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## UNITARY REPRESENTATIONS AND A GENERAL VANISHING THEOREM FOR $(0, r)$ -COHOMOLOGY

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### 1. Introduction

Let  $X=G/K$  be a Hermitian symmetric space where  $K$  is a maximal compact subgroup of a connected non-compact semisimple Lie group  $G$ . We assume that  $G$  has a finite center. Let  $H\subset K$  be a Cartan subgroup of  $G$ , let  $g, k, h$  be the complexifications of the Lie algebras  $g_0, k_0, h_0$  of  $G, K, H$ , let  $\Delta=\Delta(g, h)$  be the set of non-zero roots of  $(g, h)$ , and let  $\Delta^+\subset\Delta$  be a system of positive roots compatible with the  $G$ -invariant complex structure on  $X$ . That is, if  $g_0=k_0+p_0$  is a Cartan decomposition of  $g_0$  then the splitting of the complexified tangent space  $p=p^g$  of  $X$  at the origin is given by

$$(1.1) \quad p = p^+ \oplus p^- \quad \text{where} \quad p^\pm = \sum_{\alpha \in \Delta_\pm^+} g_{\pm\alpha}$$

for  $g_\beta$  the root space of  $\beta \in \Delta$  and  $\Delta_n^+ = \Delta^+ \cap \Delta_n$ ,  $\Delta_n$  = the set of noncompact roots in  $\Delta$ . Let  $\Delta_k^+ = \Delta^+ \cap \Delta_k$  where  $\Delta_k$  = the set of compact roots in  $\Delta$  and let  $\langle Q \rangle$  be the sum of roots in  $Q$  for  $Q \subset \Delta$ . In particular we set  $2\delta = \langle \Delta^+ \rangle$  as usual, and then we can define the following subset of the dual space  $h^*$  of  $h$ : for  $L$  the character lattice of  $H$ :

$$(1.2) \quad F'_0 = \{ \text{integral forms } \Lambda \text{ in } L \mid (\Lambda + \delta, \alpha) \neq 0 \text{ for each } \alpha \text{ in } \Delta \text{ and } (\Lambda + \delta, \alpha) > 0 \text{ for each } \alpha \text{ in } \Delta_k^+ \} .$$

Now let  $\tau \in \hat{K}$  be an irreducible unitary representation of  $K$  with highest weight  $\Lambda$  relative to the positive system  $\Delta_k^+$  for  $(k, h)$ . The induced homogeneous vector bundle  $E_\tau = G \times_K V_\tau$  over  $X$  has a holomorphic structure (here  $V_\tau$  is the representation space of  $\tau$ ). Let  $\Gamma$  be a fixed torsion free, co-compact, discrete subgroup of  $G$ . Then given  $\tau \in \hat{K}$ , there is a natural sheaf  $\theta_\tau(\Gamma)$  over  $X_\Gamma \stackrel{\text{def.}}{=} \Gamma \backslash X$  generated by the presheaf:  $U \mapsto$  abelian group of  $\Gamma$ -invariant holomorphic sections of  $E_\tau$  on the inverse image of  $U$  under the map  $X \rightarrow X_\Gamma$ , where  $U \subset X_\Gamma$  is an open set. The cohomology groups  $H^*(X_\Gamma, \theta_\tau(\Gamma))$  of  $X_\Gamma$  with coefficients in

$\theta_\tau(\Gamma)$  were introduced (and described somewhat differently) and studied by Matsushima and Murakami in [2], [3]. In [4] they obtained the formula

$$(1.3) \quad \dim H^r(X_\Gamma, \theta_\tau(\Gamma)) = \sum_{\substack{\pi \in \hat{G} \\ \pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1}} m_\pi(\Gamma) \dim \text{Hom}_K(H_\pi, \Lambda^r \mathfrak{p}^+ \otimes V_\tau)$$

where  $m_\pi(\Gamma)$  is the multiplicity of  $\pi$  in  $L^2(\Gamma \backslash G)$ ,  $\Omega$  is the Casimir operator of  $G$ ,  $H_\pi$  is the space of  $K$ -finite vectors in the representation space of  $\pi$ , and  $(, )$  is the Killing form of  $g$ ; also see [1].

