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UNITARY REPRESENTATIONS AND A GENERAL VANISHING THEOREM FOR $(0, r)$-COHOMOLOGY

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1. Introduction

Let $X=G/K$ be a Hermitian symmetric space where K is a maximal compact subgroup of a connected non-compact semisimple Lie group G. We assume that G has a finite center. Let $H\subset K$ be a Cartan subgroup of G, let g, k, h be the complexifications of the Lie algebras g_0, k_0, h_0 of G, K, H, let $\Delta=\Delta(g, h)$ be the set of non-zero roots of (g, h), and let $\Delta^+\subset\Delta$ be a system of positive roots compatible with the G-invariant complex structure on X. That is, if $g_0=k_0+p_0$ is a Cartan decomposition of g_0 then the splitting of the complexified tangent space $p=p_0^C$ of X at the origin is given by

$$p = p^+\oplus p^- \quad \text{where} \quad p^\pm = \sum_{\alpha\in\Delta_n^+} g_{\pm\alpha} \tag{1.1}$$

for g_β the root space of $\beta\in\Delta$ and $\Delta_n^+=\Delta^+\cap\Delta_n$, Δ_n=the set of noncompact roots in Δ. Let $\Delta_k^+=\Delta^+\cap\Delta_k$ where Δ_k=the set of compact roots in Δ and let $\langle Q\rangle$ be the sum of roots in Q for $Q\subset\Delta$. In particular we set $2\delta=\langle\Delta^+\rangle$ as usual, and then we can define the following subset of the dual space h^* of h: for L the character lattice of H:

$$\begin{aligned} F_0' = \{&\text{integral forms } \Lambda \text{ in } L \mid (\Lambda+\delta, \alpha)\neq 0 \quad \text{for each} \\ &\alpha \text{ in } \Delta \text{ and } (\Lambda+\delta, \alpha)>0 \quad \text{for each} \quad \alpha \text{ in } \Delta_k^+\} \,. \end{aligned} \tag{1.2}$$

Now let $\tau\in\hat{K}$ be an irreducible unitary representation of K with highest weight Λ relative to the positive system Δ_k^+ for (k, h). The induced homogeneous vector bundle $E_\tau=G\times_K V_\tau$ over X has a holomorphic structure (here V_τ is the representation space of τ). Let Γ be a fixed torsion free, co-compact, discrete subgroup of G. Then given $\tau\in\hat{K}$, there is a natural sheaf $\theta_\tau(\Gamma)$ over $X_\Gamma \overset{\text{def.}}{=} \Gamma\backslash X$ generated by the presheaf: $U\mapsto$abelian group of Γ-invariant holomorphic sections of E_τ on the inverse image of U under the map $X\to X_\Gamma$, where $U\subset X_\Gamma$ is an open set. The cohomology groups $H^*(X_\Gamma, \theta_\tau(\Gamma))$ of X_Γ with coefficients in

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$\theta_\tau(\Gamma)$ were introduced (and described somewhat differently) and studied by Matsushima and Murakami in [2], [3]. In [4] they obtained the formula

$$(1.3)\quad \dim H^r(X_\Gamma, \theta_\tau(\Gamma)) = \sum_{\substack{\pi\in\hat{G} \\ \pi(\Omega)=(\Lambda,\Lambda+2\delta)1}} m_\pi(\Gamma) \dim \mathrm{Hom}_K(H_\pi, \Lambda^r p^+\otimes V_\tau)$$

where $m_\pi(\Gamma)$ is the multiplicity of π in $L^2(\Gamma\backslash G)$, Ω is the Casimir operator of G, H_π is the space of K-finite vectors in the representation space of π, and $(\ ,\ )$ is the Killing form of g; also see [1].

