Komiya, K.
Osaka J. Math.
13 (1976), 537–546

A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF NON-SINGULAR G-VECTOR FIELDS ON G-MANIFOLDS

KATSUHIRO KOMIYA

(Received July 30, 1975)

1. Introduction

Throughout this paper G always denotes a compact Lie group and G-manifolds are smooth manifolds with smooth G-actions. In this paper we will give a necessary and sufficient condition for the existence of non-singular G-vector fields on closed G-manifolds.

Let $E\to X$ be a G-vector bundle. After choosing a G-invariant Riemannian metric on E, we denote by $S(E)\to X$ the associated G-sphere bundle of E. We abbreviate *continuous G-equivariant cross section* of E to *G-cross section* of E.

Let M be a compact G-manifold, and s: $M\to\tau(M)$ a G-cross section of the tangent bundle $\tau(M)$ of M. s is called a *non-singular G-vector field* on M, if s is not zero at each point of M. For a positive integer k, by a *G-k-field* on M we will mean k G-cross sections of $\tau(M)$ which are linearly independent at each point of M.

For a closed subgroup H of G we set

$$M_H=\{x\in M\,|\,G_x=H\}\,,$$

where G_x is the isotropy group at x. Let $N(H)$ be the normalizer of H in G. Then M_H is an $N(H)$-manifold and also a free $N(H)/H$-manifold.

For a topological space X we define $|\chi|(X)$ to be the sum $\Sigma_Y|\chi(Y)|$, where $\chi(Y)$ is the Euler characteristic of Y and Y runs over the connected components of X.

In this paper we will obtain the following results:

Theorem 1.1. *Let M be a compact G-manifold. Let s: $\partial M\to S(\tau(\partial M))$ be a G-cross section of $S(\tau(\partial M))$. Then s is extendible to a G-cross section of $S(\tau(M))$ if and only if*

$$|\chi|(M_{G_x}/N(G_x))=0, \qquad \text{or} \quad \dim N(G_x)-\dim G_x>0$$

for all $x\in M$.

From this theorem we immediately obtain the main result of this paper:

Corollary 1.2. *A closed G-manifold M admits a non-singular G-vector field if and only if*

$$|\chi|(M_{G_x}/N(G_x)) = 0\,, \qquad \text{or} \quad \dim N(G_x) - \dim G_x > 0$$

for all $x \in M$.

We also obtain the following corollary:

Corollary 1.3. *Let M be a compact G-manifold, F the stationary point set of M, and k a positive integer. We suppose that*

$$\dim N(G_x) - \dim G_x \geq k$$

for all $x \in M - F$. *Then M admits a G-k-field if and only if F admits a k-field.*

Note. The existence of a non-singular *continuous* G-vector field on a compact G-manifold implies the existence of a non-singular *smooth* G-vector field. This fact is assured by the differentiable approximation theorem [3; (6.7)] and the usual process of averaging cross sections.

2. Preliminaries

(2-1) For a closed subgroup H of G let (H) be the conjugacy class of H in G. If H is an isotropy group occuring on a G-manifold M, (H) is called an isotropy type on M.

Set

$$M_H = \{x \in M \mid G_x = H\}$$
$$M_{(H)} = \{x \in M \mid (G_x) = (H)\}\ .$$

Then M_H and $M_{(H)}$ are submanifolds of M, but, in general, not compact. If (H) is a maximal isotropy type on a compact G-manifold M, M_H and $M_{(H)}$ are compact.

(2-2) Let $\pi\colon E \to X$, $\pi'\colon E' \to X'$ be G-fibre bundles, and $f\colon E \to E'$ a G-bundle map covering $\bar{f}\colon X \to X'$. Let $s'\colon X' \to E'$ be a G-cross section of E'. s' induces a G-cross section

$$s\colon X \to \bar{f}^*E' = \{(x, e) \in X \times E' \mid \bar{f}(x) = \pi'(e)\}$$

of the induced G-fibre bundle $\bar{f}^*E'$ which sends $x \in X$ to $(x, s'\bar{f}(x)) \in X \times E'$. E is canonically isomorphic to $\bar{f}^*E'$. So s induces a G-cross section of E. We denote this G-cross section by f^*s', and call induced G-cross section from s' by f.

