-simple ([28]). In the proof of these statements, it was important that we have a dual action $\hat{\alpha}$ of α on $G\times_\alpha A$ and we consider the Connes spectrum $\Gamma(\hat{\alpha})$ for the C^*-dynamical system ($G\times_\alpha A$, $\hat{G}$, $\hat{\alpha}$). Unless the group G is abelian, we can not construct the dual action. Instead, a co-action δ on $G\times_\alpha A$ can be constructed, which then plays the same role as the dual action in the case of abelian groups.

In this chapter, for a C^*-dynamical system (A, G, α) we introduce the notion of δ-invariantness for C^*-subalgebras of $G\times_\alpha A$ and, using this, we define the essential spectrum $\Gamma(\delta)$, which is coincident with $\Gamma(\hat{\alpha})$ in the case of abelian groups. It is shown that A is prime if and only if $\Gamma(\delta) = G$ and A is G-prime. D. E. Evans and T. Sund [10] investigated C^*-dynamical systems with compact groups and mentioned that $\Gamma(\alpha)$ is not invariant under exterior equivalence. We prove that $\Gamma(\hat{\alpha})$ is invariant under exterior equivalence ($\hat{\alpha}$ is the bidual action of α) by characterizing $\Gamma(\hat{\alpha})$ in terms of the dual co-action δ of α. Our final result is that a von Neumann algebra should be hyperfinite when a compact group acts on it ergodically. This was proposed as a problem in the preprint of [17] of R. Høegh-Krohn, M. B. Landstad and E. Størmer, and, when the paper appeared, it was proved independently with us.

1. The relation between $\Gamma(\delta)$ and $\Gamma(\hat{\alpha})$ for an abelian group.

Now take a C^*-algebra A and a locally compact group G with a fixed left Haar measure dg on G. Suppose that there is a homomorphism α of G into the group Aut(A) of all $*$-automorphisms of A such that each function $g \longrightarrow \alpha_g(a)$ is norm continuous for $a \in A$. The triple (A, G, α) is called a C^*-dynamical system. Let (A, G, α) be a C^*-dynamical system and assume $A \subset B(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. We denote by K(G, A) the space of continuous functions from G to A with compact support. Define a faithful representation of K(G, A) on $L^2(G, \mathfrak{H})$ by

$$(x\xi)(g) = \int_G \alpha_{h^{-1}}(x(g))\xi(h^{-1}g)\, dg \qquad (3.1)$$

for $x \in K(G, A)$ and $\xi \in L^2(G, \mathfrak{H})$. We identify K(G, A) with its image in $B(L^2(G, \mathfrak{H}))$ and denote by $G \times_\alpha A$ the C^*-algebra generated by K(G, A). We say that $G \times_\alpha A$ is a crossed product of G with A. We define a representation ν of G on $L^2(G, \mathfrak{H})$ and a faithful representation ι of A on $L^2(G, \mathfrak{h})$ by

$$(\nu(h)\xi) = \xi(h^{-1}g), \quad (\iota(x)\xi)(h) = \alpha_{h^{-1}}(x)\xi(h)$$

for $h, g \in G$, $x \in A$ and $\xi \in L^2(G, \mathfrak{h})$. This covariant representation (ι, ν) is (π_α, λ) in (1.1). We also use a unitary operator W (called the Kac-Takesaki operator) on $L^2(G \times G)$

$$(W\xi)(g,h) = \xi(g,gh)$$

for $\xi \in L^2(G \times G)$ and we denote by λ a left regular representation of G and by $m(G)$ the von Neumann algebra generated by $\{\lambda(g); g \in G\}$. Note that this λ is different from λ in (1.1). Then we define an isomorphism δ_G of $m(G)$ into $m(G) \bar{\otimes} m(G)$ by $\delta_G(x) = W^*(x \otimes 1)W$ for $x \in m(G)$.

When A and B are C^*-algebras, we denote by $M(A)$ its multiplier algebra. If A is a concrete C^*-algebra, we may define $M(A) = \{a \in A''; ab + ca \in A \text{ for } b, c \in A\}$ ([1]). We put

$$\widetilde{M}_L(A \otimes B) = \{x \in M(A) \otimes M(B);$$

$$x(1 \otimes b) + (1 \otimes c)x \in A \otimes B,\ L_\phi(x) \in A \text{ for } b, c \in B,\ \phi \in B^*\}$$

where L_ϕ is the left slice map of ϕ and the symbol $\otimes$ means the spatial tensor product.

Proposition 1.1. The map δ;

$$\delta(x) = (1 \otimes W^*)(x \otimes 1)(1 \otimes W) \qquad \text{for } x \in G \times_\alpha A,$$

is a *-isomorphism of $G \times_\alpha A$ into $\widetilde{M}_L(G \times_\alpha A \otimes C_r^*(G))$, where $C_r^*(G)$ is the reduced group C^*-algebra of G. It satisfies the following relation,

$$\delta(\nu(g)) = \nu(g) \otimes \lambda(g),\ (g \in G), \quad \delta(\iota(a)) = \iota(a) \otimes 1,\ (a \in A) \qquad (3.2)$$

$$(\delta \otimes i)\delta = (i \otimes \delta_G)\delta \tag{3.3}$$

where i's are identity mappings of $M(C_r^*(G))$ or $M(G \times_\alpha A)$.

Proof. We get (3.2) by an easy calculation. It follows from (3.1) and (3.2) that

$$\delta(x) = \int_G (\iota(x(g)) \otimes 1)(\nu(g) \otimes \lambda(g))dg$$

for $x \in K(G, A)$. Since $\iota(A)$ and $\nu(G)$ are contained in $M(G \times_\alpha A)$ and $\lambda(G)$ is contained in $M(C_r^*(G))$, $\delta(x)$ is an element of $M(G \times_\alpha A) \otimes M(C_r^*(G))$, which implies $\delta(G \times_\alpha A) \subset M(G \times_\alpha A) \otimes M(C_r^*(G))$.

