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ON THE EXTENSION OF $G_2(3^{2n+1})$ BY THE EXCEPTIONAL GRAPH AUTOMORPHISM

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Abstract

The main aim of this paper is to compute the character table of $G_2(3^{2n+1}) \rtimes \langle\sigma\rangle$, where σ is the graph automorphism of $G_2(3^{2n+1})$ such that the fixed-point subgroup $G_2(3^{2n+1})^\sigma$ is the Ree group of type G_2. As a consequence we explicitly construct a perfect isometry between the principal p-blocks of $G_2(3^{2n+1})^\sigma$ and $G_2(3^{2n+1}) \rtimes \langle\sigma\rangle$ for prime numbers dividing $q^2 - q + 1$.

1. Introduction

In representation theory, we are interested in irreducible characters of the simple groups and of their cyclic extensions. For example, the character tables of sporadic groups and of their extensions by an automorphism appear in the *Atlas of Finite Groups* [5]. The classification of finite simple groups illustrates the importance of the finite groups of Lie type. Deligne-Lusztig Theory gives many methods to compute their character tables [4]. Moreover, we explicitly know the automorphisms of these groups; they are mainly of three types: diagram automorphisms, field automorphisms and diagonal automorphisms [3, §12]. G. Malle studied the cyclic extensions (by a diagram automorphism) of some finite groups of Lie type; he interpreted the extension as a fixed-point subgroup under a Frobenius map of a non-connected algebraic group and used a theory of Deligne-Lusztig for non-connected reductive groups developed by F. Digne and J. Michel [7]. More precisely Malle computed in [13] the almost characters (a family of class functions introduced by Lusztig) of extensions by a diagram automorphism of finite simple groups of type A_l, D_l with $l \leq 5$, and E_6 (for the twisted and untwisted cases).

Let G be a finite simple group of type B_2, F_4 (over a finite field of order an odd 2-power) or G_2 (resp. an odd 3-power). In these cases, the exceptional symmetry of the Dynkin diagram allows to define an automorphism σ of G of order 2 (see [3, §12]). The fixed-point subgroup G^σ of G under σ is the so-called Suzuki group, Ree group of type G_2 and Ree group of type F_4 respectively. These three families of twisted groups are still finite simple groups.

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We are interested in the character table of the cyclic extension $G \rtimes \langle\sigma\rangle$ of the untwisted simple group by the automorphism defining the twisted group. Let n be a non-negative integer; in [2] the character table of $\mathrm{B}_2(2^{2n+1}) \rtimes \langle\sigma\rangle$ is computed. The main aim of the present work is to compute the character table of $\tilde{\mathrm{G}} = \mathrm{G}_2(3^{2n+1}) \rtimes \langle\sigma\rangle$.

This paper is organized as follows: in Section 2 we recall some notations. In Section 3 we give generalities about the conjugacy classes and the character table of an extension of degree 2 of a finite group. We compute the conjugacy classes of $\tilde{\mathrm{G}}$ in Section 4 and its character table in Section 5. The main result of this article is:

Theorem 1.1. *Let n be a non-negative integer. We set $q = 3^{2n+1}$ and let σ be the automorphism of order* 2 *of $\mathrm{G}_2(q)$ such that its fixed-point subgroup $\mathrm{R}(q)$ is the Ree group of type G_2 with parameter q. Then the group $\mathrm{G}_2(q) \rtimes \langle\sigma\rangle$ has $(q+8)$ outer classes* (*given in* Table 2) *and* $(2q+16)$ *outer characters which are extensions of* $(q+8)$ *irreducible σ-stable characters of $\mathrm{G}_2(q)$. These σ-stable characters are described in* Section 5 *and in the equations* (8), (9) *and* (12). *The values of the outer characters on the outer classes are given in* Table 11 (*with notations in* Table 10).

In Section 6 we explicitly construct a perfect isometry between the principal p-blocks of $\tilde{\mathrm{G}}$ and $\mathrm{R}(q)$ for a prime divisor of $q^2 - q + 1$ (that is a prime number that divides the order of $\mathrm{G}_2(q)$ and that is prime to the index of $\mathrm{R}(q)$ in $\mathrm{G}_2(q)$). More precisely we will prove:

Theorem 1.2. *Let n be a non-negative integer*; *we set $\theta = 3^n$ and $q = 3\theta^2$. Let p be a prime divisor of $q^2 - q + 1$ and let $(\mathrm{K}, \mathcal{O}, p)$ be a p-modular system which is large enough for both $\mathrm{R}(q)$ and $\tilde{\mathrm{G}}$. Then*:

– *If p is a divisor of $(q - 3\theta + 1)$, we write $q - 3\theta + 1 = p^d\alpha$, where α is prime to p. We define on $\mathbb{Z}/\alpha\mathbb{Z}$ an equivalence relation by $\pm i \sim \pm qi \sim \pm q^2 i$ $(i \in \mathbb{Z}/\alpha\mathbb{Z})$ and we denote by $\mathfrak{R}_0$ its non-zero classes. The principal p-blocks are*:

$$B_0(\mathrm{R}(q)) = \{1, \xi_3, \xi_5, \xi_7, \xi_9, \xi_{10}, \eta^-_{k\alpha}; k \in \mathfrak{R}_0\},$$
$$B_0(\tilde{\mathrm{G}}) = \{1, \tilde{\theta}_1, \varepsilon\tilde{\theta}_5, \varepsilon\tilde{\theta}_{11}, \varepsilon\tilde{\theta}_{12}(1), \varepsilon\tilde{\theta}_{12}(-1), \varepsilon\tilde{\chi}_a(\alpha k); k \in \mathfrak{R}_0\}.$$

There exists a perfect isometry between $B_0(\mathrm{R}(q))$ and $B_0(\tilde{\mathrm{G}})$ defined by:

$$I = \begin{cases} 1 & \quad 1 \\ \xi_3 & \quad -\varepsilon\tilde{\theta}_5 \\ \xi_5 & \quad \tilde{\theta}_1 \\ \xi_7 \quad \mapsto & \quad -\varepsilon\tilde{\theta}_{11} \\ \xi_9 & \quad -\varepsilon\tilde{\theta}_{12}(-1) \\ \xi_{10} & \quad -\varepsilon\tilde{\theta}_{12}(1) \\ \eta^-_{\alpha k} & \quad -\varepsilon\tilde{\chi}_a(\alpha k) \end{cases}.$$

– *If p is a divisor of $(q+3\theta+1)$, we write $q+3\theta+1 = p^d\alpha$, where α is prime to p. As above, we denote by $\Re_0$ the non-trivial classes modulo the equivalence relation on $\mathbb{Z}/\alpha\mathbb{Z}$ defined by $\pm i \sim \pm qi \sim \pm q^2 i$ $(i \in \mathbb{Z}/\alpha\mathbb{Z})$. The principal p-blocks are*:

$$B_0(\mathrm{R}(q)) = \{1, \xi_3, \xi_6, \xi_8, \xi_9, \xi_{10}, \eta^+_{k\alpha}; k \in \Re_0\},$$
$$B_0(\tilde{\mathrm{G}}) = \{1, \varepsilon\tilde{\theta}_1, \varepsilon\tilde{\theta}_5, \varepsilon\tilde{\theta}_{11}, \tilde{\theta}_{12}(1), \tilde{\theta}_{12}(-1), \varepsilon\tilde{\chi}_{ababa}(\alpha k); k \in \Re_0\}.$$

There exists a perfect isometry between $B_0(\mathrm{R}(q))$ and $B_0(\tilde{\mathrm{G}})$ defined by:

$$I = \begin{cases} 1 & & 1 \\ \xi_3 & & -\varepsilon\tilde{\theta}_5 \\ \xi_6 & & -\varepsilon\tilde{\theta}_1 \\ \xi_8 & \mapsto & \varepsilon\tilde{\theta}_{11} \\ \xi_9 & & \tilde{\theta}_{12}(-1) \\ \xi_{10} & & \tilde{\theta}_{12}(1) \\ \eta^+_{\alpha k} & & -\varepsilon\tilde{\chi}_{ababa}(\alpha k) \end{cases}.$$

2. Notations

Let $\mathbf{G}$ be a connected reductive algebraic group of type G_2 over $\overline{\mathbb{F}}_3$, an algebraic closure of the finite field $\mathbb{F}_3$. For generalities on algebraic groups, we refer to [4] or [10]. We denote by $(\ ,\ )$ the canonical scalar product of $\mathbb{R}^2$. We set $\zeta_1 = (-1/2, \sqrt{3}/2)$, $\zeta_2 = (1, 0)$ and $\zeta_3 = (-1/2, -\sqrt{3}/2)$. Let

$$\Sigma = \{\pm\zeta_i,\ \zeta_i - \zeta_j \mid 1 \le i,\ j \le 3,\ i \ne j\},$$

be the root system of type G_2 (detailed in Fig. 1). We define $a = \zeta_2$, $b = \zeta_1 - \zeta_2$ and we choose $\{a, b\}$ as a fundamental system of roots. We denote by Σ^+ the set of corresponding positive roots. Precisely, we have $\Sigma^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$.

Let r be a root, we denote by w_r the reflection in the hyperplane orthogonal to r. More precisely, for all $x \in \mathbb{R}^2$ we have

$$w_r(x) = x - 2\frac{(r, x)}{(r, r)}r.$$

The Weyl group W of Σ is the group generated by all the w_r $(r \in \Sigma)$.

For every $r \in \Sigma$, let $\mathbf{X}_r = \{x_r(t) \mid t \in \overline{\mathbb{F}}_3\}$ be the one-parameter subgroup of $\mathbf{G}$ corresponding to r. The group $\mathbf{G}$ is generated by the $x_r(t)$ $(r \in \Sigma,\ t \in \overline{\mathbb{F}}_3)$ and we

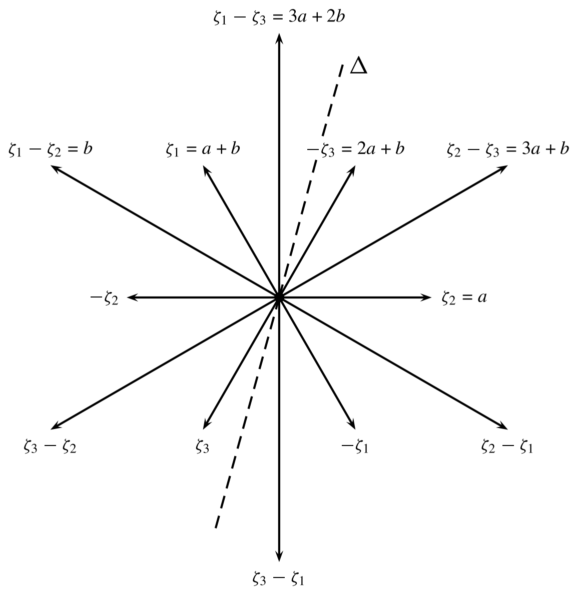

Fig. 1. Root system of type G_2

have the commutator relations (see [3]):

$$
\begin{aligned}
x_a(t)x_b(u) &= x_b(u)x_a(t)x_{a+b}(-tu)x_{3a+b}(t^3u)x_{2a+b}(-t^2u)x_{3a+2b}(t^3u^2),\\
x_a(t)x_{a+b}(u) &= x_{a+b}(u)x_a(t)x_{2a+b}(tu),\\
x_b(t)x_{3a+b}(u) &= x_{3a+b}(u)x_b(t)x_{3a+2b}(tu),\\
x_{a+b}(t)x_{3a+b}(u) &= x_{3a+b}(u)x_{a+b}(t),\\
x_{a+b}(t)x_{2a+b}(u) &= x_{2a+b}(u)x_{a+b}(t),\\
x_{a+b}(t)x_{3a+2b}(u) &= x_{3a+2b}(u)x_{a+b}(t),\\
x_{2a+b}(t)x_{3a+b}(u) &= x_{3a+b}(u)x_{2a+b}(t),\\
x_{2a+b}(t)x_{3a+2b}(u) &= x_{3a+2b}(u)x_{2a+b}(t).
\end{aligned}
\tag{1}
$$

Let $\mathcal{P} = \mathbb{Z}\Sigma$ and let χ be a homomorphism from $\mathcal{P}$ into the multiplicative group $\overline{\mathbb{F}}_3^\times$. We set $z_i = \chi(\zeta_i)$, where $i = 1, 2, 3$. We have $z_1z_2z_3 = 1$. Furthermore, the element $h(\chi)$ of the Cartan subgroup associated with χ will be denoted by $h(z_1, z_2, z_3)$. Let $\chi \in \mathrm{Hom}\big(\mathcal{P}, \overline{\mathbb{F}}_3^\times\big)$, $r \in \Sigma$ and $t \in \overline{\mathbb{F}}_3$, we have

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t). \tag{2}$$

We set

$$\mathbf{U} = \prod_{r \in \Sigma^+} \mathbf{X}_r,$$

and we denote by $\mathbf{B} = N_{\mathbf{G}}(\mathbf{U})$ a Borel subgroup of $\mathbf{G}$. We have $\mathbf{B} = \mathbf{TU}$, where $\mathbf{H} = \{h(z_1, z_2, z_3) \mid z_i \in \overline{\mathbb{F}}_3,\ z_1z_2z_3 = 1\}$ is a maximal torus of $\mathbf{G}$.