(2-3) Recall known results:

Proposition 2.1 (Segal [2; Proposition 1.3]). *Let X, Y be G-spaces, and X compact. If f_0, f_1: $X \to Y$ are G-homotopic G-maps, and $E \to Y$ is a G-vector bundle, then there is a G-bundle isomorphism*

$$\varphi: f_0^*E \cong f_1^*E .$$

From this proposition we easily obtain the following. I denotes the interval $[0,1]$ with trivial G-action.

Proposition 2.2. *If X is a compact G-space, then any G-vector bundle E over $X \times I$ is isomorphic to $(E|X \times \{0\}) \times I$ as G-vector bundles.*

Proposition 2.1 may be stated in a more precise form as the following: *If f_0, f_1 are G-homotopic relative to a closed G-invariant subspace A of X, and if we consider f_0^*E, f_1^*E subspaces of $X \times E$, then the G-bundle isomorphism φ satisfies $\varphi(x, e)=(x, e)$ for all $x \in A$ and $e \in E$.*

From this fact we obtain

Proposition 2.3. *Let $E_i \to X_i$, $i=1,2$, be G-vector bundles with X_1 compact. Let f: $X_1 \times I \to X_2$ be a G-homotopy which is constant on a closed G-invariant subspace A of X_1. Let $\bar{f}_0$: $E_1 \to E_2$ be a G-bundle map over $f_0=f|X_1 \times \{0\}$. Then there is a G-bundle map $\bar{f}$: $E_1 \times I \to E_2$ over f which is a homotopy of $\bar{f}_0$ and is constant on $E_1|A$.*

Corollary 2.4. *Let $E \to X$ be a G-vector bundle with X compact, A a closed G-invariant subspace of X, and i: $A \to X$ the inclusion map. Let f: $X \to A$ be a G-map such that if is G-homotopic to the identity of X relative to A. Then there is a G-bundle map $\bar{f}$: $E \to E|A$ over f which is the identity on $E|A$.*

(2-4) The following result has been obtained by U. Koschorke in his paper [1; §1].

Proposition 2.5. *Let M be a compact manifold, and s: $\partial M \to S(\tau(\partial M))$ a cross section of $S(\tau(\partial M))$. Then s is extendible to a cross section of $S(\tau(M))$ if and only if $|\chi|(M)=0$.*

3. Proof of Theorem 1.1

The proof will proceed by an induction for the number of isotropy types on M.

(3-1) In the first place, let M be of one isotropy type, and let (H) be the isotropy type on M.

"*if*" *part*: M_H is a compact $N(H)$-manifold with boundary $M_H \cap \partial M$. Consider the sphere bundle

$$S(\tau(M_H))/N(H)\to M_H/N(H).$$

The G-cross section s induces a cross section

$$s_1\colon \partial(M_H/N(H))\to S(\tau(M_H))/N(H)\,|\,\partial(M_H/N(H)).$$

This is assured by the G-equivariancy of s and the fact

$$\partial(M_H/N(H))=\partial M_H/N(H)=M_H\cap\partial M/N(H).$$

We may extend s_1 to a cross section of $S(\tau(M_H))/N(H)$ as follows. If

$$\dim N(H)-\dim H>0,$$

then the dimension as a cell complex of $M_H/N(H)=M_H/(N(H)/H)$ is less than or equal to the dimension of fibre of $S(\tau(M_H))/N(H)$. Therefore the obstruction to extending s_1 over $M_H/N(H)$ vanishes. If

$$\dim N(H)-\dim H=0,$$

then

$$|\chi|(M_H/N(H))=0$$

by the assumption, and then

$$S(\tau(M_H))/N(H)\cong S(\tau(M_H/N(H)))$$

for $N(H)/H$ is a finite group and $M_H/N(H)=M_H/(N(H)/H)$. The image of s_1 is in $S(\tau(\partial M_H))/N(H)$. Then s_1 can be extended over $M_H/N(H)$ by Proposition 2.5.

