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A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF NON-SINGULAR G-VECTOR FIELDS ON G-MANIFOLDS

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1. Introduction

Throughout this paper G always denotes a compact Lie group and G-manifolds are smooth manifolds with smooth G-actions. In this paper we will give a necessary and sufficient condition for the existence of non-singular G-vector fields on closed G-manifolds.

Let $E \rightarrow X$ be a G-vector bundle. After choosing a G-invariant Riemannian metric on $E$, we denote by $S(E) \rightarrow X$ the associated G-sphere bundle of $E$. We abbreviate continuous G-equivariant cross section of $E$ to G-cross section of $E$.

Let $M$ be a compact G-manifold, and $s: M \rightarrow \tau(M)$ a G-cross section of the tangent bundle $\tau(M)$ of $M$. $s$ is called a non-singular G-vector field on $M$, if $s$ is not zero at each point of $M$. For a positive integer $k$, by a G-k-field on $M$ we will mean $k$ G-cross sections of $\tau(M)$ which are linearly independent at each point of $M$.

For a closed subgroup $H$ of $G$ we set $M_H = \{ x \in M | G_x = H \}$, where $G_x$ is the isotropy group at $x$. Let $N(H)$ be the normalizer of $H$ in $G$. Then $M_H$ is an $N(H)$-manifold and also a free $N(H)/H$-manifold.

For a topological space $X$ we define $|\chi|(X)$ to be the sum $\sum \chi(Y)$, where $\chi(Y)$ is the Euler characteristic of $Y$ and $Y$ runs over the connected components of $X$.

In this paper we will obtain the following results:

Theorem 1.1. Let $M$ be a compact G-manifold. Let $s: \partial M \rightarrow S(\tau(\partial M))$ be a G-cross section of $S(\tau(\partial M))$. Then $s$ is extendible to a G-cross section of $S(\tau(M))$ if and only if

$$|\chi|(M_{G_x} N(G_x)) = 0, \quad \text{or} \quad \dim N(G_x) - \dim G_x > 0$$

for all $x \in M$. 
From this theorem we immediately obtain the main result of this paper:

**Corollary 1.2.** A closed $G$-manifold $M$ admits a non-singular $G$-vector field if and only if

$$|\chi|(M_{G_x}N(G_x)) = 0, \quad \text{or} \quad \dim N(G_x) - \dim G_x > 0$$

for all $x \in M$.

We also obtain the following corollary:

**Corollary 1.3.** Let $M$ be a compact $G$-manifold, $F$ the stationary point set of $M,$ and $k$ a positive integer. We suppose that

$$\dim N(G_x) - \dim G_x \geq k$$

for all $x \in M - F$. Then $M$ admits a $G$-$k$-field if and only if $F$ admits a $k$-field.

**Note.** The existence of a non-singular continuous $G$-vector field on a compact $G$-manifold implies the existence of a non-singular smooth $G$-vector field. This fact is assured by the differentiable approximation theorem [3; (6.7)] and the usual process of averaging cross sections.

### 2. Preliminaries

(2-1) For a closed subgroup $H$ of $G$ let $(H)$ be the conjugacy class of $H$ in $G$. If $H$ is an isotropy group occurring on a $G$-manifold $M$, $(H)$ is called an isotropy type on $M$.

Set

$$M_H = \{x \in M | G_x = H\}$$

$$M_{(H)} = \{x \in M | (G_x) = (H)\}.$$ 

Then $M_H$ and $M_{(H)}$ are submanifolds of $M$, but, in general, not compact. If $(H)$ is a maximal isotropy type on a compact $G$-manifold $M$, $M_H$ and $M_{(H)}$ are compact.

(2-2) Let $\pi: E \to X$, $\pi': E' \to X'$ be $G$-fibre bundles, and $f: E \to E'$ a $G$-bundle map covering $f: X \to X'$. Let $s': X' \to E'$ be a $G$-cross section of $E'$. $s'$ induces a $G$-cross section

$$s: X \to f^*E' = \{(x, e) \in X \times E' | f(x) = \pi'(e)\}$$

of the induced $G$-fibre bundle $f^*E'$ which sends $x \in X$ to $(x, s'f(x)) \in X \times E'$. $E'$ is canonically isomorphic to $f^*E'$. So $s$ induces a $G$-cross section of $E$. We denote this $G$-cross section by $f^*s'$, and call induced $G$-cross section from $s'$ by $f$.

