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ON SOME LAWS OF ITERATED LOGARITHM FOR BURGERS TURBULENCE WITH BROWNIAN INITIAL DATA BASED ON THE CONCAVE MAJORANT

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Abstract

We study the shock structure and the asymptotic behaviour of some flux across the origin in one-dimensional Burgers turbulence, the entropy solution to the inviscid Burgers equation, with random initial velocity for the uniformly distributed particles on the positive half line. We assume, in contrast to other works on Burgers turbulence, initially a vacuum state on the negative half line. We also obtain some asymptotic estimates for the concave majorant of Brownian motion.

1. Introduction

There has been much interest concerning the one-dimensional Burgers turbulence (or equivalently the ballistic aggregation) formed by the particles which started with random velocity. We suppose the sticky particles (infinitesimal or not) get stuck together upon collision according to the law of conservation of mass and momentum. If a point mass is created and it is isolated, its mass and velocity are unchanged as long as it meets no other particles.

The former researches assume the particles are initially distributed uniformly, i.e., the mass distribution is proportional to the Lebesgue measure which we interprete as the initial state of the particles is two-sided. If the particles on $(-\infty, 0)$ are at rest at the initial time, the initial velocity field has been called one-sided. Otherwise it is called two-sided. The one-sided or two-sided initial velocity given by a white noise is studied in [6], [16], [1], [15], [11] etc. There are also works [20] and [10] on the initial velocity given by a white noise supported on a finite interval. The one-sided initial velocity given by a Brownian motion is studied in [16], [17], [2], [3] and [4].

In the present paper, we focus on the case when the initial mass distribution is the Lebesgue measure supported on $(0, \infty)$. Specifically, we consider the initial velocity field given by either a white noise or a Brownian motion.

In the white noise case, the particles are clumped into locally finitely many clusters, shocks, immediately after the initial time. On the negative half line $(-\infty, 0)$, we have infinitely many clusters that are travelling very fast in the negative direction which

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will be eventually isolated in the sense that they will experience no collision after certain moment. In this case, we will be interested in the structure of limit clusters and the magnitude at finite time $t > 0$ of the total mass that lie in $(-\infty, -x]$ for large x. On our way, we will analyze the convex minorant of Brownian motion studied first by Groeneboom [13]; We show a limit theorem which we have not found in the literature yet.

On the other hand, in the Brownian case, we will find the left-most cluster, at the location denoted by $\xi(t)$ at time $t > 0$, travelling slowly in the negative direction and the countably many clusters that are located densely over the interval $(\xi(t), \infty)$. Some of them have a positive velocity and others have a negative one. But all the particles located in $(\xi(t), 0)$ have a negative velocity since they have crossed the origin from the right to the left. We will be interested in the long time asymptotic behaviour of this flux i.e. the mass that has crossed the origin. As a matter of fact, it has exactly the same law as the flux for the two-sided initial mass distribution studied in [4, §4] and we will depend heavily on their result.

This article is organized as follows. In Section 2, we introduce the model of sticky particles in terms of the so-called Hopf-Cole method. The results concerning the white noise and the Brownian motion are stated and proven in Sections 3 and 4 respectively.

2. The model of Burgers turbulence with one-sided initial mass distribution

If we initialize the particle system with the two-sided uniform mass distribution, it is well-known (see [9], [3] and their references) that the mass and the velocity field at time $t > 0$ is described by the Hopf-Cole solution. In the present article, we define the state of the system at $t > 0$ by the limit of some sequence of Hopf-Cole solutions. We refer the reader to [9], [12], [5], [10] or [19] for solutions to the equations of conservation of mass and momentum obtained as limits of the discrete ballistic inelastic particles.

Let $(u(y,0)\,;\ y \geq 0)$ be our initial velocity for the particles whose mass distribution is the Lebesgue measure on $[0,\infty)$. We then define the following initial velocity fields $(u_n(y,0); y \geq 0)_{n\in\mathbb{N}}$ on the entire $\mathbb{R}$:

$$u_n(y,0) = -n \quad \text{for} \quad y < 0; \quad u_n(y,0) = u(y,0) \quad \text{for} \quad y \geq 0. \tag{1}$$

To elaborate the Hopf-Cole solution, we introduce

$$U(y) = \int_0^y u(\eta,0)\,d\eta \quad \text{for} \quad y \geq 0, \quad U_n(y) = \int_0^y u_n(\eta,0)\,d\eta \quad \text{for} \quad y \in \mathbb{R} \tag{2}$$

and assume $U(\cdot)$, hence also $U_n(\cdot)$, is continuous and satisfies

$$\liminf_{y\to+\infty} \frac{U(y)}{y^2} \geq 0, \quad \liminf_{|y|\to\infty} \frac{U_n(y)}{|y|^2} \geq 0.$$

Note that $U_n(y) = U(y)$ for $y \geq 0$ and $U_n(y) = n|y|$ for $y < 0$.

We then define

$$a_n(x,t) = \max\left\{ y \in \mathbb{R}\,;\; U_n(y) + \frac{(x-y)^2}{2t} = \min_{\eta\in\mathbb{R}}\left(U_n(\eta) + \frac{(x-\eta)^2}{2t} \right) \right\} \tag{3}$$

It is easily seen that $x \mapsto a_n(x,t)$ is a right-continuous increasing function. We refer to $a_n(x,t)$ as the inverse Lagrangian function. This quantity represents the right-most initial location of the particles that lie in $(-\infty, x]$ at time t. The mass field at t is given by

$$\rho_n((x_1,x_2],t) = a_n(x_2,t) - a_n(x_1,t) \quad \text{for} \quad x_1 < x_2 \tag{4}$$

and we refer to an interval (x_1, x_2) as a rarefaction interval if $a(x_1,t) = a(x_2,t)$. A discontinuity point x for $x \mapsto a(x,t)$ corresponds to a point mass located at x with a mass $a_n(x,t) - a_n(x-,t)$ and a velocity

$$u_n(x,t) = \frac{2x - a_n(x,t) - a_n(x-,t)}{2t}. \tag{5}$$

If we define the function $u_n(x,t)$ by

$$u_n(x,t) = \frac{x - a_n(x,t)}{t} \tag{6}$$

elsewhere, u_n gives the velocity field. It coincides with the entropy solution to the inviscid Burgers equation $\partial_t(u_n) + u_n\,\partial_x(u_n) = 0$.

Now we turn our attention to the limit when $n \to \infty$. Since $U(\,\cdot\,)$ is continuous, $a_n(x,t)$ converges to the right-most location of the overall minimum on $[0,\infty)$ of the function $U(y)+(x-y)^2/(2t)$ as is easily seen if we note $U_n(y)+(x-y)^2/(2t) > x^2/(2t)$ for any $y < 0$ and $n > |x|/t$. There is an obvious physical interpretation: The particles located initially on $(-\infty, 0)$ escape immediately from our sight.

Henceforth, we set

$$a(x,t) = \max\left\{ y \in [0,\infty)\,;\; U(y) + \frac{(x-y)^2}{2t} = \min_{\eta\in[0,\infty)}\left(U(\eta) + \frac{(x-\eta)^2}{2t} \right) \right\} \tag{7}$$

and

$$u(x,t) = \frac{2x - a(x,t) - a(x-,t)}{2t} \tag{8}$$

for $x \in \mathbb{R}$ and $t > 0$. Provided that we neglect all the particles located initially on $(-\infty, 0)$, $a(x,t)$ for $x \leq 0$ clearly corresponds to the total mass of the particles that have crossed the point x from the right to the left up to time $t > 0$.