If $x \in K(G, A)$, $f \in K(G) \equiv K(G, \mathbb{C})$, We have

$$\begin{aligned}\delta(x)(1 \otimes \lambda(f)) &= \int_G (\iota(x(g)) \otimes 1)(\nu(g) \otimes \lambda(g))(1 \otimes \lambda(f))\, dg \\ &= \iint_{G \times G} f(h)(\iota(x(g)) \otimes 1)(\nu(g) \otimes \lambda(gh))\, dgdh \\ &= \iint_{G \times G} (\iota(f(g^{-1}h)x(g)) \otimes 1)(\nu(g) \otimes \lambda(h))\, dgdh\end{aligned}$$

where $\lambda(f) = \int_G f(h)\lambda(h)\, dh$, therefore $\delta(G \times_\alpha A)(1 \otimes C_r^*(G)) \subset G \times_\alpha A \otimes C_r^*(G)$. Similarly we have $(1 \otimes C_r^*(G))\delta(G \times_\alpha A) \subset (G \times_\alpha A) \otimes C_r^*(G)$.

Take $x \in K(G,A)$ and $\phi \in C_r^*(G)^*$, then

$$\begin{aligned}L_\phi(\delta(x)) &= L_\phi\left(\int_G (\iota(x(g)) \otimes 1)(\nu(g) \otimes \lambda(g))\, dg\right) \\ &= \int_G \iota(x(g))\phi(\lambda(g))\nu(g)\, dg \\ &= \int_G \iota(\phi(\lambda(g))x(g))\nu(g)\, dg = \phi x,\end{aligned}$$

where $(\phi x)(g) = \phi(\lambda(g))x(g) \in K(G, A)$. We have therefore

$$L_\phi \circ \delta(G \times_\alpha A) \subset G \times_\alpha A \qquad \text{for } \phi \in C_r^*(G)^*,$$

because $\| L_\phi \circ \delta \| \leq \| \phi \|$.

The relation (3.3) follows from (3.2) and $\delta_G(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ for $g \in G$.

We will introduce the essential spectrum of a co-action following Y. Nakagami [33] and Y. Nakagami-M. Takesaki [34]. To do this we first recall some definitions.

A co-action δ of G on a C^*-algebra A is an isomorphism of A into $\widetilde{M}_L(A\otimes C_r^*(G))$ satisfying $(\delta\otimes i)\delta = (i\otimes \delta_G)\delta$. Then we define δ_u by

$$\delta_u(a) = L_u\circ\delta(a) \qquad \text{for } u\in B_r(G), \quad a\in A,$$

where $B_r(G)$ is defined in [8] to be regular ring, and we identify $B_r(G)$ with the dual space $C_r^*(G)^*$ of $C_r^*(G)$. It follows from (3.3) that $\delta_{u\cdot v} = \delta_u\circ\delta_v$ for $u,v\in B_r(G)$.

We set

$$Sp_\delta(a) = \{g\in G;\ u(g) = 0 \quad \text{for } \delta_u(a) = 0,\ u\in B_r(G)\},$$

$$Sp(\delta) = \{g\in G;\ u(g) = 0 \quad \text{for } \delta_u = 0,\ u\in B_r(G)\}$$

and

$$\Gamma(\delta) = \bigcap\{Sp(\delta|B);\ \ B\in\mathcal{H}^\delta(A)\},$$

where $\mathcal{H}^\delta(A)$ is the family of non-zero hereditary C^*-subalgebras B of A such that $\delta_u(B)\subset B$ for $u\in B_r(G)$, which is called δ-invariant. Let E be a closed subset of G, we set

$$A^\delta(E) = \{a\in A;\ \ Sp_\delta(a)\subset E\}.$$

Lemma 1.2. If $g\in G$, then $g\in Sp(\delta)$ if and only if $A^\delta(V)\neq 0$ for every compact neighbourhood V of g.

Proof. Let V be a compact neighbourhood of g with $A^\delta(V) = \{0\}$. Take an element $g_0\in V^c$ and $v\in B_r(G)$ with (the support of v)$\cap V = \phi$ and $v(g_0)=1$. If $u\in B_r(G)$ with (the support of u)$\subset V$ and $u(g) = 1$, then,

$$\delta_v\circ\delta_u(a) = \delta_{v\cdot u}(a) = 0 \qquad \text{for } a\in A,$$

which implies $g_0 \notin Sp_\delta(\delta_u(a))$ that is $\delta_u(a) \in A^\delta(V)$. Therefore $\delta_u(a) = 0$ for all $a \in A$. As $u(g) \neq 0$, we see $g \in Sp(\delta)$.

Suppose that $g \notin Sp(\delta)$. Take a compact neighbourhood V of g with $V \cap Sp(\delta) = \phi$ and take $a \in A^\delta(V)$, then it follows from $Sp_\delta(a) \subset Sp(\delta)$ that $Sp_\delta(a) = \phi$. Since

$$I_a \equiv \{u \in B_r(G), \quad \delta_u(a) = 0\}$$

is a closed ideal of $B_r(G)$ with $\{g \in G;\ u(g) = 0 \text{ for all } u \in I_a\} = \phi$ and $B_r(G)$ is a regular ring ([11]), I_a contains the Fourier algebra A(G) of G because it contains K(G)([11]). For $\omega \in A^*$, we have

$$0 = \langle \delta_u(a), \omega \rangle = \langle \delta(a), \omega \otimes u \rangle \qquad \text{for } u \in A(G) \subset B_r(G).$$

Since, by [1, Proposition 2.4], the algebraic tensor product $A^* \odot A(G)$ of A^* and A(G) is dense in $(M(A) \otimes M(C_r^*(G)))^*$ with respect to the w^*-topology of $M(A) \otimes M(C_r^*(G))$, we have $\delta(a) = 0$, that is $a = 0$. We have therefore $A^\delta(V) = 0$.