For every $r \in \Sigma$, there exists a unique homomorphism φ_r from $SL_2(\overline{\mathbb{F}}_3)$ to $\mathbf{G}$ such that

$$\forall t \in \overline{\mathbb{F}}_3, \quad \varphi_r\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) = x_r(t),\ \varphi_r\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right) = x_{-r}(t).$$

We set

$$h_r(t) = \varphi_r\left(\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right) \quad \text{and} \quad n_r(t) = \varphi_r\left(\begin{bmatrix} 0 & t \\ -t^{-1} & 0 \end{bmatrix}\right) = x_r(t)x_{-r}(-t^{-1})x_r(t).$$

We have $h_r(t) = h(\chi_{r,t})$, where $\chi_{r,t}(a) = t^{2(a,r)/(r,r)}$. We put $n_r = n_r(1)$. For $r, s \in \Sigma$ and $t \in \overline{\mathbb{F}}_3$, we have

(3) $$n_r x_s(t) n_r^{-1} = x_{w tr(s)}(\eta_{r,s} t) \quad \text{and} \quad n_r n_s n_r^{-1} = h_{w_r(s)}(\eta_{r,s}) n_{w_r(s)},$$

where the constants $\eta_{r,s} \in \{\pm 1\}$ are given in Table 1.

Let $\mathbf{N}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{H}$ and n_r ($r \in \Sigma$). We denote by $\pi_W : \mathbf{N} \to W$ the homomorphism such that $\ker \pi_W = \mathbf{H}$ and $\pi_W(n_r) = w_r$.

We define a permutation of the root system of $\mathbf{G}$ as follows: for every $r \in \Sigma$, the image $\bar{r}$ of r is the root in the direction obtained by reflecting r in the line Δ (see Fig. 1). More precisely, we have

$$\pm a \leftrightarrow \pm b,$$
$$\pm(a+b) \leftrightarrow \pm(3a+b),$$
$$\pm(2a+b) \leftrightarrow \pm(3a+2b).$$

Table 1. The constants $\eta_{r,s}$ of G_2

	ζ_1	ζ_2	ζ_3	$\zeta_1-\zeta_2$	$\zeta_2-\zeta_3$	$\zeta_3-\zeta_1$
ζ_1	−1	−1	1	−1	1	1
ζ_2	1	−1	−1	1	−1	1
ζ_3	−1	1	−1	1	1	−1
$\zeta_1-\zeta_2$	−1	1	1	−1	1	−1
$\zeta_2-\zeta_3$	1	−1	1	−1	−1	1
$\zeta_3-\zeta_1$	1	1	−1	1	−1	−1

As in [3, §12.4], we associate to this symmetry an automorphism α of $\mathbf{G}$ defined on the generators by

$$\begin{aligned} x_{\pm a}(t) &\mapsto x_{\pm b}(t^3),\\ x_{\pm b}(t) &\mapsto x_{\pm a}(t),\\ x_{\pm(a+b)}(t) &\mapsto x_{\pm(3a+b)}(t^3),\\ x_{\pm(3a+b)}(t) &\mapsto x_{\pm(a+b)}(t),\\ x_{\pm(2a+b)}(t) &\mapsto x_{\pm(3a+2b)}(t^3),\\ x_{\pm(3a+2b)}(t) &\mapsto x_{\pm(2a+b)}(t). \end{aligned}$$

We have in particular $\alpha(h(z_1, z_2, z_3)) = h\left(z_2 z_3^{-1}, z_1 z_2^{-1}, z_3 z_1^{-1}\right)$, $\alpha(n_a) = n_b$ and $\alpha(n_b) = n_a$.

For every non-negative integer m, the standard Frobenius map F_{3^m} of $\mathbf{G}$ is defined on the generators by $x_r(t) \mapsto x_r(t^{3^m})$ ($r \in \Sigma$ and $t \in \overline{\mathbb{F}}_3$).

Let n be a non-negative integer and $F = F_{3^n} \circ \alpha$. The map F is an endomorphism of $\mathbf{G}$ such that $F^2 = F_{3^{2n+1}}$. That implies that F is a generalized Frobenius map. We denote by $\mathbf{G}^F$ (resp. $\mathbf{G}^{F^2}$) the finite fixed-point subgroup of $\mathbf{G}$ under F (resp. F^2).

We set $q = 3^{2n+1}$, $\mathrm{R}(q) = \mathbf{G}^F$ and $\mathrm{G}_2(q) = \mathbf{G}^{F^2}$. We have $\mathrm{R}(q) \subseteq \mathrm{G}_2(q)$. We denote by σ the restriction of F to $\mathrm{G}_2(q)$. It is an automorphism of $\mathrm{G}_2(q)$ of order 2. The subgroups $\mathbf{N}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{U}$ are F-stable and for all $r \in \Sigma$, the subgroup $\mathbf{X}_r$ is F^2-stable; we set $N = \mathbf{N}^{F^2}$, $B = \mathbf{B}^{F^2}$, $H = \mathbf{H}^{F^2}$, $U = \mathbf{U}^{F^2}$ and $X_r = \mathbf{X}_r^{F^2}$.

3. Generalities on extensions of degree 2

Let G be a finite group. For generalities on representation theory, we refer to [12]. We denote by $\mathrm{C}(G)$ the complex space of class functions of G and $\mathrm{Irr}(G)$ the set of irreducible characters of G. The set of characters (resp. generalized characters) is denoted by $\mathbb{N}\,\mathrm{Irr}(G)$ (resp. $\mathbb{Z}\,\mathrm{Irr}(G)$). We denote by $\mathrm{Cl}(G)$ the set of conjugacy classes of G and $\mathrm{C}_G(g)$ the centralizer of the element $g \in G$. We denote by $\langle\ ,\ \rangle_G$ the usual scalar product on $\mathrm{C}(G)$. Let H be a subgroup of G. If $\varphi \in \mathrm{C}(G)$ we denote by $\mathrm{Res}_H^G \varphi$ the restriction of φ to H. If $\phi \in \mathrm{C}(H)$, we denote by $\mathrm{Ind}_H^G \phi$ the induced class function of ϕ from H to G.

Let σ be an automorphism of order 2 of G. We set $\tilde{\mathrm{G}} = G \rtimes \langle\sigma\rangle$ and denote by (g, x) any element of this group, where $g \in G$ and $x \in \langle\sigma\rangle$. We identify G with a subgroup of index 2 of $\tilde{\mathrm{G}}$ by $g \mapsto (g, 1)$.

We are now interested in the character table of $\tilde{\mathrm{G}}$. We suppose that the conjugacy classes and the irreducible characters of G are known. What can we deduce on the conjugacy classes and character table of $\tilde{\mathrm{G}}$? We will see that the Clifford theory allows to give partial answers. First, we give general results on the conjugacy classes of $\tilde{\mathrm{G}}$ and then general results on the irreducible characters.

3.1. Conjugacy classes of $\tilde{G}$. Since the group G is a normal subgroup of $\tilde{G}$, a conjugacy class of $\tilde{G}$ is either contained in G, or it does not contain any element of G. We say in the first case that the class is an inner class of $\tilde{G}$ and in the second case, that it is an outer class. An element (g, σ) (where $g \in G$) is called outer element of $\tilde{G}$. Outer classes only contain outer elements. We notice that two outer elements are conjugate in $\tilde{G}$ if and only if there are conjugate in G. Moreover, for all $g, g' \in G$, we have

$$(g, \sigma)(g', 1)(g, \sigma)^{-1} = (g\sigma(g')g^{-1}, 1). \tag{4}$$

Let $\tilde{c}$ be an inner class of $\tilde{G}$, then $\tilde{c}$ is a union of classes of G. The automorphism σ permutes the classes of G and relation (4) shows that $\tilde{c}$ is either a single class or the union of two classes of G. More precisely, if $(g, 1) \in \tilde{c}$, we denote by c the class of g in G. If $\sigma(g) \in c$, we have $\tilde{c} = (c, 1)$ and $|C_{\tilde{G}}(g, 1)| = 2|C_G(g)|$. If $\sigma(g) \notin c$, we have $\tilde{c} = (c \cup \sigma(c), 1)$ and $|C_{\tilde{G}}(g, 1)| = |C_G(g)|$.

This study shows that the inner classes are entirely determined by the classes of G. That is why the main difficulty is to characterize the outer classes. Clifford theory allows to evaluate the number of outer classes (see [9, p.64]). We have:

Proposition 3.1. *The number of σ-stable classes of G is the same as the number of outer classes of* $\tilde{G}$.

There is no systematic parameterization for the outer classes of $\tilde{G}$. However we will give in the following a method based on Jordan decomposition of the elements of a finite group.

3.1.1. Preliminary result. Let p be a prime number. We say that an element $g \in G$ is a p-element (resp. p-regular) if its order is a power of p (resp. prime to p). Let $g \in G$. We recall that there exists a p-regular element g_1 and a p-element g_2 such that g_1 and g_2 commute and $g = g_1 g_2$. Moreover, g_1 and g_2 are unique. Then $g_1 g_2$ is called the p-decomposition of Jordan of g. The element g_1 (resp. g_2) is the p'-part (resp. p-part) of g. We define the order of a class by the order of any element in the class. A class is called a *p-regular class* if its order is prime to p. It is said a *p-class* if its order is a power of p. We have:

Lemma 3.1. *Let G be a finite group and p a prime number. Let $x_1, x_2, \ldots, x_r$ be a system of representatives of p-regular classes of G and $y_{i,1}, y_{i,2}, \ldots, y_{i,r_i}$ a system of representatives of p-classes of* $C_G(x_i)$ $(i \in \{1, \ldots, r\})$.
1. *Then the set*

$$\{x_i y_{i,j} \mid 1 \le i \le r,\ 1 \le j \le r_i\}$$

is a system of representatives of conjugacy classes of G.