3. The white noise case: on the long reach of the particle system

In this section we put

$$u(x,0)=\frac{dB(x)}{dx}\quad\text{or equivalently}\quad U(x)=B(x) \tag{9}$$

for $x \geq 0$ where $B(\cdot)$ is the standard Brownian motion started at 0. Although the initial velocity field is not a classical function, the Hopf-Cole methodology enables us to analyze the Burgers turbulence via $U(\cdot)=B(\cdot)$. In fact, the rough features of the velocity field disappear in an instant and $u(\cdot,t)$ is a piecewise affine function for any $t>0$ as is known from the works [16], [1] etc. We refer to such $u(\cdot,t)$ as the discrete shock structure.

We will show some clusters (of small mass) can have arbitrarily large velocity in the negative direction at any time $t>0$. Moreover, we will see $a(x,t)\to 0$ as $x\to-\infty$ and investigate the speed of this convergence. We relate the analysis of $a(x,t)$ to the problem of the convex minorant of Brownian motion studied first by Groeneboom [13] and then by Pitman [14], Cinlar [7] and Carol–Dykstra [8]. To be precise, let $C(\cdot)$ be the convex minorant, i.e., the greatest convex function that satisfies $C(y)\leq B(y)$ for $y\geq 0$. Then let $A(x)$ be the right-most location where $C(\cdot)$ touches the greatest affine function $y\mapsto xy+k$ that satisfies $xy+k\leq C(y)$ for all $y\geq 0$. This quantity is also interpreted as the right-continuous inverse for $C'(\cdot)$:

$$A(x)=\inf\{y\geq 0 \mid C'(y)>x\}. \tag{10}$$

Note that $C'(\cdot)$ is defined except countably many y's and we have $B(A(x))=C(A(x))$ for all $x<0$. Moreover, it is straightforward to observe that $A(\cdot)$ and $-C(\cdot)$ are increasing, $A(\cdot)$ is right-continuous and that

$$C(0)=0,\ \ C(\infty)=-\infty,\ \ A(-\infty)=0\quad\text{and}\quad A(0-)=\infty. \tag{11}$$

The law of the jumps of $A(\cdot)$ is determined by using Theorem 2.1 in Groeneboom [13] as follows. Let

$$P(dx\times dl) \tag{12}$$

be a Poisson point process on $(-\infty,0)\times(0,\infty)$ with intensity

$$(2\pi l)^{-1/2}\exp\left(-\frac{l|x|^2}{2}\right)dx\times dl. \tag{13}$$

Then $(A(x);\ -\infty<x<0)$ has the same law as $(\int_{0<l<\infty} l P((-\infty,x]\times dl);\ -\infty<x<0)$. The marginal law has the Laplace transform

$$E[\exp(-\lambda A(x))]=\frac{2}{1+\sqrt{1+2\lambda/|x|^2}}. \tag{14}$$

Note that

$$A(x) \quad \text{has the same law as} \quad c^2 A(cx) \tag{15}$$

for any constant $c > 0$.

Theorem 3.1. *For white noise initial velocity, we have for any fixed* $t > 0$,

$$\limsup_{x\to-\infty} \frac{a(x,t)}{2t^2|x|^{-2}\log(\log|x|)} = 1$$

and

$$\lim_{x\to-\infty} \left(u(x,t) - \frac{x}{t}\right) = 0$$

with probability 1. *Moreover, for any positive increasing function* $m(\cdot)$ *on* $(-\infty,-1)$, *we have*

$$\liminf_{x\to-\infty} \frac{a(x,t)}{t^2|x|^{-2}m(x)} = \infty \quad \text{or} \quad = 0$$

with probability 1 *according as* $\int_{-\infty}^{-1} \sqrt{m(x)}(dx/|x|) < \infty$ *or* $= \infty$, *respectively.*

REMARK 3.1. This result reminds the author of the famous experiment performed in 1930's by Zartman and Ko to prove the Maxwell-Boltzman velocity distribution for gas particles, where the elastic particles escape through a pin-hole to the vacuum side. The distribution of the particles after time t is comparable with the initial velocity distribution. In contrast, our particles are completely inelastic and we have a good reason to believe the clusters have tempered velocitites. Theorem 3.1 gives the quantitative nature of this sticky-jet; it implies the intensity of the jet is finite even when the reservoir has the infinite volume.

Proof. Let $C^{(t)}(y)$ be the convex minorant of $B(y) + y^2/2t$ and $A^{(t)}(x)$ be the right-most location where $C^{(t)}(y)$ touches the greatest affine function $y \mapsto xy + k$ that lie below $C^{(t)}(y)$. Note that $A^{(t)}(x)$ is a right-continuous increasing function that satisfies $A^{(t)}(-\infty) = 0$, and $A^{(t)}(\infty) = \infty$. Note also that $C^{(t)}(0) = 0$, $C^{(t)}(\infty) = \infty$. The quantity $A^{(t)}(x)$ is related to the inverse Lagrangian function via

$$a(x,t) = A^{(t)}\left(\frac{x}{t}\right). \tag{16}$$

Indeed, we have

$$\frac{(x-y)^2}{2t} + B(y) = \left(\frac{y^2}{2t} + B(y)\right) - \left(\frac{x}{t}\right) y + \frac{x^2}{2t}$$

and the affine function $(x/t)y + k$ lies below and touches $C^{(t)}(y)$ with an adequeate choice of k, which implies $A^{(t)}(x/t)$ is equivalent to the right-most location of the overall minimum of the function $(x-y)^2/(2t) + U(y)$ and we have (16).

The following lemma reveals similarity between $A(x)$ and $A^{(t)}(x)$ and will be useful when x tends to $-\infty$.

Lemma 3.1. *For any $x < 0$, we have*

$$A^{(t)}\left(x + \frac{A(x-)}{t}\right) = A(x-),$$
$$A^{(t)}\left(x + \frac{A(x+)}{t}\right) = A(x+).$$

Proof. Since $B(y) \geq C(y)$,

$$B(y) + \frac{y^2}{2t} \geq C(y) + \frac{y^2}{2t}$$

and the last expression is convex in y. Hence by the definition of $C^{(t)}(y)$,

$$C^{(t)}(y) \geq C(y) + \frac{y^2}{2t}. \tag{17}$$

Moreover if $y_0 = A(x_0)$ or $y_0 = A(x_0 - 0)$ for some x_0, we have $B(y_0) = C(y_0)$ and

$$B(y_0) + \frac{y_0^2}{2t} = C^{(t)}(y_0) = C(y_0) + \frac{y_0^2}{2t}.$$

By the definition of $A(\cdot)$, $B(y) \geq x_0(y - y_0) + C(y_0)$ for any $y \geq 0$, and by $y^2/(2t) \geq (2y_0(y - y_0) + y_0^2)/(2t)$ we have

$$\begin{aligned} B(y) + \frac{y^2}{2t} &\geq x_0(y - y_0) + C(y_0) + \frac{2y_0}{2t}(y - y_0) + \frac{y_0^2}{2t} \\ &= \left(x_0 + \frac{y_0}{t}\right)(y - y_0) + C(y_0) + \frac{y_0^2}{2t}. \end{aligned}$$

Since the equality holds if and only if $y = y_0$, $A^{(t)}(\cdot)$ is continuous at $x_0 + y_0/t$ and eventually we have $A^{(t)}(x_0 + y_0/t) = y_0$ for any $x_0 < 0$ and $y_0 := A(x_0)$. $\square$

Relying on Groeneboom's result, we will prove the following lemma, which is of its own interest (see Remark 3.4 at the end of this section).