Lemma 1.3. Let E_i be a compact set in G $(i = 1,2)$, then $A^\delta(E_1)A^\delta(E_2) \subset A^\delta(E_1E_2)$.

This lemma is proved by a usual argument (See [34, IV, Lemma 1.2]), and we leave its verification to the reader.

Proposition 1.4. $\Gamma(\delta)$ is a closed subgroup of G.

Proof. Since $Sp(\delta)$ is a closed set, $\Gamma(\delta)$ is a closed set of G. We want to see that $Sp(\delta)\Gamma(\delta) \subset Sp(\delta)$. Take $g_1 \in Sp(\delta), g_2 \in \Gamma(\delta)$ and compact neighbourhoods V, V_1, and V_2 of g_1g_2, g_1, and g_2, respectively such that $V_1V_2 \subset V$. For $a_1 \in A^\delta(V_1)$, $a_1 \neq 0$, B denote the smallest δ-invariant hereditary C^*-subalgebra generated by $\{\delta_u(a_1);\ u \in B_r(G)\}$. Then we can find an $a_2 \in B \cap A^\delta(V_2)$, $a_2 \neq 0$. Let I be the closed linear span of $\{a\delta_u(a_1);\ a \in A,\ u \in B_r(G)\}$, then I is a closed left ideal of A such that $B = I^* \cap I$. Therefore if $\delta_u(a_1)a_2 = 0$ for any

$u \in B_r(G)$, it implies $B a_2 = 0$ that is $a_2 = 0$. Hence there is a $u \in B_r(G)$ such that $\delta_u(a_1)a_2 \neq 0$. By Lemma 1.3,

$$0 \neq \delta_u(a_1)a_2 \in A^\delta(V_1)A^\delta(V_2) \subset A^\delta(V_1V_2) \subset A^\delta(V).$$

By Lemma 1.2, we conclude from this that $Sp(\delta)\Gamma(\delta) \subset Sp(\delta)$. As this is true for $\delta|B$ in place of δ, we see that $\Gamma(\delta)$ is a semi-group and it is easy to prove $\Gamma(\delta) = \Gamma(\delta)^{-1}$. Therefore $\Gamma(\delta)$ is a closed subgroup of G.

From now on, G is supposed to be abelian and we study relations between the co-action δ on $G \times_\alpha A$ and the dual action $\hat{\alpha}$ of the dual group Γ of G on $G \times_\alpha A$ (See [42]).

Proposition 1.5. Let (A, G, α) be a C^*-dynamical system and B is a C^*-subalgebra of $G \times_\alpha A$. Then B is δ-invariant if and only if B is $\hat{\alpha}_\gamma$-invariant for $\gamma \in \Gamma$ (Γ-invariant).

Proof. Take $\xi, \eta \in K(\Gamma)$ and $x \in K(G, A)$,

$$\begin{aligned}
&\int_\Gamma \xi(\gamma)\eta(\gamma)\hat{\alpha}_\gamma(x)d\gamma \\
&= \int_\Gamma \xi(\gamma)\eta(\gamma)\hat{\alpha}_\gamma\left(\int_G \iota(x(g))\nu(g)dg\right)d\gamma \\
&= \int_\Gamma \int_G \xi(\gamma)\eta(\gamma)\iota(x(g))\nu(g)\overline{\langle g,\gamma\rangle}d\gamma \\
&= \int_G\left(\int_\Gamma \xi(\gamma)\eta(\gamma)\overline{\langle g,\gamma\rangle}d\gamma\right)\iota(x(g))\nu(g)dg \\
&= \int_G \tilde{\xi}*\tilde{\eta}(-g)\iota(x(g))\nu(g)dg,
\end{aligned}$$

where $\tilde{\xi}$ is the inverse Fourier transform of ξ and the symbol * means the convolution in $L^1(G)$. On the other hand, set

$$\omega(\xi,\eta)(x) = \langle x\tilde{\xi}, \tilde{\eta}^b\rangle, \qquad \text{for } x \in C_r^*(G)$$

where $\tilde{\eta}^b(g) = \overline{\tilde{\eta}(-g)}$, then we have

$$\omega(\xi,\eta)(\lambda(g)) = \langle\lambda(g)\tilde{\xi},\tilde{\eta}^b\rangle = \int_G \tilde{\xi}(h - g)\tilde{\eta}(-h)dh = \tilde{\xi}*\tilde{\eta}(-g).$$

Then we have,

$$= \int_G \omega(\xi,\eta)(\lambda(g))\iota(x(g))\nu(g)dg$$
$$= \delta_{\omega(\xi,\eta)}\left(\int_G \iota(x(g))\nu(g)dg\right) = \delta_{\omega(\xi,\eta)}(x).$$

Therefore

$$\int_\Gamma \xi(\gamma)\eta(\gamma)\hat{\alpha}_\gamma(x)d\gamma = \delta_{\omega(\xi,\eta)}(x) \quad \text{for } x \in G\times_\alpha A. \quad (3.4)$$

The set $\{\omega(\xi,\eta);\ \xi,\eta \in K(\Gamma)\}$ is dense in $(C_r^*(G))^*$ with respect to $\sigma(C_r^*(G)^*, M(C_r^*(G)))$-topology and the map

$$\phi \in (C_r^*(G))^* \longrightarrow \delta_\phi(x) \in G\times_\alpha A$$

is norm continuous with respect to $\sigma((C_r^*(G))^*, M(C_r^*(G)))$-topology for each $x \in G\times_\alpha A$. Hence if B is Γ-invariant, then B is δ-invariant.

