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LOCALIZATION OF THE SPECTRAL SEQUENCE CONVERGING TO THE COHOMOLOGY OF AN EXTRA SPECIAL *P*-GROUP FOR ODD PRIME *P*

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1. Introduction

Extra special p-groups are groups which are central extensions of Z/p by elementary abelian p-groups. The cohomology ring of these groups occupies an important place in equivariant cohomology and in representation theories. Quillen [13] decided the cohomology for $p = 2$. However for odd prime p cases, it seems very difficult to decide the cohomology completely ([5], [14], [15], [17]). Therefore, in this paper, we study the cohomology with localization for multiplicative sets defined by a maximal split elementary abelian p-subgroup.

We consider the groop $\widetilde{G}$ which is the central product of the circle S^1 and the extra special p-group G constructed by Leary, Kropholler, Huebschmann and Moselle [12]. Let $E_r{}^{*,*}$ be the Hochschild-Serre spectral sequence induced from the central extension

$$0 \longrightarrow S^1 \longrightarrow \widetilde{G} \longrightarrow V = \oplus^{2n} Z/p \longrightarrow 0.$$

Let A be a split quotient group of $\widetilde{G}$ with $A = \oplus^n Z/p$ such that $S^1 \oplus A$ is a maximal abelian subgroup of $\widetilde{G}$, and let

$$e_A = \prod_{0 \neq x \in A*} \beta x \in H^{2(p^n-1)}(A).$$

One of our observations is that the nonzero differentials in $[e_A{}^{-1}]E_r{}^{*,*}$ are only Cartan-Serre and Kudo's transgressions. Hence we get $[e_A{}^{-1}]H^*(G)$ easily.

In the paper [17], the author studied the spectral sequence $E_r{}^{*,*}$ and applied the results to the representation theory and the group actions theory. However the proof of the main lemma (Lemma 2.4 in [17]) using Araki's base-wise reduced powers is not correct. We correct this with Corollary 2.8 in Section 2. Indeed, the spectral sequence becomes quite simple and easier to understand with the localization. Moreover we can give wider applications to representation theory and equivariant cohomology.

In Section 2, we study the behaviour of the localized spectral spectral sequence whose E_2-term is isomorphic to $[e_A{}^{-1}]E_2{}^{*,*}$. We recall the extra special p-groups in §3. In §4, we study the case $n=1,2$ with the localization by a smaller multiplicative set. The cohomology of other similar groups are studied in §5, and a result of this section is used in §7 for actions on $CP^t \times CP^s$. In §6, we construct the periodic modules with period $2p^i$ for $i \leq n$, for extra special p-groups of exponent p^2 and for similar other groups. We use the arguments by Benson-Carlson [6] in this section. In section 7, elementary abelian p-group actions on CP^t without fixed points are studied by using the fact that its equivariant cohomology is almost same as the comology of the group $\widetilde{G}$ constructed from the extra special p-group, according to the idea of Allday [3]. In §8, we compute the cohomology of a Sylow p-subgroup of $GL_4(F_p)$ with some localizations. The Brown-Peterson cohomology is studied in the last section.

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2. Hochschild-Serre spectral sequence

We consider the Serre spectral sequence such that the E_2-term is

$$E_2{}^{*,*} = H^*\left(\oplus^{2n}Z/p; H^*(BS^1)\right). \tag{2.1}$$

induced from a fibering $X \longrightarrow Y \longrightarrow Z$ with $H^*(X) \cong H^*(BS^1)$ and $H^*(Z) \cong H^*(\oplus^{2n}Z/p)$. In this paper cohomology $H^*(-)$ always means the Z/p-coefficient $H^*(-;Z/p)$ for an odd prime p. Let us write

$$H(\oplus^{2n}Z/p) = S_{2n} \otimes \wedge_{2n}, \quad H^*(BS^1) \cong Z/p[u]$$

with $S_{2n} = Z/p[y_1, \cdots y_{2n}]$, $\wedge_{2n} = \wedge(x_1, \cdots x_{2n})$, $\mathcal{B}x_i = y_i$. We assume that the first non-zero differential is

$$d_3 u = \mathcal{B}f \quad \text{with} \quad f = \sum_{k=1}^{n} x_{2k-1}x_{2k}. \tag{2.2}$$

Then by the Cartan-Serre and Kudo transgression theorems, we know

$$d_{2J+1}(u^i) = z(i), \quad d_{2J(p-1)+1}(z(i) \otimes u^{J(p-1)}) = w(i), \tag{2.3}$$

$$\begin{aligned} \text{with} \quad & z(i) = \mathcal{P}^{p^{i-2}} \cdots \mathcal{P}^1\mathcal{B}f = \textstyle\sum y_{2k-1}{}^J x_{2k} - y_{2k}{}^J x_{2k-1}, && \text{for } J = p^{i-1} \\ & w(i) = \mathcal{B}\mathcal{P}^J z(i) = \textstyle\sum y_{2k-1}{}^I y_{2k} - y_{2k}{}^I y_{2k-1} && \text{for } I = p^i \end{aligned}$$

Let us write $S(i) = S_{2n}/(w(1), \cdots, w(i))$. Recall that $(w(1), \cdots, w(n))$ is a regular sequence in S_{2n}[14].

Let B_1 (resp. B_2) be the $n \times n$-matrix with (k,i)-entry $({y_{2k-1}}^J)$ (resp. $({y_{2k}}^J)$) so that $(x_2,\cdots,x_{2n})B_1 - (x_1,\cdots,x_{2n-1})B_2 = (z(1),\cdots,z(n))$.

Lemma 2.4. *The determinant of B_1 is $((-1)^n e)^{1/(p-1)}$ where*

$$e = \prod(\lambda_1 y_1 + \lambda_3 y_3 + \cdots + \lambda_{2n-1}y_{2n-1})$$

with

$$(\lambda_1,\cdots,\lambda_{2n-1}) \neq (0,\cdots,0) \in (Z/p)^n.$$

Proof. Let us write the determinant $|B_1| \in Z/p[y_1,\cdots,y_{2n-1}]$. If we take $y_{2i-1} = \lambda_1 y_1 + \cdots + \hat{\lambda}_i\hat{y}_{2i-1} + \cdots + \lambda_n y_{2n-1}$, then $|B_1| = 0$. Hence $e^{1/(p-1)}||B_1|$. Since $\deg(e^{1/(p-1)}) = \deg(|B_1|)$ and $\prod_{0\neq\lambda\in Z/p}\lambda = -1$, we get the lemma. $\square$

Lemma 2.5. *By multiplying an upper triangular matrix with diagonal entries 1 in $SL_n(S_{2n})$, we can change B_1 to a lower triangular matrix ${B_1}'$ with (i,i)-entry $Y_{i,2i-1}$ where $Y_{i,k} = \prod(y_k + \lambda_{2i-3}y_{2i-3} + \cdots + \lambda_1 y_1)$, $\lambda_{2k-1} \in Z/p$.*

Proof. It is immediate that we can change B_1 to a lower triangular matrix ${B_1}'$ by a matrix in $SL_n(e^{-1}S_{2n})$ localized by e, since $(Y_{1,1}\ldots Y_{n,2n-1})^{p-1} = (-1)^n e$. We will show that we need not the localization. Suppose that by multiplying an upper triangular matrix $C = (c_{ij})$, $c_{jj} = 1$, $c_{ij} \in Z/p[y_1,\cdots,y_{2i-3}]$ from the right hand side, we can change B_1 to a matrix $B' = ({b_{ij}}')$ with ${b_{ij}}' = 0$ for $j > i$ and $i > k$. We can take B' when $i = 1$, because $b_{11} = y_1|b_{1i} = {y_1}^{p^{i-1}}$. Think ${b_{kj}}'$ in $Z/p[y_1,\cdots y_{2k-1}]$, for $k < j$. If we take $y_{2k-1} = y_{2s-1}$ for $s < k$, then ${b_{kj}}' = {b_{sj}}' = 0$ by the supposition. Since ${b_{kj}}'$ is a linear combination of ${y_{2k-1}}^{p^i}$ with coefficients in $Z/p[y_1,\cdots,y_{2k-3}]$, we also see if $y_{2k-1} = \lambda_1 y_1 + \cdots + \lambda_{k-1}y_{2k-3}$, then ${b_{kj}}' = 0$. Hence $Y_{k,2k-1}|{b_{kj}}'$. Therefore we can take a matrix C' with entries in $Z/p[y_1,\cdots y_{2k-1}]$, such that ${b_{kj}}' = 0$ $\square$

Note that if we take $y_{2k-1} = y_{2t-1}$, then ${b_{kk}}' = {b_{tk}}'$ also for $t > k$. Hence we have

$$z(i) = Y_{i,2i-1}x_{2i} + \cdots + Y_{i,2n-1}x_{2n} - Y_{i,2}x_1 - \cdots - Y_{i,2n}x_{2n-1} \mod (z(1),\cdots,z(i-1)). \tag{2.6}$$

let us write $e_i = Y_{1,1}Y_{2,3}\cdots Y_{i,2i-1}$. One of our main theorems is

Theorem 2.7. *Let R be an S_{2n}-algebra such that $(w(1),\cdots,w(i))$ is regular in R and ${e_i}^{-1} \in R$. Let ${E_r}^{*,*}$ be a Serre spectral sequence such that ${E_2}^{*,0} = R \otimes \wedge_{2n}$ and ${E_2}^{*,*} = R \otimes \wedge_{2n} \otimes Z/p[u]$ with $\beta x_j = y_j$ and d_3 is given by (2.2). Then for $I = p^i$, $J = p^{i-1}$ and $R(k) = R/(w(1),\cdots,w(k))$, $k < i$, we get*

$$E_{2r+1}{}^{*,*} \cong \begin{cases} R(i-1)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots x_{2n-1})\{1, z(i)u^{J(p-1)}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1+J < r \leq (p-1)J \\ R(i)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots x_{2n-1}) \quad \text{for } (p-1)J < r \leq I. \end{cases}$$

Corollary 2.8. *Let $E_r{}^{*,*}$ be a spectral sequence whose E_2-term is isomorphic to* (2.1) *and d_3 is given by* (2.2). *Then for $i \leq n$*

$$[e_i{}^{-1}]E_{2r+1}{}^{*,*}$$
$$\cong \begin{cases} [e_i{}^{-1}]S(i-1)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots x_{2n-1})\{1, z(i)u^{J(p-1)}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1+J < r \leq (p-1)J \\ [e_i{}^{-1}]S(i)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots x_{2n-1}) \quad \text{for } (p-1)J < r \leq I = p^i \end{cases}$$

Corollary 2.9 (see Yagita [17]). *Suppose the same assumption as Corollary* 2.8. *If $r < p^{n-1}(p-1)$, then $E_{2r+1}{}^{*,*}$ contains the subalgebra*

$$\begin{cases} S(i-1)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots, x_{2n-1}) & \text{for } 1+J < r \leq (p-1)J \\ S(i)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots, x_{2n-1}) & \text{for } (p-1)J < r \leq I \end{cases}$$

Corollary 2.10. *Suppose the same assumption as Corollary* 2.8. *Then*

$$[e^{-1}]E_\infty{}^{*,*} \cong [e^{-1}]S(n)[u^{p^n}] \otimes \wedge(x_1, \cdots, x_{2n-1}).$$

For proofs of Theorem 2.7 to Corollary 2.10, we need lemmas. We recall some facts from algebraic geometry. Let k be an algebraic closed field over F_p and $\mathrm{Var}(f_1 \cdots, f_r) \subset k^N$ be the variety defined by the ideal $(f_1, \cdots, f_r)$ in S_N.

Lemma 2.11 ([13]). *$(f_1, \cdots, f_r)$ is regular in S_N if and only if*

$$\dim\ \mathrm{Var}(f_1, \cdots, f_r) = N - r.$$

Lemma 2.12. *If $s < n$ and $t \leq n$, then $(w(1), \cdots, w(s), e_t)$ is regular in S_{2n}.*

Proof. We will prove that $J = (w(1), \cdots, w(s), y_{2k}, y_{2k-1} + \lambda_1 y_1 + \cdots + \lambda_{2k-3} y_{2k-3})$ is regular in S_{2n}. The variety is

$$\begin{aligned} &\mathrm{Var}(\mathrm{Ideal}\ J) \\ &\cong \mathrm{Var}(w(1), \cdots, w(s)) \cap \{y_{2k-1} = -\lambda_1 y_1 - \cdots - \lambda_{2k-3} y_{2k-3}\} \cap \{y_{2k} = 0\} \\ &\cong \mathrm{Var}(w'(1), \cdots, w'(s)) \subset k^{2n-2} = k\{y_1, \cdots, \hat{y}_{2k-1}, \hat{y}_{2k}, \cdots y_{2n}\}. \end{aligned}$$

where $w'(i) = w(i) - (y_{2k-1}{}^I y_{2k} - y_{2k}{}^I y_{2k-1})$. Since $(w'(1), \cdots, w'(s))$ is regular in $Z/p[y_1, \cdots, \hat{y}_{2k-1}, \hat{y}_{2k}, \cdots, y_{2n}]$, we get $\dim_k \mathrm{Var}(\text{Ideal } J) = 2n - 2 - i$.

Hence J is regular and so is its subsequence. Since $e_t = Y_{1,1} \ldots Y_{t,2t-1}$, we have the lemma. □

Corollary 2.13. *If $i < n$, then $S(i) \subset [e_{i+1}{}^{-1}]S(i)$.*

Proof of Corollary 2.9. It is immediate from the above corollary and Corollary 2.8. □

Corollary 2.14. *If $i \leq n$, $w(i)$ is non zero divisor in $[e_i{}^{-1}]S(i-1)$.*

Proof. If $w(i)$ $(e_i{}^N a) = 0$ in $S(i-1)$, then $(e_i{}^N a) = 0$ from the regularity of $w(i)$ and $a = 0$ from Lemma 2.12. □

For an odd degree element z in some graded algebra A, the homology $H(A, z)$ is defined by $d(a) = za$ for all a in A. Let R be an S_{2n}-algebra satisfying the assumption of Theorem 2.7, e.g., $e_i{}^{-1} \in R$. Let $A_i = R(i) \otimes \wedge_{2n}/(z(1), \cdots, z(i))$.