Lemma 3.2. *Let $A(\cdot)$ be the right-continuous inverse of $C'(\cdot)$ as in* (10).
(i) *We have with probability* 1,

$$\limsup_{x\to-\infty} \frac{A(x)}{2|x|^{-2}\log(\log|x|)} = 1. \tag{18}$$

(ii) *For any positive increasing function $m(\cdot)$ on $(-\infty, -1)$, with probability* 1,

$$\liminf_{x\to-\infty} \frac{A(x)}{|x|^{-2}m(x)} = \infty \quad or \quad = 0, \tag{19}$$

according as $\int_{-\infty}^{-1} \sqrt{m(x)}(dx/|x|) < \infty$ or $= \infty$, respectively.

Before proving Lemma 3.2, we complete the proof of Theorem 3.1. Note that, on one hand,

$$a(x,t) = A^{(t)}\left(\frac{x}{t}\right) \le A^{(t)}\left(\frac{x}{t} + \frac{A(x/t-)}{t}\right) = A\left(\frac{x}{t}-\right)$$

by Lemma 3.1 and on the other hand, since $A(x/t+) \to 0$,

$$A\left(\frac{x}{t}+\right) = a\left(x + A\left(\frac{x}{t}+\right), t\right) \le a\left(x+1, t\right)$$

for all x with large $|x|$. Then $a(x,t)$ have the asymptotic behaviour comaprable to that of $A(x/t)$ as $x \to -\infty$. Finally note that

$$2\left|\frac{x}{t}\right|^{-2}\log\left(\log\left|\frac{x}{t}\right|\right) \sim 2t^2|x|^{-2}\log(\log|x|)$$

and $\int_{-\infty}^{\cdot} \sqrt{m(tx)}(dx/|x|) < \infty$ if and only if $\int_{-\infty}^{\cdot} \sqrt{m(x)}(dx/|x|) < \infty$ for any fixed t.

The asymptotics for $u(x,t)$ follows immediately from its definition since $a(x,t)$ and $a(x-,t)$ tends to 0. □

Proof of Lemma 3.2. To bound the left hand side of (18) by 1, note first that (14) implies

$$P[A(x) > \alpha] = \int_0^\infty 2ze^{-z}\,dz \int_{|x|^2\alpha}^\infty \frac{\exp\left(-z^2/2T - T/2\right)}{\sqrt{2\pi T^3}}\,dT.$$

Let $m(x) = 2\log\log|x|$ for $x < -e$, fix $\gamma > 1$ and set $x_n = -\gamma^n$. Then

$$\begin{aligned} P\left[A(x) > |x|^{-2}m(x)\right] &= \int_0^\infty 2ze^{-z}\,dz \int_{m(x)}^\infty \frac{\exp\left(-z^2/2T - T/2\right)}{\sqrt{2\pi T^3}}\,dT \\ &\le \left(\int_0^\infty 2ze^{-z}\,dz\right) \int_{m(x)}^\infty \frac{\exp(-T/2)}{\sqrt{2\pi T^3}}\,dT \end{aligned}$$

$$\sim \text{const}\, m(x)^{-3/2} e^{-m(x)/2}$$
$$\sim \text{const}(\log\log|x|)^{-3/2}(\log|x|)^{-1},$$

where "const" stands for some constant that depends on γ and varies from line to line and "$\sim$" means the ratio of the both sides tends to 1 as $x \to -\infty$. Now we have

$$\sum_{n=1}^{\infty} P\left[A(x_n) > |x_n|^{-2} m(x_n)\right] \le \text{const} \sum_{n=1}^{\infty} (\log n)^{-3/2} n^{-1} < \infty$$

and by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty} \frac{A(x_n)}{|x_n|^{-2} m(x_n)} \le 1$$

with probability 1. Since $A(x)$ is increasing, we have, for any x that lies between x_{n+1} and x_n,

$$\frac{A(x)}{|x|^{-2} m(x)} \le \frac{A(x_n)}{|x_{n+1}|^{-2} m(x_n)} = \gamma^2 \frac{A(x_n)}{|x_n|^{-2} m(x_n)}$$

and by making γ close to 1,

$$\limsup_{x\to-\infty} \frac{A(x)}{|x|^{-2} m(x)} \le 1.$$

To bound the left hand side of (18) from below, let $I(x)\,dx$ be the intensity for the jumps of $A(x)$ with magnitude greater than $|x|^{-2}m(x)$, which is another Poisson point process. By some calculations, we have

$$I(x) := \int_{|x|^{-2}m(x)}^{\infty} (2\pi l)^{-1/2} \exp\left(-\frac{l|x|^2}{2}\right) dl$$
$$\sim \text{const}|x|^{-1}(\log|x|)^{-1}(\log\log|x|)^{-1/2}$$

and $\int_{-\infty}^{x} I(z)\,dz = \infty$ for any $x < 0$. Then a version of Borel-Cantelli lemma asssures, with probability 1, the existence of a sequence (x_n) such that $x_n \to -\infty$ and

$$A(x_n) - A(x_n-) > |x_n|^{-2} m(x_n).$$

Hence it follows

$$\limsup_{x\to-\infty} \frac{A(x)}{|x|^{-2} m(x)} \ge \limsup_{n\to\infty} \frac{A(x_n)}{|x_n|^{-2} m(x_n)} \ge 1.$$

We now prove the first half of (19). The integrability condition on $m(\,\cdot\,)$ is equivalent to $\int_0^\infty \sqrt{m(-e^s)}\,ds < \infty$ and also to

$$\sum_{n=1}^{\infty} \sqrt{m(-\delta\gamma^n)} < \infty \quad \text{for all} \quad \gamma > 1 \quad \text{and} \quad \delta > 0.$$

We then set $x_n = -\gamma^n$ for $n \geq 1$ and fix $c > 0$. Since $m(\delta x_n) \to 0$, we have by the scaling property (15),

$$P\left[A(x_n) < c|x_n|^{-2}m(\delta x_n)\right] = P[A(1) < cm(\delta x_n)] \\ \sim \text{const}\sqrt{m(\delta x_n)}.$$

Here we applied the Tauberian theorem to (14) and "const" depends on c and varies from line to line. Since the right most side is summable, we have by the Borel-Cantelli lemma

$$\liminf_{n\to\infty} \frac{A(x_n)}{|x_n|^{-2}m(\delta x_n)} \geq c$$

with probability 1. By setting $\delta = 1/\gamma$ and making c arbitrarily large, we conclude that

$$\liminf_{n\to\infty} \frac{A(x_n)}{|x_n|^{-2}m(x_n/\gamma)} = \infty.$$

Since $A(\cdot)$ is monotone, we have for any x with $x_{n+1} = \gamma x_n < x < x_n$,

$$\frac{A(x)}{|x|^{-2}m(x)} \geq \frac{A(x_{n+1})}{|x_n|^{-2}m(x_n)} = \frac{A(x_{n+1})}{\gamma^2|x_{n+1}|^{-2}m(x_{n+1}/\gamma)} \to \infty$$

with probability 1.