2. *Moreover, we have*:

$$\mathrm{C}_G(x_i y_{i,j}) = \mathrm{C}_G(x_i) \cap \mathrm{C}_G(y_{i,j}), \quad i = 1, \ldots, r, \ j = 1, \ldots, r_i.$$

3.1.2. Parameterization of outer classes of $\tilde{\mathrm{G}}$. Let us go back to our main problem. We give a method to parameterize the conjugacy classes of $\tilde{\mathrm{G}}$:

METHOD 3.1. The outer classes of $\tilde{\mathrm{G}}$ can be describe as follows:

- First, we determine representative of the σ-stable classes of G with odd order.
- Then, we determine the centralizers in $\tilde{\mathrm{G}}$ of these elements.
- Finally, we determine the outer classes of the 2-elements of these centralizers.

Proof. The outer classes of $\tilde{\mathrm{G}}$ have even order. We apply Lemma 3.1 with $p = 2$. The 2-regular elements are the elements of $\tilde{\mathrm{G}}$ which have odd order; in this case they belong to G. If g is a 2-regular part of an outer element, we notice that g and $\sigma(g)$ are conjugate in G. Thus g belongs to a σ-stable class of G with odd order. Furthermore, if h is such that (g, h) is a Jordan p-decomposition, then $h \in \mathrm{C}_{\tilde{\mathrm{G}}}(g, 1)$. Conversely, every element with odd order in a σ-stable class of G forms a Jordan's 2-decomposition with any 2-element in his centralizer in $\tilde{\mathrm{G}}$. □

REMARK 3.1. Let g be a σ-stable element of G with odd order. Then $\mathrm{C}_G(g)$ is σ-stable and we have

$$\mathrm{C}_{\tilde{\mathrm{G}}}(g, 1) = \mathrm{C}_G(g) \rtimes \langle \sigma \rangle.$$

To finish, we give a result when the centralizer of a σ-stable element with odd order is abelian:

Lemma 3.2. *Let g be a σ-stable element of G with odd order such that* $\mathrm{C}_G(g)$ *is abelian. We denote by* $(x_1, \sigma), \ldots, (x_r, \sigma)$ *a system of representatives of the outer classes of* 2*-elements of* $\mathrm{C}_{G\rtimes\langle\sigma\rangle}(g, 1)$. *Then* $\mathrm{C}_{G\rtimes\langle\sigma\rangle}(gx_i, \sigma)$ *is abelian and we have*

$$\mathrm{C}_{G\rtimes\langle\sigma\rangle}(gx_i, \sigma) = \mathrm{C}_G(g)^{\sigma}.\langle(x_i, \sigma)\rangle.$$

Proof. Since g is invariant under σ, we have $C_{G\rtimes\langle\sigma\rangle}(g) = C_G(g) \rtimes \langle\sigma\rangle$. Thus $x_i \in C_G(g)$. Let $k \in C_G(g)^{\sigma}$. Then $k(x_i, \sigma)k^{-1} = (kx_i k^{-\sigma}, \sigma) = (x_i, \sigma)$ because $C_G(g)$ is abelian. This proves that $C_G(g)^{\sigma}.\langle(x_i, \sigma)\rangle$ is abelian and that

$$C_G(g)^{\sigma}.\langle(x_i, \sigma)\rangle \subseteq C_{G\rtimes\langle\sigma\rangle}(gx_i, \sigma).$$

Conversely, we have two cases:

CASE 1. If $(k, 1) \in C_{G\rtimes\langle\sigma\rangle}(gx_i, \sigma)$ then $k \in C_G(g)$ and $(k, 1)(x_i, \sigma) = (x_i, \sigma)(k, 1)$, i.e.

$$kx_i = x_i\sigma(k).$$

Therefore, $x_i \in C_G(g)$ and $\sigma(k) \in C_G(g)$. Since $C_G(g)$ is abelian, it follows that $k = \sigma(k)$ (i.e. $k \in C_G(g)^\sigma$).

CASE 2. If $(k, \sigma) \in C_{G \rtimes \langle\sigma\rangle}(gx_i, \sigma)$ then we have $(k, \sigma)(g, 1) = (g, 1)(k, \sigma)$ (i.e. $k\sigma(g) = gk$) and $k \in C_G(g)$ (because $\sigma(g) = g$). Furthermore, we have $(k, \sigma)(x_i, \sigma) = (x_i, \sigma)(k, \sigma)$, i.e. $k\sigma(x_i) = x_i\sigma(k)$. Since k, $\sigma(k)$, x_i and $\sigma(x_i)$ commute, and we have

$$\sigma(kx_i^{-1}) = kx_i^{-1}.$$

Hence $kx_i^{-1} \in C_G(g)^\sigma$ (i.e. $k \in C_G(g)^\sigma x_i$). Thus there exists $h \in C_G(g)^\sigma$ such that $k = hx_i$. Therefore, $(k, \sigma) = (h, 1)(x_i, \sigma)$. We have proved that

$$C_{G \rtimes \langle\sigma\rangle}(gx_i, \sigma) = C_G(g)^\sigma \langle (x_i, \sigma)\rangle.$$

Since $C_G(g)^\sigma$ and $\langle (x_i, \sigma)\rangle$ commute, we deduce that

$$C_{G \rtimes \langle\sigma\rangle}(gx_i, \sigma) = C_G(g)^\sigma \times \langle (x_i, \sigma)\rangle / C_G(g)^\sigma \cap \langle (x_i, \sigma)\rangle. \qquad \square$$

3.2. Character table. Let $\phi \in \mathrm{C}(G)$. We define $\phi^\sigma \in \mathrm{C}(G)$ by $\phi^\sigma(g) = \phi(\sigma(g))$ $(g \in G)$. A class function $\phi \in \mathrm{C}(G)$ is called *σ-stable* if $\phi^\sigma = \phi$. Let ε be the linear character of $\tilde{\mathrm{G}}$ such that $\ker \varepsilon = G$. By Clifford theory we have (see [9, p.64]):

Proposition 3.2. *Let $\chi \in \mathrm{Irr}(G)$.*

– *If $\chi \neq \chi^\sigma$, then $\mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi) \in \mathrm{Irr}(\tilde{\mathrm{G}})$. Moreover, we have*

$$\forall g \in G, \quad \mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi)(g, \sigma) = 0 \quad \textit{and} \quad \mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi)(g, 1) = \chi(g) + \chi^\sigma(g).$$

– *If $\chi = \chi^\sigma$, then $\mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi)$ is the sum of exactly two distinct irreducible characters of $\tilde{\mathrm{G}}$ whose the restriction to G is χ. Moreover, if $\tilde{\chi}$ is a constituent of $\mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi)$, then we have $\mathrm{Ind}_G^{\tilde{\mathrm{G}}}(\chi) = \tilde{\chi} + \tilde{\chi}\varepsilon$.*

This proposition shows that the inner characters of $\tilde{\mathrm{G}}$ are all determined by the values of the non σ-stable characters of G. These characters are invariant under the multiplication by ε. Every σ-stable character of G has exactly two extensions. We obtain in this way all the irreducible characters of $\tilde{\mathrm{G}}$. Thus, to compute the character table of $\tilde{\mathrm{G}}$ it suffices to know the values of the extensions of the σ-stable characters of G on the outer classes of $\tilde{\mathrm{G}}$.

CONVENTION 3.1. Let $\psi \in \mathrm{Irr}(\tilde{\mathrm{G}})$. Then $\psi(1, \sigma) \in \mathbb{Z}$. Let χ be a σ-stable irreducible character of G such that its extensions to $\tilde{\mathrm{G}}$ is non-zero on $(1, \sigma)$. Then we denote by $\tilde{\chi}$ the extension of χ such that $\tilde{\chi}(1, \sigma) > 0$.

We now complete Proposition 3.1. We evaluate the number of σ-stable characters of $\tilde{\mathrm{G}}$ (and thus the number of outer irreducible characters of $\tilde{\mathrm{G}}$); see [9, p.65]:

Proposition 3.3 (bis). *The number of σ-stable irreducible characters of G is the number of σ-stable classes.*

3.2.1. The σ-reduction. We are interested in the following problem: suppose a generalized character of $\tilde{G}$ has non-trivial values on the outer classes, is it possible to work only on the outer classes? To answer this question, we introduce the notion of *σ-reduction*.

Let ψ and ψ' be two class functions on $\tilde{G}$. We set

$$\langle \psi, \psi' \rangle_\sigma = \frac{1}{|G|} \sum_{g \in G} \psi(g, \sigma) \overline{\psi'(g, \sigma)}. \tag{5}$$

Let ϕ be a σ-stable irreducible character of G. We denote by χ_ϕ an extension of ϕ to $\tilde{G}$. The irreducible characters of $\tilde{G}$ are

$$\{\chi \in \mathrm{Irr}(\tilde{G}) \mid \chi = \chi\varepsilon\} \cup \{\chi_\phi, \chi_\phi\varepsilon \mid \phi \in \mathrm{Irr}(G), \ \phi^\sigma = \phi\}.$$

We decompose ψ in the basis of irreducible characters:

$$\psi = \sum_{\chi = \chi\varepsilon} a_\chi \chi + \sum_{\phi = \phi^\sigma} (a_{\psi,\phi} \chi_\phi + a_{\psi,\phi\varepsilon} \chi_\phi \varepsilon).$$

We define *the σ-reduction of ψ*, denoted by $\rho(\psi)$, by

$$\rho(\psi) = \sum_{\substack{\phi = \phi^\sigma \\ a_{\psi,\phi} - a_{\psi,\phi\varepsilon} \geq 0}} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon}) \chi_\phi + \sum_{\substack{\phi = \phi^\sigma \\ a_{\psi,\phi} - a_{\psi,\phi\varepsilon} < 0}} (a_{\psi,\phi\varepsilon} - a_{\psi,\phi}) \chi_\phi \varepsilon.$$

Lemma 3.3. *Let $\psi \in \mathrm{C}(\tilde{G})$.*

– *Then, for every $g \in G$, we have $\rho(\psi)(g, \sigma) = \psi(g, \sigma)$. Moreover, we have*

$$\langle \psi, \psi' \rangle_\sigma = \sum_{\phi = \phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon})(a_{\psi',\phi} - a_{\psi',\phi\varepsilon}).$$

– *If $\psi \in \mathbb{Z}\,\mathrm{Irr}(\tilde{G})$ (resp. $\psi \in \mathbb{N}\,\mathrm{Irr}(\tilde{G})$), then $\rho(\psi) \in \mathbb{Z}\,\mathrm{Irr}(\tilde{G})$ (resp. $\rho(\psi) \in \mathbb{N}\,\mathrm{Irr}(\tilde{G})$).*

Proof. We have

$$\psi = \sum_{\chi = \chi\varepsilon} a_\chi \chi + \sum_{\phi = \phi^\sigma} a_{\psi,\phi\varepsilon} (\chi_\phi + \chi_\phi \varepsilon) + \sum_{\phi = \phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon}) \chi_\phi,$$

$$\psi\varepsilon = \sum_{\chi = \chi\varepsilon} a_\chi \chi + \sum_{\phi = \phi^\sigma} a_{\psi,\phi\varepsilon} (\chi_\phi + \chi_\phi \varepsilon) + \sum_{\phi = \phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon}) \chi_\phi \varepsilon.$$

Let $g \in G$, then

$$\psi(g, \sigma) = \sum_{\phi=\phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon})\chi_\phi(g, \sigma).$$

Since $(\chi_\phi\varepsilon)(g, \sigma) = -\chi_\phi(g, \sigma)$, it follows that $\psi(g, \sigma) = \rho(\psi)(g, \sigma)$. We have

$$\psi - \psi\varepsilon = \sum_{\phi=\phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon})(\chi_\phi - \chi_\phi\varepsilon).$$

Since

$$\langle \chi_\phi - \chi_\phi\varepsilon, \chi_{\phi'} - \chi_{\phi'}\varepsilon\rangle_{\tilde{G}} = 2\delta_{\phi,\phi'},$$

we deduce that

$$\langle \psi - \psi\varepsilon, \psi' - \psi'\varepsilon\rangle_{\tilde{G}} = 2\sum_{\phi=\phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon})(a_{\psi',\phi} - a_{\psi',\phi\varepsilon}).$$