Conversely suppose that B is δ-invariant. Take $\gamma \in \Gamma$, the positive definite function $<\cdot,\gamma>$ is an element of $B_r(G)$. Then by an easy calculation, we have $\delta_{<\cdot,\gamma>} = \hat{\alpha}_\gamma$, therefore B is Γ-invariant.

Given a C^*-dynamical system (A, G, α), we denote by $\mathcal{H}^\alpha(A)$ the family of non-zero, G-invariant, hereditary C^*-subalgebra of A. The Connes spectrum $\Gamma(\alpha)$ of α is defined as

$$\Gamma(\alpha) = \bigcap \{Sp(\alpha|B),\ B \in \mathcal{H}^\alpha(A)\},$$

cf. [37].

Theorem 1.6. Let (A, G, α) be a C^*- dynamical system with an abelian group G. Starting from this, we have a C^*-dynamical system $(G\times_\alpha A, \Gamma, \hat{\alpha})$ and a dual system $(G\times_\alpha A, \delta)$. Then $\Gamma(\delta)$ and $\Gamma(\hat{\alpha})$ coincide.

Proof. At first we prove $A^{\hat{\alpha}}(V) = A^{\delta}(-V)$ for every compact neighbourhood V of $g \in G$, where $A^{\hat{\alpha}}(V) = \{x \in G \times_{\alpha} A,\ Sp_{\hat{\alpha}}(x) \subset V\}$. Take $x \in A^{\hat{\alpha}}(V)$, $g_0 \notin V$ and compact neighbourhood V_0 of g_0 with $V_0 \cap V = \phi$, we can find $\xi, \eta \in K(\Gamma)$ with $\tilde{\xi} * \tilde{\eta}(g_0) = 1$ and $\tilde{\xi} * \tilde{\eta} \equiv 0$ on V_0^C. The inverse Fourier transform of $\xi(\gamma)\eta(\gamma)$ is $\tilde{\xi} * \tilde{\eta}$, so

$$\delta_{\omega(\xi,\eta)}(x) = \int_{\Gamma} \xi(\gamma)\eta(\gamma)\hat{\alpha}_{\gamma}(x)\, d\gamma = 0, \quad \text{as } x \in A^{\hat{\alpha}}(V).$$

As $\omega(\xi,\eta)(\lambda(-g_0)) = \tilde{\xi} * \tilde{\eta}(g_0) = 1$, we have $-g_0 \in Sp_{\delta}(x)$ that is $Sp_{\delta}(x) \subset -V$.

Conversely, take $x \in A^{\delta}(-V)$, $g_0 \notin V$ and a compact neighbourhood V_0 of g_0 with $V_0 \cap (-V) = \phi$, and take $\xi, \eta \in K(\Gamma)$ as above. Put $y = \delta_{\omega(\xi,\eta)}(x)$, then we have $Sp_{\delta}(y) = \phi$, since $Sp_{\delta}(\delta_u(y)) \subset$ (the support of $u) \cap Sp_{\delta}(y)$, hence we get $y = 0$. By (3.4) we have

$$\int_{\Gamma} \xi(\gamma)\eta(\gamma)\hat{\alpha}_{\gamma}(x)d\gamma = 0.$$

which implies $g_0 \notin Sp_{\hat{\alpha}}(x)$ that is $Sp_{\hat{\alpha}}(x) \subset V$. As $g \in Sp(\hat{\alpha})$ if and only if $A^{\hat{\alpha}}(V) \neq \{0\}$ for each compact neighbourhood V of g (See [37]), by Lemma 1.2, we have $Sp(\hat{\alpha}) = -Sp(\delta)$. We conclude that $\Gamma(\hat{\alpha}) = \Gamma(\delta)$.

2. Primeness of C^*-algebra with a co-action.

The statements in this section are some generalizations of those in [39] for arbitrary locally compact groups. The arguments which we do is a modification of those which D. Olesen and G.K. Pedersen did there.

Lemma 2.1. Let (A, G, α) be a C^*-dynamical system and δ be a co-action of G on $G\times_{\alpha}A$. Then an element $g \in G$ belongs to $\Gamma(\delta)$ if and only if $I \cap \alpha_g(I)$ is non-zero for every non-zero closed ideal I of A.

Proof. Suppose that $I \cap \alpha_g(I) = \{0\}$ for some non-zero closed ideal I of A. As D. Olesen and G.K. Pedersen did in [39], choose (by spectral theory) non-zero positive elements $b, c \in I$ with $bc = b$.

There is then a compact neighbourhood Ω of the identity e in G such that $\|\alpha_h(c) - c\| < 1$ for every $h \in \Omega$.

$$\alpha_h(b) = \alpha_h(b)(1 - \alpha_h(c) + c)(\sum_{n=0}^{\infty}(\alpha_h(c) - c)^n)$$

$$= \alpha_h(b)c(\sum_{n=0}^{\infty}(\alpha_h(c) - c)^n) \in I$$

, whence $bA\alpha_{h^{-1}g}(b) = \alpha_h^{-1}(\alpha_h(b)A\alpha_g(b)) \subset \alpha_h^{-1}(IA\alpha_g(I)) = \{0\}$ (3.5) for every $h \in \Omega$.

Let B be the hereditary C^*-subalgebra of $G\times_\alpha A$ generated by $\iota(b)(G\times_\alpha A)\iota(b)$ and note that $B \in \mathcal{H}^\delta(G\times_\alpha A)$ since $\delta(\iota(b)) = \iota(b)\otimes 1$.

For $\xi, \eta \in K(G)$ with $\bar{\eta}*\check{\xi}(g) \neq 0$ and supp $\bar{\eta}*\check{\xi} \subset \Omega^{-1}g$ where $\bar{\eta}(t) = \overline{\eta(t)}$ and $\check{\xi}(t) = \xi(t^{-1})$, put $x = \iota(b)\nu(f)^*\iota(a)\nu(f)\iota(b)$ for $a \in A$, $f \in K(G)$ where $\nu(f) = \int_G f(t)\nu(t)dt$ and put $\omega[\xi,\eta](d) = \langle d\xi,\eta\rangle$ for $d \in C_r^*(G)$.