The purpose of this paper is two-fold: (i) We find the most general and precise vanishing theorem possible for the cohomology $H^*(X_\Gamma, \theta_\tau(\Gamma))$, for any $\tau\in\hat{K}$ with highest weight $\Lambda\in F_0'$; (ii) We describe precisely the unitary representations π and the integers r which contribute to the formula (1.3)–i.e., those $\pi\in\hat{G}$ and those integers r such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$, $\mathrm{Hom}_K(H_\pi, \Lambda^r p^+\otimes V_\tau)\neq 0$. Actually we do a bit more in Lemma 2.2. The results are formulated in Theorems 1.9 and 1.10 below. Theorem 1.9 solves a problem (one of four problems) raised by M. Harris at the recent 834th meeting of the A.M.S. in New Jersey. If one assumes that the positive root system

$$(1.4)\qquad P^{(\Lambda)} \overset{\text{def.}}{=} \{\alpha\in\Delta \mid (\Lambda+\delta, \alpha)>0\}$$

(corresponding to the regular element $\Lambda+\delta$) is also compatible with some G-invariant complex structure on X, here we write $X\leftrightarrow P^{(\Lambda)}$, then the results (i), (ii) are already obtained in [12], [14] respectively. Thus the complete generality of this paper amounts to the removal of the assumption $X\leftrightarrow P^{(\Lambda)}$. In view of (1.3), clearly Theorem 1.9 implies Theorem 1.10.

Before stating the main results we introduce a bit more notation. For $\Lambda\in F_0'$ let

$$(1.5)\qquad \begin{aligned} &Q_\Lambda = \{\alpha\in\Delta_n^+ \mid (\Lambda+\delta, \alpha)>0\}\ , \quad Q_\Lambda' = \Lambda_n^+ - Q_\Lambda \\ &b_\Lambda = h + \sum_{\alpha\in P^{(\Lambda)}} g_\alpha \quad \text{(a Borel subalgebra of } g). \end{aligned}$$

If $q=l+u$ is a Levi decomposition of a parabolic subalgebra of g with unipotent radical u and a reductive complement l, we let $\Delta(u)$, $\Lambda(l)$ denote the roots of u, l and we set

$$(1.6)\qquad q_{u,n} = \text{the set of non-compact roots in } \Delta(u)\,.$$

In particular if θ is the Cartan involution of $g_0=k_0+p_0$ we shall consider θ-stable parabolic subalgebras $q\supset h$ in the sense of [9]. That is, for some $\lambda\in h^*$ which is pure-imaginary valued on h_0 one has

$$(1.7)\qquad \begin{aligned} &\Delta(l) = \{\alpha\in\Delta \mid (\lambda, \alpha) = 0\}\ , \quad \Delta(u) = \{\alpha\in\Delta \mid (\lambda, \alpha)>0\}\,; \\ &l = h + \sum_{\alpha\in\Delta(l)} g_\alpha\,. \end{aligned}$$

One then has $l=(l_0)^C$ where l_0 is the Lie algebra of a connected Lie subgroup L of G. For $2\delta^{(\Lambda)}=\langle P^{(\Lambda)}\rangle$, $2\delta_n^{(\Lambda)}=\langle P_n^{(\Lambda)}\rangle$, $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$-dominant since $\Lambda+\delta$ is $P^{(\Lambda)}$-regular and we assume that the finite-dimensional irreducible g-module F^Λ with $P^{(\Lambda)}$-highest weight $\Lambda+\delta-\delta^{(\Lambda)}$ integrates to a smooth G-module. In particular if $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$ then the highest weight space of F^Λ is l_0-invariant and hence is L-invariant (as L is connected). Thus this weight space can be regarded as an $(l, L\cap K)$ module which we shall denote by $\boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}$.[1] We let R_q^* denote Zuckerman's parabolic induction functor from the category of $(l, L\cap K)$ modules to the category of (g, K) modules [7]. Then we have [8]

Theorem 1.8. *For a θ-stable parabolic subalgebra $q=l+u$ (as above) and $\Lambda\in F_0'$ with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$, let $\pi(q)=R_q^s \boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}$, where $s=\dim u\cap k$. Then $\pi(q)\in\hat{G}$*[2] *and $\pi(q)(\Omega)=(\Lambda, \Lambda+2\delta)\,1$. Also $\pi(q)|_K$ contains $\mu(q)\overset{\text{def.}}{=}\Lambda+\delta-\delta^{(\Lambda)}+\langle q_{u,n}\rangle$ as its lowest highest weight (relative to Δ_k^+); see (1.6). For $j\neq s$, $R_q^j \boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}=0$.*

Also see [10]. We can now state the main results.