To prove the second half of (19), we first note that $\sum_{n=1}^{\infty}\sqrt{m(-\gamma^n)} = \infty$ for any $\gamma > 1$ and set $x_n = -\gamma^n$. We may assume $m(\cdot) < 1$ without loss of generality. We now introduce a sequence of events: For fixed $c > 0$, let

$$E_n = \left\{A(x_n) < c\,|x_n|^{-2}\,m(x_n)\right\}.$$

By the Tauberian theorem applied to (14),

$$P[E_n] \asymp \text{const}\sqrt{m(x_n)} \tag{20}$$

and hence

$$\sum_{n=1}^{\infty} P[E_n] = \infty. \tag{21}$$

Then according to Proposition 26.3 in Spitzer [18, p.317], if we have a constant $C > 0$ such that the inequalities

$$\sum_{n,m<M} P[E_n \cap E_m] \leq C \sum_{n,m<M} P[E_n]P[E_m] \tag{22}$$

hold for infinitely many $M \in \mathbb{N}$, the events E_n occur for infinitely many n's with a probability at least $1/C$. Then this is the case with probability 1 by the 0-1 law since

$A(\cdot)$ is a process with independent increments with $\lim_{x\to-\infty} A(x) = 0$ almost surely (already verified in (i)) and $\limsup_n E_n$ is a tail event.

If E_n's occur infinitely many times with probability 1, then

$$\liminf_{n\to\infty} \frac{A(x_n)}{|x_n|^{-2}m(x_n)} \leq c$$

and the left hand side actually vanishes since $c > 0$ is arbitrary. Hence (22) implies the second statement of (ii).

Let us then prove (22) for all large $M \in \mathbb{N}$. For any $n < m$, if we set $k = m - n$, we have $x_m = \gamma^k x_n < x_n < 0$. Then

$$E_m \cap E_n \subset \left\{A(x_n) - A(x_m) < c|x_n|^{-2}m(x_n),\ A(x_m) < c|x_m|^{-2}m(x_m)\right\}.$$

Since $A(\cdot)$ has independent increments,

$$\begin{aligned} P[E_m \cap E_n] &\leq P[E_m] \cdot P\left[A(x_n) - A(x_m) < c|x_n|^{-2}m(x_n)\right] \\ &= P[E_m] \cdot \Big(P[A(x_n) - A(x_m) = 0] \\ &\qquad + P\left[0 < A(x_n) - A(x_m) < c|x_n|^{-2}m(x_n)\right]\Big). \end{aligned}$$

Now we need the following estimates: For $-\infty < y < x < 0$ and $\alpha > 0$,

$$P[A(x) = A(y)] = \frac{|x|}{|y|}, \tag{23}$$

$$P[0 < A(x) - A(y) < \alpha] \sim \text{const}\frac{|x|}{|y|}(|y| - |x|)\sqrt{\alpha}, \tag{24}$$

as $\alpha \to 0$.

To see (23), note first that (14) and independence of increments imply

$$E\left[e^{-\lambda(A(x)-A(y))}\right] = \frac{1 + \sqrt{1 + 2\lambda/y^2}}{1 + \sqrt{1 + 2\lambda/x^2}}.$$

Making $\lambda \to \infty$, we have

$$P[A(x) = A(y)] = \lim_{\lambda\to\infty} E\left[e^{-\lambda(A(x)-A(y))}\right] = \frac{|x|}{|y|}.$$

We deduce (24) by the Tauberian theorem applied to the following estimate:

$$\begin{aligned} E\left[e^{\lambda(A(x)-A(y))};\ A(x) - A(y) > 0\right] &= \frac{1 + \sqrt{1 + 2\lambda/y^2}}{1 + \sqrt{1 + 2\lambda/x^2}} - \frac{|x|}{|y|} \\ &= \frac{|x|}{|y|}\left(\frac{\sqrt{1 + |y|^2/(2\lambda)} + |y|/\sqrt{2\lambda}}{\sqrt{1 + |x|^2/(2\lambda)} + |x|/\sqrt{2\lambda}} - 1\right) \end{aligned}$$

$$\sim \text{const}\frac{|x|}{|y|}\frac{|y|-|x|}{\sqrt{\lambda}},$$

where in the last step we make $\lambda \to \infty$.

Let us resume proving (22). By (23), (24) and then (20), we have

$$\begin{aligned} P[E_m \cap E_n] &< P[E_m] \cdot \left(\gamma^{-k} + \text{const}(1-\gamma^{-k})\sqrt{m(x_n)}\right) \\ &< P[E_m] \cdot \left(\gamma^{-k} + \text{const}\, P[E_n]\right). \end{aligned}$$

Combining these estimates together,

$$\begin{aligned} &\sum_{n<m<M} P[E_m \cap E_n] \\ &\leq \sum_{m<M} P[E_m] \sum_{k<m} \gamma^{-k} + \text{const} \sum_{m<M} P[E_m] \sum_{n<M} P[E_n] \\ &\leq \text{const} \sum_{m<M} P[E_m] \sum_{n<M} P[E_n]. \end{aligned}$$

In the last inequality, we used $\sum_{n<M} P[E_n] > \sum_{k=1}^{\infty} \gamma^{-k}$ which is valid for all large M by (21). Now the proofs of (22), Lemma 3.2 and hence Theorem 3.1 are complete. □

REMARK 3.2. When $x \to -0$, we have for the right-continuous inverse $A(\cdot)$ of $C'(\cdot)$ as in (10),

$$\limsup_{x\to -0} \frac{A(x)}{2|x|^{-2}\log(\log(1/|x|))} = 1.$$

and for any positive decreasing function $m(x)$ on $(-1, 0)$,

$$\liminf_{x\to -0} \frac{A(x)}{|x|^{-2}m(x)} = \infty \quad \text{or} \quad = 0,$$

according as $\int_{-1}^{0} \sqrt{m(x)}(dx/|x|) < \infty$ or $= \infty$, respectively. These results can be proven by the same techniques.

Now we turn our attention to the long-time asymptotic behaviour. We will see the particles form "the limit clusters" in the sense of Winkel [21]. To state the result, we introduce the so-called Lagrangian function $x_t(a)$ by

(25) $$x_t(a) := \inf\{x \in \mathbb{R};\ a(x,t) > a\} \quad \text{for} \quad a \geq 0$$

and also

(26) $$u_t(a) := \frac{2x - a(x_t(a), t) - a(x_t(a)-, t)}{2t}.$$

Note that $C'(a) = \inf\{x < 0\,;\ A(x) > a\}$ where $C'(a)$ is the right derivative.

Proposition 3.3. *For white noise case, we have the following.*
(i) *For each jump location u of $A(\cdot)$, i.e. $u < 0$ such that $A(u) > A(u-)$, there corresponds a limit cluster with mass $A(u) - A(u-)$ travelling at the speed u. Moreover, this cluster is formed at a finite time $t(u)$ and thereafter it meets no other cluster.*
(ii) *For a continuous point u of $A(\cdot)$, there is no limit cluster travelling at the speed u.*
(iii) *For any $u < 0$ and $a > 0$, as $t \to \infty$, the limits of $a(tu, t)$ and $u_t(a)$ exist and are equal to $A(u-)$ and $C'(a)$ respectively. More precisely, $u_t(a) = C'(a)$ for all $t > t(C'(a))$.*

REMARK 3.3. The law of $\{(u, A(u) - A(u-))\,;\ u \in (-\infty, 0), (A(u) - A(u-) > 0\}$ is identified with that of $P(du \times dl)$ in (12).