Moreover, we have

$$\langle \psi - \psi\varepsilon, \psi' - \psi'\varepsilon\rangle_{\tilde{G}} = 2\langle \psi, \psi'\rangle_\sigma.$$

Finally, we deduce that

$$\langle \psi, \psi'\rangle_\sigma = \sum_{\phi=\phi^\sigma} (a_{\psi,\phi} - a_{\psi,\phi\varepsilon})(a_{\psi',\phi} - a_{\psi',\phi\varepsilon}).$$ □

Proposition 3.3. *Let χ be a character of $\tilde{G}$.*

– *If $\langle \chi, \chi\rangle_\sigma = 1$, then $\rho(\chi)$ is the extension of $\phi \in \mathrm{Irr}(G)$ such that*

$$\left\langle \mathrm{Res}_G^{\tilde{G}}\, \chi, \phi\right\rangle_G \equiv 1 \mod 2.$$

– *If $\langle \chi, \chi\rangle_\sigma = 2$, then $\rho(\chi)$ is the sum of exactly two distinct irreducible characters. Moreover $\phi \in \mathrm{Irr}(G)$ is a constituent of $\rho(\chi)$ if and only if $\left\langle \mathrm{Res}_G^{\tilde{G}}\, \chi, \phi\right\rangle_G \equiv 1 \bmod 2$.*

Proof. Let χ be a character of $\tilde{G}$. For every $\phi \in \mathrm{Irr}(G)$, we have

$$\left\langle \mathrm{Res}_G^{\tilde{G}}\, \chi, \phi\right\rangle_G = 2a_{\chi,\phi\varepsilon} + \langle \chi, \chi_\phi\rangle_\sigma.$$

Thus, if $\langle \chi, \chi\rangle_\sigma = 1$, then there exists $\phi \in \mathrm{Irr}(G)$ such that $a_{\chi,\phi} - a_{\chi,\phi\varepsilon} = \pm 1$ and for all $\phi' \neq \phi$, we have $a_{\chi,\phi'} - a_{\chi,\phi'\varepsilon} = 0$. Thus $\left\langle \mathrm{Res}_G^{\tilde{G}}\, \chi, \phi\right\rangle_G = 2a_{\chi,\phi\varepsilon} \pm 1$ and $\left\langle \mathrm{Res}_G^{\tilde{G}}\, \chi, \phi'\right\rangle_G = 2a_{\chi,\phi\varepsilon}$. We argue in a similar way if $\langle \chi, \chi\rangle_\sigma = 2$. Indeed, in this case, there exists ϕ_1 and ϕ_2 such that $a_{\chi,\phi_1} - a_{\chi,\phi_1\varepsilon} = \pm 1$ and $a_{\chi,\phi_2} - a_{\chi,\phi_2\varepsilon} = \pm 1$ and for all $\phi' \neq \phi_1, \phi_2$ we have $a_{\chi,\phi'} - a_{\chi,\phi'\varepsilon} = 0$. □

3.2.2. Two lemmas. We now give two lemmas that will be useful in the construction of the character table of $\tilde{\mathrm{G}}$. The first one in a consequence of Mackey's Theorem (see [12]) and the proof of the second one is immediate.

Lemma 3.4. *Let K be a subgroup of G invariant under σ. We define the subgroup $\tilde{K} = K \rtimes \langle\sigma\rangle$ of $\tilde{\mathrm{G}}$. Let $\tilde{\phi} \in \mathrm{C}(\tilde{K})$ and $\phi = \mathrm{Res}_G^{\tilde{\mathrm{G}}}\,\tilde{\phi}$. Then we have*

$$\mathrm{Res}_G^{\tilde{\mathrm{G}}}\,\mathrm{Ind}_{\tilde{K}}^{\tilde{\mathrm{G}}}\,\tilde{\phi} = \mathrm{Ind}_K^G\,\phi.$$

Lemma 3.5 (Construction of linear characters). *Let χ be a σ-stable linear character of G, then*

$$\tilde{\chi} : \tilde{\mathrm{G}} \to \mathbb{C}; \quad (g, x) \mapsto \chi(g)$$

is a linear character of $\tilde{\mathrm{G}}$.

4. Outer classes of $\mathrm{G}_2(q) \rtimes \langle\sigma\rangle$

We return to the situation in Section 2. Let $\mathbf{G}$ be a connected reductive group of type G_2 over $\overline{\mathbb{F}}_3$ with Weyl group W and $n \in \mathbb{N}$. We set $q = 3^{2n+1}$ and $\theta = 3^n$. We denote by F the endomorphism of $\mathbf{G}$ such that the fixed-point subgroup is the Ree group $\mathrm{R}(q)$. Then $\mathrm{G}_2(q) = \mathbf{G}^{F^2}$ is the finite group of Lie type type G_2 and the restriction σ of F to $\mathrm{G}_2(q)$ is an automorphism of order 2. We set $\tilde{\mathrm{G}} = \mathrm{G}_2(q) \rtimes \langle\sigma\rangle$. We use the notation of Section 2, completed by Enomoto's [8]. The conjugacy classes of $\mathrm{G}_2(q)$ (resp. $\mathrm{R}(q)$) are given in [8] (resp. [15]). We recall the classes of $\mathrm{G}_2(q)$ in Table 13.

In the general situation where $\mathbf{G}$ is a connected reductive group and F is a map on $\mathbf{G}$ such that $\mathbf{G}^F$ and $\mathbf{G}^{F^2}$ are finite subgroups, we recall that there exists a correspondence between the conjugacy classes of $\mathbf{G}^F$ and the outer classes of $\mathbf{G}^{F^2} \rtimes \langle F\rangle$ (the restriction of F to $\mathbf{G}^{F^2}$ is denoted here by the same symbol); this is the so-called *Shintani correspondence* [6]. Since the group $\mathrm{R}(q)$ has $(q+8)$ classes, it then follows that $\tilde{\mathrm{G}}$ has $(q + 8)$ outer classes. In this section, we propose to parameterize these classes.

4.1. The classes of $\mathrm{R}(q)$ and their distribution in $\mathrm{G}_2(q)$.

In [8], the conjugacy classes are parameterized with methods of algebraic groups. Thus, we will give a parameterization of this type for the classes of $\mathrm{R}(q)$, in order to be able to give their distribution in $\mathrm{G}_2(q)$.

4.1.1. Semisimple classes of $\mathrm{R}(q)$. First, we give representatives of conjugacy classes of maximal tori of $\mathrm{R}(q)$. Since the maximal torus $\mathbf{H}$ of $\mathbf{G}$ is F-stable, it follows that F induces an automorphism on W (denoted by F). We say that w, w' in W

are F-conjugate if $w' = xwF(x^{-1})$ for some $x \in W$. This defines an equivalence relation on W; the equivalence classes are called F-conjugacy classes of W.There are four F-conjugacy classes in W, with representatives is 1, w_a, $w_aw_bw_a$ and $w_aw_bw_aw_bw_a$. We denote by W_F this set.

In [10, §4.3.7], it is shown that there is a 1-1 correspondence between the F-classes of W and the $\mathbf{G}^F$-classes of maximal tori of $\mathbf{G}$. Let $w \in W_F$ and $n_w \in N$ such that $\pi_W(n_w) = w$. We denote by T_w a maximal torus of $\mathbf{G}$ associated to w. Then, we have

$$T_w^F \cong \mathbf{H}^{[w]} = \left\{t \in \mathbf{H} \;\middle|\; F(t) = n_w^{-1}tn_w\right\}. \tag{6}$$

We set $\mathrm{C}_{W,F}(w) = \{x \in W \mid x^{-1}wF(x) = w\}$.

Proposition 4.1. *We have*:

$$\begin{aligned}
T_1^F &\cong \left\{h\left(z^{3^{n+1}+1}, z, z^{-(3^{n+1}+2)}\right) \;\middle|\; z^{q-1} = 1\right\},\\
T_{w_aw_bw_a}^F &\cong \left\{h\left(z, \epsilon z^{-(3^{n+1}+1)/2}, \epsilon z^{(3^{n+1}-1)/2}\right) \;\middle|\; z^{(q+1)/2} = 1,\ \epsilon = \pm 1\right\},\\
T_{w_a}^F &\cong \left\{h\left(z, z^{1-3^{n+1}}, z^{3^{n+1}-2}\right) \;\middle|\; z^{q-3^{n+1}+1} = 1\right\},\\
T_{w_aw_bw_aw_bw_a}^F &\cong \left\{h\left(z^{1+3^{n+1}}, z, z^{-(3^{n+1}+2)}\right) \;\middle|\; z^{q+3^{n+1}+1} = 1\right\}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
C_{W,F}(1) &= \{1, w_aw_bw_aw_bw_aw_b\},\\
C_{W,F}(w_a) &= \{1, w_aw_b, w_aw_bw_aw_b, w_aw_bw_aw_bw_aw_b, w_bw_aw_bw_a, w_bw_a\},\\
C_{W,F}(w_aw_bw_a) &= C_{W,F}(w_a),\\
C_{W,F}(w_aw_bw_aw_bw_a) &= C_{W,F}(w_a).
\end{aligned}$$

Proof. We have:

w	$n_w^{-1}h(z_1, z_2, z_3)n_w$
w_a	$h(z_3^{-1}, z_2^{-1}, z_1^{-1})$
$w_aw_bw_a$	$h(z_1, z_3, z_2)$
$w_aw_bw_aw_bw_a$	$h(z_2^{-1}, z_1^{-1}, z_3^{-1})$

Then, by using (6), we compute $\mathbf{H}^{[1]}$, $\mathbf{H}^{[w_a]}$, $\mathbf{H}^{[w_aw_bw_a]}$ and $\mathbf{H}^{[w_aw_bw_aw_bw_a]}$.

To compute $C_{W,F}(w)$ (where $w \in W_F$) we use that $W = \langle w_a, w_b\rangle$ and that $F(w_a) = w_b$ and $F(w_b) = w_a$. □

A semisimple element $s \in \mathbf{G}$ is called regular if $\mathrm{C}_{\mathbf{G}}(s)$ is a maximal torus in $\mathbf{G}$. Given a regular semisimple element $s \in \mathbf{G}^F$, we say that s is of type w if s is conjugate to an element of T_w^F. Note that w is unique up to F-conjugation. A class of $\mathrm{R}(q)$ is of type w if it has a representative of type w

Proposition 4.2. *The classes of semisimple elements of* $\mathrm{R}(q)$ *are given as follows*:

- $(1/2)(q-3)$ *regular semisimple conjugacy classes of type* 1.
- $(1/6)(q-3)$ *regular semisimple conjugacy classes of type* $w_a w_b w_a$.
- $(1/6)(q-3^{n+1})$ *regular semisimple conjugacy classes of type* w_a.
- $(1/6)(q+3^{n+1})$ *regular semisimple conjugacy classes of type* $w_a w_b w_a w_b w_a$.
- *The class with representative* $J = h(1,-1,-1)$.
- *The class of identity.*