(2-3) Recall known results:
Proposition 2.1 (Segal [2; Proposition 1.3]). Let $X, Y$ be $G$-spaces, and $X$ compact. If $f_0, f_1: X \to Y$ are $G$-homotopic $G$-maps, and $E \to Y$ is a $G$-vector bundle, then there is a $G$-bundle isomorphism

$$\varphi: f_*^*E \cong f_1^*E.$$ 

From this proposition we easily obtain the following. $I$ denotes the interval $[0,1]$ with trivial $G$-action.

Proposition 2.2. If $X$ is a compact $G$-space, then any $G$-vector bundle $E$ over $X \times I$ is isomorphic to $(E | X \times \{0\}) \times I$ as $G$-vector bundles.

Proposition 2.1 may be stated in a more precise form as the following: If $f_0, f_1$ are $G$-homotopic relative to a closed $G$-invariant subspace $A$ of $X$, and if we consider $f_0^*E, f_1^*E$ subspaces of $X \times E$, then the $G$-bundle isomorphism $\varphi$ satisfies $\varphi(x, e) = (x, e)$ for all $x \in A$ and $e \in E$.

From this fact we obtain

Proposition 2.3. Let $E_i \to X_i$, $i=1,2$, be $G$-vector bundles with $X_1$ compact. Let $f: X_1 \times I \to X_2$ be a $G$-homotopy which is constant on a closed $G$-invariant subspace $A$ of $X_1$. Let $f_0: E_1 \to E_2$ be a $G$-bundle map over $f_0 = f | X_1 \times \{0\}$. Then there is a $G$-bundle map $f: E_1 \times I \to E_2$ over $f$ which is a homotopy of $f_0$ and is constant on $E_1 \times A$.

Corollary 2.4. Let $E \to X$ be a $G$-vector bundle with $X$ compact, $A$ a closed $G$-invariant subspace of $X$, and $i: A \to X$ the inclusion map. Let $f: X \to A$ be a $G$-map such that $f$ is $G$-homotopic to the identity of $X$ relative to $A$. Then there is a $G$-bundle map $\tilde{f}: E \to E \mid A$ over $f$ which is the identity on $E \mid A$.

(2-4) The following result has been obtained by U. Koschorke in his paper [1; §1].

Proposition 2.5. Let $M$ be a compact manifold, and $s: \partial M \to S(\tau(\partial M))$ a cross section of $S(\tau(\partial M))$. Then $s$ is extendible to a cross section of $S(\tau(M))$ if and only if $|X\mid(M) = 0$.

3. Proof of Theorem 1.1

The proof will proceed by an induction for the number of isotropy types on $M$.

(3-1) In the first place, let $M$ be of one isotropy type, and let $(H)$ be the isotropy type on $M$.

"if" part: $M_H$ is a compact $N(H)$-manifold with boundary $M_H \cap \partial M$. Consider the sphere bundle
The $G$-cross section $s$ induces a cross section

$$s_1: \partial(M_H/N(H)) \to S(\tau(M_H))/N(H) \mid \partial(M_H/N(H)).$$

This is assured by the $G$-equivariancy of $s$ and the fact

$$\partial(M_H/N(H)) = \partial M_H/N(H) = M_H \cap \partial M/N(H).$$

We may extend $s_1$ to a cross section of $S(\tau(M_H))/N(H)$ as follows. If

$$\dim N(H) - \dim H > 0,$$

then the dimension as a cell complex of $M_H/N(H) = M_H/(N(H)/H)$ is less than or equal to the dimension of fibre of $S(\tau(M_H))/N(H)$. Therefore the obstruction to extending $s_1$ over $M_H/N(H)$ vanishes. If

$$\dim N(H) - \dim H = 0,$$

then

$$|\chi(M_H/N(H))| = 0$$

by the assumption, and then

$$S(\tau(M_H))/N(H) \cong S(\tau(M_H/N(H)))$$

for $N(H)/H$ is a finite group and $M_H/N(H) = M_H/(N(H)/H)$. The image of $s_1$ is in $S(\tau(\partial M_H))/N(H)$. Then $s_1$ can be extended over $M_H/N(H)$ by Proposition 2.5.