Proof. We only prove (i) here since (ii) and (iii) can be shown by a similar argument in Winkel [21, Lemma 1] where the initial velocity is assumed to have a cádlág path.

Groeneboom [13, Lemma 2.1] observed that if $A(\cdot)$ is discontinuous at u, the process

$$X(y) := B(y + A(u-)) - C(y + A(u-))$$

for $0 \le y \le A(u) - A(u-)$, conditionally on $\tau(u) := A(u) - A(u-)$, is a Brownian excursion with the duration $\tau(u)$. There are some known facts on the Brownian excursions: With probability 1, $X(y)$ is non-negative and vanishes only if $y = 0$ or $y = \tau(u)$; $X(y) \ge \text{const}\, y^{(1/2)+\varepsilon}$ and $X(\tau(u) - y) \ge \text{const}\, y^{(1/2)+\varepsilon}$ for any $\varepsilon > 0$ and small $y > 0$. Then there exists an $s(u) > 0$ such that

$$X(y) \ge \frac{s}{2} y(\tau(u) - y) \quad \text{for any} \quad s \in [0, s(u)] \quad \text{and any} \quad y \in (0, \tau(u)),$$

and

$$X(y) = \frac{s(u)}{2} y(\tau(u) - y) \quad \text{for some} \quad y = y(u) \in (0, \tau(u))$$

and the uniqueness of $y(u)$ is a standard fact. This is equivalent to the following. For any $t > 1/(s(u))$, the overall minimum of the function

$$y \mapsto B(y) + \frac{1}{2t}\left(y - \frac{A(u) + A(u-)}{2} - tu\right)^2 \tag{27}$$

is attained exactly twice on $[0, \infty)$ when $y = A(u)$ or $y = A(u-)$. Indeed, it is easy to see the minimum is not attained on $[0, A(u-)) \cup (A(u), \infty)$ as follows. Let $\gamma(y)$ be the affine function that touches $C(y)$ tangentially on the interval $[A(u-), A(u)]$, i.e.

$$\gamma(y) := u(y - A(u-)) + B(A(u-)).$$

By the definition of $A(\cdot)$, $C(y) > \gamma(y)$ if $y \notin [A(u-), A(u)]$. Combining this with

$$\frac{1}{2t}(y - A(u-))(A(u) - y) < 0$$

which is valid for $y \notin [A(u-), A(u)]$, we have

$$\begin{aligned}
&B(y) - \gamma(y) - \frac{1}{2t}(y - A(u-))(A(u) - y)\\
&\geq C(y) - \gamma(y) - \frac{1}{2t}(y - A(u-))(A(u) - y)\\
&> -\frac{1}{2t}(y - A(u-))(A(u) - y)\\
&> 0
\end{aligned}$$

for any $y \notin [A(u-), A(u)]$. But the left hand side vanishes if $y = A(u)$ or $y = A(u-)$.

On the other hand, some elementary calculations reveal the following.

$$\begin{aligned}
&B(y) - \gamma(y) - \frac{1}{2t}(y - A(u-))(A(u) - y)\\
&= B(y) - uy + uA(u-) - B(A(u-)) + \frac{y^2}{2t} - \frac{(A(u) + A(u-))\,y}{2t} + \frac{A(u)A(u-)}{2t}\\
&= B(y) + \frac{1}{2t}\left(y^2 - 2tuy - (A(u) + A(u-))\,y\right) + (\text{terms not containing } y)\\
&= B(y) + \frac{1}{2t}\left(y - \frac{A(u) + A(u-)}{2} - tu\right)^2 + (\text{terms not containing } y).
\end{aligned}$$

Hence the latter cannot attain its minimum if $y \notin [A(u-), A(u)]$.

Replacing y by $y + A(u-)$, we have, for $y \in (0, \tau(u))$, $\gamma(y + A(u-)) = C(y + A(u-))$ and

$$B(y + A(u-)) - \gamma(y + A(u-)) - \frac{1}{2t}y(\tau(u) - y) = X(y) - \frac{1}{2t}y(\tau(u) - y).$$

Now recall that the right hand side is positive if $t > 1/s(u)$, which implies the left hand side and hence (27) cannot attain its minimum if $A(u-) < y < A(u)$, but attains exactly twice when $y = A(u-)$ or $y = A(u)$. The same method applies to the case when $t \leq 1/s(u)$ and assures the minimum of (27) is attained exactly three times when $y = A(u-)$, $A(u-) + y(u)$, $A(u)$ if $t = 1/s(u)$. But if $t < 1/s(u)$, these three y's cannot be at the same time the locations of the minimum of the function

$$B(y) + \frac{1}{2t}(y - x)^2$$

for any choice of x. By the physical interpretation, at time $t \in [1/s(u), \infty)$, there is a cluster consisting of the particles located initially on exactly $(A(u-), A(u))$. But it is not the case at any time before $1/s(u)$. □

REMARK 3.4. The celebrated law of the iterated logarithm for Brownian motion states

$$\liminf_{y \to +0} \frac{B(y)}{\sqrt{2y \log\log(1/y)}} = -1.$$

Our Lemma 3.2 supplement it by pointing out that there are sometimes very few y's where $B(y)\big/\sqrt{2y \log\log y}$ walks up to -1. In fact, by neglecting the $\log\log$ factor, one is tempted to approximate as $C(y) \approx -\sqrt{2y}$, which implies $A(x) \approx 2^{-1}|x|^{-2}$ (this comes from solving $C'(A(x)) = x$).

Clearly, this is not the case in Lemma 3.2.

If we reverse this course and suppose, for some fixed $x > 0$, $A(x) \approx 2/|x|^2$ and $C(A(x)) = -\sqrt{2A(x)}$, we have $C(A(x)) = xA(x)$ and $C(y)$ is linear on the interval $[0, A(x)]$. If we recall a path-property of Brownian motion, we easily deduce the tangential line of slope x to the graph $C(\,\cdot\,)$ never crosses the origin.

However, the above inspection still suggests the tangential line comes much closer to the origin, more precisely the right-most location y where $B(y) = C(y)$ and $0 < y < A(x)$ is very close to the origin than $A(x)$. In other words, the ratio of two successive y's where $B(y) = C(y)$ can be very large.

4. The Brownian case: on the flux across the origin

If we assume $u(y, 0) = B(y)$, we will find no rarefaction intervals in $(0, \infty)$ as is the case in [16], [17], [2], [3] and [4]. Moreover we can prove a dichotomy in Theorem 4.1. To state the result, it is convenient to introduce the first passage process

$$T(x) := \inf\{y \geq 0\,;\ tB(y) + y \geq x\} \tag{28}$$

for a Brownian motion with drift.