The purpose of this paper is two-fold: (i) We find the most general and precise vanishing theorem possible for the cohomology  $H^*(X_\Gamma, \theta_\tau(\Gamma))$ , for any  $\tau \in \hat{K}$  with highest weight  $\Lambda \in F'_0$ ; (ii) We describe precisely the unitary representations  $\pi$  and the integers  $r$  which contribute to the formula (1.3)—i.e., those  $\pi \in \hat{G}$  and those integers  $r$  such that  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ ,  $\text{Hom}_K(H_\pi, \Lambda^r \mathfrak{p}^+ \otimes V_\tau) \neq 0$ . Actually we do a bit more in Lemma 2.2. The results are formulated in Theorems 1.9 and 1.10 below. Theorem 1.9 solves a problem (one of four problems) raised by M. Harris at the recent 834<sup>th</sup> meeting of the A.M.S. in New Jersey. If one assumes that the positive root system

$$(1.4) \quad P^{(\Lambda)} \stackrel{\text{def.}}{=} \{ \alpha \in \Delta \mid (\Lambda + \delta, \alpha) > 0 \}$$

(corresponding to the regular element  $\Lambda + \delta$ ) is also compatible with some  $G$ -invariant complex structure on  $X$ , here we write  $X \leftrightarrow P^{(\Lambda)}$ , then the results (i), (ii) are already obtained in [12], [14] respectively. Thus the complete generality of this paper amounts to the removal of the assumption  $X \leftrightarrow P^{(\Lambda)}$ . In view of (1.3), clearly Theorem 1.9 implies Theorem 1.10.

Before stating the main results we introduce a bit more notation. For  $\Lambda \in F'_0$  let

$$(1.5) \quad \begin{aligned} Q_\Lambda &= \{ \alpha \in \Delta_n^+ \mid (\Lambda + \delta, \alpha) > 0 \}, \quad Q'_\Lambda = \Lambda_n^+ - Q_\Lambda \\ b_\Lambda &= \mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha \quad (\text{a Borel subalgebra of } \mathfrak{g}). \end{aligned}$$

If  $q = l + u$  is a Levi decomposition of a parabolic subalgebra of  $\mathfrak{g}$  with unipotent radical  $u$  and a reductive complement  $l$ , we let  $\Delta(u)$ ,  $\Lambda(l)$  denote the roots of  $u$ ,  $l$  and we set

$$(1.6) \quad q_{u,n} = \text{the set of non-compact roots in } \Delta(u).$$

In particular if  $\theta$  is the Cartan involution of  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$  we shall consider  $\theta$ -stable parabolic subalgebras  $q \supset \mathfrak{h}$  in the sense of [9]. That is, for some  $\lambda \in \mathfrak{h}^*$  which is pure-imaginary valued on  $\mathfrak{h}_0$  one has

$$(1.7) \quad \begin{aligned} \Delta(l) &= \{ \alpha \in \Delta \mid (\lambda, \alpha) = 0 \}, \quad \Delta(u) = \{ \alpha \in \Delta \mid (\lambda, \alpha) > 0 \}; \\ l &= \mathfrak{h} + \sum_{\alpha \in \Delta(l)} \mathfrak{g}_\alpha. \end{aligned}$$

One then has  $l = (l_0)^{\mathcal{C}}$  where  $l_0$  is the Lie algebra of a connected Lie subgroup  $L$  of  $G$ . For  $2\delta^{(\Lambda)} = \langle P^{(\Lambda)} \rangle$ ,  $2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)} \rangle$ ,  $\Lambda + \delta - \delta^{(\Lambda)}$  is  $P^{(\Lambda)}$ -dominant since  $\Lambda + \delta$  is  $P^{(\Lambda)}$ -regular and we assume that the finite-dimensional irreducible  $g$ -module  $F^\Lambda$  with  $P^{(\Lambda)}$ -highest weight  $\Lambda + \delta - \delta^{(\Lambda)}$  integrates to a smooth  $G$ -module. In particular if  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$  then the highest weight space of  $F^\Lambda$  is  $l_0$ -invariant and hence is  $L$ -invariant (as  $L$  is connected). Thus this weight space can be regarded as an  $(l, L \cap K)$  module which we shall denote by  $\mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}^{1)}$ . We let  $R_q^*$  denote Zuckerman's parabolic induction functor from the category of  $(l, L \cap K)$  modules to the category of  $(g, K)$  modules [7]. Then we have [8]