Lemma 2.15. $H([e_{i+1}{}^{-1}]A_i; z(i+1)) \cong \{0\}$.

Proof. From (2.6) and $Y_{k,2k-1}{}^{-1} \in [e_i{}^{-1}]S_{2n}$ for $k \leq i$, we inductively see

$$A_i \cong R(i) \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots, x_{2n-1}).$$

Hence we get

$$\begin{aligned} &H([e_{i+1}{}^{-1}]R(i) \otimes \wedge(x_{2i+2}, \cdot, x_{2n}, x_1, \cdot\cdot, x_{2n-1}), z(i+1) = Y_{i+1,2i+1}x_{2i+2} + \cdots) \\ &\quad \cong [e_{i+1}{}^{-1}]R(i) \otimes H(\wedge(z(i+1), x_{2i+4}, \cdot\cdot, x_{2n}, x_1, \cdot\cdot, x_{2n-1}), z(i+1)) = \{0\}, \end{aligned}$$

since the homology $H(\wedge(z, x, \cdots), z)$ is always zero, from the definition. □

Proof of Theorem 2.7. Suppose that $E_{2J+2}{}^{*,*}$ is isomorphic to

$$R(i-1)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots, x_{2n-1})\{1, z(i)u^{J(p-1)}\}$$

Since $E_{2r+1}{}^{*,k} = 0$ for $0 < k < 2J(p-1)$, we know $d_{k+1} = 0$ for these k.

The next differential is the Kudo's transgression $d_{2J(p-1)+1}(z(i)u^{J(p-1)}) = w(i)$. Since $w(i)$ is non zero divisor in $R(i-1)$,

$$\begin{aligned} (1) \qquad E_{2J(p-1)+2}{}^{*,*} &\cong (E_{2J+2}{}^{*,0}/(w(i)))[u^I] \\ &\cong R(i)[u^I] \otimes \wedge(x_{2i+2}, \cdots, x_{2n}, x_1, \cdots, x_{2n-1}). \end{aligned}$$

So $E_{2J(p-1)+2}{}^{*,0} \cong A_i[u^I]$. Since $E_{2J+2}{}^{*,k} = 0$ for $2J(p-1) < k < 2I$, the next non zero differential is the Cartan-Serre transgression $d_{2I+1}(u^I) = z(i+1)$. Hence

$$E_{2I+2}{}^{*,2Ir} \cong \begin{cases} E_{2I+1}{}^{*,0}/(z(i+1)) & \text{for} \quad r=0 \\ H(E_{2I+1}{}^{*,0}, z(i+1)) & \text{for} \quad 0<r<p-1 \\ \mathrm{Ker} z(i+1)|E_{2I+1}{}^{*,0} & \text{for} \quad r=p-1 \end{cases}$$

$$E_{2I+2}{}^{*,k} \cong \{0\} \qquad k \neq 0 \bmod I.$$

From Lemma 2.15, $[e_{i+1}{}^{-1}]H(E_{2J+1}{}^{*,0}, z(i+1)) = \{0\}$. For each odd degree element $z \in A$, we see that $\mathrm{Ker} z|A = H(A,z) + \mathrm{Image}(z)$. Hence we get

$$[e_{i+1}{}^{-1}]E_{2I+2}{}^{*,*} \cong [e_{i+1}{}^{-1}]A_i/(z(i+1))[u^{Ip}]\{1, z(i+1)u^{I(p-1)}\}.$$

Thus we can complete the proof of Theorem 2.7. □

Proof of Corollary 2.10. To see this corollary, we only need to show that u^{p^n} is permanent, i.e,

$$d_{2p-1}(u^{p^n}) = z(n+1) = 0 \quad \text{in} \quad [e^{-1}]S(n) \otimes \wedge(x_1, \cdots, x_{2n-1}).$$

From (2.6), we can easily see

$$z(n+1) = -Y_{n+1,2}x_1 - \cdots - Y_{n+1,2n}x_{2n} \bmod (z(1), \cdots, z(n))$$

where $Y_{n+1,2i} = \prod(y_{2i} + \lambda_{2n-1}y_{2n-1} + \cdots + \lambda_1 y_1)$. We want to show that each $Y_{n+1,2i}$ is in the ideal $J = (w(1), \cdots, w(n))$ of $[e^{-1}]S_{2n}$. For this we recall that $J = \sqrt{J}$ and its variety $\mathrm{Var}(J)$ has the decomposition

$$\mathrm{Var}(J) = \cup W \otimes k$$

(see [14] or Theorem 5.1 below) where W ranges over the maximal B-isotropic subspaces of the vector space $V = Z/p\{y_1, \cdots, y_{2n}\} \cong (Z/p)^{2n}$ with $B(y,y') = \sum y_{2k}y'_{2k-1} - y_{2k-1}y'_{2k}$. As a subspace of $[e^{-1}]S_{2n}(k)$, each $W \otimes k$ is expressed by

$$[e^{-1}]W \otimes k = \bigcap_{1 \leq i \leq n} \{(y_1, \cdots, y_{2n}) | y_{2i} = \lambda_{1i}y_1 + \cdots + \lambda_{2n-1,i}y_{2n-1}\}$$

otherwise W is defined by linear forms not involving y_{2i} for some i, which would imply that $y_{2i-1} = 0$ by the B-isotropic condiction, but this is ruled out by the localization e^{-1}. On the otherhand

$$\mathrm{Var}(Y_{n+1,2i}) = \bigcup_{(\lambda_1, \cdots \lambda_{2n-1})} \{(y_1, \cdots, y_{2n}) | y_{2i} = \lambda_1 y_1 + \cdots + \lambda_{2n-1}y_{2n-1}\}.$$

Thus $\mathrm{Var}(Y_{n+1,2i})$ contains all $[e^{-1}]W \otimes k$. Since $J = \sqrt{J}$, we get the result. $\square$

REMARK 2.16. Corollary 2.10 is also proved easily by using the cohomology of the extra special p-group $\widetilde{E_n}$ defined in the next section (see [14] Proposition 4.7). Let M be a maximal abelian subgroup of $\widetilde{E_n}$ and z be the 1-dimentional representation of M of which is the dual of non zero element in the center $Z(E_n)$. Then the Chern class of the induced representation of z gives

$$c_p{}^n(\mathrm{Ind}_M{}^{\widetilde{E_n}}(z))|Z(E_n) = u^{p^n}.$$

REMARK 2.17. Let A be an elementary abelian p-group and suppose that there exists a continuous map $X \longrightarrow BA$ for some space X. Define $e_A = \prod y$ where y ranges over all Bockstein images of non zero elements in $H^1(A)$. Then we can consider the localized cohomology $[e_A{}^{-1}]H^*(X)$. Since $e_A = e = (-1)^n(\det B_1)^{p-1}$ for $\mathrm{rank}_p(A) = n$, we know $\mathcal{P}^i(e_A) \in \mathrm{ideal}(e_A)$ for all i. Let $\mathcal{P}_t : H^*(X) \longrightarrow H^*(X)[\,[\,t\,]\,]$ be the total reduced powers defined by $\mathcal{P}_t(x) = \sum \mathcal{P}^i(x)t^i$. Then this is a ring homomorphism and easily extends to $[e^{-1}]H^*(X)$ by $\mathcal{P}_t(e_A{}^{-1}) = \mathcal{P}_t(e_A)^{-1}$, e.g.,

$$\mathcal{P}_t(y^{-1}) = y^{-1}(1 + y^{p-1}t)^{-1} = y^{-1} - y^{p-2}t + y^{2p-3}t^2 + \cdots \quad \text{for } 0 \neq y \in H^2(A).$$

Thus $[e^{-1}]H^*(X)$ is a $\mathcal{A}_p$-algebra in which holds the Cartan formula. Of course the Cartan-Serre and Kudo's transgression theorems hold for the localized Hochschild-Serre spectral sequence, however it is not unstable, e.g. in general $\mathcal{P}^i x \neq x^p$ for $i = 2\deg(x)$. Given an $\mathcal{A}_p$-module M, the unstable module $Un(M)$ is defined by elements $x \in M$ such that $\mathcal{P}^i(x) = 0$ for all $2i > \deg(x)$. It is immediate that Image $(H^*(X) \longrightarrow [e^{-1}]H^*(X)) \subset Un([e^{-1}]H^*(X))$. (For more details about $Un([e^{-1}]H^*(X))$, see [7] or Corollary 7.10 bellow.)

3. Extra special p-groups

An extra special p-group G is a group such that its center is Z/p and there is a central extension

$$(3.1) \qquad 1 \longrightarrow Z/p \longrightarrow G \longrightarrow V \longrightarrow 1 \quad \text{where } V = \oplus^{2n} Z/p.$$

Such a group is isomorphic to the n-th central product $E \cdots E = E_n$ or $E_{n-1}M$ where E (resp. M) is the non abelian group of the order p^3 and exponent p (resp. p^2). Hence we can explicitly write

$$(3.2) \qquad E_n = \langle a_1, \cdots a_{2n}, c \mid \begin{array}{l} [a_{2i-1}, a_{2i}] = c, \quad c \in \mathrm{Center} \\ [a_i, a_j] = 1 \text{ for } i < j, \quad (i,j) \neq (2k-1, 2k) \\ a_k{}^p = c^p = 1 \end{array} \rangle.$$

The group $E_{n-1}M$ is written similarly except for $a_{2n}{}^p = c$.

Let us write by $x_i \in H^1(V) = \mathrm{Hom}(V, Z/p)$ the dual of a_i and write $y_i = \mathcal{B}x_i$ Then the cohomology of V is $H^*(V) \cong S_{2n} \otimes \wedge_{2n}$.

Proposition 3.3 (Proposition 2.4 in [14]). *The extension* (3.1) *represent the element in* $H^2(V)$

$$f = \sum_{i=1}^{n} x_{2i-1}x_{2i} \quad (\text{resp. } \sum_{i=1}^{n} x_{2i-1}x_{2i} + y_{2n}) \quad \text{for } G = E_n \ (\text{resp. } E_{n-1}M).$$

We consider the spectral sequence induced from (3.1)

$$(3.4) \qquad \begin{aligned} E_2{}^{*,*} &= H^*(V; H^*(Z/p)) \\ &\cong S_{2n} \otimes \wedge_{2n} \otimes Z/p[u] \otimes \wedge(z) \Longrightarrow H^*(G) \end{aligned}$$

with $\mathcal{B}z = u$. From Proposition 3.3, we know (Lemma 2.5 in [14])

$$(3.5) \qquad d_2 z = f.$$

Then $E_3{}^{*,*}$ is not isomorphic to (2.1), while $d_{2J+1}(u^J) = z(i)$ and $d_{2(p-1)J+1}(z(i) \otimes u^{J(p-1)}) = w(i)$. This spectral sequence seems quite difficult.

Hence we consider other arguments which are used by Kropholler, Leary, Huebschmann and Moselle. Embed $\langle c \rangle \cong Z/p \subset S^1$ and consider the central product

$$(3.6) \qquad \widetilde{G} = G \times_{\langle c \rangle} S^1.$$

Note that $\widetilde{E_n} \cong \widetilde{E_{n-1}M}$, indeed, take $a_{2n}c^{-1/p}$ as a_{2n}, if $a_{2n}{}^p = c$. Then we have the exact sequence

$$(3.7) \qquad 1 \longrightarrow S^1 \longrightarrow \widetilde{G} \longrightarrow V \longrightarrow 1$$

and the induced spectral sequence

$$(3.8) \qquad E_2{}^{*,*} \cong H^*(V; H^*(BS^1)) \Longrightarrow H^*(\widetilde{G})$$

This spectral sequence satisfies (2.1) and (2.2), hence we can apply all results in Section 2. In particular, from Corollary 2.10, we get;

Theorem 3.9. $[e^{-1}]H^*(\widetilde{E_n}) \cong [e^{-1}]S(n)[u^{p^n}] \otimes \wedge(x_1, x_3, \cdots, x_{2n-1})$.

Given $H^*(\widetilde{G})$, to see $H^*(G)$ we use the following fibration induced from (2.1)

$$(3.10) \qquad S^1 = \widetilde{G}/G \longrightarrow BG \longrightarrow B\widetilde{G}.$$

The induced spectral sequence is

$$E_2{}^{*,*} = H^*(\widetilde{G}; H^*(S^1)) = H^*(\widetilde{G}) \otimes \wedge(z) \Longrightarrow H^*(G) \tag{3.11}$$

with $d_2 z = f$. Therefore

Proposition 3.10. *There is an $S(n)$-module isomorphism*

$$H^*(G) \cong (\mathrm{Ker}(f)|H^*(\widetilde{G}))\{z\} \oplus H^*(\widetilde{G})/(f).$$

Since $(x_2, \cdots, x_{2n}) = (x_1, \cdots, x_{2n-1})B_2 B_1{}^{-1} + (z(1), \cdots, z(n))B_1{}^{-1}$, $f = \sum x_{2i-1}x_{2i}$ is expressed as

$$f = \sum b_{ij} x_{2i-1} x_{2j-1} \quad \text{for} \quad B_2 B_1{}^{-1} = (b_{ij}).$$

In particular, when $n \leq 2$, we can compute that $f = 0$ in $[e^{-1}]S(n) \otimes \wedge(x_1, \cdots, x_{2n-1})$.