Proposition 4.1. *Fix $t > 0$ and let $\xi(t)$ be $\inf\{x < 0\,;\ a(x, t) > 0\}$. Then for the Brownian initial velocity,*
(i) *$\xi(t)$ lies in $(-\infty, 0)$ with probability 1 and it is in fact the minimum of the set $\{x < 0\,;\ a(x, t) > 0\}$ and*
(ii) *the process $(a(x + \xi(t), t) - a(\xi(t), t)\,;\ x \geq 0)$ has the same law as $(T(x); x \geq 0)$ and is independent of $\Xi(t) := (\xi(s)\,;\ 0 \leq s \leq t)$ and $\Omega(t) := (a(\xi(s), s)\,;\ 0 \leq s \leq t)$.*

In particular,
(iii) *the shocks at time $t > 0$ are dense in $(\xi(t), \infty)$ and there are no shocks in $(-\infty, \xi(t))$.*

REMARK 4.1. Unfortunately, the author is not able to obtain the law of $\xi(t)$ nor that of $a(\xi(t), t)$. By the above proposition and (33), obtaining the law of $\xi(t)$ is equivalent to obtaining that of $a(\xi(t), t) \in (0, a(0, t))$.

Proof. Note first that for fixed $x < 0$, $a(x,t) > 0$ if and only if $\int_0^y (tB(\eta) + \eta)\,d\eta \le xy$ for some $y > 0$. Then

$$\xi(t) = \min_{y>0} \frac{1}{y} \int_0^y (tB(\eta) + \eta)\,d\eta,$$

and

$$a(\xi(t), t) = \max\left\{ y \ge 0\,;\ \int_0^y (tB(\eta) + \eta)\,d\eta = y\xi(t) \right\}. \tag{29}$$

By the argument in Sinai [17, p.605], we have $\int_0^y B(\eta)\,d\eta < 0$ for some $y \in (0,1)$ with probability 1, which implies $\min_{0<y<1}(1/y)\int_0^y tB(\eta)\,d\eta < 0$. It is well-known by Girsanov's theorem that the law of the Brownian motion with a drift $tB(y) + y$ is absolutely continuous with respect to that of $tB(y)$ and hence $\min_{0<y<1}(1/y)\int_0^y (tB(\eta) + \eta)\,d\eta < 0$ is also valid with probability 1. From the other side, since $tB(y)+y$ is transient to ∞, we have $\min_{y>1}(1/y)\int_0^y (tB(\eta) + \eta)\,d\eta > -\infty$. Finally, $\int_0^y (tB(\eta) + \eta)\,d\eta = o(y)$ for small y and hence $\xi(t) > -\infty$ and $a(\xi(t), t) > 0$ by (29); the statement (i) is proven.

To prove (ii), we will show the following: For any $d \in \mathbb{N}$, $z \in (-\infty, 0)$, any bounded Borel functional f on the path space, any bounded continuous funtion F on $\mathbb{R}^d$ and any $(x_1, \ldots, x_d) \in (0,\infty)^d$, we have

$$\begin{aligned} &E[f(\Xi(t), \Omega(t))F(a(x_1 + z, t) - a(z,t), \ldots, a(x_d + z, t) - a(z,t))\,;\ \xi(t) \le z] \\ &= E[f(\Xi(t), \Omega(t))\,;\ \xi(t) \le z]E[F(T(x_1), \ldots, T(x_d))]. \end{aligned}$$

This equality is verified if we note the following facts (a)–(e).

(a) Let $z < 0$. Then $\xi(t) \le z$ if and only if $\int_0^y (tB(\eta) + \eta)\,d\eta \le zy$ for some $y \ge 0$. If we define $\tau(z)$ by

$$\inf\left\{ y \ge 0\,;\ \int_0^y (tB(\eta) + \eta)\,d\eta \le zy \right\} \tag{30}$$

where $\inf\emptyset = \infty$, then $\tau(z)$ is clearly a stopping time.

(b) Subsequently,

$$\sigma(z) := \inf\{y \ge \tau(z)\,;\ tB(y) + y \ge z\} \tag{31}$$

is a stopping time such that $\sigma(z) < \infty$ if and only if $\xi(t) \le z$.

(c) Conditionally on the event $\{\sigma(z) < \infty\}$,

$$W(y) := B(y + \sigma(z)) + \frac{\sigma(z) - z}{t} \tag{32}$$

has the same law as $(B(y)\,;\ y \ge 0)$ and is independent of $(B(y)\,;\ 0 \le y \le \sigma(z))$. Moreover, by the definition of $a(x+z,t)$, $a(x+z,t) - \sigma(z)$ is exactly the same as the right-most location of the overall minimum of the function

$$y \mapsto \int_0^y (tW(\eta) + \eta - x)\, d\eta$$

for any $x \geq 0$,

(d) By Lemma 1 in Bertoin [2] applied to $W(\,\cdot\,)$, conditionally on the event $\{\sigma(z) < \infty\}$, $(a(x+z,t) - a(z,t);\ x \geq 0)$ has the same law as $(T(x);\ x \geq 0)$ and independent of $(W(y);\ 0 \leq y \leq a(z,t) - \sigma(z))$ and at the same time independent of $(B(y);\ 0 \leq y \leq \sigma(z))$ by (c), hence also of $(B(y);\ 0 \leq y \leq a(z,t))$.

(e) If $\xi(t) \leq z$, the path-valued random variables $\Xi(t)$ and $\Omega(t)$ are functionals of the killed process $(B(y);\ 0 \leq y \leq a(z,t))$.

Combining (a)–(e), we have

$$\begin{aligned}
&E[f(\Xi(t), \Omega(t))F(a(x_1+z,t) - a(z,t), \dots, a(x_d+z,t) - a(z,t));\ \xi(t) \leq z] \\
&= E[f(\Xi(t), \Omega(t))F(a(x_1+z,t) - a(z,t), \dots, a(x_d+z,t) - a(z,t));\ \sigma(z) < \infty] \\
&= E[f(\Xi(t), \Omega(t));\ \sigma(z) < \infty]E[F(T(x_1), \dots, T(x_d))] \\
&= E[f(\Xi(t), \Omega(t));\ \xi(t) \leq z]E[F(T(x_1), \dots, T(x_d))].
\end{aligned}$$

Now we show how this equality implies the statement (ii). On one hand,

$$\begin{aligned}
\sum_{n=0}^{\infty} E\Bigg[f(\Xi(t), \Omega(t))F\bigg(a\left(x_1 - \frac{n}{2^k}, t\right) - a\left(-\frac{n}{2^k}, t\right), \dots, \\
a\left(x_d - \frac{n}{2^k}, t\right) - a\left(-\frac{n}{2^k}, t\right)\bigg);\ -\frac{n+1}{2^k} < \xi(t) \leq -\frac{n}{2^k} \Bigg]
\end{aligned}$$

is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} E\left[f(\Xi(t), \Omega(t));\ -\frac{n+1}{2^k} < \xi(t) \leq -\frac{n}{2^k} \right] E\left[F(T(x_1), \dots, T(x_d))\right] \\
&= E[f(\Xi(t), \Omega(t))]E[F(T(x_1), \dots, T(x_d))].
\end{aligned}$$

On the other hand, since $x \mapsto a(x,t)$ is right-continuous, it converges as $k \to \infty$ to

$$\begin{aligned}
E[f(\Xi(t), \Omega(t))F(a(x_1 + \xi(t), t) - a(\xi(t), t), \dots, \\
a(x_d + \xi(t), t) - a(\xi(t), t))]
\end{aligned}$$

and the proof of (ii) is complete.