**Theorem 1.8.** *For a  $\theta$ -stable parabolic subalgebra  $q = l + u$  (as above) and  $\Lambda \in F'_0$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ , let  $\pi(q) = R_q^s \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ , where  $s = \dim u \cap k$ . Then  $\pi(q) \in \hat{G}^{2)}$  and  $\pi(q)(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ . Also  $\pi(q)|_K$  contains  $\mu(q) \stackrel{\text{def.}}{=} \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle$  as its lowest highest weight (relative to  $\Delta_k^+$ ); see (1.6). For  $j \neq s$ ,  $R_q^j \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}} = 0$ .*

Also see [10]. We can now state the main results.

**Theorem 1.9.** *For  $\Lambda \in F'_0$ , let  $\pi \in \hat{G}$  with  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ , and let  $r \geq 0$  be an integer with  $\text{Hom}_K(H_\pi, \Lambda^r P^+ \otimes V_\tau) \neq 0$  where (as above)  $H_\pi$  is the space of  $K$ -finite vectors in the representation space of  $\pi$  and  $V_\tau$  is the representation space of  $\tau \in \tilde{K}$  with  $\Delta_k^+$ -highest weight  $\Lambda$ . Then (i)  $\pi = R_q^{\dim u \cap k} \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$  for some  $\theta$ -stable parabolic  $q = l + u \supset b_\Lambda$  (where as above we assume  $F^{(\Lambda)}$  integrates smoothly to  $G$ ) with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$  and (ii)  $r = |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Lambda \cap q_{u,n}|$ ; <sup>3)</sup> see (1.5), (1.6). Conversely let  $q = l + u \supset b_\Lambda$  be a  $\theta$ -stable parabolic subalgebra of  $g$  and assume  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ . Let  $\pi(q) = R_q^{\dim u \cap k} \mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$  so that  $\pi(q) \in \hat{G}$  such that  $\pi(q)(\Omega) = (\Lambda, \Lambda + 2\delta) 1$  by Theorem 1.8. Let  $r_{q,\Lambda} \stackrel{\text{def.}}{=} |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Lambda \cap q_{u,n}|$ , and let  $\mu(q) = \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle$  be the lowest  $K$ -type of  $\pi(q)$ . Then if, at least,  $\mu(q) - \delta_n^{(\Lambda)}$  is  $\Delta_k^+$ -dominant,  $\text{Hom}_K(H_{\pi(q)} \Lambda^{r_{q,\Lambda}} p^+ \otimes V_\tau) \neq 0$  and in fact is one-dimensional.*

**Theorem 1.10.** *Let  $\tau \in \tilde{K}$  be an irreducible representation of  $K$  with  $\Delta_k^+$ -highest weight  $\Lambda \in F'_0$ . Then the sheaf cohomology  $H^r(X_\Gamma, \theta_\tau(\Gamma))$  vanishes unless  $r = |Q'_\Lambda| - |q_{u,n}| + 2|Q_\Lambda \cap q_{u,n}|$  for some  $\theta$ -stable parabolic subalgebra  $q = l + u \supset b_\Lambda$  of  $g$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ .*

As observed earlier the first half of Theorem 1.9 immediately implies Theorem 1.10, given the formula (1.3).

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1).  $\mathcal{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$  is unitary  
 2).  $\pi \in \hat{G}$  and its  $(g, K)$  module are denoted by the same symbol.  
 3).  $|S|$  is the cardinality of a set  $S$ .