Corollary 3.11. *If $n \leq 2$, then there is an $S(n)$-algebra isomorphism*

$$[e^{-1}]H^*(E_n) \cong [e^{-1}]S(n)[u^{p^n}] \otimes \wedge(x_1, \cdots, x_{2n-1}) \otimes \wedge(z).$$

Corollary 3.12. *If $n \leq 2$, then there is an $S(n)$-module isomorphism*

$$\begin{aligned} &[e^{-1}]H^*(E_{n-1}.M) \\ &\quad \cong [e^{-1}]((\mathrm{Ker}(y_{2n})|S(n))\{z\} \oplus S(n)/(y_{2n}))[u^{p^n}] \otimes \wedge(x_1, \cdots, x_{2n-1}). \end{aligned}$$

Proof. From Proposition 3.3, for this case, $f = y_{2n}$. □

Next consider other similar groups. Let $\widetilde{E}(s)_n = E_n \times_{\langle c \rangle} Z/p^s$ be the central extension by Z/p^s, $s \geq 2$. Then the central extention

$$0 \longrightarrow Z/p^s \longrightarrow \widetilde{E}(s)_n \longrightarrow V \longrightarrow 0.$$

induces the spectral sequence $E(s)_r{}^{*,*}$ converging to $H^*(\widetilde{E}(s)_n)$. Let us write $H^*(Z/p^s) = Z/p[u] \otimes \wedge(z')$.

Proposition 3.13 ([17]). $\underset{\sim}{E}(s)_r{}^{*,*} \cong E_r{}^{*,*} \otimes \wedge(z')$ *where $E_r{}^{*,*}$ is the spectral sequence* (3.8) *converging $H^*(\widetilde{E}_n)$.*

Proof. Let $d_2(z') = \sum \lambda_{ij}x_i x_j + \mu_k y_k$. Then $\lambda_{ij} = 0$ since $\mathcal{B}(z') = 0$. Consider the automorphism A of $\widetilde{E}(s)_n$ defined by $a_1 \longmapsto a_1 a_2, a_j \longmapsto a_j$ (for $j > 1$) and

$c \longmapsto c$. The induced automorphism A^* of $E(s)_r{}^{*,*}$ is given by $y_2 \longmapsto y_2 + y_1$, $y_j \longmapsto y_j$ (for $j \neq 2$) and $z' \longmapsto z'$. Hence $d_2(z')$ is invariant and we see $\mu_2 = 0$. Similarly we see all $\mu_k = 0$. Therefore $d_2(z') = 0$

Since $\widetilde{E}(s)_n \subset \widetilde{E}_n$, there is the natural map $E_r{}^{*,*} \longrightarrow E(s)_r{}^{*,*}$. By induction on r, we see this proposition. □

Finally for this section, we look at the spectral sequence (3.4) for $E_{n-1}.M$. Let us write this spectral sequence as $'E_r{}^{*,*}$ and write as $E_r{}^{*,*}$ the spectral sequence converging to $H^*(\widetilde{E_n})$. Recall that

$$d_2(z) = f' + y_{2n} \quad \text{with} \quad f' = x_1x_2 + \cdots + x_{2n-1}x_{2n}.$$

Hence $'E_3{}^{*,*} \cong S_{2n-1} \otimes \wedge_{2n} \otimes Z/p[u]$ where $S_{2n-1} = Z/p[y_1, \cdots, y_{2n-1}]$. Now we consider a filtration of $'E_r{}^{*,*}$ by the ideal $I = (x_1, \cdots, x_{2n})$, and its graded algebra $gr'E_r{}^{*,*} = \oplus_{s=0} I^s/I^{s+1}$. Of corse $gr'E_3{}^{*,*} = S_{2n} \otimes \wedge_{2n} \otimes Z/p[u]/(y_{2n})$. Then almost all arguments in Section 2 work.

Theorem 3.14. *For*

$$r < p^i < p^{n-1}, \quad [e_i{}^{-1}]gr'E_{2r+1}{}^{*,*} \cong [e_i{}^{-1}]E_{2r+1}{}^{*,*}/(y_{2n})$$

For the proof of this theorem we recall the following lemmas.

Lemma 3.15. *Let F_1 be a submodule of a module F and $w \in F$. If the multiplication by w on F/F_1 is injective, then $F_1/wF_1 \subset F/wF$.*

Lemma 3.16 (Lemma E in [15]). *Let $F_1 \subset F$ and $zF_1 \subset F_1$. If the spectral sequence $H(F/F_1 \oplus F_1, z) \Longrightarrow H(F, z)$ collapses at the E_1-level, then $F_1/zF_1 \subset F/zF$ and $(F/zF)/(F_1/zF_1) \cong (F/F_1)/z(F/F_1)$.*

Proof. If $zF \cap F_1 \neq zF_1$, then $H(F, z) \supset (\text{Ker } z|F_1)/zF$ but $\not\supset (\text{Ker } d|F_1)/zF_1$. This means the spectral sequence does not collapse. □

Proof of Theorem 3.14. Suppose the statement for $r = J + 1$. Then by Lemma 3.15 and Kudo's transgression theorem, we get the statement for $r \leq I$. By the reason similar to the proof of Lemma 2.15, we get $H([e_i{}^{-1}]gr'E_{2I+1}{}^{*,0}, z(i+1)) = 0$. Of course the spectral sequence

$$H([e_{i+1}{}^{-1}]gr'E_{2I+1}{}^{*,0}, z(i+1)) \Longrightarrow H([e_{i+1}{}^{-1}]E_{2I+1}{}^{*,0}, z(i+1))$$

collapses, hence we have the statement for $r = I + 1$ from Lemma 3.16. □

4. The cases n small

In this section, we study the spectral sequence (2.1) without or with less localization when n is small. More strong results are given in [15].

Suppose $n < p$. The first non zero differential is $d_3 u = z_n(1)$. At first we want to compute $H(s_{2n} \otimes \wedge_{2n}, z(1))$. For this, we use the following lemma taken from [15].

Lemma A. *Let* $z, y \in A$ *be elements of* $|z| =$ odd *and* $|y| =$ even. *Then for* $|x| = |z| - |y|$, *we have additive isomorphism*

$$H(A \otimes \wedge(x), yx + z) \cong (H(A, z)/y)\{x\} \oplus \mathrm{Ker}(y|H(A, z)).$$

From Lemma A, we have $H(S_{2n} \otimes \wedge_1, y_2 x_1) \cong S_{2n}/(y_2)\{x_1\}$. By induction on n

$$H(S_{2n} \otimes \wedge_{2n}, z(1)) = Z/p\{x_1 \cdots x_{2n}\} = Z/p\{f^n\} \quad \text{since} \quad n < p. \tag{4.1}$$

Since $\mathrm{Ker}\, z \cong \mathrm{Im}\, z \oplus H(A, z)$ for $z \in A^{\mathrm{odd}}$, it is immediate that

Lemma B. *There is an isomorphism* $(A/z)/H(A, z) \cong \mathrm{Im}\, z \subset A$. *In particular, if* A *is* w*-free for* $w \in A^{\mathrm{even}}$, *then so is* $(A/z)/H(A, z)$.

Apply this lemma with $A = S_{2n} \otimes \wedge_{2n}, z = z(1), w = y_1$. Since w is injective on A for this case, we know that y_i is injective on $A/(z \oplus H(A, z))$. Since f^n is y_i-torsion, there is no non zero differential $d_r : Z/p\{f^n u^s\} \longrightarrow A/z$ for $r < 2p - 1$.

Next recall the Kudo's transgression $d_{2p-1}(z(1) \otimes u^{p-1}) = w(1)$.
By Lemma B with $w = w(1)$, we know $\mathrm{Ker}(d_{2p-1}|\mathrm{Im}\, z(1)) = 0$.

Lemma 4.2. $d_{2p-1}(f^n u^{p-1}) = nz(2) f^{n-1}$.

Proof. Since $E_r{}^{*,\mathrm{odd}} = 0$, the Bockstein maps from $E_r{}^{*,\mathrm{even}}$ to $E_r{}^{*+1,\mathrm{even}}$. The element $\mathcal{B}(f^n u^{p-1}) = n\mathcal{B}(f) f^{n-1} u^{p-1} = nz(1) f^{n-1} u^{p-1}$ maps to $nw(1) f^{n-1}$ by d_{2p-1}. Since $\mathcal{B}(z(2)) = w(1)$, we know that $d_{2p-1}(f^n u^{p-1}) = nz(2) f^{n-1} + a$ with $a \in \mathrm{Ker}\, \mathcal{B}$. Since $x_i f = 0$ in $S_{2n} \otimes \wedge_{2n}$, we know $x_i a = 0$ in $S_{2n} \otimes \wedge_{2n}/(z(1))$ and hence $\mathcal{B}(x_i a) = y_i a = 0$ but $\mathrm{Ker}\, y_i = Z/p\{f^n\}$ from Lemma B with $w = y_i$. $\square$

Therefore we get ([15])

Theorem 4.3. $$E_{2p}{}^{*,2j} \cong \begin{cases} S_{2n} \otimes \wedge_{2n}/(z(1), w(1), z(2) f^{n-1}) & j = 0 \bmod p \\ Z/p\{f^n\} & 1 \le j < p - 1 \\ 0 & j = p - 1. \end{cases}$$

The next differential is $d_{2p+1}(u^p) = z(2)$. Let $E = S_{2n} \otimes \wedge_{2n}/(z(1), w(1))$. We want know $H(E/z(2)f^{n-1}, z(2))$. First we note the additive isomorphism

$$(4.4) \qquad H(E/z(2)f^{n-1}, z(2)) \cong H(E, z(2)) \oplus Z/p\{f^{n-1}\}.$$

The computations in [15] for the cohomology $H(E, z(2))$ is very long. Hence in this paper, we give a computation with $[y_1{}^{-1}]$, which is somewhat shorter. Recall (2.4) which shows that with mod $(z(1))$

$$z(2) = y_{2,1}x_1 + \sum_{i=2} y_{2i,1}x_{2i-1} - y_{2i-1,1}x_{2i}$$

where $y_{k,1} = \prod_{\lambda \in Z/p}(y_k - \lambda y_1) = y_k{}^p - y_1{}^{p-1}y_k$.

Let $A = S_{2n}/(w(1))$. Applying Lemma A we get $H(A \otimes \wedge(x_3); y_{4,1}x_3) \cong A/(y_{4,1}) \cdot \{x_3\}$, Applying Lemma A and induction on n, we have

$$(4.5) \qquad \begin{aligned} &H(A \otimes \wedge(x_3, \cdots, x_{2n}), \sum_{i=2} y_{2i,2n-1}x_{2i-1} - y_{2i-1,2n-1}x_{2i}) \\ &= A/(y_{3,1}, \cdots, y_{2n,1})\{x_3 \cdots x_{2n}\} \end{aligned}$$

if we can see the following lemma.

Lemma 4.6. $(w(1), y_{3,1}, \cdots, y_{2n,1})$ *is regular in* $[y_1{}^{-1}]S_{2n}$.

Proof. Since the variety is expressed by

$$\mathrm{Var}(y_{i,1}) = \mathrm{Var}\left(\prod_{\lambda \in Z/p}(y_i - \lambda y_1)\right) = \bigcup_{\lambda \in Z/p}\{(y_1, \cdots, y_{2n}) | y_i = \lambda y_1\},$$

we easily see $\mathrm{Var}(y_{3,1}, \cdots, y_{2n,1}) = \cup V_{(\lambda_3, \cdots \lambda_{2n})}$
with $V_{(\lambda_3, \cdots, \lambda_{2n})} = \{(y_1, \cdots, y_{2n}) | y_i = \lambda_i y_1 | i \geq 3\}$. Then

$$(1) \qquad \mathrm{Var}(w(1)) \cap V_{(\lambda_3, \cdots, \lambda_{2n})} \cong \mathrm{Var}(y_1 y_{2,1}) \subset k\{y_1, y_2\} = k^2$$

since

$$(2) \quad \begin{aligned} w(1) &= \sum_{i=2} y_{2i}y_{2i-1,2i} + y_1 y_{2,1} = \sum_{i=2} y_{2i}y_{2i-1,1} - y_{2i-1}y_{2i,1} + y_1 y_{2,1} \\ &= y_1 y_{2,1} \quad \text{on} \quad V_{(\lambda_3, \cdots, \lambda_{2n})}. \end{aligned}$$

Therefore $(1) = \bigcup_{\lambda \in Z/p}\{(y_1, y_2) | y_2 = \lambda y_1\}$ and this has dimension 1. □

Since $y_{2,1} = 0$ in $\mathrm{Ideal}(w(1), y_{3,1}, \cdots, y_{2n,1})$, we have from Lemma A

Proposition 4.7. $[y_1{}^{-1}]H(E, z(2)) \cong [y_1{}^{-1}]S_{2n}/I_n\{x_3\cdots x_{2n}\} \otimes \wedge(x_1)$ *with* $I_n = (y_{2,1}, \cdots, y_{2n,1})$.

For arguments without localization, after long calculations the following is given in [15],

Proposition 4.8 ([15]). $\mathcal{B} : H(E, z(2))^{\mathrm{odd}} \cong H(E, z(2))^{\mathrm{even}} - Z/p\{f^n\}$ *and* $H(E, z(2))^{\mathrm{odd}} \cong S_{2n}\{x_1', \cdots, x_{2n}'\}/(y_{ij}x_j', y_ix_k' = y_kx_i')$ *where* $x_i' = x_if^{n-1}$.

Note that the cocycle in $[y_1{}^{-1}]E/z(2)$ which is represented by $x_3\cdots x_{2n}$ in the righthand side module in Proposition 4.7 is $y_1{}^{-1}\mathcal{B}(x_1x_3\cdots x_{2n})$.

The fact that $E_{2p+2}{}^{*,*} \cong E_{2p(p-1)+1}{}^{*,*}$ is also proved in [15]. When $n = 2$, we have

$$(4.9) \qquad d_{2(p-1)p+1}\{f^{n-1}u^{p(p-1)}\} = (y_{12}' - y_{34}')\mathcal{B}(x_1x_2).$$

where $y_{ij}' = (y_i{}^{p^2}y_j - y_j{}^{p^2}y_i)/y_iy_{ji} = y_i{}^{p(p-1)} + y_i{}^{(p-1)(p-1)}y_j{}^{p-1} + \cdots + y_j{}^{p(p-1)}$ so that $w(2) = \sum y_{2i}y_{2i-1,2i}y_{2i-1,2i}'$. Moreover $d_{2p^3-3}\{f^2u^{p^3-2}\} = z(3)$, and these are all of the non-zero differentials for the case $n = 2$.