The statement (iii) follows immediately from (ii) since $a(x,t)$ vanishes for any x in $(-\infty, \xi(t))$ and $T(x)$, being a Lévy process with non-finite Lévy measure, has the dense jump times. □

REMARK 4.2. By exploiting the technique of the "delayed solution" in [4, §4], one can prove $t \mapsto (\xi(t), a(\xi(t), t))$ is a time-inhomogeneous Markov process.

Let us turn our attention to the evolution in time. In the rest of this section, we see the following. At the initial time, all the particles are in the positive side and sufficiently many of them have the negative velocity so that some clusters will cross the origin from the right to the left. The left-most cluster at $\xi(t)$ is accompanied by the clusters that are travelling in the interval $(\xi(t), 0)$ in the negative direction.

Then $a(0, t)$ is interpreted as the total mass of this flux and the initial vacancy on $(-\infty, 0)$ is irrelevant concerning this quantity. So if we focus on the long or short time asymptotics of $a(0, t)$, there is no difference between our setting and those in [2] and [4] where they assumed the particles are initially uniformly distributed over $\mathbb{R}$ and at rest on $(-\infty, 0)$. In fact, we will heavily depend on the formula obtained there.

Let $0 < s < t$. Then we have by Theorem 1 in [2],

$$P[a(0,t) \in dy] = 2^{-1/4}\Gamma\left(\frac{1}{4}\right)^{-1}\left(\frac{y}{t^2}\right)^{-3/4}\exp\left(-\frac{y}{2t^2}\right)d\left(\frac{y}{t^2}\right). \tag{33}$$

According to Lemma 3 and the equation (12) in [4], the increments of $t \mapsto a(0, t)$ are decomposed to give

$$a(0,t) - a(0,s) = \tau_{s,t}(a(0,s)) + A(s,t) \tag{34}$$

with a positive random variable $A(s, t)$ specified via its Laplace transform

$$E[\exp(-\lambda A(s,t))] = \sqrt{\frac{s}{t} + \frac{t-s}{t\sqrt{1+2t^2\lambda}}} \tag{35}$$

and a subordinator $\tau_{s,t}(\,\cdot\,)$ (increasing process with stationary and independent increments) with Laplace transform

$$E\left[\exp\left(-\lambda\tau_{s,t}(x)\right)\right] = \exp\left(-\frac{x(t-s)}{st^2}\left(\sqrt{1+2t^2\lambda}-1\right)\right). \tag{36}$$

where the three random components

$$(a(0,r);\ 0 \le r \le s),\quad \tau_{s,t}(\,\cdot\,)\quad \text{and}\quad A(s,t)\quad \text{are independent of each other.} \tag{37}$$

Note that the random variable

$$\frac{A(cs,ct)}{c^2}\quad \text{has the same law as}\quad A(s,t) \tag{38}$$

for any constant $c > 0$ by (35). We now state the main result in this section.

Theorem 4.1. *For the Brownian initial velocity, we have*

$$\limsup_{t\to+\infty}\frac{a(0,t)}{2t^2\log(\log t)}=1 \tag{39}$$

with probability 1*. Moreover, for any positive decreasing function* $m(x)$ *on* $(1,\infty)$,

$$\liminf_{t\to\infty}\frac{a(0,t)}{t^2m(t)}=\infty \quad \textit{or} \quad =0 \tag{40}$$

with probability 1 *according as* $\int_1^\infty m(x)^{1/4}(dx/x)<\infty$ *or* $=\infty$.

REMARK 4.3. When $t\to+0$, we have by the same techniques

$$\limsup_{t\to+0}\frac{a(0,t)}{2t^2\log(\log(1/t))}=1$$

and for any positive increasing function $m(x)$ on $(0,1)$,

$$\liminf_{t\to+0}\frac{a(0,t)}{t^2m(t)}=\infty \quad \text{or} \quad =0$$

according as $\int_0^1 m(x)^{1/4}(dx/x)<\infty$ or $=\infty$.

Proof. To bound the left hand side of (39) by 1, we fix $\gamma>1$ and $c>0$, set $t_n=\gamma^n$ for $n\in\mathbb{N}$ and $m(t)=c\log\log t$. By (33) and $m(t_n)\to\infty$, we have

$$\begin{aligned}P\left[a(0,t_n)>t_n^2m(t_n)\right]&=\int_{m(t_n)}^\infty \text{const}\, y^{-3/4}e^{-y/2}\,dy\\&\sim \text{const}\, m(t_n)^{-3/4}e^{-m(t_n)/2}\\&\sim \text{const}(\log n)^{-3/4}n^{-c/2},\end{aligned}$$

which is summable if $c>2$. Hence by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\frac{a(0,t_n)}{t_n^2m(t_n)}\le 1$$

with probability 1. For any t that lies between t_n and t_{n+1}, we have

$$\frac{a(0,t)}{t^2m(t)}\le\frac{a(0,t_{n+1})}{t_n^2m(t_n)}\lesssim\gamma^2\frac{a(0,t_{n+1})}{t_{n+1}^2m(t_{n+1})}.$$

Hence $\limsup_{t\to+\infty}a(0,t)/(2t^2\log\log t)\le(c\gamma^2)/2$ and by making γ arbitrarily close to 1 and c arbitrarily close to 2, we have the upper bound by 1.

To bound the left hand side of (39) from below, we consider the sequence of independent random variables

$$\{A(t_n, t_{n+1})\,;\; n \geq 1\}$$

and we will show in the sequel

$$\sum_{n=1}^{\infty} P\left[A(t_{n-1}, t_n) > t_n^2 m(t_n)\right] \equiv \sum_{n=1}^{\infty} P\left[A\left(\gamma^{-1}, 1\right) > m(t_n)\right] = \infty \tag{41}$$

if $c \leq 2$. It is easy to see that (41) implies the statement (39). Indeed, since $a(0, t_n) \geq A(t_{n-1}, t_n)$, we have

$$\begin{aligned}\limsup_{t\to+0} \frac{a(0,t)}{t^2 m(t)} &\geq \limsup_{n\to\infty} \frac{a(0,t_n)}{t_n^2 m(t_n)} \\ &\geq \limsup_{n\to\infty} \frac{A(t_{n-1},t_n)}{t_n^2 m(t_n)} \\ &\geq 1\end{aligned}$$

with probability 1 and we only have to set $c = 2$. Now it remains to prove (41). To begin with, note that the random variables

$$\left\{t_n^{-2} A(t_{n-1}, t_n)\,;\; n \geq 1\right\}$$

have identical laws by (38), which is in particular the same as that of $A(\gamma^{-1}, 1)$. According to (35), we have

$$E\left[\exp\left(-\lambda A\left(\gamma^{-1}, 1\right)\right)\right] = \sqrt{\frac{1}{\gamma} + \frac{\gamma - 1}{\gamma\sqrt{1 + 2\lambda}}},$$

which implies the following dichotomy:

$$E\left[\exp\left(\mu A\left(\gamma^{-1}, 1\right)\right)\right] \begin{cases} = \infty, & \text{if} \quad \mu \geq \dfrac{1}{2}, \\ < \infty, & \text{if} \quad \mu < \dfrac{1}{2}. \end{cases}$$

We then note $m(t_n) = c\log(n\log\gamma)$ and

$$c_1 e^{x/c} - c_2 < \sum_{n=1}^{\infty} 1\{x > m(t_n)\} < c_3 e^{x/c} + c_4$$

where c_1, c_2, c_3, c_4 are positive constants depending on γ and c. Hence the left hand side of (41) is bounded from below by

$$\sum_{n=1}^{\infty} P\left[A\left(\gamma^{-1},1\right)\geq m(t_n)\right] = E\left[\sum_{n=1}^{\infty} 1\left\{A\left(\gamma^{-1},1\right)\geq m(t_n)\right\}\right] \geq c_1 E\left[\exp\left(c^{-1}A\left(\gamma^{-1},1\right)\right)-c_2\right]$$

and the latter diverges if $c \leq 2$.