**2. Proof of the main lemma**

In this section  $G/K$  need *not* be assumed to be Hermitian. We assume only  $\text{rank}_R G = \text{rank}_R K$ . We shall make use of the  $1/2$ -spin modules  $S^\pm$  for  $k$ . Its weights are of the form  $\delta_n - \langle T \rangle$ , where  $T \subset \Delta_n^+$ ,  $(-1)^{|T|} = \pm 1$ . If  $V_{\Lambda + \delta_n}$  is the irreducible  $k$ -module with  $\Delta_k^+$ -highest weight  $\Lambda + \delta_n$ ,  $\Lambda \in F'$ , then  $S^\pm \otimes V_{\Lambda + \delta_n}$  integrates<sup>1)</sup> smoothly to a  $K$ -module, and we have the following main lemma. Here  $W, W_k$  are the Weyl groups of  $(g, h), (k, h)$  and for  $(w, \tau) \in W \times W_k$

$$(2.1) \quad \Phi_w^{(\Lambda)} = w(-P^{(\Lambda)}) \cap P^{(\Lambda)}, \quad \Phi_\tau = \tau(-\Delta_k^+) \cap \Delta_k^+.$$

**Lemma 2.2.** *Let  $\mu$  be a common  $K$ -type (relative to  $\Delta_k^+$ ) of  $\pi|_K, S^\pm \otimes V_{\Lambda + \delta_n}$ , where  $\pi \in \hat{G}$  such that  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$ . Then (even without the assumption that  $G/K$  is Hermitian symmetric) there is a  $\theta$ -stable parabolic subalgebra  $q = l + u \supset b_\Lambda$  such that  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0, \mu = \mu(q)$  and  $\pi = R_q^s \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ ,  $s = \dim u \cap k$ .  $q$  can be chosen such that  $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)}$  for some  $(w, \tau) \subset W \times W_k$  with  $wP^{(\Lambda)} \supset \Delta_k^+$  (see above notation). Conversely let  $q = l + u \supset b_\Lambda$  be a  $\theta$ -stable parabolic subalgebra of  $g$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ , let  $\pi = \pi(q) = R_q^{\dim u \cap k} \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ , and let  $\mu = \mu(q)$ . Then  $\pi \in \hat{G}$  such that  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$  and if (at least)  $\mu - \delta_n^{(\Lambda)}$  is  $\Delta_k^+$ -dominant then  $\mu$  is a common  $K$ -type of  $\pi|_K, S^\pm \otimes V_{\Lambda + \delta_n}$  for  $\det \sigma(-1)^{|P^{(\Lambda)} - q_{u,n}|} = \pm 1$  where  $\sigma \in W$  is the unique element  $\ni P^{(\Lambda)} = \sigma \Delta^+$ .*

*Proof.* The hypotheses on  $\Lambda, \mu, \pi$  are exactly those of Theorem 2.8 of [15]. Thus by that theorem  $\mu$  has the form

$$(2.2) \quad \mu = \Lambda + \delta_n + \tau^{-1}(w\delta^{(\Lambda)} - \delta_k)$$

for some  $(w, \tau) \in W \times W_k$  with  $\Delta_k^+ \subset wP^{(\Lambda)}$ . We re-interpret (2.3) as follows. Let  $P$  be the positive root system  $\tau^{-1}wP^{(\Lambda)}$ . Thus (as  $\Delta_k^+ \subset wP^{(\Lambda)}$ ),  $\tau^{-1}\Delta_k^+ \subset P \Rightarrow P = \tau^{-1}\Delta_k^+ \cup P_n, P_n \stackrel{\text{def.}}{=} \Delta_n \cap P, \Rightarrow \delta(P) (= \frac{1}{2}$  the sum of roots in  $P) = \tau^{-1}\delta_k + \delta_n(P)$ , where  $2\delta_n(P) = \langle P_n \rangle$ . On the other hand  $\delta(P) = \tau^{-1}w\delta^{(\Lambda)}$ ; i.e.  $\tau^{-1}(w\delta^{(\Lambda)} - \delta_k) = \delta_n(P)$ . Then since  $\delta - \delta^{(\Lambda)} = \delta_n - \delta_n^{(\Lambda)}$  for  $2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)} \rangle$ , (2.3) can be written

$$(2.4) \quad \mu = (\Lambda + \delta - \delta^{(\Lambda)}) + \delta_n^{(\Lambda)} + \delta_n(P).$$

Thus by Kumaresan's 2nd lemma, Proposition 5.16 of [9], there is a  $\theta$ -stable parabolic subalgebra  $q = l + u \supset b_\Lambda$  of  $g$  such that

$$(2.6) \quad \begin{aligned} \mu &= \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle \stackrel{\text{def.}}{=} \mu(q) \\ &= \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle, \\ (\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) &= 0. \end{aligned}$$

Thus  $\pi|_K$  contains the lowest highest weight  $\mu(q)$ , and since  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$

1). Since  $\Lambda \in L$  implies  $(\Lambda + \delta_n) + \delta_n = \Lambda + 2\delta_n \in L$ .