Theorem 1.9. *For $\Lambda\in F_0'$, let $\pi\in\hat{G}$ with $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$, and let $r\geq 0$ be an integer with $\mathrm{Hom}_K(H_\pi, \Lambda^r P^+\otimes V_\tau)\neq 0$ where (as above) H_π is the space of K-finite vectors in the representation space of π and V_τ is the representation space of $\tau\in\hat{K}$ with Δ_k^+-highest weight Λ. Then (i) $\pi=R_q^{\dim u\cap k}\boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}$ for some θ-stable parabolic $q=l+u\supset b_\Lambda$ (where as above we assume $F^{(\Lambda)}$ integrates smoothly to G) with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$ and (ii) $r=|Q_\Lambda'|-|q_{u,n}|+2|Q_\Lambda\cap q_{u,n}|$;*[3] *see (1.5), (1.6). Conversely let $q=l+u\supset b_\Lambda$ be a θ-stable parabolic subalgebra of g and assume $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$. Let $\pi(q)=R_q^{\dim u\cap k}\boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}$ so that $\pi(q)\in\hat{G}$ such that $\pi(q)(\Omega)=(\Lambda, \Lambda+2\delta)\,1$ by Theorem 1.8. Let $r_{q,\Lambda}\overset{\text{def.}}{=}|Q_\Lambda'|-|q_{u,n}|+2|Q_\Lambda\cap q_{u,n}|$, and let $\mu(q)=\Lambda+\delta-\delta^{(\Lambda)}+\langle q_{u,n}\rangle$ be the lowest K-type of $\pi(q)$. Then if, at least, $\mu(q)-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant, $\mathrm{Hom}_K(H_{\pi(q)}\Lambda^{r_{q,\Lambda}}p^+\otimes V_\tau)\neq 0$ and in fact is one-dimensional.*

Theorem 1.10. *Let $\tau\in\hat{K}$ be an irreducible representation of K with Δ_k^+-highest weight $\Lambda\in F_0'$. Then the sheaf cohomology $H^r(X_\Gamma, \theta_\tau(\Gamma))$ vanishes unless $r=|Q_\Lambda'|-|q_{u,n}|+2|Q_\Lambda\cap q_{u,n}|$ for some θ-stable parabolic subalgebra $q=l+u\supset b_\Lambda$ of g with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$.*

As observed earlier the first half of Theorem 1.9 immediately implies Theorem 1.10, given the formula (1.3).

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1). $\boldsymbol{C}_{\Lambda+\delta-\delta^{(\Lambda)}}$ is unitary

2). $\pi\in\hat{G}$ and its (g, K) module are denoted by the same symbol.

3). $|S|$ is the cardinality of a set S.

2. Proof of the main lemma

In this section G/K need *not* be assumed to be Hermitian. We assume only $\mathrm{rank}_R G=\mathrm{rank}_R K$. We shall make use of the 1/2-spin modules $S^{\pm}$ for k. Its weights are of the form $\delta_n-\langle T\rangle$, where $T\subset\Delta_n^+$, $(-1)^{|T|}=\pm 1$. If $V_{\Lambda+\delta_n}$ is the irreducible k-module with Δ_k^+-highest weight $\Lambda+\delta_n$, $\Lambda\in F_0'$, then $S^{\pm}\otimes V_{\Lambda+\delta_n}$ integrates [1] smoothly to a K-module, and we have the following main lemma. Here W, W_k are the Weyl groups of (g, h), (k, h) and for $(w, \tau)\in W\times W_k$

$$\Phi_w^{(\Lambda)}=w(-P^{(\Lambda)})\cap P^{(\Lambda)}\,,\quad \Phi_\tau=\tau(-\Delta_k^+)\cap\Delta_k^+\,. \tag{2.1}$$