Proof. Let $w \in W_F$. Let $g \in \mathrm{R}(q)$ be a regular semisimple element of type w. Since g is semisimple, there exists a maximal torus $\mathbf{T}$ of $\mathbf{G}$ containing g, such that $F(\mathbf{T}) = \mathbf{T}$. Since g is regular, it follows that $\mathrm{C}_{\mathbf{G}}(g) = \mathbf{T}$. Thus $\mathbf{T}$ is the unique maximal torus of $\mathbf{G}$ containing g. Moreover $\mathbf{T}$ and T_w are conjugate by an element of $\mathrm{R}(q)$. Then, g is conjugate in $\mathrm{R}(q)$ to a unique element of T_w^F. Moreover, two elements of T_w^F are conjugate in $\mathrm{R}(q)$ if and only if they are conjugate in $\mathrm{N}_{\mathbf{G}}(T_w)^F$. Indeed, let $h, h' \in T_w^F$ and $g \in \mathrm{R}(q)$ such that $h' = ghg^{-1}$. Then $\mathrm{C}_{\mathbf{G}}(h') = g\mathrm{C}_{\mathbf{G}}(h)g^{-1}$, that is $T_w = gT_wg^{-1}$. It follows that $g \in \mathrm{N}_{\mathbf{G}}(T_w)$. Since $g \in \mathrm{R}(q)$, then $g \in \mathrm{N}_{\mathbf{G}}(T_w)^F$. Assume now that there exists $g, g' \in \mathrm{N}_{\mathbf{G}}(T_w)^F$ and $h \in T_w^F$ such that $ghg^{-1} = g'hg'^{-1}$. Then $g'^{-1}g \in \mathrm{C}_{\mathbf{G}}(h) = T_w$. Since $g, g' \in \mathrm{R}(q)$, we then have $g = g'T_w^F$. It follows that two elements of T_w^F are conjugate in $\mathrm{R}(q)$ if and only if they are in the same orbit for the action of $\mathrm{N}_{\mathbf{G}}(T_w)^F/T_w^F$ on T_w^F. We have $\mathrm{N}_{\mathbf{G}}(T_w)^F/T_w^F \cong \mathrm{C}_{W,F}(w)$ (see [10, §4.3.7]) and $T_w^F \cong \mathbf{H}^{[w]}$ (in $\mathbf{G}$). The group $\mathrm{C}_{W,F}(w)$ acts on $\mathbf{H}^{[w]}$; in [10, §4.3.7], it is proved that two elements of T_w^F are in the same $\mathrm{N}_{\mathbf{G}}(T_w)^F/T_w^F$-orbit if and only if their conjugate in $\mathbf{H}^{[w]}$ are in the same $\mathrm{C}_{W,F}(w)$-orbit. Proposition 4.1 gives the result. There are two other semisimple classes in $\mathrm{R}(q)$, which have 1 and $J = h(1,-1,-1)$ as representatives. By using [15], we see that $|\mathrm{C}_{\mathrm{R}(q)}(J)| = q(q-1)(q+1)$. ☐

4.1.2. Unipotent classes of R(q). For every $t, u, v \in \mathbb{F}_q$, we set

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})x_{2a+b}(t^{2\theta+1}),$$
$$\beta(u) = x_{a+b}(u^\theta)x_{3a+b}(u),$$
$$\gamma(v) = x_{2a+b}(v^\theta)x_{3a+2b}(v).$$

We have (see [3, Prop. 13.6.4]):

Proposition 4.3. *Every element of $U^\sigma = \mathbf{U}^F$ is uniquely written in the form $\alpha(t)\beta(u)\gamma(v)$ (t, u, v in $\mathbb{F}_q$) Moreover, we have the relations*:

$$
(7)\qquad
\begin{aligned}
\alpha(u)\alpha(v) &= \alpha(u+v)\beta(-uv^{3\theta})\gamma(-u^2v^{3\theta} + uv^{3\theta+1}),\\
\beta(u)\alpha(v) &= \alpha(v)\beta(u)\gamma(-uv),\\
\beta(u)\beta(v) &= \beta(u+v),\\
\gamma(u)\alpha(v) &= \alpha(v)\gamma(u),\\
\gamma(u)\beta(v) &= \beta(v)\gamma(u),\\
\gamma(u)\gamma(v) &= \gamma(u+v).
\end{aligned}
$$

We then have:

Lemma 4.1. *We have*:

$$
\begin{aligned}
\alpha(t)^{-1} &= \alpha(-t)\beta(-t^{3\theta+1})\gamma(t^{3\theta+2}),\\
\beta(t)^{-1} &= \beta(-t),\\
\gamma(t)^{-1} &= \gamma(-t),\\
\alpha(v)^{-1}\alpha(u)\alpha(v) &= \alpha(u)\beta(vu^{3\theta} - uv^{3\theta})\gamma(-v^2u^{3\theta} - u^2v^{3\theta} - uv^{3\theta+1} - vu^{3\theta+1}).
\end{aligned}
$$

Moreover, for every $h(z) = h(z^{3\theta+1}, z, z^{-(3\theta+2)}) \in \mathbf{H}^F = H^\sigma$, we have

$$\forall u, v, w, z \in \mathbb{F}_q, \quad h(z)\alpha(u)\beta(v)\gamma(w)h(z)^{-1} = \alpha(uz^{3\theta})\beta(vz^{3\theta+3})\gamma(wz^{6\theta+3}).$$

Proof. This is consequence of the relations (7). We only prove

$$
\begin{aligned}
&\alpha(v)^{-1}\alpha(u)\alpha(v)\\
&= \alpha(v)^{-1}\alpha(u+v)\beta(-uv^{3\theta})\gamma(-u^2v^{3\theta} + uv^{3\theta+1})\\
&= \alpha(-v)\beta(-v^{3\theta+1})\gamma(v^{3\theta+2})\alpha(u+v)\beta(-uv^{3\theta})\gamma(-u^2v^{3\theta} + uv^{3\theta+1})\\
&= \alpha(-v)\alpha(u+v)\beta(-uv^{3\theta} - v^{3\theta+1})\gamma(-u^2v^{3\theta} + uv^{3\theta+1} + v^{3\theta+2} + v^{3\theta+1}(u+v))\\
&= \alpha(u)\beta(v(u+v)^{3\theta})\gamma(-v^2(u+v)^{3\theta} - v(u+v)^{3\theta+1})\\
&\quad\times \beta(-uv^{3\theta} - v^{3\theta+1})\gamma(-u^2v^{3\theta} + uv^{3\theta+1} + v^{3\theta+2} + v^{3\theta+1}(u+v))\\
&= \alpha(u)\beta(-uv^{3\theta} - v^{3\theta+1} + v(u+v)^{3\theta})\\
&\quad\times \gamma(-u^2v^{3\theta} + uv^{3\theta+1} + v^{3\theta+2} + v^{3\theta+1}(u+v) - v^2(u+v)^{3\theta} - v(u+v)^{3\theta+1})\\
&= \alpha(u)\beta(-uv^{3\theta} + vu^{3\theta})\gamma(-u^2v^{3\theta} - v^2u^{3\theta} - uv^{3\theta+1} - vu^{3\theta+1}). \qquad \square
\end{aligned}
$$

We set $E = \{t^3 - t \mid t \in \mathbb{F}_q\}$. Then E is an additive subgroup of $\mathbb{F}_q$. We denote by $\pi_E : \mathbb{F}_q \to \mathbb{F}_q/E$ the homomorphism such that $\ker \pi_E = E$. Let $\xi \in \mathbb{F}_q$ such that $\pi_E(\xi) = 1$.

Proposition 4.4. *The elements* $\gamma(1)$, $\beta(1)$, $\beta(-1)$, $\alpha(1)$, $\alpha(1)\beta(\xi)$ *and* $\alpha(1)\beta(-\xi)$ *are representatives of the unipotent classes of* $\mathrm{R}(q)$.

Proof. Using the Bruhat decomposition (see [3, p.117]) we prove that two elements in U^σ are conjugate in $\mathrm{R}(q)$ if and only if they are conjugate in B^σ. Moreover if $u \in U^\sigma$, then $\mathrm{C}_{\mathrm{R}(q)}(u)$ is contained in B^σ. Consequently it is sufficient to determine the conjugacy classes of elements of U^σ in B^σ. Then, using the Lemma 4.1 and the fact that U^σ is normal in B^σ, we first prove that the elements 1, $\gamma(1)$, $\beta(1)$ and $\beta(-1)$ are not conjugate. Moreover, if $u \neq 0$ we prove that $\alpha(u)\beta(v)\gamma(w)$ ($v, w \in \mathbb{F}_q$) is conjugate to $\alpha(1)\beta(\pi_E(v))$ and the result follows. □

REMARK 4.1. The three classes of $R(q)$ parameterized in [15] by Y, YT and YT^{-1} are the same as the classes in this paper with representatives $\alpha(1)$, $\alpha(1)\beta(\xi)$ and $\alpha(1)\beta(-\xi)$. But we do not *a priori* make the identification between these two parameterizations.

4.1.3. Conjugacy classes of R(*q*). Now, we give the conjugacy classes of the Ree group.

Theorem 4.1. *The Ree group* $\mathrm{R}(q)$ *has* $(q+8)$ *conjugacy classes. The representatives of the non trivial classes are*

- *The representatives of classes of type* w *(with* $w \in W_F$*) with centralizer of order* $\left|T_w^F\right|$.
- *The element* $J = h(1, -1, -1)$ *with centralizer of order* $q(q-1)(q+1)$.
- *The element* $\gamma(1)$ *with centralizer* $\mathbf{U}^F = U^\sigma$ *of order* q^3.
- *The elements* $\beta(1)$ *and* $\beta(-1)$ *with centralizer (of order* $2q^2$*):*

$$\langle J\rangle\langle \beta(t), \gamma(u) \mid t, u \in \mathbb{F}_q\rangle.$$

- *The elements* $\alpha(1)$, $\alpha(1)\beta(\xi)$ *and* $\alpha(1)\beta(-\xi)$ *with respective centralizer (of order* $3q$*):*

$$\langle\alpha(1)\rangle\Gamma, \quad \langle\alpha(1)\beta(\xi)\rangle\Gamma \quad \text{and} \quad \langle\alpha(1)\beta(-\xi)\rangle\Gamma,$$

where $\Gamma = \langle\gamma(t), t \in \mathbb{F}_q\rangle$.

- *The elements* $\beta(1)J$ *and* $\beta(-1)J$ *with centralizer (of order* $2q$*):*

$$\langle J\rangle\langle\beta(t), t \in \mathbb{F}_q\rangle.$$

Proof. In [15] it is proved that $\mathrm{R}(q)$ has $(q+8)$ classes. It is then sufficient to find $(q+8)$ elements that are not conjugate. Proposition 4.2 gives $(q-2)$ elements. Using Proposition 4.4, we obtain 6 new elements. These elements have odd order. We find easily their centralizer. Only the centralizers of $\beta(1)$ and $\beta(-1)$ have even order, and J is a representative of the only 2-semisimple class of this group. By Proposition 3.1, we obtain two new classes of R(q), with representatives $\beta(1)J$ and $\beta(-1)J$. The trivial class gives one more class, thus there is one class missing, with representative J. □

4.1.4. Distribution of the classes of R(q) in $G_2(q)$. We use the notation of Table 13. We set $h_0 = h(-1, 1, -1)$. We have:

Proposition 4.5. *The elements $\beta(1)$ and $\beta(-1)$ (resp. $\beta(1)J$ and $\beta(-1)J$) belong in the class A_{42} of $G_2(q)$ (resp. in B_5); see* Table 13 *for the notations. The other representatives of classes of* R(q) *are not conjugate in* $G_2(q)$.