$=(\Lambda + \delta - \delta^{(\Lambda)}, \Lambda + \delta - \delta^{(\Lambda)} + 2\delta^{(\Lambda)}) 1$  we may also conclude from Proposition 6.1 of [9] that  $\pi = R_q^{\dim u \cap k} \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ . From (2.3), (2.6),  $\tau^{-1}(w\delta^{(\Lambda)} - \delta_k) = -\delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$ . Then by (2.1),  $\langle (P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} \rangle = 2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle + (\delta_k - \tau^{-1}\delta_k) = \delta_n^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} + \tau^{-1}\delta_k + (\delta_k - \tau^{-1}\delta_k) = \delta^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} = \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle$ , from whence we may conclude that  $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)}$ . Conversely let  $q = l + u \supset b_\Delta$  be a  $\theta$ -stable parabolic subalgebra of  $g$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ , let  $\pi = \pi(q) = R_q^s \mathbf{C}_{\Lambda + \delta - \delta^{(\Lambda)}}$ ,  $s = \dim u \cap k$ , and let  $\mu = \mu(q)$ . Then  $\pi \in \hat{G}$  such that  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$  by Theorem 1.8, and by that same theorem  $\mu$  occurs in  $\pi$ . It suffices therefore to show that  $\mu$  occurs in  $S^\pm \otimes V_{\Lambda + \delta_n}$ , at least if  $\mu - \delta_n^{(\Lambda)}$  is  $\Delta_k^+$ -dominant.