In this paper, we give a proof of the above fact with $[y_1{}^{-1}]$-localization but for general general n

Lemma 4.10.

$$\begin{aligned} d_{2p(p-1)+1}\{x_1f^{n-1} \otimes u^{p(p-1)}\} &= y_{2,1}'y_1(x_3\cdots x_{2n}) \\ &+x_1\sum_{i=2} y_{2i-1,2i}'(y_{2i}(x_3\cdots \hat{x}_{2i}\cdots x_{2n}) - y_{2i-1}(x_3\cdots \hat{x}_{2i-1}\cdots x_{2n})) \end{aligned}$$

Proof. Recall $x_1f^{n-1} = x_1x_3\cdots x_{2n}$. The element $y_{2,1}x_1f^{n-1} = z(2)$ $x_3\cdots x_{2n}$ go to $w(2)x_3\cdots x_{2n}$ via Kudo's transgression $d_{2p(p-1)+1}$. The target is

$$\begin{aligned} (1) \qquad w(2)x_3\cdots x_{2n} &= \sum y_{2i-1,2i}'y_{2i}y_{2i-1}x_3\cdots x_{2n} \\ &= \sum y_{2i-1,2i}'(y_{2i}y_{2i-1,1} - y_{2i-1}y_{2i,1})x_3\cdots x_{2n} \end{aligned}$$

From (2.6), $y_{2,1}x_1 = \sum_{i=2} y_{2i-1,1}x_{2i} - y_{2i,1}x_{2i-1} \bmod (z(1), z(2))$, since $Y_{2,k} = y_{k,1}$. Thus we know with the same modulo, $y_{2,1}x_1x_3\cdots \hat{x}_{2i}\cdots x_{2n} = y_{2i-1,1}x_3\cdots x_{2n}$. Hence

$$\begin{aligned} (1) = \ & y_{2,1}(y_{12}'x_3\cdots x_{2n} \\ &+\sum_{i=2} y_{2i-1,2i}'x_1(y_{2i}(x_3\cdots \hat{x}_{2i}\cdots x_{2n}) - y_{2i-1}(x_3\cdots \hat{x}_{2i-1}\cdots x_{2n}))) \end{aligned}$$

Since $y_{2,1}$-torsion elements in $E/(z(2))$ are also contained in $H(E,z(2))$ from Lemma B, we get the lemma. $\square$

Since the target element of the differential in Lemma 4.10 and its Bockstein are not in $H(E,z(2))$, they are $[y_1{}^{-1}]S(1)$-free. Hence we get

Theorem 4.11.

$$\begin{aligned}[y_1{}^{-1}]E_{2p^2+1}{}^{*,0} \cong [y_1{}^{-1}]&(S(2)\otimes\wedge(x_1,x_2,\cdots,x_{2n})/(z(2),d,\mathcal{B}(d))\\ &\oplus S_{2n}/I_n\{x_1f^{n-1},\mathcal{B}(x_1f^{n-1})\}\{u^p,u^{2p},\cdots,u^{p(p-2)}\})\otimes Z/p[u^{p^2}],\end{aligned}$$

where $I_n=(y_{2,1},\cdots,y_{2n-1,1})$ and d is the image of $d_{2p(p-1)+1}$ given by Lemma 4.10.

5. Cohomology of other similar p-groups

In this section we study some applications for arguments in §2. Let A be a commutative ring and let $A(k)$ denote the variety, that is the set of ring homomorphisms from A to k endowed with the Zarisky topology. Let us write by $H^*(G)(k)$ the variety $(H^*(G)/\sqrt{0})(k)$. For example, $H^*(V)(k)=S_{2n}(k)=V\otimes k$ and $S(n)(k)=(S_{2n}/J)(k)=\mathrm{Var}(J)$ for $J=(w(1),\cdots,w(n))$.

Theorem 5.1 ([14]). *Let $B: V\times V\longrightarrow Z/p$ be the alternating from defined by $B(a,b)=\sum a_{2i-1}b_{2i}-a_{2i}b_{2i-1}$. Then $\mathrm{Var}(J)=\cup W\otimes k$ where W ranges over the set I of maximal B-isotropic subspaces of V and the cardinality of I is $(p+1)\cdots(p^n+1)$.*

Theorem 5.2. *The ideal J has a prime decomposition $J=\sqrt{J}=\bigcap_{W\in I}\mathcal{P}_W$, where $\mathcal{P}_W=\mathrm{Ker}(H^*(V)/\sqrt{0}\longrightarrow H^*(W)/\sqrt{0})$.*

We consider a group G which is an extension of $\widetilde{E}_n\oplus V'$ for $V'=\oplus^m Z/p$ by S^1. Such a group is represented by elements in $H^2(\widetilde{E}_n\oplus V',S^1)=H^3(\widetilde{E}_n\oplus V',Z)$. Consider the spectral sequence

$$E_2=H^*(\widetilde{E}_n\oplus V')\otimes Z/p[u']\Longrightarrow H^*(G). \tag{5.3}$$

First assume the case $d_3u'=\sum_{i=1}^{s}\mathcal{B}(x_{4i-1}x_{4i-3})$ for $2s\le n$. Let J' be the ideal in $Z/p[y_1,\cdots,y_{2n}]$ generated by $\mathcal{B}\mathcal{P}^{p^{i-2}}\cdots\mathcal{P}(d_3u)$ and let $\mathcal{P}'_W$ be the corresponding prime ideal. Of course $e\in\mathcal{P}'_W$ for all W, so $e\in J'$. Hence we only have the trivial result, namely, $[e^{-1}]H^*(G)=0$.

Next we consider the case

$$d_3u'=\sum_{i=1}^{m}\beta(x_{2i-1}x_{2i}{}') \quad \text{with} \tag{5.4}$$

$$H^*(V') = Z/p[y_2', \cdots, y_{2m}'] \otimes \wedge(x_2', \cdots, x_{2m}').$$

This group is represented as $\widetilde{G}' = G' \times_{\langle c \rangle} S^1$ with

$$\begin{aligned} G' = \langle \widetilde{E}(n), a_2', \cdots, a_{2m}', c' \mid & [a_{2j-1}, a_k'] = c'\delta_{2j,k} \\ & c'^p = a_k'^p = [a_i', a_j'] = 1, \ c' \in \mathrm{Center}(G) \rangle \end{aligned}$$

Theorem 5.5. *Let $J' = (w'(1), \cdots, w'(m))$ with $w(i)' = \mathcal{B}\mathcal{P}^{p^{i-2}} \cdots \mathcal{P}(d_3(u'))$ for (5.4) and let $\widetilde{G}'$ be the group above. Then*

$$[e^{-1}]H^*(\widetilde{G}') = [e^{-1}]H^*(\widetilde{E}_n) \otimes Z/p[y_2', \cdots, y_{2m}']/(J') \otimes Z/p[u'^{p^n}].$$

Proof. From Theorem 2.7, we only need to prove the regularity of $(w(1)', \cdots, w(m)')$ in $[e^{-1}]S(n) \otimes Z/p[y_2', \cdots, y_{2m}']$ For this, we study the map

$$i : [e^{-1}]S(n)(k) \longrightarrow S(n)(k) = \bigcup_{W \in I} W \otimes k.$$

Suppose that $0 \neq x \in \mathrm{Image}(i) \cap W$ for some $W \in I$. This means that there are non zero maps x_1 and x_2 such that the following diagram commutes

$$\begin{tikzcd} S_{2n}/\mathcal{P}_W = S(W) \arrow[dr, "x_2"] & \\ S(n) = S_{2n}/J \arrow[u] \arrow[r, "x = i(x_1)"] \arrow[d] & k \\ {[e^{-1}]S(n)} \arrow[ur, "x_1"'] & \end{tikzcd}$$

Hence $e \notin \mathcal{P}_W$. Conversely if $e \notin \mathcal{P}_W$, then it is easy to see $[e^{-1}]S(n)(k) \supset W \otimes k$. Therefore $[e^{-1}]S(n)(k) = \cup \widetilde{W} \otimes k$ where $\widetilde{W}$ ranges I with $e \notin \mathcal{P}_{\widetilde{W}}$. Hence

$$S_{odd} = Z/p[y_1, \cdots, y_{2n-1}] \longrightarrow S(n) \longrightarrow S(\widetilde{W})$$

is an isomorphism. Therefore $\widetilde{W}$ is

$$Z/p\{y_1, \cdots, y_{2n} | y_{2j} = \lambda_{j1}y_1 + \cdots + \lambda_{jn}y_{2n-1}\} \subset V \quad \text{for some } (\lambda_{jk}) \in (Z/p)^n.$$

Similar arguments can be applied for y_{even}' instead of y_{even}. Then we have

$$([e^{-1}]S(n)[y_2', \cdots, y_{2n}']/J')(k) = \bigcup \widetilde{W}' \otimes k$$

with $\widetilde{W}' = Z/p\{y_i, y_{2k}' | y_{2j} = \lambda_{j1}y_1 + \cdots + \lambda_{jn}y_{2n-1}, y_{2j}' = \lambda_{j1}'y_1 + \cdots + \lambda_{jm}'y_{2m-1}\}$. In particular $\dim_k \widetilde{W} \otimes k = n$. Hence $(w(1)', \cdots, w(m)')$ is regular.

□

6. Periodic modules with large period

Let $\Omega^r(M)$ be the r-th kernel in the minimal resolution of $k(G)$-module M, i.e. if

$$(6.1) \qquad 0 \longrightarrow M_r \longrightarrow Q_{r-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

is exact and if each Q_i is projective, then $M_r \cong \Omega^r(M) \oplus Q$ for some projective module Q. A G-module M is said to be periodic if $\Omega^m(M) \cong M$ for some $m > 0$. The smallest such m is called the period of M.

For a G-module M, let $I_G(M)$ be the annihilator ideal in $H^*(G;k)$ of $\mathrm{Ext}_{k*(G)}{}^*(M,M) \cong H^*(G, \mathrm{Hom}_k(M,M))$. Let $V_G(M)$ be the subvariety of $H(G)(k)$ associated with $I_G(M)$, e.g., $V_G(k) = H(G)(k)$. Remark that if V is a closed homogeneous subvariety of $V_G(k)$, then there is a $k(G)$-module M with $V_G(M) = V$ (Proposition 2.1 (vii) in [6]).

We recall arguments of Benson-Carlson [6]. Consider a central extension of a finite group

$$(6.2) \qquad 1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where $N = Z/p^s$ for $s \geq 1$ and Q is a p-group. Remark that the paper [6] is written assuming that $s = 1$, however all arguments in [6] work also in the case $s \geq 2$. Let $\bar{N}$ denote the sum $\sum_{g \in N} g$ as an element of the group ring $k(N)$. Then for $r > 0$, $\bar{N}\Omega^{2r}(k)$ is a $k(G)$-module with N-acting trivially, so we may regard it as a $k(Q)$-module. We set $V_r = V_Q(\bar{N}\Omega^{2r}(k)) \subset H(Q)(k)$.

Theorem 6.3 (Andrews [6]). *Let M be an indecomposable $k(Q)$-module regarded as a $k(G)$-module by inflation. Then M is a periodic $k(G)$-module of period dividing $2r$ if and only if $V_Q(M) \bigcap V_r = \{0\}$.*

Theorem 6.4 (Benson-Carlson [6]). *Let $E_r{}^{*,*}$ be the spectral sequence induced from* (6.2). *Let $K_I \subset H^*(Q)$ be the kernel of the induced map $E_2{}^{*,0} \to E_{2I+1}{}^{*,0}$ for $I = p^i$. Then $V_I = V_Q(K_I)$.*

Theorem 6.5 ([17]). *Let G be the p-group $\widetilde{E}(s)_n$, $s \geq 2$ or $E_{n-1}.M$. Then there are periodic $k(G)$-modules of period $2p^i$ for all $i \leq n$, and no higher period.*

Proof (See the proof of Crollary 6.2 in [6]). By Proposition 3.13, Theorem 3.14, Corollary 2.9 and Theorem 6.4, we may find a closed homogeneous subvariety V of $H(Q)(k)$ with $V \bigcap V_J \neq \{0\}$ but $V \bigcap V_I = \{0\}$ for $I = p^i$ and $J = p^{i-1}$. By the remark after the definition of $V_G(M)$, we may find a $k(Q)$-module M with $V_Q(M) = V$. Then by Theorem 6.3, M has period $2I = 2p^i$. □

Corollary 6.6. *Let G and G' be p-groups such that there is a commutative diagram of central extensions*

$$\begin{array}{llcccccccc}
(1) & 0 & \longrightarrow & Z/p^s & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1 \\
 & & & \Big\| & & \Big\downarrow g & & \Big\downarrow f & & \\
(2) & 0 & \longrightarrow & Z/p^s & \longrightarrow & \widetilde{E}(s)_n & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

where $s \geq 2$ and $g : G \to \widetilde{E}(s)_n$ is a split epimorphism. Then there are periodic $k(G)$-modules of period $2i$ for all $i \leq n$, and no higher period.

Proof. Let us write by ${}_1E_r{}^{*,*}$ and ${}_2E_r{}^{*,*}$ the spectral sequences induced from (1) and (2) respectively. Then the following diagram is commutative

$$\begin{array}{ccc}
{}_1E_{2I+1}{}^{*,0}(k) & \xrightarrow[i(k)]{\hspace{2em}} & {}_1E_{2J+1}{}^{*,0}(k) \\
\Big\downarrow f(k) & & \Big\downarrow f(k) \\
{}_2E_{2I+1}{}^{*,0}(k) = S(i)(k) & \overset{\hookrightarrow}{\not\cong} & S(i-1)(k) = {}_2E_{2J+1}{}^{*,0}(k)
\end{array}$$

Here $f(k)$ is split epic but there is not a split epimorphism $S(i-1)(k) \to S(i)(k)$. Hence $i(k)$ is not an isomorphism. □

For example, the group $G' \times_{\langle c \rangle} Z/p^s$ for G' in Theorem 5.5 satisfies the above corollary.