Let

$$s_2: M_H/N(H) \to S(\tau(M_H))/N(H)$$

be an extension of $s_1$. Let

$$\pi: S(\tau(M_H)) \to S(\tau(M))/N(H)$$

be the canonical projection, and let

$$\pi^*s_2: M_H \to S(\tau(M_H))$$

be the induced $N(H)$-cross section. $\pi^*s_2$ may be considered an $N(H)$-cross section of $S(\tau(M))/M_H$, since $S(\tau(M_H))$ is a subbundle of $S(\tau(M))/M_H$. Moreover $\pi^*s_2$ is an extension of $s \mid \partial M_H$. Since $M$ is of one isotropy type, the $G$-action

$$G \times (S(\tau(M))/M_H) \to S(\tau(M))$$

induces a $G$-bundle isomorphism
Then $\pi^*s_2$ induces a $G$-cross section

$$s_2: M \to S(\tau(M))$$

which is an extension of $s$.

"only if" part: Let

$$t: M \to S(\tau(M))$$

be an extension of $s$. By the $G$-equivariancy of $t$, $t$ is restricted to an $N(H)$-cross section

$$t_1: M_H \to S(\tau(M_H)) .$$

$t_1$ induces a cross section

$$t_2: M_H/N(H) \to S(\tau(M_H))/N(H) .$$

If

$$\dim N(H) - \dim H = 0 ,$$

then

$$S(\tau(M_H))/N(H) \simeq S(\tau(M_H)/N(H)) ,$$

and

$$t_2(\partial(M_H/N(H))) \subset S(\tau(\partial(M_H/N(H)))) .$$

Therefore

$$\dim N(H) - \dim H = 0$$

implies

$$|X|(M_H/N(H)) = 0$$

by Proposition 2.5.

This completes the proof for the case in which $M$ is of one isotropy type.

(3-2) Let the theorem be true for the case in which the number of isotropy types is $k - 1$. Let $M$ be a compact $G$-manifold with $k$ isotropy types.

"if" part: Let $(H)$ be a maximal isotropy type on $M$. Then $M_{(H)}$ is a compact $G$-submanifold of $M$ with one isotropy type. From the preceding argument we obtain a $G$-cross section

$$s_1: M_{(H)} \to S(\tau(M))|_{M_{(H)}}$$

such that the image of $s_1$ is in $S(\tau(M_{(H)}))$ and $s_1|_{\partial M_{(H)}} = s|_{\partial M_{(H)}}$.

Let $T(M_{(H)})$ be a closed $G$-invariant tubular neighborhood of $M_{(H)}$ in $M$. 
By Corollary 2.4 we obtain a $G$-bundle map
$$
\pi : S(\tau(M))| T(M_{(H)}) \to S(\tau(M))| M_{(H)}
$$
such that $\pi$ covers the canonical projection of $T(M_{(H)})$ to $M_{(H)}$ and $\pi$ is the identity on $S(\tau(M))| M_{(H)}$. $\pi$ induces a $G$-cross section
$$
\pi* s_1 : T(M_{(H)}) \to S(\tau(M))| T(M_{(H)})
$$
from $s_1$ such that $\pi* s_1| M_{(H)} = s_1$.

To obtain a $G$-cross section
$$
L_i = \partial M \cup T(M_{(H)}) \to S(\tau(M))| L_i
$$
extending $s$, we may apply Lemma 5.1 (stated in the last section) as follows. Apply $E \to X, A, B, D, S\ A$ and $S\ B$ in Lemma 5.1 to $\tau(M) \to M, T(M_{(H)}), \partial M, \partial M_{(H)}, \pi* s_1$ and $s$, respectively. So, in this case,
$$
C = A \cap B = T(M_{(H)}) \cap \partial M = T(M_{(H)})| \partial M_{(H)},
$$
and this is compact. Moreover this is equivariantly deformable to $\partial M_{(H)}$, and
$$
\pi* s_1| \partial M_{(H)} = s| \partial M_{(H)}.
$$
Let $K$ be a closed $G$-invariant collar of $\partial M_{(H)}$ in $M_{(H)}$. By Proposition 2.2, $T(M_{(H)})| K$ has the desired property as $U$ in Lemma 5.1. Therefore we can apply Lemma 5.1 to this case. So we obtain a $G$-cross section
$$
s_2 : L_i \to S(\tau(M))| L_i
$$
extending $s$.