Define  $Q = P_n^{(\Lambda)} - q_{u,n} \subset P_n^{(\Lambda)}$  so that  $\mu = \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n + \delta_n^{(\Lambda)} - \langle Q \rangle = \Lambda + \delta + \delta_n^{(\Lambda)} - \langle Q \rangle - \delta_k$ . That is,  $\pi, \mu, Q$  satisfy the conditions of Theorem 2.6 of [15]. By that theorem  $\exists (w, \tau) \in W \times W_k$  therefore such that  $\Phi_w^{(\Lambda)} = Q \cup \Phi_\tau, \tau^{-1}wP^{(\Lambda)} \supset \Delta_k^+, w(\Lambda + \delta - \delta^{(\Lambda)}) = \Lambda + \delta - \delta^{(\Lambda)}$ . Let  $P = \tau^{-1}wP^{(\Lambda)}$  so that  $\delta_n(P) = \tau^{-1}w\delta^{(\Lambda)} - \delta_k$  by the same argument preceding (2.4). Note also that for  $\alpha \in \Delta_k^+ \subset \tau^{-1}wP^{(\Lambda)}, w^{-1}\tau\alpha \in P^{(\Lambda)} \Rightarrow (\Lambda + \delta - \delta^{(\Lambda)}, w^{-1}\tau\alpha) \geq 0$ ; i.e.  $0 \leq (\tau^{-1}w(\Lambda + \delta - \delta^{(\Lambda)}), \alpha) = (\tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)}), \alpha)$ ; i.e. both  $\Lambda + \delta - \delta^{(\Lambda)}$  and  $\tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)})$  are  $\Delta_k^+$ -dominant  $\Rightarrow \Lambda + \delta - \delta^{(\Lambda)} \stackrel{(v)}{=} \tau^{-1}(\Lambda + \delta - \delta^{(\Lambda)})$  (since  $\tau \in W_k$ ). Now  $\delta^{(\Lambda)} - w\delta^{(\Lambda)} = \langle \Phi_w^{(\Lambda)} \rangle = \langle Q \rangle + \delta_k - \tau\delta_k \Rightarrow \tau^{-1}\langle Q \rangle = \tau^{-1}\delta_n^{(\Lambda)} - \tau^{-1}w\delta^{(\Lambda)} + \delta_k = \tau^{-1}\delta_n^{(\Lambda)} - \delta_n(P) \Rightarrow \mu = \Lambda + \delta - \delta^{(\Lambda)} + 2\delta_n^{(\Lambda)} - \langle Q \rangle = \tau[\Lambda + \delta - \delta^{(\Lambda)} + \tau^{-1}(\delta_n^{(\Lambda)} - \langle Q \rangle)] + \delta_n^{(\Lambda)}$  (by (iv), (v))  $= \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)] + \delta_n^{(\Lambda)} \Rightarrow \mu - \delta_n^{(\Lambda)} = \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)]$ , where  $\delta_n(P)$  is  $\Delta_k^+$ -dominant since  $P \supset \Delta_k^+$ . If we assume that  $\mu - \delta_n^{(\Lambda)}$  is  $\Delta_k^+$ -dominant then we see that both  $\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)$  and  $\tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)]$  are both  $\Delta_k^+$ -dominant with  $\tau \in W_k$ . Hence  $\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P) \stackrel{(vi)}{=} \tau[\Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P)] \Rightarrow \mu - \delta_n^{(\Lambda)} = \Lambda + \delta - \delta^{(\Lambda)} + \delta_n(P) = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \delta_n(P) \Rightarrow \mu = \Lambda + \delta_n + \delta_n(P)$ . But (v) and (vi) imply that  $\delta_n(P) = \tau\delta_n(P) = w\delta^{(\Lambda)} - \tau\delta_k$ ; i.e.,  $\tau^{-1}w\delta^{(\Lambda)} - \delta_k = w\delta^{(\Lambda)} - \tau\delta_k \Rightarrow \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle = \langle \Phi_w^{(\Lambda)} \rangle - \langle \Phi_\tau \rangle = \langle Q \rangle \Rightarrow \Phi_{\tau^{-1}w}^{(\Lambda)} \stackrel{a.}{=} \stackrel{def.}{=} Q = P_n^{(\Lambda)} - q_{u,n}$ . As  $\Delta_k^+ \subset P = \tau^{-1}w\sigma\Delta^+, \delta_n(P)$  is a  $k$ -type of  $S^\pm$  for  $\det \tau^{-1}w\sigma = \pm 1$ ; i.e.  $\mu = \Lambda + \delta_n + \delta_n(P)$  is a  $K$ -type of  $S^\pm \otimes V_{\Lambda + \delta_n}$  for  $\det \sigma(-1)^{|P_n^{(\Lambda)} - q_{u,n}|} (= \det \sigma \det \tau^{-1}w \text{ by } a.) = \pm 1$ , which concludes the proof of Lemma 2.2.

REMARKS. Lemma 2.2 refines and completes the results in [11], [13]. In the converse statement in Lemma 2.2 the preceding proof shows that  $P_n^{(\Lambda)} - q_{u,n} = \Phi_w^{(\Lambda)}$  for some  $w \in W$  such that  $\Delta_k^+ \subset wP^{(\Lambda)}$ . Also since the  $K$ -type  $\mu(q)$  occurs exactly once in  $\pi(q)$  one has

$$(2.7) \quad \dim \text{Hom}_K(H_{\pi(q)}, S^\pm \otimes V_{\Lambda + \delta_n}) = 1$$

for  $\det \sigma(-1)^{|P_n^{(\Lambda)} - q_{u,n}|} = \pm 1$ .

### 3. Proof of Theorem 1.9.

Assume now, as above,  $G/K$  has a  $G$ -invariant complex structure com-

patible with the positive system  $\Delta^+$ . Then (see 1.1)

$$(3.7) \quad \begin{aligned} \Sigma \oplus \Lambda^{n-j} P^+ &= S^\pm \otimes V_{\delta_n} \\ (-1)^j &= \pm 1 \end{aligned}$$