Lemma 2.2. *Let μ be a common K-type (relative to Δ_k^+) of $\pi|_K$, $S^{\pm}\otimes V_{\Lambda+\delta_n}$, where $\pi\in\hat{G}$ such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$. Then (even without the assumption that G/K is Hermitian symmetric) there is a θ-stable parabolic subalgebra $q=l+u\supset b_\Lambda$ such that $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$, $\mu=\mu(q)$ and $\pi=R_q^s C_{\Lambda+\delta-\delta^{(\Lambda)}}$, $s=\dim u\cap k$. q can be chosen such that $(P_n^{(\Lambda)}-q_{u,n})\cup\Phi_{\tau^{-1}}=\Phi_{\tau^{-1}w}^{(\Lambda)}$ for some $(w, \tau)\subset W\times W_k$ with $wP^{(\Lambda)}\supset\Delta_k^+$ (see above notation). Conversely let $q=l+u\supset b_\Lambda$ be a θ-stable parabolic subalgebra of g with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$, let $\pi=\pi(q)=R_q^{\dim u\cap k}C_{\Lambda+\delta-\delta^{(\Lambda)}}$, and let $\mu=\mu(q)$. Then $\pi\in\hat{G}$ such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$ and if (at least) $\mu-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant then μ is a common K-type of $\pi|_K$, $S^{\pm}\otimes V_{\Lambda+\delta_n}$ for $\det\sigma(-1)^{|P_n^{(\Lambda)}-q_{u,n}|}=\pm 1$ where $\sigma\in W$ is the unique element $\ni P^{(\Lambda)}=\sigma\Delta^+$.*

Proof. The hypotheses on Λ, μ, π are exactly those of Theorem 2.8 of [15]. Thus by that theorem μ has the form

$$\mu=\Lambda+\delta_n+\tau^{-1}(w\delta^{(\Lambda)}-\delta_k) \tag{2.3}$$

for some $(w, \tau)\in W\times W_k$ with $\Delta_k^+\subset wP^{(\Lambda)}$. We re-interpret (2.3) as follows. Let P be the positive root system $\tau^{-1}wP^{(\Lambda)}$. Thus (as $\Delta_k^+\subset wP^{(\Lambda)}$), $\tau^{-1}\Delta_k^+\subset P\Rightarrow$ $P=\tau^{-1}\Delta_k^+\cup P_n$, $P_n\overset{\text{def.}}{=}\Delta_n\cap P$, $\Rightarrow\delta(P)$ $(=\frac{1}{2}$ the sum of roots in $P)=\tau^{-1}\delta_k+\delta_n(P)$, where $2\delta_n(P)=\langle P_n\rangle$. On the other hand $\delta(P)=\tau^{-1}w\delta^{(\Lambda)}$; i.e. $\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=\delta_n(P)$. Then since $\delta-\delta^{(\Lambda)}=\delta_n-\delta_n^{(\Lambda)}$ for $2\delta_n^{(\Lambda)}=\langle P_n^{(\Lambda)}\rangle$, (2.3) can be written

$$\mu=(\Lambda+\delta-\delta^{(\Lambda)})+\delta_n^{(\Lambda)}+\delta_n(P)\,. \tag{2.4}$$

Thus by Kumaresan's 2nd lemma, Proposition 5.16 of [9], there is a θ-stable parabolic subalgebra $q=l+u\supset b_\Lambda$ of g such that

$$\begin{aligned}\mu&=\Lambda+\delta-\delta^{(\Lambda)}+\langle q_{u,n}\rangle\overset{\text{def.}}{=}\mu(q)\\&=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle\,,\\(\Lambda+\delta-\delta^{(\Lambda)},\ &\Delta(l))=0\,.\end{aligned} \tag{2.6}$$

Thus $\pi|_K$ contains the lowest highest weight $\mu(q)$, and since $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$

1). Since $\Lambda\in L$ implies $(\Lambda+\delta_n)+\delta_n=\Lambda+2\delta_n\in L$.