Then

$$\delta_{\omega[\xi,\ \eta]}(x) = \delta_{\omega[\xi,\ \eta]}(\iota(b)\int_G f^{\#}(t)\nu(t)dt\,\iota(a)\int_G f(s)\nu(s)ds\,\iota(b))$$

$$= \delta_{\omega[\xi,\ \eta]}(\iint_{G\times G} f^{\#}(t)f(s)\iota(b\,\alpha_t(a)\alpha_{ts}(b))\nu(ts)dtds)$$

$$= \iint_{G\times G}\bar{\eta}*\check{\xi}(ts)f^{\#}(t)f(s)\iota(b\alpha_t(a)\alpha_{ts}(b))\nu(ts)dtds$$

$$\overset{(3.5)}{=} 0$$

where $f^{\#}(t) = \frac{1}{\Delta(t)}\alpha_t(x(t))^*$, Δ is the modular function of G.

Since $\{x \in G\times_\alpha A;\ x = \iota(b)\nu(f)^*\iota(a)\nu(f)\iota(b),\ a \in A.\ f \in K(G)\}$ is dense in B, we have $\delta_{\omega[\xi,\ \eta]}(B) = \{0\}$. Since $\bar{\eta}*\check{\xi}(g) \neq 0$, this implies $g \notin Sp(\delta|_B)$ i.e. $g \notin \Gamma(\delta)$.

Conversely, if $g \notin \Gamma(\delta)$, there is a $B \in \mathcal{H}^\delta(G\times_\alpha A)$ such that $g \in Sp(\delta|_B)$. Therefore there is a compact neighbourhood Ω of g in G and $B \in \mathcal{H}^\delta(G\times_\alpha A)$ such that $\delta_u(B) = \{0\}$ whenever $u \in B_r(G)$ with $\Omega \subset \text{supp}(u) \equiv$ the closure of $\{g \in G;\ u(g) \neq 0\}$. Choose a compact neighbourhood Ω_1 of g and a symmetric compact neighbourhood Ω_0 of e with $\Omega_0\Omega_1\Omega_0 \subset \Omega$ and $\Omega_0\ g\ \Omega_0 \subset \Omega_1$. (3.6)

Let L be a δ-invariant closed left ideal of $G\times_\alpha A$ such that $B = L^*\cap L$ and L_0 be the δ-invariant closed left ideal generated by $\{x\nu(f);\ x\in L,\ f\in K(G),\ \mathrm{supp}\ f\subset\Omega_0\}$. We show that the norm-closure L_{00} of $\{\int_G x(t)\nu(t)dt;\ x\in K(G;\ L),\ \mathrm{supp}\ x\subset\Omega_0\}$ is a δ-invariant closed ideal containing L_0.

L_{00} is a closed left ideal of $G\times_\alpha A$ since L is a closed left ideal of $G\times_\alpha A$. It contains the element of the form $x\nu(f)$. So we only need to show that L_{00} is δ-invariant. For $x\in K(G;\ L)$ with $\mathrm{supp}\ x\subset\Omega_0$, we have

$$< \delta_\omega(\int_G x(t)\nu(t)dt),\ \phi >$$

$$= <\delta(\int_G x(t)\nu(t)dt),\ \phi\otimes\omega>$$

$$=\int_G <\delta(x(t))\nu(t)\otimes\lambda(t),\ \phi\otimes\omega>dt$$

$$=\int_G <\delta_{\lambda(t)\omega}(x(t))\nu(t),\ \phi>dt$$

for $\omega\in(C_r^*(G))^*$ and $\phi\in(G\times_\alpha A)^*$ where $<d,\ \lambda(t)\omega> = <d\lambda(t),\omega>$ for $d\in(C_r^*(G))$.

Put $z(t) = \delta_{\lambda(t)\omega}(x(t))$, then $z(t)$ is an element of L since $x(t)$ is an element of L and L is δ-invariant, moreover z is an element of $K(G;\ L)$ with $\mathrm{supp}\ z\subset\Omega_0$. Therefore L_{00} is δ-invariant and contains L_0.

Take $x,\ y\in K(G;\ L)$ with $\mathrm{supp}\ x\cup\mathrm{supp}\ y\subset\Omega_0$, then we have for $u\in B_r(G)$ with $\mathrm{supp}\ u\subset\Omega_1$ and $\phi\in(G\times_\alpha A)^*$,

$$<\delta_u((\int_G y(t)\nu(t)dt)^*(\int_G x(s)\nu(s)ds)),\ \phi>$$

$$= <\iint_{G\times G}\nu(t)^*\otimes\lambda(t)^*\delta(y(t)^*x(s))\nu(s)\otimes\lambda(s)dtds,\ \phi\otimes u>$$

$$=\iint_{G\times G}<\delta_{\lambda(s)u\lambda(t)^*}(y(t)^*x(s)),\ \nu(s)\phi\nu(t)^*>dtds,$$

Since $\text{supp}\, x \cup \text{supp}\, y \subset \Omega_0$, $\text{supp}\, \lambda(s)u\lambda(t)^* \subset \Omega$ for $t, s \in \Omega_0$ and $y(t)^*x(s) \in B$, this is

$$= \iint_{\Omega_0 \times \Omega_0} <\delta_{\lambda(s)u\lambda(t)^*}(y(t)^*x(s)),\ \nu(s)\phi\nu(t)^*> dtds = 0.$$

Therefore $\delta_u(y^*x) = 0$ for $x, y \in L_0$ and $u \in B_r(G)$ with $\text{supp}\, u \subset \Omega_1$.