for  $n = |\Delta_n^+|$ . Here  $\dim V_{\delta_n} = 1$  by Weyl's dimension formula (since  $(\delta_n, \Delta_k^+) = 0$ ) and for  $(-1)^{n-r} = \pm 1$  we have  $r = n - (n-r) \Rightarrow \Lambda^r P^+ \otimes V_\Delta \stackrel{(i)}{\subset} S^\pm \otimes V_{\delta_n} \otimes V_\Delta = S^\pm \otimes V_{\Delta+\delta_n}$ . Suppose  $\text{Hom}_K(H_\pi, \Lambda^r P^+ \otimes V_\tau) \neq 0$  as in the statement of Theorem 1.9. Then there is a common  $K$ -type  $\mu$  of  $\pi|_K, \Lambda^r P^+ \otimes V_\tau$ . Since  $\Lambda$  is the highest weight of  $\tau, \mu = \Lambda + \langle T \rangle$ , where  $T \subset \Delta_n^+, |T| = r$ , and since  $\Lambda^r P^+ \otimes V_\tau \subset S^\pm \otimes V_{\Delta+\delta_n}$  (by (i)) Lemma 2.2 gives  $\mu = \mu(q), \pi = R_q^s \mathbf{C}_{\Delta+\delta-\delta^{(\Lambda)}}$ ,  $s = \dim u \cap k$ , for some  $\theta$ -stable parabolic  $q = l + u \supset b_\Delta$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ . We claim that

$$(3.8) \quad r = 2|q_{u,n} \cap Q_\Delta| + |Q'_\Delta| - |q_{u,n}|.$$

To see this we use the following set-theoretic observation.

**Lemma 3.9.** *There is a bijection  $b$  of the subsets of  $P_n^{(\Lambda)}$  onto the subsets of  $\Delta_n^+$  given by  $bQ = (Q'_\Delta \cap -Q) \cup (Q_\Delta - Q) \subset \Delta_n^+$  for  $Q \subset P_n^{(\Lambda)}, b^{-1}T = (Q_\Delta - T) \cup [-(T \cap Q'_\Delta)] \subset P_n^{(\Lambda)}$  for  $T \subset \Delta_n^+$ .  $b$  satisfies  $\langle bQ \rangle = \langle Q_\Delta \rangle - \langle Q \rangle$  and  $|Q| = |bQ| - 2|(bQ) \cap Q_\Delta| + |Q_\Delta|$  for  $Q \subset P_n^{(\Lambda)}$ .*

In the following application we need only to know that  $b$  is onto. Namely, we can write  $T = bQ_1$  for some  $Q_1 \subset P_n^{(\Lambda)}$  with  $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q_1 \rangle$ , and

$$(3.10) \quad |Q_1| = |T| - 2|T \cap Q_\Delta| + |Q_\Delta|.$$

By Lemma 2.2 we may also assume that  $(P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}} \stackrel{(ii)}{=} \Phi_{\tau^{-1}w}^{(\Lambda)}$  for a suitable  $(w, \tau) \in W \times W_k$ . As  $\mu = \Lambda + \langle T \rangle$  and  $\mu = \mu(q) \stackrel{\text{def.}}{=} \Lambda + \delta - \delta^{(\Lambda)} + \langle q_{u,n} \rangle = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$ , we have  $\langle T \rangle = \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$ . But also  $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q_1 \rangle = \delta_n + \delta_n^{(\Lambda)} - \langle Q_1 \rangle$  and hence  $\langle Q_1 \rangle = 2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle = \langle P_n^{(\Lambda)} \rangle - \langle q_{u,n} \rangle \Rightarrow \langle Q_1 \cup \Phi_{\tau^{-1}} \rangle = \langle \Phi_{\tau^{-1}w}^{(\Lambda)} \rangle$  (by (ii))  $\Rightarrow Q_1 \cup \Phi_{\tau^{-1}} = \Phi_{\tau^{-1}w}^{(\Lambda)} = (P_n^{(\Lambda)} - q_{u,n}) \cup \Phi_{\tau^{-1}}$  (again by (ii))  $\Rightarrow Q_1 = P_n^{(\Lambda)} - q_{u,n}$ . Finally since  $T = bQ_1, T \cap Q_\Delta = Q_\Delta - Q_1$  by definition of  $b$ ; i.e.  $T \cap Q_\Delta = Q_\Delta \cap q_{u,n}$  by (iii), so that  $r = |T| = |Q_1| + 2|Q_\Delta \cap q_{u,n}| - |Q_\Delta|$  (by (3.10))  $= |Q'_\Delta| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$ , by (1.5) and (iii), which proves (3.8) and which proves the first half of Theorem 1.9. Conversely let  $q = l + u \supset b_\Delta$  be a  $\theta$ -stable parabolic subalgebra of  $g$  with  $(\Lambda + \delta - \delta^{(\Lambda)}, \Delta(l)) = 0$ , let  $\pi = \pi(q) = R_q^s \mathbf{C}_{\Delta+\delta-\delta^{(\Lambda)}}$ ,  $s = \dim u \cap k$ , and let  $r_{q,\Delta} = |Q'_\Delta| - |q_{u,n}| + 2|Q_\Delta \cap q_{u,n}|$ . As observed in Lemma 2.2,  $\pi \in \hat{G}$  such that  $\pi(\Omega) = (\Lambda, \Lambda + 2\delta) 1$  and for  $\det \sigma (-1)^{|P_n^{(\Lambda)} - q_{u,n}|} = \pm 1, \mu = \mu(q)$  is a common  $K$ -type of  $\pi|_K, S^\pm \otimes V_{\Delta+\delta_n}$  at least if  $\mu - \delta_n^{(\Lambda)}$  is  $\Delta_k^+$ -dominant, which we assume. The point therefore is to show that  $\mu$  is a  $K$ -type of  $\Lambda^r P^+ \otimes V_\Delta$  for  $r = r_{q,\Delta}$ . Since  $V_\mu$  occurs in  $S^\pm \otimes V_{\Delta+\delta_n}, V_\mu$  occurs in  $\Lambda^{n-j} P^+ \otimes V_\Delta$  for some  $j$