$=(\Lambda+\delta-\delta^{(\Lambda)},\ \Lambda+\delta-\delta^{(\Lambda)}+2\delta^{(\Lambda)})\ 1$ we may also conclude from Proposition 6.1 of [9] that $\pi=R_q^{\dim u\cap k}C_{\Lambda+\delta-\delta^{(\Lambda)}}$. From (2.3), (2.6), $\tau^{-1}(w\delta^{(\Lambda)}-\delta_k)=-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle$. Then by (2.1), $\langle(P_n^{(\Lambda)}-q_{u,n})\cup\Phi_{\tau^{-1}}\rangle=2\delta_n^{(\Lambda)}-\langle q_{u,n}\rangle+(\delta_k-\tau^{-1}\delta_k)=\delta_n^{(\Lambda)}-\tau^{-1}w\delta^{(\Lambda)}+\tau^{-1}\delta_k+(\delta_k-\tau^{-1}\delta_k)=\delta^{(\Lambda)}-\tau^{-1}w\delta^{(\Lambda)}=\langle\Phi_{\tau^{-1}w}^{(\Lambda)}\rangle$, from whence we may conclude that $(P_n^{(\Lambda)}-q_{u,n})\cup\Phi_{\tau^{-1}}=\Phi_{\tau^{-1}w}^{(\Lambda)}$. Conversely let $q=l+u\supset b_\Lambda$ be a θ-stable parabolic subalgebra of g with $(\Lambda+\delta-\delta^{(\Lambda)},\ \Delta(l))=0$, let $\pi=\pi(q)=R_q^sC_{\Lambda+\delta-\delta^{(\Lambda)}}$, $s=\dim u\cap k$, and let $\mu=\mu(q)$. Then $\pi\in\hat{G}$ such that $\pi(\Omega)=(\Lambda,\Lambda+2\delta)\ 1$ by Theorem 1.8, and by that same theorem μ occurs in π. It suffices therefore to show that μ occurs in $S^\pm\otimes V_{\Lambda+\delta_n}$, at least if $\mu-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant. Define $Q=P_n^{(\Lambda)}-q_{u,n}\subset P_n^{(\Lambda)}$ so that $\mu\overset{(\mathrm{iv})}{=}\Lambda+\delta-\delta^{(\Lambda)}+\langle q_{u,n}\rangle=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle=\Lambda+\delta_n+\delta_n^{(\Lambda)}-\langle Q\rangle=\Lambda+\delta+\delta_n^{(\Lambda)}-\langle Q\rangle-\delta_k$. That is, π, μ, Q satisfy the conditions of Theorem 2.6 of [15]. By that theorem $\exists(w,\tau)\in W\times W_k$ therefore such that $\Phi_w^{(\Lambda)}=Q\cup\Phi_\tau$, $\tau^{-1}wP^{(\Lambda)}\supset\Delta_k^+$, $w(\Lambda+\delta-\delta^{(\Lambda)})=\Lambda+\delta-\delta^{(\Lambda)}$. Let $P=\tau^{-1}wP^{(\Lambda)}$ so that $\delta_n(P)=\tau^{-1}w\delta^{(\Lambda)}-\delta_k$ by the same argument preceding (2.4). Note also that for $\alpha\in\Delta_k^+\subset\tau^{-1}wP^{(\Lambda)}$, $w^{-1}\tau\alpha\in P^{(\Lambda)}\Rightarrow(\Lambda+\delta-\delta^{(\Lambda)},\ w^{-1}\tau\alpha)\geq0$; i.e. $0\leq(\tau^{-1}w(\Lambda+\delta-\delta^{(\Lambda)}),\alpha)=(\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)}),\alpha)$; i.e. both $\Lambda+\delta-\delta^{(\Lambda)}$ and $\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)})$ are Δ_k^+-dominant $\Rightarrow\Lambda+\delta-\delta^{(\Lambda)}\overset{(\mathrm{v})}{=}\tau^{-1}(\Lambda+\delta-\delta^{(\Lambda)})$ (since $\tau\in W_k$). Now $\delta^{(\Lambda)}-w\delta^{(\Lambda)}=\langle\Phi_w^{(\Lambda)}\rangle=\langle Q\rangle+\delta_k-\tau\delta_k\Rightarrow\tau^{-1}\langle Q\rangle=\tau^{-1}\delta_n^{(\Lambda)}-\tau^{-1}w\delta^{(\Lambda)}+\delta_k=\tau^{-1}\delta_n^{(\Lambda)}-\delta_n(P)\Rightarrow\mu=\Lambda+\delta-\delta^{(\Lambda)}+2\delta_n^{(\Lambda)}-\langle Q\rangle=\tau[\Lambda+\delta-\delta^{(\Lambda)}+\tau^{-1}(\delta_n^{(\Lambda)}-\langle Q\rangle)]+\delta_n^{(\Lambda)}$ (by (iv), (v)) $=\tau[\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)]+\delta_n^{(\Lambda)}\Rightarrow\mu-\delta_n^{(\Lambda)}=\tau[\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)]$, where $\delta_n(P)$ is Δ_k^+-dominant since $P\supset\Delta_k^+$. If we assume that $\mu-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant then we see that both $\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)$ and $\tau[\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)]$ are both Δ_k^+-dominant with $\tau\in W_k$. Hence $\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)\overset{(\mathrm{vi})}{=}\tau[\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)]\Rightarrow\mu-\delta_n^{(\Lambda)}=\Lambda+\delta-\delta^{(\Lambda)}+\delta_n(P)=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\delta_n(P)\Rightarrow\mu=\Lambda+\delta_n+\delta_n(P)$. But (v) and (vi) imply that $\delta_n(P)=\tau\delta_n(P)=w\delta^{(\Lambda)}-\tau\delta_k$; i.e., $\tau^{-1}w\delta^{(\Lambda)}-\delta_k=w\delta^{(\Lambda)}-\tau\delta_k\Rightarrow\langle\Phi_{\tau^{-1}w}^{(\Lambda)}\rangle=\langle\Phi_w^{(\Lambda)}\rangle-\langle\Phi_\tau\rangle=\langle Q\rangle\Rightarrow\Phi_{\tau^{-1}w}^{(\Lambda)}\overset{a.}{=}Q\overset{\mathrm{def.}}{=}P_n^{(\Lambda)}-q_{u,n}$. As $\Delta_k^+\subset P=\tau^{-1}w\sigma\Delta^+$, $\delta_n(P)$ is a k-type of $S^\pm$ for $\det\tau^{-1}w\sigma=\pm1$; i.e. $\mu=\Lambda+\delta_n+\delta_n(P)$ is a K-type of $S^\pm\otimes V_{\Lambda+\delta_n}$ for $\det\sigma(-1)^{|P_n^{(\Lambda)}-q_{u,n}|}$ ($=\det\sigma\det\tau^{-1}w$ by $a.$) $=\pm1$, which concludes the proof of Lemma 2.2.