Put $B_0 = L_0^* \cap L_0$. Then we get $B_0 \in \mathcal{H}^\delta(G\times_\alpha A)$ and $\delta_u(B_0) = 0$ whenever $u \in B_r(G)$ with $\text{supp}\, u \subset \Omega_1$. So, for any element x in B_0, we have $Sp(\delta|_{B_0}) \cap \Omega_1 = \phi$.

Choose a non-zero positive element y in B and $f \in K(G)$ with $\text{supp}\, f \subset \Omega_0$ such that the element $y_0 = \nu(f)^*y\nu(f)$ of B_0 is non-zero and choose positive linear functionals $u_i \in B_r(G)$ with $\text{supp}\, u_i \subset \Omega_0$ such that $\sup u_i = \phi_e$ (with respect to the order in $(C_r^*(G))^*$) where ϕ_e is the Plancherel weight on $C_r^*(G) \subset m(G)$, that is the canonical weight of a generalized Hilbert algebra K(G). (See [29]).

Then we have $Sp_\delta(\delta_{u_i}(y_0)) \subset \text{supp}\, u_i \subset \Omega_0$, $\sup \delta_{u_i}(y_0) = \delta_{\phi_e}(y_0)$ and $\delta_{\phi_e}(y_0) \in \iota_\alpha(A)$ by [29] Lemma 3.2 ~ 3.5 and Theorem 3.

Put $x_{i,j} = \delta_{u_i}(y_0)\iota(a)\nu(g)\delta_{u_j}(y_0) \in B_0 (a \in A)$. Then $x_{i,j} \in A^\delta(\Omega_0)A^\delta(g\Omega_0) \subset A^\delta(\Omega_1)$ by Lemma 1.3 and (3.6). Therefore $Sp_\delta(x_{i,j}) \subset \Omega_1 \cap Sp(\delta_{B_0}) = \phi$ ie $x_{i,j} = 0$. Let a_0 be an element of A such that $\iota(a_0) = \delta_{\phi_e}(y_0)$. Since $x_{i,j}$ converges weakly to $\iota(a_0)\iota(a)\nu(g)\iota(a_0)$ for $a \in A$, we get $a_0 A \alpha_g(a_0) = \{0\}$. Then we find a non-zero closed ideal I of A (viz the closed ideal generated by the non-zero element a_0) such that $I \cap \alpha_g(I) = \{0\}$.

Let (A, G, α) be a C^*-dynamical system. We say that A is G-prime (resp prime) if any two non-zero G-invariant closed ideals(resp closed ideals) of A have a non-zero intersection and that $G\times_\alpha A$ is δ-prime if any two non-zero δ-invariant closed ideals of $G\times_\alpha A$ have a non-zero intersection.

Theorem 2.2. Let (A, G, α) be a C^*-dynamical system. Then the following two conditions are equivalent;

(i) A is prime

(ii) (a) A is G-prime and (b) $\Gamma(\delta) = G$.

Proof. (i) → (ii) (a) is obvious, (i) → (ii) (b) follows immediately from Lemma 2.1.

(ii) → (i) If A is not prime, there are orthogonal non-zero closed ideals I and J of A. Take $g \in G$ and assume that $\alpha_g(I) \cap J \neq \{0\}$. Then by (ii) (b) and Lemma 2.1, we have $\{0\} \neq \alpha_g(I) \cap J \cap \alpha_{g^{-1}}(\alpha_g(I) \cap J) \subset J \cap I = \{0\}$, which is absurd. Consequently $\alpha_g(I) \cap J = \{0\}$ for any $g \in G$. Let I_G be the closed ideal generated by $\bigcup_{g \in G} \alpha_g(I)$, then $I_G \cap J = \{0\}$ and I_G is G-invariant, which contradicts (ii) (a). Thus A is prime.

Proposition 2.3. Let (A, G, α) be a C^*-dynamical system. Then A is G-prime if and only if $Gx_\alpha A$ is δ-prime.

Proof. If J_1 and J_2 are orthogonal non-zero δ-invariant closed ideals of $Gx_\alpha A$, take non-zero positive elements x_1 and x_2 in J_1 and J_2 respectively, and choose $f \in K(G)$ with $\nu(f)^* x_i \nu(f) \neq 0$ $(i = 1, 2)$. Since $J_i (i = 1, 2)$ is δ-invariant, we have $\delta_\omega(\nu(f)^* x_i \nu(f)) \subset J_i$ $(i = 1, 2)$, and $\delta_\omega(\nu(f)^* x_1 \nu(f)) M(Gx_\alpha A) \delta_{\omega'} \cdot (\nu(f)^* x_2 \nu(f)) = \{0\}$ for $\omega, \omega' \in (C_r^*(G))^*$. By [29] Lemmas 3.2 ∿ 3.5 and Theorem 3, $\delta_{\phi_e}(\nu(f)^* x_i \nu(f))$ is a non-zero element of $\iota(A)$ $(i = 1, 2)$ and we put $\iota(a_i) = \delta_{\phi_e}(\nu(f)^* x_i \nu(f))$ $(i = 1, 2)$. we have then that $\iota(a_1) M(Gx_\alpha A) \iota(a_2) = \{0\}$, and in particular, we have $\iota(\alpha_g(a_1) a \alpha_h(a_2)) = \nu(g)\iota(a_1)\nu(g)^{-1}\iota(a)\nu(h)\iota(a_2)\nu(h)^{-1} = 0$ for $a \in A$. In this way, we have found orthogonal non-zero G-invariant closed ideals of A viz the closed ideals generated by the orbits $\alpha_G(a_1)$ and $\alpha_G(a_2)$.

Conversely, suppose that we have two orthogonal non-zero G-invariant closed ideals I_1, I_2 of A. Let J_i be the closed subspace of $Gx_\alpha A$ generated by $\{\int_G \iota(x(g))\nu(g)dg;\ x \in K(G; I_i)\}$ $(i = 1, 2)$.