Proof. By using Table 13, we show that the elements of type w ($w \in W_F$) and the elements 1, J and $\gamma(1)$ lie in different classes of $G_2(q)$. Moreover, we have $h_0\beta(1)h_0^{-1} = \beta(-1)$ and $h_0\beta(1)Jh_0^{-1} = \beta(-1)J$. By using the Chevalley relations (1), we show that for every $u, v, w, t \in \mathbb{F}_q$, the element $x_a(1)x_b(1)x_{a+b}(u)x_{3a+b}(v)x_{2a+b}(w)x_{3a+2b}(t)$ is conjugate in $G_2(q)$ to

$$x_a(1)x_b(1)x_{3a+b}(\pi_E(u+v)).$$

Therefore, we have

$$\begin{aligned}\alpha(1) &= x_a(1)x_b(1)x_{a+b}(1)x_{2a+b}(1),\\ \alpha(1)\beta(\xi) &= x_a(1)x_b(1)x_{a+b}(1+\xi^\theta)x_{3a+b}(\xi)x_{2a+b}(1),\\ \alpha(1)\beta(-\xi) &= x_a(1)x_b(1)x_{a+b}(1-\xi^\theta)x_{3a+b}(-\xi)x_{2a+b}(1).\end{aligned}$$

Moreover, we have

$$\xi^\theta = (\xi + \xi^3 + \cdots + \xi^{\theta/3})^3 - (\xi + \xi^3 + \cdots + \xi^{\theta/3}) + \xi.$$

Then, we have $\pi_E(\xi^\theta) = \pi_E(\xi)$. It follows that $\pi_E(\xi + \xi^\theta + 1) = \pi_E(1) - 1$ and $\pi_E(-\xi - \xi^\theta + 1) = \pi_E(1) + 1$. The result is proved. □

4.2. Maximal σ-stable torus of $G_2(q)$ of order $(q+1)^2$. In this section, we construct a maximal σ-stable torus of order $(q+1)^2$, which is contained in $C_{G_2(q)}(J)$. We use the same notation as previously. We have

$$C_{G_2(q)}(J) = \langle H, X_{\pm(a+b)}, X_{\pm(3a+b)}\rangle.$$

Let $r \in \{a+b, 3a+b\}$. We set $S_r = \langle X_{-r}, X_r\rangle$ and $L = S_{a+b}S_{3a+b}$. For every x, y, z, t in $\mathbb{F}_q$ such that $xz - yt = 1$, we set

$$\varphi_r(x, y, z, t) = \varphi_r\left(\begin{bmatrix} x & y \\ t & z \end{bmatrix}\right).$$

We set $\lambda(r) = 1$ if r is short and $\lambda(r) = 3$ if r is long.

Proposition 4.6. *We have*:

1. *The groups* S_{a+b} *and* S_{3a+b} *commute. Then the set* L *is a subgroup of* $C_{G_2(q)}(J)$.
2. *Let* $r \in \{a+b, 3a+b\}$ *and* $x, y, z, t \in \mathbb{F}_q$. *We have the relations*:

$$\varphi_r(x, y, z, t) = x_r((x-1)t^{-1})x_{-r}(t)x_r((z-1)t^{-1}) \quad \text{if} \quad t \neq 0.$$

$$\varphi_r(x, y, z, t) = x_{-r}((z-1)y^{-1})x_r(y)x_{-r}((x-1)y^{-1}) \quad \text{if} \quad y \neq 0.$$

$$\varphi_r(x, 0, x^{-1}, 0) = h_r(x).$$

3. *Moreover, if* $\chi \in \mathcal{P}$ *and* $r, s \in \{a+b, 3a+b\}$, *then we have the relations*:

$$h(\chi)\varphi_r(x, y, z, t)h(\chi)^{-1} = \varphi_r(x, \chi(r)y, z, \chi(r)^{-1}t),$$

$$n_r\varphi_r(x, y, z, t)n_r^{-1} = \varphi_r(z, -t, x, -y),$$

$$n_s\varphi_r(x, y, z, t)n_s^{-1} = \varphi_r(x, y, z, t) \quad \text{if} \quad s \neq r.$$

4. *Finally, if* $r \in \{a+b, 3a+b\}$, *we have*

$$\varphi_r(x, y, z, t)^{\sigma} = \varphi_{\overline{r}}(x^{\lambda(\overline{r})\theta}, y^{\lambda(\overline{r})\theta}, z^{\lambda(\overline{r})\theta}, t^{\lambda(\overline{r})\theta}).$$

Proof. This is the a consequence of the relations (1), (2) and (3). □

Since q is odd, every generator μ of $\mathbb{F}_q^{\times}$ is not a square in $\mathbb{F}_q$. Let $\sqrt{\mu} \in \mathbb{F}_{q^2}$ be a square root of μ. Thus we have the identification:

$$K_{\mu} = \left\{ \begin{bmatrix} x & \mu y \\ y & x \end{bmatrix} \right\} \leftrightarrow \zeta = x + y\sqrt{\mu},$$

where x and y are in $\mathbb{F}_q$ such that $x^2 - \mu y^2 = 1$. It follows that K_{μ} is a cyclic subgroup of $SL_2(\mathbb{F}_q)$ with order $q+1$.

Let γ be a generator of $\mathbb{F}_q^{\times}$. We fix $\sqrt{\gamma}$ a square root of γ in $\mathbb{F}_{q^2}$. Since $(q-1)$ is prime to 3, it follows that $\gamma^{3\theta}$ is a generator of $\mathbb{F}_q^{\times}$ and $(\sqrt{\gamma})^{3\theta}$ is a square root of $\gamma^{3\theta}$. Let $x, y \in \mathbb{F}_q$ such that $x^2 - \gamma y^2 = 1$. We set $t_{a+b}(x, y) = \varphi_{a+b}(x, \gamma y, x, y)$ and $t_{3a+b}(x, y) = \varphi_{3a+b}(x, \gamma^{3\theta} y, x, y)$. By using the Proposition 4.6, we show:

Lemma 4.2. *We have*

$$t_r(x, y)^{-1} = t_r(x, -y),$$

$$h(\chi)t_r(x, y)h(\chi)^{-1} = t_r(x, \chi(r)y),$$

$$t_r(x, y)^{\sigma} = t_{\overline{r}}(x^{\lambda(\overline{r})\theta}, y^{\lambda(\overline{r})\theta}).$$

We set

$$T_{a+b} = \varphi_{a+b}(K_{\gamma}),$$

$$T_{3a+b} = \varphi_{3a+b}(K_{\gamma^{3\theta}}).$$

The subgroup $T = T_{a+b}T_{3a+b}$ of $C_{G_2(q)}(J)$ is abelian and $T_{a+b} \cap T_{3a+b} = \{1, J\}$. Thus it follows that $|T| = (1/2)(q+1)^2$.

We set $T_\sigma = \{t_{a+b}(x, y)t_{3a+b}(x^{3\theta}, y^{3\theta}) \mid x, y \in \mathbb{F}_q x^2 - y^2\gamma = 1\}$. Let $x_0 \in T_\sigma$ with a non-trivial odd order (such an element exists when $q > 3$. In the case where $q = 3$, $G_2(q)$ has no tori of order $(q+1)^2$). We set $H' = C_{G_2(q)}(x_0)$.

Proposition 4.7. *We have*:

- *The set T_σ is an abelian subgroup of* $R(q)$ *of order* $(1/2)(q+1)$.
- *The element x_0 lies in* $R(q)$ *and is a semisimple regular element of type $w_a w_b w_a$.*
- *The subgroup H' is a maximal σ-stable torus of* $G_2(q)$ *of order* $(q+1)^2$, *which is contained in* $C_{G_2(q)}(J)$.

Proof. Obviously, T_σ is an abelian subgroup of order $(1/2)(q+1)$ and we have $T_\sigma \subseteq R(q)$. Thus $x_0 \in R(q)$. Comparing the order of x_0 and the order of elements in Theorem 4.1, we see that x_0 is a semisimple regular element of type $w_a w_b w_a$. Thus, using Table 13, we see that semisimple regular elements of non-trivial odd order dividing $(q+1)$ have a maximal torus of $G_2(q)$ of order $(q+1)^2$ as centralizer. Since $\sigma(x_0) = x_0$, it follows that H' is σ-stable. Moreover, since $x_0 \in C_{G_2(q)}(J)$, it follows that $J \in H'$. But H' is abelian, then every elements of H' commute with J. Thus, $H' \subseteq C_{G_2(q)}(J)$. ☐

We now will describe explicitly H'. The elements γ^{-1} and -1 are not square in $\mathbb{F}_q$. It follows that $-\gamma^{-1}$ is a square. Then there exists $a_0 \in \mathbb{F}_q$ such that $a_0^2\gamma = -1$, and we have $a_0^{6\theta}\gamma^{3\theta} = -1$. We set

$$A_0 = \begin{bmatrix} 1 & -\gamma a_0 \\ a_0 & -1 \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} 1 & -\gamma^{3\theta} a_0^{3\theta} \\ a_0^{3\theta} & -1 \end{bmatrix}.$$

We have $\det(A_0) = \det(B_0) = 1$. We define the element τ by

$$\tau = h(-1, 1, -1)\varphi_{a+b}(A_0)\varphi_{3a+b}(B_0).$$

Let $\zeta = x_\zeta + y_\zeta\sqrt{\gamma}$ be a generator of K_γ and $t_{a+b}(x_\zeta, y_\zeta)$ its corresponding element in T_{a+b}. Then $t_{a+b}(x_\zeta, y_\zeta)$ (resp. $\sigma(t_{a+b}(x_\zeta, y_\zeta))$) is a generator of T_{a+b} (resp. T_{3a+b}). We set $g_{a+b} = t_{a+b}(x_\zeta, y_\zeta)^4$, $g_{3a+b} = t_{3a+b}\left(x_\zeta^{3\theta}, y_\zeta^{3\theta}\right)^4$ and $\tau' = t_{3a+b}\left(x_\zeta^{3\theta}, y_\zeta^{3\theta}\right)^{(q+1)/4}$.

Proposition 4.8. *We have*

$$H' = \langle \tau g_{a+b} \rangle \times \langle \tau' g_{3a+b} \rangle.$$

Moreover we have $\sigma(\tau) = h(-1, -1, 1)\varphi_{a+b}(A_0)\varphi_{3a+b}(B_0) = J\tau$. *The σ-stable elements of H' of odd order form a cyclic subgroup of order* $(1/4)(q+1)$ *generated by* $g_{a+b}g_{3a+b}$ *and denoted by* H'_σ.

Proof. Since the elements of T commute with x_0, it follows that T is a subgroup of H' with index 2. The element τ is not in T, and commutes with the elements of T. Moreover the order of τ is 4 and $\tau^2 \neq J$. We then have proved the result. □

4.3. The σ-stable classes of $\mathrm{G}_2(q)$. We recall that the classes of $\mathrm{G}_2(q)$ are given in Table 13. We set $X = \gamma(1)$ and $T = \beta(1)$. Using Proposition 4.1, we see that $\alpha(1)$, $\alpha(1)\beta(\xi)$ and $\alpha(1)\beta(-\xi)$ are not conjugate in $\mathrm{G}_2(q)$. Their distribution in the classes of $\mathrm{G}_2(q)$ depends on the value of n modulo 3. In the following, we denote by Y_1, Y_2 and Y_3 a permutation of these elements such that Y_1 (resp. Y_2 and Y_3) belongs to A_{51} (resp. A_{52} and A_{53}). We denote by H'' a maximal σ-stable torus of G of order $(q^2 - q + 1)$.