REMARKS. Lemma 2.2 refines and completes the results in [11], [13]. In the converse statement in Lemma 2.2 the preceding proof shows that $P_n^{(\Lambda)}-q_{u,n}=\Phi_w^{(\Lambda)}$ for some $w\in W$ such that $\Delta_k^+\subset wP^{(\Lambda)}$. Also since the K-type $\mu(q)$ occurs exactly once in $\pi(q)$ one has

$$\dim \mathrm{Hom}_K(H_{\pi(q)},\ S^\pm\otimes V_{\Lambda+\delta_n})=1 \quad \text{for} \quad \det\sigma(-1)^{|P_n^{(\Lambda)}-q_{u,n}|}=\pm1\,. \tag{2.7}$$

3. Proof of Theorem 1.9.

Assume now, as above, G/K has a G-invariant complex structure compatible with the positive system Δ^+. Then (see 1.1)

$$(3.7)\qquad \begin{aligned} \Sigma\oplus\Lambda^{n-j}P^+ &= S^\pm\otimes V_{\delta_n} \\ (-1)^j &= \pm 1 \end{aligned}$$

for $n=|\Delta_n^+|$. Here dim $V_{\delta_n}=1$ by Weyl's dimension formula (since $(\delta_n, \Delta_k^+)=0$) and for $(-1)^{n-r}=\pm 1$ we have $r=n-(n-r)\Rightarrow\Lambda^r P^+\otimes V_\Lambda \overset{(i)}{\subset} S^\pm\otimes V_{\delta_n}\otimes V_\Lambda=S^\pm\otimes V_{\Lambda+\delta_n}$. Suppose $\mathrm{Hom}_K(H_\pi, \Lambda^r P^+\otimes V_\tau)\neq 0$ as in the statement of Theorem 1.9. Then there is a common K-type μ of $\pi|_K$, $\Lambda^r P^+\otimes V_\tau$. Since Λ is the highest weight of τ, $\mu=\Lambda+\langle T\rangle$, where $T\subset\Delta_n^+$, $|T|=r$, and since $\Lambda^r P^+\otimes V_\tau\subset S^\pm\otimes V_{\Lambda+\delta_n}$ (by (i)) Lemma 2.2 gives $\mu=\mu(q)$, $\pi=R_q^s C_{\Lambda+\delta-\delta^{(\Lambda)}}$, $s=\dim u\cap k$, for some θ-stable parabolic $q=l+u\supset b_\Lambda$ with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$. We claim that

$$(3.8)\qquad r = 2|q_{u,n}\cap Q_\Lambda|+|Q'_\Lambda|-|q_{u,n}|.$$

To see this we use the following set-theoretic observation.

Lemma 3.9. *There* is a bijection b of the subsets of $P_n^{(\Lambda)}$ onto the subsets of Δ_n^+ given by $bQ=(Q'_\Lambda\cap -Q)\cup(Q_\Lambda-Q)\subset\Delta_n^+$ for $Q\subset P_n^{(\Lambda)}$, $b^{-1}T=(Q_\Lambda-T)\cup[-(T\cap Q'_\Lambda)]\subset P_n^{(\Lambda)}$ for $T\subset\Delta_n^+$. b satisfies $\langle bQ\rangle=\langle Q_\Lambda\rangle-\langle Q\rangle$ and $|Q|=|bQ|-2|(bQ)\cap Q_\Lambda|+|Q_\Lambda|$ for $Q\subset P_n^{(\Lambda)}$.

In the following application we need only to know that b is onto. Namely, we can write $T=bQ_1$ for some $Q_1\subset P_n^{(\Lambda)}$ with $\langle T\rangle=\langle Q_\Lambda\rangle-\langle Q_1\rangle$, and