7. Elementary abelian p-group actions on CP^m

We recall arguments of Allday [3]. Let X be a finite complex such that

$$H^*(X) \cong H^*(CP^m) \cong Z/p[u]/(u^{m+1}).$$

Let $V' = \oplus^t Z/p$ and $H^*(BV') \cong S_t \otimes \wedge_t \cong Z/p[y_1, \cdots, y_t] \otimes \wedge(x_1, \cdots, x_t)$. Assume that X is a V'-complex. Consider the spectral sequence

$$(7.1) \qquad E_2{}^{*,*} = H^*(BV'; H^*(X)) \Longrightarrow H_{V'}{}^*(X) = H^*(X \times_{V'} EV').$$

Since $\mathcal{B}u = 0$, we can take n with $0 \leq 2n \leq t$ such that

$$(7.2) \qquad d_3 u = \sum_{i=1}^{n} \mathcal{B}(x_{2i-1}x_{2i}) \quad \text{as in (2.2)}.$$

Lemma 7.3. *If $d_{2I+1}u^I \neq 0$ for $I = p^i$, then $pI|m+1$.*

Proof. We prove this lemma by induction on i. It is clear when $i = -1$.

Suppose $m+1=Is$ and

$$E_{2I+1}{}^{*,*} \cong \left(\bigoplus_{j=0}^{2I-2} E_{2I+1}{}^{*,j}\right) \otimes Z/p[u^I]/(u^{Is}).$$

If $p \nmid s$, then $0 = d_{2I+1}u^{m+1} = s(u^I)^{s-1}d_{2I+1}(u^I) \neq 0$ in $E_{2I+1}{}^{*,*}$. This is a contradiction, so $p|s$. □

Corollary 7.4. *If* (7.2) *holds, then* $p^n|m+1$

Proof. This is immediate from Theorem 2.7. □

Theorem 7.5. *Let X be a V'-complex such that* $H^*(X) = Z/p[u]/(u^{p^n s})$ *and* (7.2) *holds. Then* $[e^{-1}]H^*{}_V(X) \cong [e^{-1}]S(n)[u^{p^n}]/(u^{p^n s}) \otimes \wedge(x_1,\cdots,x_{2n-1})$ *and* $[e^{-1}]H^*{}_{V'}(X) \cong [e^{-1}]H_V{}^*(X) \otimes Z/p[y_{2n+1},\cdots,y_t] \otimes \wedge(x_{2n+1},\cdots,x_t)$.

Hereafter we always assume (7.2) and consider only the V-action induced from the V'-action.

For a given multiplicative set $S \subset H^*(V)$ and a V-complex X, let X^S be a set of points x such that each element in S maps to non zero element in $H^*(V) \to H_V{}^*(X) \to H^*(V_x)$ where V_x is the isotropy group of V at $x \in X$. Then the localization theorem (Hsiang) is stated as $S^{-1}H_V{}^*(X) \cong S^{-1}H_V{}^*(X^S)$. Hence for a subgroup W of V, we get $S_W{}^{-1}H_V(X) = S_W{}^{-1}H_V(X^W)$ for the fixed points set X^W where S_W is the multiplicative set generated by $\mathcal{B}(V^* - \mathrm{Ker}(V^* \to W^*))$ identifying $W^* = H^1(W)$. Let $e_W = \prod \mathcal{B}x$ where x ranges all non zero elements in $H^1(W)$, e.g., $e_{V_{\mathrm{odd}}} = e$ for $V_{\mathrm{odd}} = Z/p\{y_1,\cdots,y_{2n-1}\}$. Then

(7.6) $\quad [e_V{}^{-1}]H_V{}^*(X) \cong [e_V{}^{-1}]H_V{}^*(X^V) \cong [e_V{}^{-1}]H^*(X^V) \otimes H^*(V)$.

Recall the set I of maximal B-isotropic subspaces W in V in Theorem 5.1.

Corollary 7.7. *Suppose that X is a V-complex as in Theorem* 7.5 *and* $n>0$. *Then* $[e_V{}^{-1}]H_V{}^*(X) = 0$, *($X$ is V-fixed point free),* $[e^{-1}]H_V{}^*(X) \cong [e^{-1}] H_V{}^*(X^e)$, $S_{V_{\mathrm{odd}}}{}^{-1}H_V{}^*(X)(k) \cong V_{\mathrm{odd}} \otimes k$, $[e^{-1}]H_V{}^*(X)(k) \cong \bigcup W' \otimes k$ *where W' ranges in I such that* $\pi_*(W') = V_{\mathrm{odd}}$ *for* $\pi : W' \subset V \xrightarrow[\mathrm{proj}]{} V_{\mathrm{odd}}$.

Proof. We only need to see the last statement. If $e \notin \mathcal{P}_W$, then the map $\pi : S(V_{\mathrm{odd}}) \subset S(V')/J \longrightarrow S(W)$ is injective and hence π_* is surjective. Thus we get the corollary. □

Now we recall some results of Hsiang. We say the orbit type $0(X)$ of a given G-space X is the set of conjugacy classes of isotropy subgroups G_x for each $x \in X$.

Theorem 7.8 (Hsiang [8]). *Let X be a compact $V = \oplus^{2n} Z/p$-space without fixed point. Let J be* $\mathrm{Ker}(\pi^* : H^*(BV) \to H_v{}^*(X))$, $\sqrt{J}$ *be the radical of J and* $\sqrt{J} = P_1 \cap \cdots \cap P_a$ *the irreducible decomposition of J into its prime components. Then*

(i) *There is* 1-1 *correspondence between $\{P_i\}$ and the maximal elements $\{H_i\}$ of* $0(X)$ *by* $P_i = \mathrm{Ker}(H^*(BV) \to H^*(BH_i))$.

(ii) *Let Y_j be the fixed point set of H_j, Then*

$$H_v(X)_{P_j} \cong H^*{}_v(Y_j)_{P_j} \cong H^*(Y_j/V) \otimes H^*(H_j)_{P_j}.$$

Corollary 7.9. *Let V and X satisfy the assumptions of Theorem* 7.5. *Then*

(i) *There is* 1-1 *correspondence between the set of maximal elements in* $0\ (CP^m)$ *and the set I of maximal B-isotropic subspaces of V, i.e. all maximal isotropy subgroups are isomorphic to $\oplus^n Z/p$ and the cardinal number of I is* $(p+1)(p^2+1)\cdots(p^n+1)$.

(ii)
$$\begin{aligned} S_{V_{\mathrm{odd}}}{}^{-1} H_V{}^*(X) &\cong S_{V_{\mathrm{odd}}}{}^{-1} H_V{}^*(X^{V_{\mathrm{odd}}}) \\ &\cong S_{V_{\mathrm{odd}}}{}^{-1}(H^*(X^{V_{\mathrm{odd}}}/V) \otimes H^*(V_{\mathrm{odd}})) \end{aligned}$$

Proof. From Theorem 5.1 and Theorem 5.2, the corollary is immediate. □

Recall that we can extend the Steenrod algebra action to the localized equivariant cohomology (Remark 2.16). Dwyer-Wilkerson [7], [4] proved $H_V{}^*(X^A) \cong Un(S_A{}^{-1} H_V{}^*(X))$ for each finite V-complex X and each subgroup A of V.

Corollary 7.10. *Let X and V satisfy the assumption of Theorem* 7.5. *Then we have* $H_V{}^*(X^{V_{\mathrm{odd}}}) \cong Un(S_{V_{\mathrm{odd}}}{}^{-1} S(n)[u^{p^n}]/(u^{p^n s}) \otimes \wedge(x_1, \cdots, x_{2n-1}))$.

Next we consider the case $X = CP^t \times CP^s$ and $V' = \oplus^{2n+m} Z/p$ acts on X such that the projection onto the first factor is equivariant with respect to an action of V' on CP^t; and supposed that V' acts trivially on $H^*(X)$. Then we get the fibering

$$CP^s \longrightarrow (EV' \times_{V'} (CP^t \times CP^s)) \longrightarrow (EV' \times_{V'} CP^t)$$

which induces the spectral sequence

$$E_2{}^{*,*} = H^*{}_{V'}(CP^t) \otimes H^*(CP^s) \Longrightarrow H_{V'}{}^*(CP^t \times CP^s).$$

Then by the same arguments as in the proof of Theorem 5.5, we can see

Theorem 7.11. *Let* $V' = V \oplus Z/p\{y_2', \cdots, y_{2m}'\}$ *and* $t = t'p^n$ *and* $s = s'p^m$. *Consider a V'-action on* $X = CP^t \times CP^s$ *such that the projection onto the first factor is equivarent with respect to an action of V' on CP^t ; and suppose that V'*

acts trivially on $H^*(X)$. *Suppose also* d_3u *is as in* (7.2) *and* d_3u' *is as* (5.4). *Then*

$$[e^{-1}]H_{V'}{}^*(X)=[e^{-1}]S(n)\otimes Z/p[y_2{}',\cdots,y_{2m}{}']/(J')\otimes Z/p[u^{p^n},u'^{p^m}]/(u^t,u'^s)$$

where u *and* u' *are ring generators of* $H^*(CP^t)$ *and* $H^*(CP^s)$ *respectively.*

Remark that we can construct V' actions which satisfies Theorem 7.5 and Theorem 7.11, by using skeletons of classifying spaces of $\widetilde{E}_n$ and $\widetilde{G}'$ in §5.

Finally we give the example for $n=1$. The ideal $J=(w(1))=(y_1{}^py_2-y_1y_2{}^p)$ has the primary decomposition $(y_z)\cap\bigcap_{i\in z/p}(y_1-iy_2)$. Hence there are $p+1$ maximal isotropy subgroups, which are isomorphic to Z/p. On the other hand, there is a $\widetilde{E}_1$-action on C^p such that

$$\begin{aligned} a_1 &: (z_1,\cdots,z_p)\longrightarrow(\xi^1z_1,\cdots,\xi^pz_p) \quad \text{with} \quad \xi=\exp 2\pi\sqrt{-1}/p\\ a_2 &: (z_1,\cdots,z_p)\longrightarrow(z_2,\cdots,z_p,z_1)\\ \theta &: (z_1,\cdots,z_p)\longrightarrow(\eta z_1,\cdots,\eta z_\theta) \qquad \text{with} \quad \eta=\exp 2\pi\sqrt{-1}\theta. \end{aligned}$$

Consider the induced $(Z/p\oplus Z/p)$-action on $CP^{p-1}=(C-\{0\})/\langle\theta\rangle$. The fixed points under the $\langle a_1\rangle$-action are $(1,0,\cdots,0),\cdots,(0,\cdots,0,1)$. For $x=(1,0,\cdots,0)$, we see $G_x=\langle a_1\rangle\cong Z/p$. Since we can take $p_{ij}\in GL_p(C)$ such that $p_{ij}{}^{-1}a_1{}^ia_2{}^jp_{ij}=a_1$ in $GL_p(C)$, all maximal isotropy groups are $\langle a_1\rangle$, $\langle a_2a_1{}^i\rangle$ for $0\le i\le p-1$, which correspond to (y_2), and (y_1-iy_2) respectively by $(y_1-iy_2)=\mathrm{Ker}(H^*(G)\to H^*(\langle a_2a_1{}^i\rangle)$.

We also see equivariant cohomologies for $n=1$.

$$\begin{aligned} &[e^{-1}]H_V{}^*(CP^{p-1})\cong[e^{-1}]S_2\otimes\wedge(x_1)/(y_2{}^p-y_1{}^{p-1}y_2)\\ &S_{V\mathrm{odd}}{}^{-1}H_V{}^*(CP^{p-1})\cong H_V{}^*(CP^{p-1})_{(V\mathrm{odd})}\cong Z/p[y_1{}^{-i},y_1]\otimes\wedge(x_1)\\ &[e^{-1}]H_V(CP^{p-1})(k)\cong\bigcup_{i\in Z/p}\mathrm{Var}(y_2-iy_1),\quad S_{V\mathrm{odd}}{}^{-1}H_V(CP^{p-1})(k)=V_{\mathrm{odd}}\otimes k. \end{aligned}$$

8. Cohomology of a Sylow p-subgroup of $GL_4(F_p)$

Let $GL_n(F_p)$ be the general linear group over F_p and U_n be its p-Sylow subgroup generated by upper triangular matrices with diagonal entries 1. Let a_{ij} be the element in U_n such that all entries are zero except for diagonal entries and the (i,j)-entry, which are 1. Then it is well known

$$U_n=\langle a_{ij}|1\le i<j\le n\rangle$$

and

$$[a_{ij},a_{hk}]=\begin{cases} I & \text{if} \quad j\neq h\\ a_{ik} & \text{if} \quad j=h.\end{cases}$$

Hereafter we compute $H(U_4)$. When $p=2$ the cohomology is computed in [16] and it is used to compute $H^*(GL_4(F_2))$. The cohomology is also important to decide the cohomology of the sporadic simple groups M_{12}, $O'N$ [2], [1].

We assume p odd. For ease of argument we simply write the subscripts(12) (resp. (23), (34), (13), (24), (14)) as 1 (resp. 2, 3, 4, 5, 6), for example $a_1 = a_{12}$, $x_2 = x_{23}, \cdots$

$$\begin{pmatrix} & 1 & 4 & 6 \\ & & 2 & 5 \\ & & & 3 \\ & & & \end{pmatrix}$$

Let us write by $U(i_1 \cdots i_k)$ the subgroup of U generated by $a_{i_1}, \cdots, a_{i_k}$. The subgroup $U(124)$ is isomorphic to the extra-special p-group E_1. Hence we know from Corollary 3.11.