For $x_i \in K(G; I_i)$ $(i = 1, 2)$, we have

$$\int_G \iota(x_1(g))\nu(g)dg \int_G \iota(x_2(g))\nu(g)dg$$

$$= \iint_{G\times G} \iota(x_1(g)\alpha_g(x_2(h)))\nu(gh)dgdh = 0$$

because I_i is G-invariant $(i = 1, 2)$, and this implies $J_1 J_2 = \{0\}$. Since I_i is G-invariant, J_i is a closed ideal of $G \times_\alpha A$.

For $\omega \in (C_r^*(G))^*$ and $x \in K(G; I_i)$,

$$\delta_\omega\left(\int_G \iota(x(g))\nu(g)dg\right) = \int_G \omega(\lambda(g))\iota(x(g))\nu(g)dg,$$

therefore J_1 and J_2 are δ-invariant.

3. C^*-dynamical system with the action of a compact group.

Throughout this section, we assume that G is compact and dg is the normalized Haar measure on G. Let (A,G,α) be a C^*-dynamical system and $\hat{G}$ be the space of isomorphism classes of all irreducible representations of G. If $\pi \in \hat{G}$, we denote by χ_π the associated "modified character" $\chi_\pi(g) = (1/\dim \pi)\mathrm{Tr}(\pi(g))$, and $u(i,j,\pi)$ the associated "coordinate functions" $u(i,j,\pi)(g) = \langle \pi(g)\xi_i, \xi_j\rangle$, where $\{\xi_i\}$ is a normalized orthogonal basis for H_π. By definition $\pi \in \mathrm{Sp}(\alpha)$, iff $\alpha(\chi_\pi)(A) \neq \{0\}$, where

$$\alpha(\chi_\pi)(a) = \int_G \chi_\pi(g)\alpha_g(a)\, dg \qquad \text{for } a \in A,$$

([10]), and $\Gamma(\alpha) = \bigcap\{\mathrm{Sp}\,(\alpha|B);\ B \in \mathcal{H}^\alpha(A)\}$.

Lemma 3.1. If $\pi \notin \Gamma(\alpha)$, then there is a non-zero closed ideal I of $G \times_\alpha A$ such that

$$\bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \cap I = \{0\},$$

where $\bigvee_{i,j}\delta_{u(i,j,\pi)}(I)$ denotes the closed ideal of $G \times_\alpha A$ generated by $\delta_{u(i,j,\pi)}(I)$, $i,j=1,2,\cdots,\dim \pi$.

Proof. If $\pi \notin \Gamma(\alpha)$, there is a $B \in \mathcal{H}^\alpha(A)$ such that $\alpha(\chi_\pi)(B) = \{0\}$. Take a non-zero G-invariant positive element b of B and put a non-zero element $y = \int_G \iota(b)\nu(g)dg \in G \times_\alpha A$, then we have, for $a \in A$, $g \in G$,

$$\delta_{u(i,j,\pi)}(y)\iota(a)\nu(g)y$$
$$= \iint_{G\times G} u(i,j,\pi)(h)\iota(b\alpha_h(a)b)\nu(hgk)dkdh$$

$$= \int_G \iota\left(\int_G u(i,j,\pi)(h)\alpha_h(bab)dh\right)\nu(k)dk,$$

put

$$z = \int_G u(i,j,\pi)(h)\alpha_h(bab)dh .$$

Since B is hereditary, the element bab is in B. Therefore we have $\alpha(\chi_\pi)(bab) = 0$. By the relation $\alpha(u(i,j,\pi))[(u(k,l,\pi))(c)] = \alpha(u(i,j,\pi)*u(k,l,\pi))(c)$ for any c in A and the orthogonality relations for compact groups, $\alpha(\chi_\pi)(c) = 0$ is equivalent to

$$\int_G u(i,j,\pi)(h)\alpha_h(c)dh = 0, \quad \text{for } i,j=1,2,..,\dim \pi.$$

Therefore z=0 and so $\delta_{u(i,j,\pi)}(y)\iota(a)\nu(g)y = 0$, that is

$$\delta_{u(i,j,\pi)}(y)G\times_\alpha Ay = \{0\}, \quad \text{for } i,j=1,2,\cdots,\dim \pi.$$

Let I be the non-zero closed ideal of $G\times_\alpha A$ generated by y. By easy calculation, we have $\bigvee_{i,j}\delta_{u(i,j,\pi)}(I)\cap I = \{0\}$.

We use the definition of the crossed product $G\times_\delta(G\times_\alpha A)$ with the co-action δ, the dual action $\hat{\delta}$ of δ, and Takesaki's duality (See [18],[30] and [34]).

Lemma 3.2. If there is a non-zero closed ideal I of $G\times_\alpha A$ such that $\bigvee_{i,j}\delta_{u(i,j,\pi)}(I)\cap I = \{0\}$, then π does not belong to $\Gamma(\hat{\hat{\alpha}})$ where $\hat{\hat{\alpha}}$ is the bidual action of α (See[30] or [34]).

Proof. Take a non-zero positive element y in I. For $\sigma\in\hat{G}$, put $z=\delta(y)(1\otimes u(i,j,\sigma))\delta(x)\delta(y)$ for $x\in G\times_\alpha A$. We then have

$$\int_G \chi_\pi(g)\hat{\hat{\alpha}}_g(z)dg$$

$$= \int_G \chi_\pi(g)\delta(y)\hat{\hat{\alpha}}_g(1\otimes u(i,j,\sigma))\delta(x)\delta(y)dg$$

$$= \int_G \chi_\pi(g)\delta(y) \sum_{k,m=1}^{\dim \sigma} \overline{u(k,i,\sigma)(g)}\delta(\delta_{u(m,j,\sigma)}(xy))$$

$$(1\otimes u(k,m,\sigma))dg$$

$$= \sum_{k,m=1}^{\dim \sigma} \int_G \chi_\pi(g)\overline{u(k,i,\sigma)(g)}dg\delta(y\delta_{u(m,j,\sigma)}(xy))$$

$$(1\otimes u(k,m,\sigma))$$

$$= \begin{cases} 0, & \text{when } \pi\neq\sigma, \text{ by [16, Theorem 27.19]}, \\ 0, & \text{when } \pi=\sigma, \text{ by } y\delta_{u(m,j,\sigma)}(xy) = 0 . \end{cases}$$