Let

$$s_2\colon M_H/N(H)\to S(\tau(M_H))/N(H)$$

be an extension of s_1. Let

$$\pi\colon S(\tau(M_H))\to S(\tau(M_H))/N(H)$$

be the canonical projection, and let

$$\pi^*s_2\colon M_H\to S(\tau(M_H))$$

be the induced $N(H)$-cross section. π^*s_2 may be considered an $N(H)$-cross section of $S(\tau(M))\,|\,M_H$, since $S(\tau(M_H))$ is a subbundle of $S(\tau(M))\,|\,M_H$. Moreover π^*s_2 is an extension of $s\,|\,\partial M_H$. Since M is of one isotropy type, the G-action

$$G\times(S(\tau(M))\,|\,M_H)\to S(\tau(M))$$

induces a G-bundle isomorphism

$$G \times_{N(H)} (S(\tau(M)) \mid M_H) \cong S(\tau(M)) \,.$$

Then $\pi^* s_2$ induces a G-cross section

$$s_3 \colon M \to S(\tau(M))$$

which is an extension of s.

"only if" part: Let

$$t \colon M \to S(\tau(M))$$

be an extension of s. By the G-equivariancy of t, t is restricted to an $N(H)$-cross section

$$t_1 \colon M_H \to S(\tau(M_H)) \,.$$

t_1 induces a cross section

$$t_2 \colon M_H/N(H) \to S(\tau(M_H))/N(H) \,.$$

If

$$\dim N(H) - \dim H = 0 \,,$$

then

$$S(\tau(M_H))/N(H) \cong S(\tau(M_H/N(H))) \,,$$

and

$$t_2(\partial(M_H/N(H))) \subset S(\tau(\partial(M_H/N(H)))) \,.$$

Therefore

$$\dim N(H) - \dim H = 0$$

implies

$$|\chi|(M_H/N(H)) = 0$$

by Proposition 2.5.

This completes the proof for the case in which M is of one isotropy type.

(3-2) Let the theorem be true for the case in which the number of isotropy types is $k-1$. Let M be a compact G-manifold with k isotropy types.

"if" part: Let (H) be a maximal isotropy type on M. Then $M_{(H)}$ is a compact G-submanifold of M with one isotropy type. From the preceding argument we obtain a G-cross section

$$s_1 \colon M_{(H)} \to S(\tau(M)) \mid M_{(H)}$$

such that the image of s_1 is in $S(\tau(M_{(H)}))$ and $s_1 \mid \partial M_{(H)} = s \mid \partial M_{(H)}$.

Let $T(M_{(H)})$ be a closed G-invariant tubular neighborhood of $M_{(H)}$ in M.

By Corollary 2.4 we obtain a G-bundle map

$$\pi\colon S(\tau(M))\,|\,T(M_{(H)})\to S(\tau(M))\,|\,M_{(H)}$$

such that π covers the canonical projection of $T(M_{(H)})$ to $M_{(H)}$ and π is the identity on $S(\tau(M))\,|\,M_{(H)}$. π induces a G-cross section

$$\pi^*s_1\colon T(M_{(H)})\to S(\tau(M))\,|\,T(M_{(H)})$$

from s_1 such that $\pi^*s_1\,|\,M_{(H)}=s_1$.

To obtain a G-cross section

$$L_1=\partial M\cup T(M_{(H)})\to S(\tau(M))\,|\,L_1$$

extending s, we may apply Lemma 5.1 (stated in the last section) as follows. Apply $E\to X$, A, B, D, s_A and s_B in Lemma 5.1 to $\tau(M)\to M$, $T(M_{(H)})$, ∂M, $\partial M_{(H)}$, π^*s_1 and s, respectively. So, in this case,

$$C=A\cap B=T(M_{(H)})\cap\partial M=T(M_{(H)})\,|\,\partial M_{(H)}\,,$$

and this is compact. Moreover this is equivariantly deformable to $\partial M_{(H)}$, and

$$\pi^*s_1\,|\,\partial M_{(H)}=s\,|\,\partial M_{(H)}\,.$$

Let K be a closed G-invariant collar of $\partial M_{(H)}$ in $M_{(H)}$. By Proposition 2.2, $T(M_{(H)})\,|\,K$ has the desired property as U in Lemma 5.1. Therefore we can apply Lemma 5.1 to this case. So we obtain a G-cross section

$$s_2\colon L_1\to S(\tau(M))\,|\,L_1$$

extending s.