(8.1) $$[{y_1}^{-1}]H^*(U(124)) \cong Z/p[{y_1}^{-1}, y_1, y_2, v_4]/(w_{12}(1)) \otimes \wedge(x_1, z_4)$$

Here v_4 is defined by using the Evens' norm

(8.2) $$v_4 = \mathrm{Norm}(U(14) \subset U(124))(y_4)$$

and hence

(8.3) $$\begin{aligned} v_4|U(14) &= {y_4}^p - {y_1}^{p-1}y_4 = y_{41}, \quad z_4|U(14) = x_4 - (y_4/y_1)x_1, \\ \text{and} \quad w_{12}(1) &= {y_1}^p y_2 - y_1 {y_2}^p = y_1 y_{21}. \end{aligned}$$

Note that $x_2 = (y_2/y_1)x_1$ in $[{y_1}^{-1}]H^*(U(14))$. The conjugation map ${a_2}^*$ induced from a_2 on $[{y_1}^{-1}]H^*(U(14))$ is given by

$$y_4 \longrightarrow y_4 + y_1, \quad x_4 \longrightarrow x_4 + x_1.$$

Since the elements z_4 and v_4 must be invariant under this ${a_2}^*$, we get (8.3).

Let us write $M = U/U(6)$ and $\widetilde{M} = M \times_{U(5)} S^1$. We study the cohomology $[{y_1}^{-1}]H^*(\widetilde{M})$. We consider the spectral sequence

(8.4) $${E_2}^{*,*} = [{y_1}^{-1}]H^*(U(124) \oplus U(3)) \otimes H^*(\widetilde{U}(5)) \Longrightarrow [{y_1}^{-1}]H^*(\widetilde{M})$$

where $\widetilde{U}(5) = U(5) \times_{U(5)} S^1$. Let us write

(8.5) $R = Z/p[{y_1}^{-1}, y_1, y_3, v_4] \otimes \wedge(x_1, z_4)$ and $B = Z/p\{1, y_2, \cdots, {y_2}^{p-1}\}$.

Then ${E_2}^{*,*} = R \otimes B \otimes Z/p[y_5] \otimes \wedge(x_3)$. The first nonzero differential is

(8.6) $$d_3 y_5 = y_2 x_3 - y_3 x_2 = y_2(x_3 - (y_3/y_1)x_1).$$

Let ${x_3}' = x_3 - (y_3/y_1)x_1$. Then the homology is

$$\begin{aligned} H(R \otimes B \otimes \wedge(x_3), dy_5) &= H(R \otimes B \otimes \wedge({x_3}'), y_2 {x_3}') \\ &= \mathrm{Ker}(y_2)|(R \otimes B) \oplus R \otimes B/(y_2)\{{x_3}'\} \\ &= R\{{y_1}^{p-1} - {y_2}^{p-1}, {x_3}'\} = R\{1 - (y_2/y_1)^{p-1}, {x_3}'\}. \end{aligned}$$

since $w_{12}(1) = y_1^p y_2 - y_1 y_2^p = 0$ in $[y_1^{-1}]H^*(U(124))$. For ease of notations, we write by $1'$ simply the element $1 - (y_2/y_1)^{p-1}$. Therefore we get

$$(8.7) \qquad E_4^{*,*'} = \begin{cases} R \otimes B \otimes \wedge(x_3')/(y_2 x_3') & *' = 0 \\ R\{1', x_3'\} \quad 0 < *' < p-1 \\ R\{1', x_3'\} \oplus R \otimes (B - Z/p\{1\})\{x_3'\} & *' = p-1. \end{cases}$$

Lemma 8.8. $d_r = 0$ *for* $4 \leq r \leq 2p-2$.

Proof. We only need to show that $d_r(x \otimes y_5{}^i) = 0$ for $x = x_3'$ or $1'$. Let us write $U(i \cdots j6)/U(6)$ by $U(i \cdots j)'$. Consider the extension

$$0 \longrightarrow U(5) \longrightarrow U(1435)' \longrightarrow U(134)' \longrightarrow 0$$

and the induced spectral sequence $EE_r^{*,*}$. Since $U(1345)' = (Z/p)^4$ is abelian, all differentials in $EE_r^{*,*}$ are zero. Let $i^* : E_r^{*,*} \to EE_r^{*,*}$ be the map induced from the inclusion $i : U(1345)' \longrightarrow M$. Suppose $dx \neq 0$ in $E_r^{*,*}$ for one of the above x. Since x is y_2-torsion,

$$d_r x \in R\{1', x_3'\} \otimes y_5{}^s \quad \text{for} \quad 0 \leq s < p-1.$$

However $i^*|R\{1', x_3'\} \otimes y_5{}^s$ is injective. Hence $i^* d_r x = d_r i^* x \neq 0$ and this is a contradiction. □

Lemma 8.9. $d_{2p-1}(1' \otimes y_5{}^{p-1}) = 0$, $d_{2p-1}(x_3' \otimes y_5{}^{p-1}) = (y_2/y_1)^{p-1} y_{32}$.

Proof. Since $i^*(1') = 1$, $d_r(1' \otimes y_5{}^{p-1}) = 0$ is proved by the arguments similar to the proof of Lemma 8.8. By the Kudo's transgression theorem, we have $d(y_2 x_3') = y_2 y_{32}$, Hence we get $d_{2p-1}(x_3') = y_{32}$ modulo $\mathrm{Ker}(y_2) = \mathrm{Ideal}(1')$. Since $i^*(d_{2p-1}(x_3')) = 0$ for the map $i^* : E_r^{*,*} \longrightarrow EE_r^{*,*}$ in the proof of Lemma 8, we know $d_{2p-1}(x_3')$ must be in the ideal (y_2). Hence we get this lemma. □

Therefore we have

Lemma 8.10.

$$[y_1{}^{-1}]E_{2p}{}^{*,*'} \cong \begin{cases} R \otimes B \otimes \wedge(x_3')/(y_2 x_3', y_2 y_{32}) & *' = 0 \\ R \otimes \wedge(x_3')\{1'\} \quad 0 < *' < p-1 \end{cases}$$

Here we note that $R\{x_3', 1'\} = R \otimes \wedge(x_3')\{1'\}$ for $0 < *' < p-1$ and that additively

$$E_r{}^{*,0} \cong R/(y_3)\{y_2, \cdots, y_2{}^{p-1}\} \otimes \{1, y_3, \cdots, y_3{}^{p-1}\} \oplus R \otimes \wedge(x_3{}').$$

Since $v_5|U(5) = y_5{}^p$, $y_5{}^p$ is permanent in this spectral sequence. Thus $[y_1{}^{-1}]E_{2p}{}^{*,*} \cong [y_1{}^{-1}]E_\infty{}^{*,*}$. Next consider$[y_1{}^{-1}]H^*(M)$. From the fibering

$$S^1 \longrightarrow BM \longrightarrow B\widetilde{M},$$

we get the spectral sequence

$$E'_2{}^{*,*} \cong H^*(B\widetilde{M}) \otimes \wedge(x_5) \Longrightarrow H^*(M).$$

The differential is

$$d_2(x_5) = x_2x_3 = (y_2/y_1)x_1x_3 = (y_2/y_1)x_1x_3{}' = 0.$$

Hence this spectral sequence collapses and $[y_1{}^{-1}]H^*(M) \cong [y_1{}^{-1}]H^*(\widetilde{M}) \otimes \wedge(z_5)$. Here we can take v_5 and z_5 such that

(8.11) $$v_5|U(1345)' = y_{53}, \quad z_5|U(1345)' = x_5 - (y_4/y_1)x_3.$$

Let k_5 be an element corresponding to $1' \otimes y_5$ in $E_\infty{}^{*,*}$. Then $k_5|U(1345)' = y_5 - (y_4/y_1)y_3$ and $k_5{}^i$ corresponds $1'^i \otimes y_5{}^i = 1' \otimes y_5{}^i$ for $1 \leq i \leq p-1$. Moreover $\mathcal{B}z_5 = k_5$ on $U(1345)'$. Here we notice that for all $s > 1$

$$1'^s = 1'^{(s-1)} - (y_2/y_1)^{p-1}1'^{(s-1)} = 1'^{(s-1)} = 1'.$$

Proposition 8.12. *There is an additive isomorphism* $[y_1{}^{-1}]H^*(M) \cong Q \otimes (C \oplus K)$ *with* $Q \otimes C = \mathrm{Ker}(1')$ *and* $Q \otimes K = \mathrm{Im}(1')$, *where*

$$\begin{aligned} Q &= Z/p[y_1{}^{-1}, y_1, v_4, v_5] \otimes \wedge(x_1, z_4, z_5) \\ C &= Z/p\{y_2, \cdots, y_2{}^{p-1}\} \otimes Z/p\{1, y_3, \cdots, y_3{}^{p-1}\} \\ K &= Z/p[y_3] \otimes \wedge(x_3{}') \otimes \{1', k_5, \cdots, k_5{}^{p-1}\} \end{aligned}$$

Let $i : U(1345)' \subset M$ be the inclusion. Then it is immediate that $i^*|QK$ is injective. Let $\ell : [y_1{}^{-1}]H^*(M) \longrightarrow [y_1{}^{-1}, y_2{}^{-1}]H^*(M)$ be the localization. We can take k_5 so that $y_2k_5 = 0$ multiplying by $1'$ if neccessary. Then $[y_1{}^{-1}, y_2{}^{-1}]H^*(M) \cong [y_2{}^{-1}]Q \otimes C$. Therefore we get

Corollary 8.13. *The map* $i^* \times \ell$ *is injective.*

Let $\widetilde{U} = U \times_{U(6)} S^1$. The short exact sequence

$$1 \longrightarrow \widetilde{U}(6) \longrightarrow \widetilde{U} \longrightarrow M \longrightarrow 1$$

induces the spectral sequence

$$[y_1{}^{-1}]E_2{}^{*,*} \cong [y_1{}^{-1}]H^*(M; H^*(\widetilde{U}(6)) \Longrightarrow [y_1{}^{-1}]H^*(\widetilde{U}) \tag{8.14}$$

Since $1'$ is permanent and $\mathrm{Ker}(1') = \mathrm{Im}(y_2)$, we have a decomposition

$$[y_1{}^{-1}]E_r{}^{*,*} \cong \mathrm{Im}(1')E_r{}^{*,*} \oplus \mathrm{Ker}(1')E_r{}^{*,*}.$$

From the argument just before Corollary 8.13, we have

Lemma 8.15. $[y_1{}^{-1}, y_2{}^{-1}]E_r{}^{*,*} \cong [y_2{}^{-1}]\ \mathrm{Ker}(1')E_r{}^{*,*}$.

Write by $IE_r{}^{*,*}$ the spectral sequence induced from

$$1 \longrightarrow U(6) \longrightarrow U(13456) \longrightarrow U(1345)' \longrightarrow 1 \tag{8.16}$$

and write by $i : U(13456) \subset U$ the usual inclusion. Since $U(13456) \cong E.E$ the extra-special p-group of order p^5, we know the spectral sequence $[(y_1y_{41})^{-1}]IE_r{}^{*,*}$ well from Section 3.

Lemma 8.17. *The following map i^* of spectral sequences is injective;*

$$i^* : [y_1v_4{}^{-1}]\mathrm{Im}(1')E_r{}^{*,*} \longrightarrow [(y_1y_{41})^{-1}]IE_r{}^{*,*}.$$

Proof. First recall $\mathrm{Im}(1')E_2{}^{*,*} = Q \otimes K \otimes Z/p[y_6]$. The differntial $d_2(y_6)$ is contained in Image (i^*) and $Q \otimes K$ is a free $\wedge(z_5, x_3')$ -module. We can easily see the lemma for $r = 3$, by using the fact

$$\mathrm{Im}(1')E_3{}^{*,0} = \mathrm{Im}(1')E_2{}^{*,0}/(d_2(y_6) = y_1z_5 + \cdots).$$

For $3 < r < 2p-1$, the statement in the lemma is correct, since $i^*d_r = 0$ so $d_r = 0$. For $r = 2p-1$, in $IE_r{}^{*,*}$ the non zero differential is just the Kudo's transgression. In $IE_r{}^{*,*}$, $w(1) = y_1y_{51} + y_4y_{43}$ is a non-zero-divisor, and so it is also in $\mathrm{Im}(1')E_r{}^{*,*}$. Thus we want to see that

$$i^* : \mathrm{Im}(1')E_r{}^{*,*}/(w(1)) \longrightarrow IE_r{}^{*,*}/(w(1)).$$

is injective. For this, it is sufficient to prove that if $w(1)a$ in $\mathrm{Im}(1')E_r{}^{*,0} = Q \otimes K/(d_2(y_5))$ for $a \in IE_r{}^{*,0}$, then $a \in Q \otimes K$. Here we note the fact that $i^*(Q \otimes$

$K) = H^*(U(1345)')^{U(2)}$ the invariant ring under a_2. This is proved by the facts that $Z/p[y_1, y_4]^{U(2)} = Z/p[y_1, v_4]$ and $i^*(k_5) = y_5 + \cdots$. If $w(1)a \in Q \otimes K$, then $w(1)a$ is invariant under $a_2{}^*$, hence $w(1)(a_2{}^* - 1)a = 0$. Since $w(1)$ is non-zero-divisor, $(a_2{}^* - 1)a = 0$ and this means a is in the invariant ring $Q \otimes K$. Using similar arguments for larger r, we can prove the lemma. □

We will study $\mathrm{Im}(1')E_r{}^{*,*}$ more explicitly. Hereafter we work only in $\mathrm{Im}(1')$ or in the restriction to $U(13456)$.

Lemma 8.18. $d_2(y_6) = y_1 z_5 - k_5 x_1 + y_3 z_4$.