$$(3.10)\qquad |Q_1|=|T|-2|T\cap Q_\Lambda|+|Q_\Lambda|.$$

By Lemma 2.2 we may also assume that $(P_n^{(\Lambda)}-q_{u,n})\cup\Phi_{\tau^{-1}}\overset{(ii)}{=}\Phi^{(\Lambda)}_{\tau^{-1}w}$ for a suitable $(w, \tau)\in W\times W_k$. As $\mu=\Lambda+\langle T\rangle$ and $\mu=\mu(q)\overset{\text{def.}}{=}\Lambda+\delta-\delta^{(\Lambda)}+\langle q_{u,n}\rangle=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle$, we have $\langle T\rangle=\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle$. But also $\langle T\rangle=\langle Q_\Lambda\rangle-\langle Q_1\rangle=\delta_n+\delta_n^{(\Lambda)}-\langle Q_1\rangle$ and hence $\langle Q_1\rangle=2\delta_n^{(\Lambda)}-\langle q_{u,n}\rangle=\langle P_n^{(\Lambda)}\rangle-\langle q_{u,n}\rangle\Rightarrow\langle Q_1\cup\Phi_{\tau-1}\rangle=\langle\Phi^{(\Lambda)}_{\tau^{-1}w}\rangle$ (by (ii))$\Rightarrow Q_1\cup\Phi_{\tau^{-1}}=\Phi^{(\Lambda)}_{\tau^{-1}w}=(P_n^{(\Lambda)}-q_{u,n})\cup\Phi_{\tau^{-1}}$ (again by (ii))$\Rightarrow Q_1\overset{(iii)}{=}P_n^{(\Lambda)}-q_{u,n}$. Finally since $T=bQ_1$, $T\cap Q_\Lambda=Q_\Lambda-Q_1$ by definition of b; i.e. $T\cap Q_\Lambda=Q_\Lambda\cap q_{u,n}$ by (iii), so that $r=|T|=|Q_1|+2|Q_\Lambda\cap q_{u,n}|-|Q_\Lambda|$ (by (3.10))$=|Q'_\Lambda|-|q_{u,n}|+2|Q_\Lambda\cap q_{u,n}|$, by (1.5) and (iii), which proves (3.8) and which proves the first half of Theorem 1.9. Conversely let $q=l+u\supset b_\Lambda$ be a θ-stable parabolic subalgebra of g with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(l))=0$, let $\pi=\pi(q)=R_q^s C_{\Lambda+\delta-\delta^{(\Lambda)}}$, $s=\dim u\cap k$, and let $r_{q,\Lambda}=|Q'_\Lambda|-|q_{u,n}|+2|Q_\Lambda\cap q_{u,n}|$. As observed in Lemma 2.2, $\pi\in\hat{G}$ such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)\,1$ and for $\det\sigma$ $(-1)^{|P_n^{(\Lambda)}-q_{u,n}|}=\pm 1$, $\mu=\mu(q)$ is a common K-type of $\pi|_K$, $S^\pm\otimes V_{\Lambda+\delta_n}$ at least if $\mu-\delta_n^{(\Lambda)}$ is Δ_k^+-dominant, which we assume. The point therefore is to show that μ is a K-type of $\Lambda^r P^+\otimes V_\Lambda$ for $r=r_{q,\Lambda}$. Since V_μ occurs in $S^\pm\otimes V_{\Lambda+\delta_n}$, V_μ occurs in $\Lambda^{n-j}P^+\otimes V_\Lambda$ for some j