with  $(-1)^j = \pm 1$  (by (3.7)) according as  $(-1)^{|P_n^{(\Lambda)} - q_{u,n}|} \det \sigma = \pm 1$ . Write  $\mu = \Lambda + \langle T \rangle$  where  $T \subset \Delta_n^+$ ,  $|T| = n - j$ . Also write  $T = bQ$  by Lemma 3.9, where  $Q \subset P_n^{(\Lambda)}$ ,  $|Q| = |T| - 2|Q_\Delta - Q| + |Q_\Delta|$  (since  $bQ \cap Q_\Delta = Q_\Delta - Q$ ), and  $\langle T \rangle = \langle Q_\Delta \rangle - \langle Q \rangle$ . By the remarks following the proof of Lemma 2.2 we can write  $P_n^{(\Lambda)} - q_{u,n} = \Phi_w^{(\Lambda)}$  for a suitable  $w \in W$ . Since  $\mu = \mu(q) = \Lambda + \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle$  we have  $\langle T \rangle = \delta_n - \delta_n^{(\Lambda)} + \langle q_{u,n} \rangle = \delta_n + \delta_n^{(\Lambda)} - (2\delta_n^{(\Lambda)} - \langle q_{u,n} \rangle) = \langle Q_\Delta \rangle - \langle \Phi_w^{(\Lambda)} \rangle$ ; i.e.  $\langle Q_\Delta \rangle - \langle \Phi_w^{(\Lambda)} \rangle = \langle Q_\Delta \rangle - \langle Q \rangle \Rightarrow \langle Q \rangle = \langle \Phi_w^{(\Lambda)} \rangle \Rightarrow Q = \Phi_w^{(\Lambda)} \Rightarrow$  (by (vii))  $n - |q_{u,n}| = |\Phi_w^{(\Lambda)}| = |T| - 2|Q_\Delta - \Phi_w^{(\Lambda)}| + |Q_\Delta| = n - j - 2|Q_\Delta \cap q_{u,n}| + |Q_\Delta| \Rightarrow n - j = 2|Q_\Delta \cap q_{u,n}| + |Q'_\Delta| - |q_{u,n}| = r_{q,\Delta}$ ; i.e.  $V_\mu \subset \Lambda^{r_{q,\Delta}} P^+ \otimes V_\Delta$ , as desired. By (2.7)  $\dim \text{Hom}_K(H_{\mu(q)}, \Lambda^{r_{q,\Delta}} P^+ \otimes V_\Delta) = 1$ , which completes the proof of Theorem 1.9.

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