Let $T^0(M_{(H)})$ be the part of $T(M_{(H)})$ corresponding to the open disc bundle. Set

$$L=M-T^0(M_{(H)})\,.$$

Then L is a compact G-manifold with *corner*. Smoothing the corner of L, let L' be the resulting smooth G-manifold. Note that L and L' are the same topological space. Let

$$\varphi\colon\partial L'\times[0,1]\to L'$$

be a G-invariant collar of $\partial L'$ in L' such that

$$\varphi(\partial L'\times\{0\})=\partial L'\,.$$

Identify $\partial L'\times[0,1]$ with the image of φ. By Proposition 2.2 there is a G-bundle isomorphism

$$S(\tau(M))\,|\,\partial L'\times[0,1]\cong(S(\tau(M))\,|\,\partial L')\times[0,1]\,.$$

$S(\tau(M))|\partial L'$ admits a G-cross section s_3 wihch is a restriction of s_2. From s_3 the above isomorphism induces a G-cross section

$$s_4\colon \partial L'\times[0, 1]\to S(\tau(M))|\partial L'\times[0, 1] .$$

Set

$$L_2 = L'-\partial L'\times[0, 1) .$$

L_2 is a compact G-manifold with $k-1$ isotropy types. $S(\tau(M))|L_2$ admits a G-cross section s_5 on $\partial L_2=\partial L'\times\{1\}$ which is the restriction of s_4. $S(\tau(M))|L_2$ and $S(\tau(L_2))$ are identical, since the smoothing process of the corner of L does not change the differentiable structure outside the corner. By careful consideration we see that the image of s_5 is in $S(\tau(\partial L_2))$. So, by the hypothesis of the induction, s_5 is extendible to a G-cross section

$$s_6\colon L_2\to S(\tau(L_2)) = S(\tau(M))|L_2 .$$

Then s_2, s_4 and s_6 give a G-cross section of $S(\tau(M))$ extending s.

"only if" part: It suffices to show that, for each isotropy type (H) on M, if

$$\dim N(H)-\dim H = 0$$

then

$$|\chi|(M_H/N(H)) = 0 .$$

Let (H) be a maximal isotropy type on M. As in the "*only if*" part for the case of one isotropy type, we see that

$$\dim N(H)-\dim H = 0$$

implies

$$|\chi|(M_H/N(H)) = 0 .$$

Let (H') be another isotropy type on M. Consider $L'_{H'}$ where L' is the compact G-manifold in the previous "*if*" part. Since L' is of $k-1$ isotropy types,

$$\dim N(H')-\dim H' = 0$$

implies

$$|\chi|(L'_{H'}/N(H')) = 0$$

by the hypothesis of the induction. Furthermore $L'_{H'}$ is equivariantly homotopy equivalent to $M_{H'}$. Then

$$\dim N(H')-\dim H' = 0$$

implies

$$|\chi|(M_{H'}/N(H')) = 0 .$$

Thus Theorem 1.1 is completely proved.

4. Proof of Corollary 1.3

If M admits a G-k-field, then the k-field is tangent to F by equivariancy. So the "*only if*" part is trivial.

Now we assume that F admits a k-field. Then there are k cross sections $s_1, s_2, \cdots, s_k$ of $\tau(F)$ which are linearly independent at each point of F. Since $\tau(F)$ is a subvector bundle of $\tau(M)|F$, we may regard $s_1, \cdots, s_k$ as G-cross sections of $\tau(M)|F$. As in the proof of Theorem 1.1 we may extend s_1 to a nowhere vanishing G-cross section s_1' of $\tau(M)$. Choosing a G-invariant Riemannian metric on $\tau(M)$, let E be the G-invariant subvector bundle of $\tau(M)$ which is orthogonal to the image of s_1'. The assumption

$$\dim N(G_x) - \dim G_x \geq k$$

enables us to extend s_2 to a nowhere vanishing G-cross section s_2' of E by the same method in the proof of Theorem 1.1. By repeating this process we may extend $s_3, \cdots, s_k$ to a G-cross sections $s_3', \cdots, s_k'$ of $\tau(M)$ such that $s_1', \cdots, s_k'$ are linearly independent at each point of M. So we obtain a G-k-field on M.

5. Concluding lemma

We will conclude this paper by proving the following lemma which was used in the proof of Theorem 1.1. G acts trivially on intervals considered.