Proof. The group $U(13456)$ is isomorphic to the extra special p-group with order p^5 and exponent p. Hence

$$\begin{aligned} d_2(y_6) &= z(1) = y_1 x_5 - y_5 x_1 + y_3 x_4 - y_4 x_3 \\ &= y_1(x_5 - (y_4/y_1)x_3) - (y_5 - (y_4/y_1)y_3)x_1 + y_3(x_4 - (y_4/y_1)x_1). \end{aligned}$$ □

Lemma 8.19. $z(2) = y_1{}^p z_5 + v_4 x_3 - (v_5 + k_5 y_3{}^{p-1})x_1 + y_3{}^p z_4,$

$$w(1) = y_1{}^p k_5 + v_4 y_3 - (v_5 + k_5 y_3{}^{p-1})y_1.$$

Proof. Since $\mathcal{B}(z(2)) = w(1)$, we only need to compute $z(2)$. Applying $\mathcal{P}^1$ to $z(1)$

$$\begin{aligned} z(2) &= \mathcal{P}^1 z(1) = \mathcal{P}^1(y_1 z_5 - k_5 x_1 + y_3 z_4) \\ &= y_1{}^p z_5 + y_1 \mathcal{P}^1(z_5) - \mathcal{P}^1(k_5)x_1 + y_3{}^p z_4 + y_3 \mathcal{P}^1(z_4). \end{aligned}$$

Here $\mathcal{P}^1(z_5) = \mathcal{P}^1(x_5 - (y_4/y_1)x_3) = -\mathcal{P}^1(y_4/y_1)x_3$. Since $\mathcal{P}^1(y^{-1}) = -y^{p-2}$, we get

$$\mathcal{P}^1(y_4/y_1) = y_4{}^p/y_1 - y_4 y_1{}^{p-2} = y_{41}/y_1.$$

Similarly $\mathcal{P}^1(z_4) = \mathcal{P}^1(x_4 - (y_4/y_1)x_1) = -(y_{41}/y_1)x_1$. Next compute

$$\begin{aligned} \mathcal{P}^1(k) &= \mathcal{P}^1(y_5 - (y_4/y_1)y_3) = y_5{}^p - (y_{41}/y_1)y_3 - (y_4/y_1)y_3{}^p \\ &= y_{53} - (y_{41}/y_1)y_3 + y_5 y_3{}^{p-1} - (y_4/y_1)y_3{}^p = v_5 - (y_{41}/y_1)y_3 + k_5 y_3{}^{p-1}. \end{aligned}$$

Hence

$$\begin{aligned} z(2) &= y_1{}^p z_5 - y_1(y_{41}/y_1)x_3 \\ &\quad -(v_5 - y_{41}y_3/y_1 + k_5 y_3{}^{p-1})x_1 + y_3{}^p z_4 - y_3(y_{41}/y_1)x_1 \\ &= y_1{}^p z_5 - v_4 x_3 - (v_5 + k_5 y_3{}^{p-1})x_1 + y_3{}^p z_4. \end{aligned}$$ □

Since $y_1{}^{p-1}z(1) = y_1{}^pz_5 - y_1{}^{p-1}k_5x_1 + y_1{}^{p-1}y_3z_4$, we have with modulo $(z(1))$

$$\begin{aligned}(8.20)\qquad z(2) &= (y_1{}^{p-1} - y_3{}^{p-1})k_5x_1 - v_4x_3 - v_5x_1 + y_{31}z_4\\ w(1) &= (y_1{}^{p-1} - y_3{}^{p-1})k_5y_1 - v_4y_3 - v_5y_1.\end{aligned}$$

Moreover, modulo $(w(1))$, we can make the change

$$(8.21)\qquad z(2) - (w(1)/y_1)x_1 = -v_4(x_3 - y_3x_1/y_1) + y_{31}z_4.$$

To compute $\mathcal{P}^pz(2)$, we prepare

$$\mathcal{P}^p(y_4/y_1) = y_4{}^py_1{}^{p^2-2p} - y_4y_1{}^{p^2-p-1} = y_{41}y_1{}^{p^2-2p}$$

since $\mathcal{P}^{p-1}(y^{-1}) = y^{p^2-2p}$ and $\mathcal{P}^p(y) = -y^{p^2-p-1}$. Hence

$$\mathcal{P}(z_5) = \mathcal{P}^p(x_5 - (y_4/y_1)x_3) = -y_{41}y_1{}^{p^2-2p}x_3.$$

Similarly $\mathcal{P}^p(z_4) = -y_{41}y_1{}^{p^2-2p}x_1$. The action for k_5 is

$$\begin{aligned}\mathcal{P}^p(k_5y_3{}^{p-1}) &= \mathcal{P}^p(y_5y_3{}^{p-1} - (y_4/y_1)y_3{}^p)\\ &= (y_5y_3{}^{p-1})^p - y_{41}y_1{}^{p^2-2p}y_3{}^p - (y_4/y_1)y_3{}^{p^2}.\end{aligned}$$

Lemma 8.22.
$$\begin{aligned}z(3) &= \mathcal{P}^pz(2)\\ &= y_1{}^{p^2}z_5 - v_4y_1{}^{p^2-p}x_3 + v_4{}^px_3 - v_5{}^px_1\\ &\quad - (v_5y_3{}^{p^2-p} + k_5y_3{}^{p^2-1})x_1 + y_3{}^{p^2}z_4.\end{aligned}$$

and $w(2) = y_1{}^{p^2}k_5 - v_4y_1{}^{p^2-p}y_3 + v_4{}^py_3 - v_5{}^py_1 - (v_5y_3{}^{p^2-p} + k_5y_3{}^{p^2-1})y_1$.

Proof. The $\mathcal{P}^p$ action for $z(2)$ is

$$\begin{aligned}\mathcal{P}^pz(2) &= y_1{}^{p^2}z_5 + y_1{}^p(-y_{41}y_1{}^{p^2-2p}x_3) + v_4{}^px_3 - v_5{}^px_1\\ &-(y_5{}^py_3{}^{p^2-p} - y_{41}y_1{}^{p^2-2p}y_3{}^p - (y_4/y_1)y_3{}^p)x_1 + y_3{}^{p^2}z_4 - y_3{}^py_{41}y_1{}^{p^2-2p}x_1.\end{aligned}$$

The sum of the above line gives

$$-(y_{51}y_3{}^{p^2-p} + y_5y_3{}^{p^2-1} - (y_4/y_1)y_3{}^{p^2})x_1 + y_3{}^{p^2}z_4. \qquad \square$$

Let $v_6(2)$ be an element such that $v_6(2)|\widetilde{U}(6) = y_6{}^{p^2}$

Theorem 8.23. $[y_1{}^{-1}, v_4{}^{-1}]H^*(\widetilde{U})\{1'\}$
$\cong Z/p[y_1, y_1{}^{-1}, v_4, v_4{}^{-1}, k_5, y_3, v_6(2)] \otimes \wedge(x_1, z_4)/(w(1), w(2))$
where $v_5 = k_5{}^p - v_4(y_3/y_1)^p + k_5y_3{}^{p-1}$ *in* $w(1)$ *and* $w(2)$.

Proof. We only need to see

$$\begin{aligned} k_5{}^p &= (y_5 - (y_4/y_1)y_3)^p \\ &= y_5{}^p - y_5 y_3{}^{p-1} + y_5 y_3{}^{p-1} - ((y_4{}^p - y_4 y_1{}^{p-1})/y_1{}^p)y_3{}^p - (y_4/y_1)y_3{}^p \\ &= v_5 - v_4(y_3/y_1)^p + k_5 y_3{}^{p-1} \end{aligned}$$

□

For the study of $[y_1{}^{-1}, y_2{}^{-1}]H^*(M)$, we study first $[y_2{}^{-1}]H^*(M)$. By arguments similar to those of the case $[y_1{}^{-1}]H^*(M)$, we get

$$[y_2{}^{-1}]H^*(U(124)) \cong Z/p[y_2{}^{-1}, y_2, y_1, v_4]/(y_{12}) \otimes \wedge(x_2, z_4)$$

Proposition 8.24. $[y_2{}^{-1}]H^*(M) \cong R'[y_1, y_3]/(y_{12}, y_{31})$
$\cong R' \otimes Z/p\{1, y_1, \cdots, y_1{}^{p-1}\} \otimes Z/p\{1, y_3, \cdots, y_3{}^{p-1}\}$
where $R' = Z/p[y_2, y_2{}^{-1}, v_4, v_5] \otimes \wedge(x_2, z_4, z_5)$.

Proof. Consider the central extension

$$1 \longrightarrow \langle a_5 \rangle \longrightarrow M \longrightarrow U(124) \oplus U(3) \longrightarrow 1. \tag{8.25}$$

The facts that $[y_2{}^{-1}]H^*(U(124) \oplus U(3))$ is $Z/p[y_2, y_2{}^{-1}, y_3] \otimes \wedge(x_3)$-free and that $d_3 y_5 = y_2 x_3 - y_3 x_2$ prove the proposition. □

Next consider the spectral sequence

$$[(y_1y_2)^{-1}]E_2{}^{*,*} \cong [(y_1y_2)^{-1}]H^*(M; H^*(\widetilde{U}(6)) \Longrightarrow [(y_1y_2)^{-1}]H^*(\widetilde{U}). \tag{8.26}$$

To study $d_3(y_6)$, we first consider the theory without any localization. In the spectral sequence induced from

$$1 \longrightarrow U(4) \longrightarrow U(124) \longrightarrow U(1) \oplus U(2) \longrightarrow 1,$$

the element $[x_2x_4] \in E_2{}^{*,*}$ is permanent since $d_2(x_4) = x_1x_2$, Write by x_{24} the element $[x_2x_4]$ in $H^2(U(124))$ and by $z_4{}'$ its Bockstein image. Similarly we can define x_{25} and $z_5{}'$ in $H^3(U(235))$ such that $\mathcal{B}(x_{25}) = z_5{}' = y_2z_5$ in $[y_2{}^{-1}]H^*(M)$.

Here we recall the weight defined by the action of diagonal elements [11]. Namely the weight $w(x) \in Z/(p-1)\{\alpha, \beta, \gamma\}$ is defined by

$$wt(x_1) \text{ (resp. } x_2, x_3, x_4, x_5, x_6) = \alpha \text{ (resp. } \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma).$$

The weight has the properties $wt(x_i) = wt(y_i)$ and $wt(yz) = wt(y) + wt(z)$.

We can show that the weight space $H^4(M)_{\alpha+2\beta+\gamma} = Z/p\{y_3x_{24}, y_1x_{25}\}$. For dimensional reasons, 4-dimensional elements are generated by $y_4y_5, y_1x_2x_5, y_3x_2x_4$ in the spectral sequence of

$$1 \longrightarrow U(4) \oplus U(5) \longrightarrow M \longrightarrow U(1) \oplus U(2) \oplus U(3) \longrightarrow 1.$$

But y_4y_5 is not a permanent cycle.

Since $wt(y_2z_6) = \alpha + 2\beta + \gamma$, we get $d_2(y_2z_6) = y_3x_{24} + y_1x_{25}$ in the spectral sequence converging to $H^*(U)$. Applying the Bockstein, $d_3(y_2y_6) = y_3z_4' + y_1z_5'$. Therefore we have

Lemma 8.27. $d_3(y_6) = y_3z_4 + y_1z_5$ *in the spectral sequence* (8.26).

We will study $\mathcal{P}^1(z_4')$. Let $A_i = \langle a_2a_1{}^i, a_4\rangle \subset U(124)$ for $0 \le i \le p-1$. Then $y_2|A_i = y$, $y_1|A_i = iy$ and $v_4|A_i = y_4{}^p - y^{p-1}y_4$ after the identification $H^*(A_i) = Z/p[y_4, y] \otimes \wedge(x_4, x)$.

Lemma 8.28. *If the restricted image* $x|A_i = 0$ *for all* $0 \le i \le p-1$, *then* $x = 0$ *in* $[y_2{}^{-1}]H^*(U(124))$.

Proof. We will prove the case $x = f(y_1, y_2) \in Z/p[y_1, y_2]$. Other cases are proved similarly. Since $x|A_i = f(iy, y) = 0$ in $Z/p[y]$, we see that $x = f(y_1, y_2)$ divides $y_1 - iy_2$, so divides $y_{12} = \Pi_{i\in z/p}(y_1 - iy_2)$, which is zero in $[y_2{}^{-1}]H^*(U(124))$. □

Lemma 8.29. $\mathcal{P}^1 z_4' = y_2{}^{p-1}z_4' - v_4x_2$.

Proof.
$$\begin{aligned}\mathcal{P}^1 z_4|A_i &= \mathcal{P}^1(yx_4 - y_4x) = y^px_4 - y_4{}^px \\ &= y^{p-1}(yx_4 - y_4x) - (y_4{}^p - y_4y^{p-1})x.\end{aligned}$$

From Lemma 8.28, we get the lemma. □

Lemma 8.30. $z(2) = y_2{}^{p-1}(y_1z_5' + y_3z_4') - y_2{}^{-1}(y_1v_5 + y_3v_4)x_2$
$w(1) = \beta(z(2)) = -y_1v_5 - y_3v_4$.