To prove the first half of (40), note that the integrability condition on $m(\,\cdot\,)$ is equivalent to $\int_1^\infty m(e^s)^{1/4}\,ds < \infty$ and also to

$$\sum_{n=1}^{\infty} m(\gamma^n)^{1/4} < \infty \quad \text{for all} \quad \gamma > 1.$$

We then set $t_n = \gamma^n$ for $n \geq 1$ and fix $c > 0$. Since $m(t_n) \to 0$, we have by (33),

$$P\left[a(0,t_n) < ct_n^2 m(t_n)\right] \sim \int_0^{cm(t_n)} \text{const}\, y^{-3/4}e^{-y/2}\,dy \sim \text{const}\, c^{1/4} m(t_n)^{1/4},$$

which is summable for any γ and c. Then by the Borel-Cantelli lemma,

$$\liminf_{n\to\infty} \frac{a(0,t_n)}{t_n^2 m(t_n)} \geq c$$

with probability 1 and by making c arbitrarily large,

$$\liminf_{n\to\infty} \frac{a(0,t_n)}{t_n^2 m(t_n)} = \infty.$$

Since $a(0,t)$ is monotone, we have for $t_n < t < t_{n+1} = \gamma t_n$,

$$\frac{a(0,t)}{t^2 m(t)} \geq \frac{a(0,t_n)}{t_{n+1}^2 m(t_n)} = \frac{a(0,t_n)}{\gamma^2 t_n^2 m(t_n)} \to \infty$$

with probability 1.

To prove the second half of (40), we assume $m(\,\cdot\,) < 1$ and $\sum_{n=1}^\infty m(\gamma^n)^{1/4} = \infty$ for any $\gamma > 1$. We fix $c > 0$, let $t_n = \gamma^n$ and

$$E_n = \left\{a(0,t_n) < ct_n^2 m(t_n)\right\}.$$

Then we have

$$P[E_n] > \text{const}\exp\left(-c\frac{m(1)}{2}\right) c^{1/4} m(t_n)^{1/4} \tag{42}$$

by (33) so that

$$\sum_{n=0}^{\infty} P[E_n] = \infty. \tag{43}$$

By the same reasoning as in the proof of Lemma 3.2, according to Proposition 26.3 in Spitzer [18, p.317], the events E_n occur for infinitely many n's with a positive probability if we have a constant $C > 0$ such that the inequalities

$$\sum_{n,m<M} P[E_n \cap E_m] \le C \sum_{n,m<M} P[E_n]P[E_m] \tag{44}$$

hold for infinitely many $M \in \mathbb{N}$. Then this is the case with probability 1 by the 0-1 law.

If E_n's occur infinitely many times, we easily see

$$\liminf_{n\to\infty} \frac{a(0,t_n)}{t_n^2 m(t_n)} \le c$$

and the left hand side actually vanishes by making c arbitrarily small. Hence (44) implies the second statement of (40).

Let us then prove (44) for, in fact, all large $M \in \mathbb{N}$. For any $n < m$, if we set $k = m - n$, we have $t_n = \gamma^{-k} t_m < t_m$. By (34), $a(0,t_m) \ge A(t_n, t_m)$ and

$$E_m \cap E_n \subset \left\{a(0,t_n) < t_n^2 m(t_n),\, A(t_n,t_m) < t_m^2 m(t_m)\right\}.$$

By making use of the independence as in (37) and the scaling property (38),

$$\begin{aligned} P[E_m \cap E_n] &\le P\left[a(0,t_n) < t_n^2 m(t_n),\, A(t_n,t_m) < t_m^2 m(t_m)\right] \\ &= P\left[a(0,t_n) < t_n^2 m(t_n)\right] P\left[A(t_n,t_m) < t_m^2 m(t_m)\right] \\ &= P\left[a(0,t_n) < t_n^2 m(t_n)\right] P\left[A\left(\gamma^{-k},1\right) < m(t_m)\right]. \end{aligned}$$

By making $\lambda \to \infty$ in (35), we have

$$P\left[A\left(\gamma^{-k},1\right) = 0\right] = \gamma^{-k/2}. \tag{45}$$

Moreover we have the following estimate.

Lemma 4.2. *For all $\xi > 0$ and $k \in \mathbb{N}$, we have*

$$P\left[0 < A\left(\gamma^{-k},1\right) < \xi\right] < \text{const}\,\xi^{1/4}. \tag{46}$$

Proof. By (35), (45) and concavity of the square-root, we have

$$E\left[\exp\left(-\xi^{-1}A\left(\gamma^{-k},1\right)\right) ;\ A\left(\gamma^{-k},1\right) > 0\right] = \sqrt{\gamma^{-k} + \frac{1-\gamma^{-k}}{\sqrt{1+2\xi^{-1}}}} - \sqrt{\gamma^{-k}}$$

$$
\begin{aligned}
&< \sqrt{\frac{1-\gamma^{-k}}{\sqrt{1+2\xi^{-1}}}} \\
&= \xi^{1/4}\sqrt{\frac{1-\gamma^{-k}}{\sqrt{2+\xi}}} < \frac{\xi^{1/4}}{2^{1/4}}
\end{aligned}
$$

for all $k \geq 1$. On the other hand,

$$
\begin{aligned}
&E\left[\exp\left(-\xi^{-1}A\left(\gamma^{-k},1\right)\right)\ ;\ A\left(\gamma^{-k},1\right) > 0\right] \\
&= \int_0^\infty \xi^{-1}e^{-\xi^{-1}a}P\left[0 < A\left(\gamma^{-k},1\right) < a\right]da \\
&> \int_\xi^{2\xi} \xi^{-1}e^{-\xi^{-1}a}P\left[0 < A\left(\gamma^{-k},1\right) < \xi\right]da \\
&= \left(e^{-1}-e^{-2}\right)P\left[0 < A\left(\gamma^{-k},1\right) < \xi\right]. \qquad \square
\end{aligned}
$$

Let us resume proving (44). By (45), (46) and then by (42),

$$
P\left[A\left(\gamma^{-k},1\right) < m(t_m)\right] \leq \gamma^{-k/2} + \text{const}\, m(t_m)^{1/4} \leq \gamma^{-k/2} + \text{const}\, P[E_m].
$$

Here and in the following, "const" depends on c and varies from line to line. Then we have, for all large M,

$$
\begin{aligned}
&\sum_{n<m<M} P[E_m \cap E_n] \\
&\leq \sum_{n<M} P[E_n] \sum_{k<M-n} \gamma^{-k/2} + \text{const} \sum_{n<M} P[E_n] \sum_{m<M} P[E_m] \\
&\leq \text{const} \sum_{n<M} P[E_n] \sum_{m<M} P[E_m].
\end{aligned}
$$

In the last inequality, we used $\sum_{m<M} P[E_m] > \sum_{k=1}^\infty \gamma^{-k/2}$ which is valid for all large M by (43). □

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