Lemma 5.1. *Let $E \to X$ be a G-vector bundle. Let A, B be G-invariant subspaces with $C = A \cap B$ compact. We assume that there is a G-invariant subspace D of C such that C is equivariantly deformable to D, i.e., there is a G-homotopy*

$$F: C \times [0, 1] \to C$$

such that $F(x, 0) = x$ and $F(x, 1) \in D$ for all $x \in C$. We also assume that there is a G-invariant neighborhood U of C in A such that there is a G-homeomorphism

$$\varphi: C \times [0, 3] \approx U$$

with $\varphi(x, 0) = x$ for all $x \in C$. Let

$$s_A: A \to S(E)|A$$

and

$$s_B: B \to S(E)|B$$

be G-cross sections which agree on D. Then there is a G-cross section

$$s: A \cup B \to S(E)|A \cup B$$

which agrees with s_A *on* $A-U$ *and* s_B *on* B.

Proof. By Proposition 2.3 there is a G-bundle map

$$\bar{F}\colon (S(E)|C)\times[0,1]\to S(E)|C$$

which covers F and is the identity on $(S(E)|C)\times\{0\}$, and also there is a G-bundle isomorphism

$$\bar{\varphi}\colon (S(E)|C)\times[0,3]\cong S(E)|U$$

which covers φ and is the identity on $(S(E)|C)\times\{0\}$. We define a G-bundle map

$$K\colon (S(E)|C)\times[0,2]\to S(E)|C$$

by

$$K(v,t)=\begin{cases}\bar{F}(v,t) & \text{if } 0\leq t\leq 1\\ \bar{F}(v,2-t) & \text{if } 1\leq t\leq 2\end{cases}$$

for $v\in S(E)|C$ and $t\in[0,2]$. Then we may define a G-cross section

$$s_1\colon C\times[0,2]\to(S(E)|C)\times[0,2]$$

of the G-bundle $(S(E)|C)\times[0,2]$ by

$$s_1(x,t)=\begin{cases}K^*s_B(x,t) & \text{if } 0\leq t\leq 1\\ K^*s_A(x,t) & \text{if } 1\leq t\leq 2\,.\end{cases}$$

This is well-defined since K^*s_A and K^*s_B agree on $C\times\{1\}$. s_1 satisfies the following equations for all $x\in C$

$$s_1(x,0)=(s_B(x),0)$$

and

$$s_1(x,2)=(s_A(x),2)\,.$$

We define a map

$$\lambda\colon [2,3]\to[0,3]$$

by $\lambda(t)=3t-6$ for $t\in[2,3]$. We denote by s_2 the G-cross section of $(S(E)|C)\times[2,3]$ which is induced from the G-cross section $s_A|U$ by the composition

$$(S(E)|C)\times[2,3]\xrightarrow{(id,\lambda)}(S(E)|C)\times[0,3]\xrightarrow{\bar{\varphi}}S(E)|U\,.$$

s_2 satisfies the following equations for all $x\in C$

$$s_2(x,2)=(s_A(x),2)$$

and

$$s_2(x, 3) = \bar{\varphi}^{-1} s_A \varphi(x, 3) .$$

Since s_1 and s_2 agree on $C\times\{2\}$, we obtain a G-cross section

$$s_3\colon C\times[0, 3]\to(S(E)|C)\times[0, 3]$$

from s_1 and s_2. Then the induced G-cross section

$$(\bar{\varphi}^{-1})^* s_3\colon U\to S(E)|U$$

satisfies the following equations

$$\begin{aligned} (\bar{\varphi}^{-1})^* s_3(x) &= s_B(x) && \text{for all } x\in C \\ (\bar{\varphi}^{-1})^* s_3(x) &= s_A(x) && \text{for all } x\in \varphi^{-1}(C\times\{3\}) . \end{aligned}$$

So we obtain a G-cross section

$$s\colon A\cup B\to S(E)|A\cup B$$

defined by

$$s(x) = \begin{cases} s_A(x) & \text{if } x\in A-U \\ (\bar{\varphi}^{-1})^* s_3(x) & \text{if } x\in U \\ s_B(x) & \text{if } x\in B . \end{cases}$$

Yamaguchi University

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