Then we get $\int_G \chi_\pi(g)\hat{\hat{\alpha}}_g(\delta(y)a\delta(y))\, dg=0$ for $a \in G\times_\delta(G\times_\alpha A)$ because the vector space generated by $\{(1\otimes u(i,j,\sigma))\delta(x);\ x \in G\times_\alpha A,\ \sigma\in\hat{G}\}$ is norm dense in $G\times_\delta(G\times_\alpha A)$. Let B be the non-zero hereditary C^*-subalgebra of $G\times_\delta(G\times_\alpha A)$ generated by $\delta(y)$. Then B is $\hat{\hat{\alpha}}_g$-invariant for each $g\in G$ because $\hat{\hat{\alpha}}_g(\delta(y)) = \delta(y)$, therefore

$$\int_G \chi_\pi(g)\hat{\hat{\alpha}}_g(B)dg = \{0\} ,$$

which implies $\pi\notin\Gamma(\hat{\hat{\alpha}})$.

Theorem 3.3. Let (A,G,α) be a C^*-dynamical system, then $\Gamma(\hat{\alpha}) = \{\pi\in\hat{G},\ \ \bigvee_{i,j}\delta_{u(i,j,\pi)}(I)\cap I \neq 0\}$, for each non-zero ideal I of $G\times_\alpha A$ and $\Gamma(\alpha)\supset\Gamma(\hat{\hat{\alpha}})$.

Proof. By Takesaki's duality, $G\times_{\hat{\alpha}}(G\times_\delta(G\times_\alpha A))$ is isomorphic to $(G\times_\alpha A)\otimes C(L^2(G))$, therefore each closed ideal I' of $(G\times_\alpha A)\otimes C(L^2(G))$ is of the form $I\otimes C(L^2(G))$, where I is a closed ideal of $G\times_\alpha A$, moreover

$$\bigvee_{i,j} \hat{\hat{\delta}}_{u(i,j,\pi)}(I') = \bigvee_{i,j} \delta_{u(i,j,\pi)}(I)\otimes C(L^2(G)).$$

Hence, if $\pi\notin\Gamma(\hat{\hat{\alpha}})$, we have a non-zero closed ideal I of $G\times_\alpha A$ such that

$$\bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \cap I = \{0\}$$

by Lemma 3.1.

Remark 3.4. Let (A,G,α) and (A,G,β) be C^*-dynamical systems. If α is exterior equivalent to β (See [39, 4.2]), then $\Gamma(\hat{\alpha})=\Gamma(\hat{\hat{\beta}})$, but $\Gamma(\alpha)$ is not always equal to $\Gamma(\beta)$ (See [10]).

Here we give an affirmative answer for the problem whether a von Neumann algebra is hyperfinite if a compact group acts ergodically on it.

Proposition 3.5. Let (M,Γ,α) be W^*-dynamical system with $M^\Gamma = \mathbb{C}1$ where Γ is a second countable compact (not necessarily abelian) group. Then the von Neumann algebra M is injective.

Proof. For $f \in L^1(\Gamma)$, we define $\alpha(f)(x) = \int_\Gamma f(\gamma)\alpha_\gamma(x)d\gamma$ for $x \in M$. It is easy to prove that $\alpha(f)$ is a normal completely positive map of M with $||\alpha(f)||$ is equal to L^1-norm $||f||$ if f is a positive function on Γ. Let $\mathcal{F}(\Gamma)$ be the linear span by $\{u(i,j,\pi);\ \pi \in \hat{\Gamma},\ i=1,2,\cdots,\dim \pi\}$ where $u(i,j,\pi)$ are coefficient functions for the irreducible representation π. Then $\mathcal{F}(\Gamma)$ is a dense subalgebra of the space $C(\Gamma)$ of all continuous functions on Γ. Choose $\{f_n\}$ in $C(\Gamma)$ such that f_n converges to the Dirac measure δ_e at e in the w*-topology with L^1-norm $||f_n|| = 1$. Then $\lim_{n\to\infty} \alpha(f_n)(x) = x$ in the σ-weak topology. Since $\mathcal{F}(\Gamma)$ contains constant functions, we can choose g_n in $\mathcal{F}(\Gamma)$ such that g_n is a positive function with L^1-norm $||g_n|| = 1$ and sup-norm $||f_n - g_n|| \leq 1/n$. Then $\alpha(g_n)$ is a unital normal completely positive map of M

such that $\alpha(g_n)$ converges to 1 in the point-σ-weak topology. Since g_n is in $\mathcal{F}(\Gamma)$, $\alpha(g_n)(M)$ is contained in the subspace generated by the finite union of $\alpha(u(i,j,\pi))(M)$ $(i,j=1,2,\cdots, \dim \pi, \pi \in \hat{\Gamma})$. Since $\alpha(u(i,j,\pi))(M)$ is contained in $M^{\alpha}(\pi)$ and $M^{\alpha}(\pi)$ is a finite dimensional subspace of M (See [17], Proposition 2.1), $\alpha(g_n)$ is of finite rank. Thus we conclude that M is semi-discrete. By [9] Corollary 5.10, M is injective.

Corollary 3.6. Let (A,Γ,α) be a C^*-dynamical system with $A^{\Gamma} = \mathbb{C}1$ and Γ is a second countable compact group. Then the C^*-algebra A is nuclear.

Proof. The proof is the same way as above by using [5].

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