with $(-1)^j=\pm 1$ (by (3.7)) according as $(-1)^{|P_n^{(\Lambda)}-q_{u,n}|}\det\sigma=\pm 1$. Write $\mu=\Lambda+\langle T\rangle$ where $T\subset\Delta_n^+$, $|T|=n-j$. Also write $T=bQ$ by Lemma 3.9, where $Q\subset P_n^{(\Lambda)}$, $|Q|\overset{\text{(vii)}}{=}|T|-2|Q_\Lambda-Q|+|Q_\Lambda|$ (since $bQ\cap Q_\Lambda=Q_\Lambda-Q$), and $\langle T\rangle=\langle Q_\Lambda\rangle-\langle Q\rangle$. By the remarks following the proof of Lemma 2.2 we can write $P_n^{(\Lambda)}-q_{u,n}=\Phi_w^{(\Lambda)}$ for a suitable $w\in W$. Since $\mu=\mu(q)=\Lambda+\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle$ we have $\langle T\rangle=\delta_n-\delta_n^{(\Lambda)}+\langle q_{u,n}\rangle=\delta_n+\delta_n^{(\Lambda)}-(2\delta_n^{(\Lambda)}-\langle q_{u,n}\rangle)=\langle Q_\Lambda\rangle-\langle\Phi_w^{(\Lambda)}\rangle$; i.e. $\langle Q_\Lambda\rangle-\langle\Phi_w^{(\Lambda)}\rangle=\langle Q_\Lambda\rangle-\langle Q\rangle\Rightarrow\langle Q\rangle=\langle\Phi_w^{(\Lambda)}\rangle\Rightarrow Q=\Phi_w^{(\Lambda)}\Rightarrow$(by (vii)) $n-|q_{u,n}|=|\Phi_w^{(\Lambda)}|=|T|-2|Q_\Lambda-\Phi_w^{(\Lambda)}|+|Q_\Lambda|=n-j-2|Q_\Lambda\cap q_{u,n}|+|Q_\Lambda|\Rightarrow n-j=2|Q_\Lambda\cap q_{u,n}|+|Q'_\Lambda|-|q_{u,n}|=r_{q,\Lambda}$; i.e. $V_\mu\subset\Lambda^{r_{q,\Lambda}}P^+\otimes V_\Lambda$, as desired. By (2.7) dim $\mathrm{Hom}_K(H_{\pi(q)},\Lambda^{r_{q,\Lambda}}P^+\otimes V_\Lambda)=1$, which completes the proof of Theorem 1.9.

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