Let $T'(M_{(H)})$ be the part of $T(M_{(H)})$ corresponding to the open disc bundle. Set
$$
L = M - T'(M_{(H)}).
$$
Then $L$ is a compact $G$-manifold with corner. Smoothing the corner of $L$, let $L'$ be the resulting smooth $G$-manifold. Note that $L$ and $L'$ are the same topological space. Let
$$
\varphi : \partial L' \times [0, 1] \to L'
$$
be a $G$-invariant collar of $\partial L'$ in $L'$ such that
$$
\varphi(\partial L' \times \{0\}) = \partial L'.
$$
Identify $\partial L' \times [0, 1]$ with the image of $\varphi$. By Proposition 2.2 there is a $G$-bundle isomorphism
$$
S(\tau(M))| \partial L' \times [0, 1] \cong (S(\tau(M))| \partial L') \times [0, 1].
$$
$S(\tau(M)) | \partial L'$ admits a $G$-cross section $s_3$ which is a restriction of $s_2$. From $s_3$ the above isomorphism induces a $G$-cross section

$$s_4 : \partial L' \times [0, 1] \to S(\tau(M)) | \partial L' \times [0, 1].$$

Set

$$L_2 = L' - \partial L' \times [0, 1].$$

$L_2$ is a compact $G$-manifold with $k - 1$ isotropy types. $S(\tau(M)) | L_2$ admits a $G$-cross section $s_4$ on $\partial L_2 = \partial L' \times \{1\}$ which is the restriction of $s_3$. $S(\tau(M)) | L_2$ and $S(\tau(L_2))$ are identical, since the smoothing process of the corner of $L$ does not change the differentiable structure outside the corner. By careful consideration we see that the image of $s_4$ is in $S(\tau(\partial L_2))$. So, by the hypothesis of the induction, $s_4$ is extendible to a $G$-cross section

$$s_5 : L_2 \to S(\tau(L_2)) = S(\tau(M)) | L_2.$$

Then $s_2$, $s_4$ and $s_5$ give a $G$-cross section of $S(\tau(M))$ extending $s$.

"only if" part: It suffices to show that, for each isotropy type $(H)$ on $M$, if

$$\dim N(H) - \dim H = 0$$

then

$$|\chi|(M_H|N(H)) = 0.$$

Let $(H)$ be a maximal isotropy type on $M$. As in the "only if" part for the case of one isotropy type, we see that

$$\dim N(H) - \dim H = 0$$

implies

$$|\chi|(M_H|N(H)) = 0.$$

Let $(H')$ be another isotropy type on $M$. Consider $L'_{H'}$ where $L'$ is the compact $G$-manifold in the previous "if" part. Since $L'$ is of $k - 1$ isotropy types,

$$\dim N(H') - \dim H' = 0$$

implies

$$|\chi|(L'_{H'}|N(H')) = 0$$

by the hypothesis of the induction. Furthermore $L'_{H'}$ is equivariantly homotopy equivalent to $M_{H'}$. Then

$$\dim N(H') - \dim H' = 0$$

implies

$$|\chi|(M_{H'}|N(H')) = 0.$$
Thus Theorem 1.1 is completely proved.

4. Proof of Corollary 1.3

If $M$ admits a $G$-$k$-field, then the $k$-field is tangent to $F$ by equivariancy. So the "only if" part is trivial.

Now we assume that $F$ admits a $k$-field. Then there are $k$ cross sections $s_1, s_2, \ldots, s_k$ of $\tau(F)$ which are linearly independent at each point of $F$. Since $\tau(F)$ is a subvector bundle of $\tau(M)|F$, we may regard $s_1, \ldots, s_k$ as $G$-cross sections of $\tau(M)|F$. As in the proof of Theorem 1.1 we may extend $s_1$ to a nowhere vanishing $G$-cross section $s_1'$ of $\tau(M)$. Choosing a $G$-invariant Riemannian metric on $\tau(M)$, let $E$ be the $G$-invariant subvector bundle of $\tau(M)$ which is orthogonal to the image of $s_1'$. The assumption

$$\dim N(G_x) - \dim G_x \geq k$$

enables us to extend $s_2$ to a nowhere vanishing $G$-cross section $s_2'$ of $E$ by the same method in the proof of Theorem 1.1. By repeating this process we may extend $s_3, \ldots, s_k$ to a $G$-cross sections $s_3', \ldots, s_k'$ of $\tau(M)$ such that $s_1', \ldots, s_k'$ are linearly independent at each point of $M$. So we obtain a $G$-$k$-field on $M$.

5. Concluding lemma

We will conclude this paper by proving the following lemma which was used in the proof of Theorem 1.1. $G$ acts trivially on intervals considered.