Proposition 4.9. *The group $\mathrm{G}_2(q)$ has $(q+8)$ σ-stable classes. More precisely*:

- *The classes with the σ-stable representatives 1, X, T, Y_1, Y_2, Y_3, J and JT.*
- *The classes A_{41} and B_4 (without σ-stable representatives).*
- *The $(1/2)(q-3)$ classes with σ-stable representative x (of type 1) in H and such that $\mathrm{C}_{\mathrm{G}_2(q)}(x) = H$.*
- *The $(1/6)(q-3)$ classes with σ-stable representative x (of type $w_a w_b w_a$) in H' and such that $\mathrm{C}_G(x) = H'$.*
- *The $(1/3)q$ classes with σ-stable representative x (of type w_a or $w_a w_b w_a w_b w_a$) in H'' and such that $\mathrm{C}_G(x) = H''$.*

Proof. Using Proposition 3.1, Shintani correspondence and Proposition 4.1, we show that $\mathrm{G}_2(q)$ has $(q+8)$ σ-stable classes. Proposition 4.5 gives $(q+6)$ σ-stable representatives of $\mathrm{G}_2(q)$. We obtain the two classes without σ-stable representatives by using Table 13). □

REMARK 4.2. The fact that $\mathrm{G}_2(q)$ has σ-stable classes wihout σ-stable elements does not contradict Lang's theorem. Indeed, let C be the class in **G** which contains A_{41} (this is similar to B_4). Then C is F-stable. Lang's theorem only says that C has F-stable elements. It follows that $C^{F^2} = A_{41} \cup A_{42}$ has F-stable elements. This is indeed the case: $\beta(1) \in C^{F^2} \cap \mathrm{R}(q)$. But Lang's theorem does not say that A_{41} and A_{42} have F-stable elements. In Proposition 4.5, we prove that $\mathrm{Cl}(\beta(-1)) \subseteq A_{42}$ and $\mathrm{Cl}(\beta(1)) \subseteq A_{42}$. Thus $C^F = \mathrm{Cl}(\beta(-1)) \cup \mathrm{Cl}(\beta(1)) \subseteq A_{42}$. Then, it follows that A_{41} has no F-stable elements.

We denote by L_1 and L_{aba} a system of representatives of odd order of classes of type 1 and $w_a w_b w_a$ given in Proposition 4.9. We denote by L_a and L_{ababa} systems of representatives of classes of type w_a and $w_a w_b w_a w_b w_a$. We have:

$$|L_1| = \frac{q-3}{4}, \quad |L_{aba}| = \frac{q-3}{24}, \quad |L_a| = \frac{q-3\theta}{6} \quad \text{and} \quad |L_{ababa}| = \frac{q+3\theta}{6}.$$

We set $\eta = x_{a+b}(1)x_{3a+b}(-1)$. Consequently, we have:

Corollary 4.1. *The elements of L_1, L_a, L_{aba}, L_{ababa} and 1, X, T, Y_1, Y_2, Y_3 and η form a system of representatives of classes of $\mathrm{G}_2(q)$ of odd order.*

4.4. Outer classes of $\mathbf{G_2(q)} \rtimes \langle\sigma\rangle$. We now can give the outer classes of $\tilde{\mathrm{G}}$. We keep the same notation as the preceding sections.

Theorem 4.2. *We set $S = \mathrm{C}_{\tilde{\mathrm{G}}}(h_0, \sigma)$. Let H_0 (resp. T_0') be the subgroup of H^σ (resp. H') whose elements have odd order. Let $x \in \{\tau^2, J, J\tau^2\}$. We set $H_x = \langle T_0', x\rangle$. Moreover, we set*

$$\begin{aligned} C_{a+b,2a+b} &= X_{a+b}X_{3a+b}X_{2a+b}X_{3a+2b}, \\ C'_{a+b,2a+b} &= \{x_{a+b}(-u)x_{3a+b}(-u^{3\theta})x_{2a+b}(-t)x_{3a+2b}(t^{3\theta}) \mid u, t \in \mathbb{F}_q\}, \\ S_{a+b,3a+b} &= \{x_{a+b}(t)x_{3a+b}(-t^{3\theta}) \mid t \in \mathbb{F}_q\}, \\ S_{a,b} &= \{x_{2a+b}(t)x_{3a+b}(t^{3\theta}) \mid t \in \mathbb{F}_q\}. \end{aligned}$$

Then Table 2 *gives representatives for the outer classes of $\tilde{\mathrm{G}}$ and their centralizers.*

Proof. We essentially apply the method of Method 3.1. The σ-stable representatives with odd order are given in Corollary 4.1. If $g \in \{X, Y_1, Y_2, Y_3\} \cup L_a \cup L_{ababa}$, then $\mathrm{C}_{\mathrm{G}_2(q)}(g)$ has odd order. Then $\mathrm{C}_{\tilde{\mathrm{G}}}(g)$ has only one 2-unipotent class with representative σ. Let $h \in L_1$. Then $\mathrm{C}_{\tilde{\mathrm{G}}}(h) = H \rtimes \langle\sigma\rangle$. This group has two 2-unipotent outer classes with representatives σ and (h_0, σ). Let $h \in L_{aba}$. Then $\mathrm{C}_{\tilde{\mathrm{G}}}(h) = H' \rtimes \langle\sigma\rangle$. Representatives of 2-unipotent outer classes of this group are σ, (τ, σ), (τ', σ) and $(\tau\tau', \sigma)$ (we use results of Section 4.2). We have $\mathrm{C}_G(T) = \mathrm{C}_{a+b,2a+b}.\langle J\rangle$. Outer 2-unipotent representatives of this group are $(1, \sigma)$ and (J, σ). Finally, $\mathrm{C}_{\tilde{\mathrm{G}}}(\eta) = C_{a+b,2a+b} \rtimes \langle(h_0, \sigma)\rangle$. Outer 2-unipotent representatives of this group are (h_0, σ) and (Jh_0, σ). We obtain in this way $(q + 6)$ outer classes of $\tilde{\mathrm{G}}$. Moreover, by using Shintani correspondence, we know that $\tilde{\mathrm{G}}$ has $(q + 8)$ outer classes. Elements σ and (h_0, σ) are 2-unipotent and not conjugate in $\tilde{\mathrm{G}}$. We then obtain representatives of outer classes of $\tilde{\mathrm{G}}$. To compute the centralizer of these elements, we use Lemma 3.2. □

5. Outer characters of $\mathbf{G_2(q)} \rtimes \langle\sigma\rangle$

We keep the notation of Section 4. We propose in this section to compute the character table of $\tilde{\mathrm{G}}$. In Section 3, we have seen that it is sufficient to determine the values of the outer characters on the outer classes of $\tilde{\mathrm{G}}$. The character table of $\mathrm{G}_2(q)$ (resp. $\mathrm{R}(q)$) is given in Table 14 (resp. in Table 15). The subgroups B, H' and $\mathrm{R}(q)$ are σ-stable. By inducing the characters of $\tilde{B} = B \rtimes \langle\sigma\rangle$ and $\mathrm{R}(q) \times \langle\sigma\rangle$, we obtain the majority of irreducible outer characters of $\tilde{\mathrm{G}}$. We use modular methods to complement

Table 2. Outer classes of $\tilde{G}$

Class	Representative		Number	Centralizer	Order of Centralizer
C_1	$(1, \sigma)$		1	$R \times \langle \sigma \rangle$	$2q^3(q^2-1)(q^2-q+1)$
C_2	(h_0, σ)		1	S	$2q(q^2-1)$
D_1	(X, σ)		1	$U^\sigma \times \langle \sigma \rangle$	$2q^3$
D_{21}	(T, σ)		1	$C_{a+b,3a+b}.\langle J \rangle \times \langle \sigma \rangle$	$4q^2$
D_{22}	(T^{-1}, σ)		1	$C'_{a+b,3a+b}.\langle J \rangle \times \langle \sigma \rangle$	$4q^2$
D_{31}	$(\eta h_0, \sigma)$		1	$S_{a+b,3a+b} \times \langle (h_0, \sigma) \rangle$	$4q$
D_{32}	$(\eta^{-1} h_0, \sigma)$		1	$S_{a+b,3a+b} \times \langle (J h_0, \sigma) \rangle$	$4q$
D_{41}	(Y_1, σ)		1	$\langle Y_1 \rangle S_{a,b} \times \langle \sigma \rangle$	$6q$
D_{42}	(Y_2, σ)		1	$\langle Y_2 \rangle S_{a,b} \times \langle \sigma \rangle$	$6q$
D_{43}	(Y_3, σ)		1	$\langle Y_3 \rangle S_{a,b} \times \langle \sigma \rangle$	$6q$
$E_1(h)$	(h, σ)	$h \in L_1$	$\frac{q-3}{4}$	$H^\sigma \times \langle \sigma \rangle$	$2(q-1)$
$E_2(h)$	(hh_0, σ)	$h \in L_1$	$\frac{q-3}{4}$	$H_0^\sigma \times \langle (h_0, \sigma) \rangle$	$2(q-1)$
$F_1(h)$	(h, σ)	$h \in L_{aba}$	$\frac{q-3}{24}$	$T^F_{\omega_a \omega_b \omega_a} \times \langle \sigma \rangle$	$2(q+1)$
$F_2(h)$	$(h\tau, \sigma)$	$h \in L_{aba}$	$\frac{q-3}{24}$	$H_{\tau^2} \times \langle (\tau, \sigma) \rangle$	$2(q+1)$
$F_3(h)$	$(h\tau', \sigma)$	$h \in L_{aba}$	$\frac{q-3}{24}$	$H_J \times \langle (\tau', \sigma) \rangle$	$2(q+1)$
$F_4(h)$	$(h\tau\tau', \sigma)$	$h \in L_{aba}$	$\frac{q-3}{24}$	$H_{J\tau^2} \times \langle (\tau\tau', \sigma) \rangle$	$2(q+1)$
$G_1(h)$	(h, σ)	$h \in L_a$	$\frac{q-3\theta}{6}$	$T^F_{\omega_a} \times \langle \sigma \rangle$	$2(q-3\theta+1)$
$H_1(h)$	(h, σ)	$h \in L_{ababa}$	$\frac{q+3\theta}{6}$	$T^F_{\omega_a \omega_b \omega_a \omega_b \omega_a} \times \langle \sigma \rangle$	$2(q+3\theta+1)$

the missing values of one character, and we can then obtain all the missing irreducible characters of $\tilde{G}$ by inducing the irreducible characters of $H' \rtimes \langle\sigma\rangle$.

By Proposition 3.1 there are $(q + 8)$ σ-stable characters. We use the notation of appendix. Using the character table of $G_2(q)$, we prove that 1, θ_1, θ_2, θ_5, θ_6, θ_7, θ_{10}, θ_{11}, $\theta_{12}(1)$ and $\theta_{12}(-1)$ are σ-stable. Moreover $(1/2)(q - 3)$ (resp. $(1/6)(q - 3)$ and $(1/6)q$) σ-stable characters belong to the families χ_9 (resp. χ_{12} and χ_{14}). The notations will be specified in relations (8), (9) and (12).

5.1. Some irreducible characters obtained by induction from $\tilde{B}$. The classes of B are given in [8, p.206].

Proposition 5.1. *We set $\tilde{B} = B \rtimes \langle\sigma\rangle$; the group $\tilde{B}$ has $(q+7)$ outer classes, given in* Table 3. *Moreover, in* Table 4 *we give the induction formula from $\tilde{B}$ to $\tilde{G}$.*

Proof. Using [8], we prove that B has $(q+7)$ σ-stable classes. Thus, Lemma 3.1 implies that $\tilde{B}$ has $(q + 7)$ outer classes. By using a similar method as in §4.4, we obtain representatives of outer classes of $\tilde{B}$ and their distribution in classes of $\tilde{G}$. Then we immediately deduce the induction formula. □

We set

$$\chi_k = \chi_9(k, (3\theta - 1)k). \tag{8}$$

We denote by E_0 the set $\{1, \ldots, (1/2)(q - 3)\}$.

Proposition 5.2. *Extensions of χ_k $(k \in E_0)$ are obtained from the induction of the linear characters of $\tilde{B}$. The values of $\tilde{\chi}_k$ are given in* Table 11.