Proof. Compute the following

$$\begin{aligned}\mathcal{P}^1(y_2{}^{-1}(y_1z_5' + y_3z_4')) &= -y_2{}^{p-2}(y_1z_5' + y_3z_4') + y_2{}^{-1}(y_1{}^pz_5' + y_3{}^pz_4') \\ &\quad + y_2{}^{-1}y_1(y_2{}^{p-1}z_5' - v_5x_2) + y_2{}^{-1}y_3(y_2{}^{p-1}z_4' - v_4x_2) \\ &= y_2{}^{-1}(y_1{}^pz_5' + y_3{}^pz_4') - y_2{}^{-1}(y_1v_5 + y_3v_4)x_2.\end{aligned}$$

Using the facts that $y_1{}^p = y_2{}^{p-1}y_1$, $y_3{}^p = y_2{}^{p-1}y_3$, we get the lemma. □

Since $z(2) = 0 \bmod (z(1), w(1))$, we have that $E_{2p-1}{}^{*,*} = E_\infty{}^{*,*}$ for the spectral sequence (8.26). Therefore we get

Theorem 8.31. $[y_1{}^{-1}, y_2{}^{-1}]H^*(\widetilde{U})$
$\cong Z/p[y_1, y_1{}^{-1}, y_2, y_2{}^{-1}, y_3, v_4, v_6] \otimes \wedge(x_2, z_4)/(y_1{}^{p-1} - y_2{}^{p-1}, y_{32})$

Theorem 8.32. $[y_1{}^{-1}, y_2{}^{-1}]H^*(U) \cong [y_1{}^{-1}, y_2{}^{-1}]H^*(\widetilde{U}) \otimes \wedge(z_6)$
$[y_1{}^{-1}, v_4{}^{-1}]H^*(U)\{1'\} \cong [y_1{}^{-1}, v_4{}^{-1}]H^*(\widetilde{U}) \otimes \wedge(z_6)\{1'\}$

Proof. First note that $d_2(z_6)$ is

$$\begin{aligned} x_1x_5 + x_3x_4 &= x_1(x_5 - (y_4/y_1)x_3) + x_3(x_4 - (y_4/y_1)x_1) \\ &= x_1z_5 + x_3z_4 \quad \text{in} \quad [y_1{}^{-1}]H^*(\widetilde{U}(13456)). \end{aligned}$$

Since $wt(z_6) = \alpha+\beta+\gamma$, for dimensional reasons x_2 and y_2 do not appear in $d_2(y_6)$. Hence $d_2(z_6) = x_1z_5 + x_3z_4$ also in $[y_1{}^{-1}]H^*(\widetilde{U})$. For the case with $[y_1{}^{-1}, y_2{}^{-1}]$, we get $y_2(x_1z_5 + x_3z_4) = x_2(y_1z_5 + y_3z_4) = 0$. For the case $[(y_1v_4)^{-1}]\mathrm{Im}\{1'\}$, we have

$$\begin{aligned} y_1(x_1z_5 + x_3z_4) &= x_1(k_5x_1 - y_3z_4) + y_1x_3z_4 \\ &= (-x_1y_3 + y_1x_3)z_4 = 0 \quad \text{from} \quad (8.21). \end{aligned}$$ □

9. Brown-Peterson cohomology theory

Let $BP^*(-)$ (resp. $K(m)^*(-)$) be the Brown-Peterson cohomology theory (resp. the Morava K-theory) with the coefficient $BP^* = Z_{(p)}[v_1, \cdots]$ (resp. $K(m)^* = Z/p[v_m, v_m{}^{-1}]$). For any compact Lie group G, it was conjectured in [9] that

$$BP^{\mathrm{odd}}(BG) = 0 \quad \text{and} \quad K(m)^{\mathrm{odd}}(BG) = 0.$$

However I. Kriz [10] claims that $K(m)^{\mathrm{odd}}(BU_4) \neq 0$ for the Sylow p-subgroup U_4 of $GL_4(F_p)$. In this section we cosider the mod p BP-theory $P(1)^*(-) = BP^*(-; Z/p)$ and show that $P(1)^{\mathrm{odd}}(BU_4)$ is zero with some localization.

We also recall the theory $P(m)^*(-)$ with the coefficient $P(m)^* = Z/p[v_m, v_{m+1}, \cdots]$.

Theorem 9.1. *There is a filtration such that*

$$gr[e_n{}^{-1}]P(m)^*(B\widetilde{E}_n) \cong [e_n{}^{-1}]P(m+n)^* \otimes S(n)[u^{p^n}].$$

Proof. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2{}^{*,*} = [e_n{}^{-1}]H^*(B\widetilde{E}_n; P(m)^*) \Longrightarrow [e_n{}^{-1}]P(m)^*(B\widetilde{E}_n).$$

Here we recall $[e_n{}^{-1}]H^*(B\widetilde{E}_n) = [e_n{}^{-1}]S(n)[u^{p^n}]\otimes\wedge(x_1,\cdots,x_{2n-1})$. First non-zero differential is

$$d_{2p^m-1}(x_i) = v_m \otimes Q_m(x_i) = v_m y_i{}^{p^m} \quad ([9],[10]).$$

where Q_m is the Milnor primitive operation inductively defined by $Q_0 = \beta$, $Q_m = \mathcal{P}^{p^{m-1}}Q_{m-1} - Q_{m-1}\mathcal{P}^{p^{m-1}}$. Let us write $x_i{}' = x_i - (y_i/y_1)^{p^m}x_1$. Then $d_{2p^m-1}(x_i{}')$ $= 0$ and $\wedge(x_1, x_3{}', \cdots, x_{2n-1}{}') = \wedge(x_1,\cdots,x_{2n-1})$. Hence we have

$$E_{2p^m-1}{}^{*,*} = [e_n{}^{-1}]P(m+1)\otimes S(n)[u^I]\otimes\wedge(x_3{}',\cdots,x_{2n-1}{}').$$

We can continue this argument for d_{2p^s-1} for all $s > m$. Let B' be the matrix whose (i,k) entry $(y_{2k-1}{}^{p^{m+i-1}}) = (Q_{m+i-1}x_{2k-1})$. By multiplying an upper trianglar matrix D with diagonal entries one from right, we can change B' to a lower triangular matrix B'', i.e. $B'D = B''$ since $|B'| = ((-1)^n e)^{p^{n-1}/(p-1)}$. Let us write $D = (d_{ij})$. Then (i,j)-entry of $B'D = B''$ is

$$\sum_k Q_{m+i-1}(x_{2k-1})d_{kj} = Q_{m+i-1}\left(\sum_k d_{kj}x_{2k-1}\right)$$

since $Q_j(d_{is}) = 0$. Let $(x_1{}',\cdots,x_{2n-1}{}') = (x_1,\cdots,x_{2n-1})D$. Then we have

$$Q_{m+1}(x_{2s-1}{}') = \begin{cases} (Y_{i,2i-1})^{p^m} & \text{for} \quad s = i \quad \text{(see Lemma 2.5)} \\ 0 & \text{for} \quad i < s. \end{cases}$$

Thus we get

$$[e^{-1}]E_{2p^{m+n}}{}^{*,*} \cong [e^{-1}]P(m+n)^*\otimes S(n)[u^{p^n}].$$

This term is even dimensionally generated and hence is isomorphic to the infinite term. □

Recall the statements and the notations in Theorem 8.23 and Theorem 8.31.

Theorem 9.2. *There is a filtration such that*

(i) $gr[(y_1v_4)^{-1}]P(1)^*(BU_4)\{1'\}$
$\cong P(3)^*[y_1, y_1{}^{-1}, v_4, v_4{}^{-1}, k_5, y_3, v_6(2)]/(w(1), w(2), v_3v_6(2)^p)$

(ii) $gr[(y_1v_4)^{-1}, y_2]P(1)^*(BU_4)$
$\cong P(3)^*[y_1, y_1{}^{-1}, y_2, y_2{}^{-1}, y_3, v_4, v_4{}^{-1}, v_6]$
$/(y_1{}^{p-1} - y_2{}^{p-1}, y_{31}, y_{32}, v_3v_6(2)^p)$ with $v_6(2) = v_6{}^p - y_2{}^{p(p-1)}v_6$.

By arguments similar to the proof of Theorem 9.1, we can easily prove the theorem if we can show

(9.3) $\qquad Q_2z_6 = v_6(2)$ in both the cases $\mathrm{Im}\{1'\}$ and $\mathrm{Ker}\{1'\}$.

At first, we study the $[(y_1v_4)^{-1}]\mathrm{Image}\{1'\}$.

Lemma 9.4. $z_6|U(146) = x_6 - (y_{64}/y_{14})x_1 - (y_{61}/y_{41})x_4$.

Proof. Let us write the restriction as

$$z_6|U(146) = x_6 + b_1x_1 + b_4x_4$$

with $b_1 \in Z/p[(y_1y_{41})^{-1}, y_1, y_4, y_6]$. The element $z_6|U(146)$ is invariant under the action $a_5{}^*$ induced from the element a_5 in $U(5)$. Since

$$a_5{}^*z_6|U(146) = x_6 + x_1 + (a_5{}^*b_1)x_1 + (a_5{}^*b_4)x_4,$$

we have $(a_5{}^*-1)b_1 = -1$ and $(a_5{}^*-1)b_4 = 0$. By the action $a_3{}^*$, we also know $(a_3{}^*-1)b_4 = -1$ and $(a_3{}^*-1)b_1 = 0$. Since $Z/p[y_1, y_6]^{U(5)} = Z/p[y_1, y_{61}]$, we have

$$b_1 = -(y_{64}/y_{14})^s \quad \text{and} \quad b_4 = -(y_{61}/y_{41})^t.$$

We will prove $s = t = 1$ by showing that $y_1y_{41}x_6$ is permanent without any localization in the spectral sequence

$$E_2{}^{*,*} = H^*(\tilde{U}(13456)) \otimes \wedge(x_6) \Longrightarrow H^*(U(13456)).$$

From (2.6), we know in $H^*(\tilde{U}(13456))$,

$$y_1x_5 = 0 \bmod (x_1, x_3, x_4) \quad \text{and} \quad y_{41}x_3 = 0 \bmod (x_1, x_4).$$

Hence $d_2(y_1y_{41}x_6) = y_1y_{41}(x_1x_5 + x_3x_4) = 0$. □

Corollary 9.5. $\mathcal{B}Z|U(146) = 0$.

Lemma 9.6. $Q_2z_6|U(146) = -y_6{}^{p^2} \bmod \{y_6{}^i|i < p^2\}$,

Proof. First we note that

$$\mathcal{P}^1y_{41} = \mathcal{P}^1(y_4{}^p - y_1{}^{p-1}y_4) = y_1{}^{2p-2}y_4 - y_1{}^{p-1}y_4{}^p = -y_1{}^{p-1}y_{41}.$$

Since $0 = \mathcal{P}^1(y_{41}y_{41}{}^{-1}) = (-y_1{}^{p-1}y_{41})(y_{41}{}^{-1}) + y_{41}\mathcal{P}^1(y_{41}{}^{-1})$, we get $\mathcal{P}^1(y_{41}{}^{-1}) = y_1{}^{p-1}y_{41}/(y_{41})^{-2} = y_1{}^{p-1}/y_{41}$. Therefore $\mathcal{P}^1(y_{64}/y_{14}) = (-y_4{}^{p-1}y_{64}/y_{14} + y_{64}y_4{}^{p-1}/y_{14}) = 0$. From Lemma 9.4, we have

$$\mathcal{P}^1z_6|U(146) = -\mathcal{P}^1(y_{64}/y_{14})x_1 - \mathcal{P}^1(y_{61}/y_{41})x_4 = 0.$$

Hence $Q_1z_6|U(146)=0$. Thus with $\mathrm{mod}\{{y_6}^i|i<p^2\}$, we get

$$\begin{aligned} Q_2z_6|U(146) &= -Q_1\mathcal{P}^p z_6|U(146) \\ &= ({y_{64}}^p/y_{14})Q_1x_1 + ({y_{61}}^p/y_{41})Q_1x_4 \\ &= {y_6}^{p^2}({y_1}^p/(y_{14}) + {y_4}^p/(y_{41})) = {y_6}^{p^2}. \end{aligned} \qquad \square$$

Next consider the case with localization $[{y_2}^{-1}]$. Recall the subgroups A_i of $U(124)$ and Lemma 8.27, In $[(y_1y_2)^{-1}]H^*(U(124))$, the element x_{24} defined before Lemma 8.23 is expressed by x_2z_4 because its restrictions to A_i are all x_2x_4. Since $d_2(y_2x_6)=(y_1x_{25}+y_3x_{24})$ in $H^*(M)$, we get

$$\begin{aligned} d_2({y_2}^2x_6) &= y_2(y_1x_{25}+y_3x_{24}) \\ &= y_2(y_1x_2z_5+y_3x_2z_4) = x_2(y_1{z_5}'+y_3z_4') = 0 \end{aligned}$$

in $H(\widetilde{U})$. Therefore ${y_2}^2x_6$ is permanent. Hence

$$z_6|U(1246) = x_6 - (y_6/y_2)x_2 + (by_6/{y_2}^2)z_4$$

since $z_6|U(124)=0$ and $z_6|U(1246)$ is invariant under the action ${a_5}^*$. But, by considering the degree and weight, $b=0$. Thus we have

Lemma 9.7. $z_6|U(1246) = x_6-(y_6/y_2)x_2$.

Hence $\mathcal{B}z_6|U(1246)=0$ and $\mathcal{P}^1z_6|U(1246)=(-y_{62}/y_2)x_2$. Therefore $Q_1z_6 = -v_6$.

Lemma 9.8. $Q_2z_6|U(1246) = {y_{62}}^p - y_{62}{y_2}^{p(p-1)}$.

Proof. The left hand side of the above formula is

$${y_{62}}^p - Q_1\mathcal{P}^p(y_6/y_2)x_2 = {y_{62}}^p - y_{62}{y_2}^{p^2-2p}{y_2}^p. \qquad \square$$

Last, we note the case $p=2$. When $p=2$, the situation is quite different. However Theorem 9.1 also holds in this case. The cohomology of extra-special 2-groups are completely determined by Quillen [13], in particular

$$H^*(D_n) = {S_{2n}}'/J \otimes Z/2[z^{2^n}]$$

where D_n is the central product of the dihedral group D of order 8, ${S_{2n}}' = Z/2[x_1,\cdots,x_{2n}]$, $J=(f, Sq^1f,\cdots,Sq^{2^{n-2}}\cdots Sq^1f)$, $f=\sum x_{2i-1}x_{2i}$, and $|z|=1$.

Let us write ${x_i}^2$ (resp. ${z_i}^2$) by y_i (resp. u) and use the filtration by y_i. Then we have

$$grH^*(D_n) = S_{2n} \otimes \wedge_{2n}/(z(1), \cdots, z(n-1), w(1), \cdots, w(n-1), f, f^2) \otimes Z/2[u^{2^{n-1}}]$$
$$grH^*(\widetilde{D}_n) = S_{2n} \otimes \wedge_{2n}/(z(1), \cdots, z(n), w(1), \cdots, w(n)) \otimes Z/2[u^{2^n}].$$

Therefore Theorem 3.9 and Theorem 9.1 also hold for $p = 2$.

Proposition 9.9. $[{e_n}^{-1}]grH^*(\widetilde{D_n}) \cong [{e_n}^{-1}]S(n) \otimes \wedge(x_1, \cdots, x_{2n-1}) \otimes Z/2[u^{2^n}]$

$$[{e_n}^{-1}]grP(m)^*(B\widetilde{D_n}) \cong [{e_n}^{-1}]P(m+n)^* \otimes S(n) \otimes Z/2[u^{2^n}].$$

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