Lemma 5.1. Let $E \to X$ be a $G$-vector bundle. Let $A, B$ be $G$-invariant subspaces with $C = A \cap B$ compact. We assume that there is a $G$-invariant subspace $D$ of $C$ such that $C$ is equivariantly deformable to $D$, i.e., there is a $G$-homotopy $F: C \times [0, 1] \to C$ such that $F(x, 0) = x$ and $F(x, 1) \in D$ for all $x \in C$. We also assume that there is a $G$-invariant neighborhood $U$ of $C$ in $A$ such that there is a $G$-homeomorphism $\varphi: C \times [0, 3] \approx U$ with $\varphi(x, 0) = x$ for all $x \in C$. Let

$$s_A: A \to S(E)|A$$

and

$$s_B: B \to S(E)|B$$

be $G$-cross sections which agree on $D$. Then there is a $G$-cross section

$$s: A \cup B \to S(E)|A \cup B$$
which agrees with $s_A$ on $A - U$ and $s_B$ on $B$.

Proof. By Proposition 2.3 there is a $G$-bundle map

$$F: (S(E) \mid C) \times [0, 1] \to S(E) \mid C$$

which covers $F$ and is the identity on $(S(E) \mid C) \times \{0\}$, and also there is a $G$-bundle isomorphism

$$\varphi: (S(E) \mid C) \times [0, 3] \cong S(E) \mid U$$

which covers $\varphi$ and is the identity on $(S(E) \mid C) \times \{0\}$.

We define a $G$-bundle map

$$K: (S(E) \mid C) \times [0, 2] \to S(E) \mid C$$

by

$$K(v, t) = \begin{cases} F(v, t) & \text{if } 0 \leq t \leq 1 \\ F(v, 2-t) & \text{if } 1 \leq t \leq 2 \end{cases}$$

for $v \in S(E) \mid C$ and $t \in [0, 2]$. Then we may define a $G$-cross section

$$s_1: C \times [0, 2] \to (S(E) \mid C) \times [0, 2]$$

of the $G$-bundle $(S(E) \mid C) \times [0, 2]$ by

$$s_1(x, t) = \begin{cases} K^*s_B(x, t) & \text{if } 0 \leq t \leq 1 \\ K^*s_A(x, t) & \text{if } 1 \leq t \leq 2 \end{cases}.$$ 

This is well-defined since $K^*s_A$ and $K^*s_B$ agree on $C \times \{1\}$. $s_1$ satisfies the following equations for all $x \in C$

$$s_1(x, 0) = (s_B(x), 0)$$

and

$$s_1(x, 2) = (s_A(x), 2).$$

We define a map

$$\lambda: [2, 3] \to [0, 3]$$

by $\lambda(t) = 3t - 6$ for $t \in [2, 3]$. We denote by $s_2$ the $G$-cross section of $(S(E) \mid C) \times [2, 3]$ which is induced from the $G$-cross section $s_A \mid U$ by the composition

$$(S(E) \mid C) \times [2, 3] \xrightarrow{(id, \lambda)} (S(E) \mid C) \times [0, 3] \xrightarrow{\varphi} S(E) \mid U.$$ 

$s_2$ satisfies the following equations for all $x \in C$

$$s_2(x, 2) = (s_A(x), 2)$$

and
\( s_\beta(x, 3) = \phi^{-1} s_\alpha \phi(x, 3) \).

Since \( s_1 \) and \( s_2 \) agree on \( C \times \{2\} \), we obtain a \( G \)-cross section
\[
s_\gamma: C \times [0, 3] \to (S(E) \mid C) \times [0, 3]
\]
from \( s_1 \) and \( s_2 \). Then the induced \( G \)-cross section
\[
(\phi^{-1})^* s_\gamma: U \to S(E) \mid U
\]
satisfies the following equations
\[
(\phi^{-1})^* s_\gamma(x) = s_B(x) \quad \text{for all } x \in C
\]
\[
(\phi^{-1})^* s_\gamma(x) = s_A(x) \quad \text{for all } x \in \phi^{-1}(C \times \{3\}) .
\]

So we obtain a \( G \)-cross section
\[
s: A \cup B \to S(E) \mid A \cup B
\]
defined by
\[
s(x) = \begin{cases} 
  s_A(x) & \text{if } x \in A - U \\
  (\phi^{-1})^* s_\gamma(x) & \text{if } x \in U \\
  s_B(x) & \text{if } x \in B .
\end{cases}
\]

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References