Proof. Since U is σ-stable, it follows that $U \lhd \tilde{B}$. Let π_U be the projection from $\tilde{B}$ to $\tilde{B}/U$. The quotient is isomorphic to $H \rtimes \langle\sigma\rangle$. Let ϕ be a linear character of $H \rtimes \langle\sigma\rangle$. Then $\phi \circ \pi_U$ is a linear character of $\tilde{B}$. Irreducible characters of $H \rtimes \langle\sigma\rangle$ are parameterized by $\phi_{k,l}$ $(k, l \in \mathbb{Z}/(q - 1)\mathbb{Z})$. The values are given by

$$\phi_{k,l}(h(\gamma^i, \gamma^j, \gamma^{-i-j})) = \gamma_0^{ik+jl}.$$

We have $\phi_{k,l}^{\sigma} = \phi_{k,l}$ if and only if $\gamma_0^{3^n(2j+i)k+3^n(i-j)l} = \gamma_0^{ik+jl}$ if and only if

$$\begin{cases} 3^n(k + l) = k \\ 3^n(2k - l) = l \end{cases}.$$

Then, we deduce that σ-stable characters of H are the $\phi_{k,(3^{n+1}-1)k}$, where $k \in \mathbb{Z}/(q - 1)\mathbb{Z}$.

Table 3. Outer classes of $\tilde{B}$

Class	reprsentative	Number	Order centraliser
c_1	$(1, \sigma)$	1	$2q^3(q-1)$
c_2	(h_0, σ)	1	$2q(q-1)$
d_1	$(x_{2a+b}(1)x_{3a+2b}(1), \sigma)$	1	$2q^3$
d_{21}	$(x_{a+b}(1)x_{3a+b}(1), \sigma)$	1	$4q^2$
d_{22}	$(x_{a+b}(1)x_{3a+b}(1)J, \sigma)$	1	$4q^2$
d_{31}	$(x_{a+b}(1)x_{3a+b}(-1)h_0, \sigma)$	1	$4q$
d_{32}	$(x_{a+b}(1)x_{3a+b}(-1)Jh_0, \sigma)$	1	$4q$
d_{41}	$(x_a(1)x_b(1)u_0, \sigma)$	1	$6q$
d_{42}	$(x_a(1)x_b(1)x_{3a+b}(\xi)u_1, \sigma)$	1	$6q$
d_{43}	$(x_a(1)x_b(1)x_{3a+b}(-\xi)u_2, \sigma)$	1	$6q$
$e_{11}(h)$	$(h, \sigma)\ h \in L_1$	$\frac{1}{4}(q-3)$	$2(q-1)$
$e_{12}(h)$	$(h^{-1}, \sigma)\ h \in L_1$	$\frac{1}{4}(q-3)$	$2(q-1)$
$e_{21}(h)$	$(hh_0, \sigma)\ h \in L_1$	$\frac{1}{4}(q-3)$	$2(q-1)$
$e_{22}(h)$	$(h^{-1}h_0, \sigma)h \in L_1$	$\frac{1}{4}(q-3)$	$2(q-1)$

Table 4. Induction formula for the outer classes of $\tilde{B}$ to $\tilde{\mathrm{G}}$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(C_1) = (q^3+1)\phi(c_1)$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(C_2) = (q+1)\phi(c_2)$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_1) = \phi(d_1)$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{21}) = \phi(d_{21})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{22}) = \phi(d_{22})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{31}) = \phi(d_{31})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{32}) = \phi(d_{32})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{41}) = \phi(d_{41})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{42}) = \phi(d_{42})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(D_{43}) = \phi(d_{43})$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(E_1(h)) = \phi(e_{11}(h)) + \phi(e_{12}(h))$$

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{\mathrm{G}}}\,\phi(E_2(h)) = \phi(e_{21}(h)) + \phi(e_{22}(h))$$

By Lemma 3.5, we construct $(q-1)$ irreducible outer characters $\tilde{\phi}_{k,(3^{n+1}-1)k}$ of $H \rtimes \langle\sigma\rangle$. We denote by

$$\phi_k = \tilde{\phi}_{k,(3^{n+1}-1)k} \circ \pi_U,$$

the $(q-1)$ linear characters of $\tilde{B}$. We denote by h_γ a generator of H^σ. The set of σ-stable elements of H with odd order is $\langle h_\gamma^2\rangle$. For every $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $\epsilon = \pm 1$, we have

$$\begin{aligned}\phi_k(h_\gamma^{2i}h_0^\epsilon, \sigma) &= \gamma_0^{(2i(3^{n+1}+1)+\epsilon(q+1)/2)k+2ik(3^{n+1}-1)}\\ &= (-1)^{\epsilon k}\gamma_0^{3^{n+1}4ik}.\end{aligned}$$

Moreover, since $3^{n+1}-1$ is even, we have $\phi_k(J) = \gamma_0^{(3^{n+1}-1)k(q-1)/2} = 1$. Thus, if $k \notin \{0, (q-1)/2\}$, we deduce from Lemma 3.4 that $\operatorname{Res}^{\tilde{G}}_{G_2(q)} \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_k = \operatorname{Ind}^{G_2(q)}_{B} \phi_k$ is irreducible. It follows that $\operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_k$ is irreducible. However

$$\operatorname{Ind}^{G_2(q)}_{B} \phi_k = \chi_9(k, (3^{n+1}-1)k).$$

Moreover, we have $\tilde{\chi}_k = \tilde{\chi}_{-k}$. Thus we obtain $(1/2)(q-3)$ outer irreducible character of $\tilde{G}$. □

Proposition 5.3. *We have* $\rho\big(\operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0\big) = 1 + \tilde{\theta}_5$. *The outer values of* $\tilde{\theta}_5$ *are given in* Table 11. *Moreover,* $\rho\big(\operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}\big)$ *is the sum of one extension of* θ_6 *and* θ_7.

Proof. Using Table 4, we compute the induction to $\tilde{G}$ of ϕ_0 and $\phi_{(q-1)/2}$. The values are given in Table 5. We have

$$\big\langle \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0, \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0\big\rangle_\sigma = 2.$$

Since $\big\langle \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0, 1\big\rangle_{\tilde{G}} = 1$ and $\big\langle \operatorname{Res}^{\tilde{G}}_{G_2(q)} \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0, \theta_5\big\rangle_{G_2(q)} = 1$ are odd, we deduce from Proposition 3.3 that $\rho\big(\operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_0 - 1\big)$ is irreducible. Since its value on $(1,\sigma)$ is q^3, it follows that it is $\tilde{\theta}_5$. We have $\big\langle \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}, \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}\big\rangle_\sigma = 2$. Moreover, $\big\langle \operatorname{Res}^{\tilde{G}}_{G_2(q)} \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}, \theta_6\big\rangle_{G_2(q)}$ and $\big\langle \operatorname{Res}^{\tilde{G}}_{G_2(q)} \operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}, \theta_7\big\rangle_{G_2(q)}$ are odd. Then, we deduce that $\operatorname{Ind}^{\tilde{G}}_{\tilde{B}} \phi_{(q-1)/2}$ is the sum of one extension of θ_6 and θ_7. □

5.2. Some irreducible characters obtained by induction from $\tilde{R}(q)$.

Note that $\tilde{R}(q) = R(q) \times \langle\sigma\rangle$ is a direct product. We immediately deduce its character table by using the Table 15. We have:

Proposition 5.4. *Induction's formula from* $\tilde{R}(q)$ *to* $\tilde{G}$ *are given in* Table 6.

Table 5. Induction of ϕ_0 and $\phi_{(q-1)/2}$

	C_1	C_2	D_1	D_{21}	D_{22}	D_{31}
$\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\,\phi_0$	(q^3+1)	$(q+1)$	1	1	1	1
$\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\,\phi_{(q-1)/2}$	(q^3+1)	$-(q+1)$	1	1	1	-1

	D_{32}	D_{41}	D_{42}	D_{43}	$E_1(h^i)$	$E_2(h^i)$
$\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\,\phi_0$	1	1	1	1	2	2
$\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\,\phi_{(q-1)/2}$	-1	1	1	1	2	-2

Table 6. Induction's formula of $\tilde{\mathrm{R}}(q)$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(C_1) = \phi(1) + q^2(q^2-q+1)\phi(J)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(D_1) = \phi(\gamma(1))$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(D_{21}) = \phi(\beta(1)) + q\phi(\beta(-1)J)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(D_{22}) = \phi(\beta(-1)) + q\phi(\beta(1)J)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(A_{51}) = \phi(Y_1)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(A_{52}) = \phi(Y_2)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(A_{51}) = \phi(Y_3)$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(E_1(h)) = \phi(h) + \phi(hJ), \quad h \in L_1$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(F_1(h)) = \phi(h) + \phi(hJ) + \phi(h\tau^2) + \phi(hJ\tau^2), \quad h \in L_{aba}$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(G_1(h)) = \phi(h), \quad h \in L_a$$

$$\mathrm{Ind}_{\tilde{\mathrm{R}}(q)}^{\tilde{G}}\,\phi(H_1(h)) = \phi(h), \quad h \in L_{ababa}$$

Proof. In Theorem 4.1, we have computed the classes of $R(q)$ and their distribution in $G_2(q)$ (see Proposition 4.5). We obtain by Proposition 4.5 the distribution of the outer classes of $\tilde{R}(q)$. Then, we deduce the induction formula. □

We denote by E_1 (resp. E_2) the non zero equivalence classes of $\mathbb{Z}/(q-3\theta+1)\mathbb{Z}$ (resp. $\mathbb{Z}/(q+3\theta+1)\mathbb{Z}$) given by $i \sim j \Leftrightarrow i = \pm j$, $i = \pm qj$ or $i = \pm q^2 j$. In the following, we denote by

$$\begin{aligned} &\forall k \in E_1, \quad \chi_{a,k} = \chi_{14}((q+3\theta+1)k), \\ &\forall k \in E_2, \quad \chi_{ababa,k} = \chi_{14}((q-3\theta+1)k). \end{aligned} \tag{9}$$

The characters $\chi_{a,k}$ $(k \in E_1)$ and $\chi_{ababa,k}$ $(k \in E_2)$ are σ-stable characters of $G_2(q)$.

We set $\pi_3 = \sigma_0^{(q+3\theta+1)^2}$ and $\pi_4 = \sigma_0^{(q-3\theta+1)^2}$. The notation in the Table 15 is chosen such that the primitive root of unity with order $(q-3\theta+1)$ (resp. $(q+3\theta+1)$) is π_3 (resp. π_4). To simplify notations, the induced character from $\tilde{R}(q)$ to $\tilde{G}$ of a character $\chi \in \mathrm{Irr}\big(\tilde{R}(q)\big)$ will be always denoted by χ.

Proposition 5.5. *Characters $\rho(\xi_9)$, $\rho(\xi_{10})$, $\rho(\eta_k^-)$ $(k \in E_1)$ and $\rho(\eta_k^+)$ $(k \in E_2)$ are irreducible extensions of $\theta_{12}(1)$, $\theta_{12}(-1)$, $\chi_{a,k}$ and $\chi_{ababa,k}$ respectively. The values of these characters on outer elements are given in* Table 11. *Moreover $\rho(\xi_1+\xi_2)$, $\rho(\xi_3+\xi_4)$, $\rho(\xi_5+\xi_6)$ and $\rho(\xi_7+\xi_8)$ have two constituents.*

Proof. We have

$$\begin{aligned} \langle \xi_1+\xi_2, \xi_1+\xi_2\rangle_\sigma &= 2, \\ \langle \xi_3+\xi_4, \xi_3+\xi_4\rangle_\sigma &= 2, \\ \langle \xi_5+\xi_6, \xi_5+\xi_6\rangle_\sigma &= 2, \\ \langle \xi_7+\xi_8, \xi_7+\xi_8\rangle_\sigma &= 2, \\ \langle \xi_9, \xi_9, \rangle_\sigma &= 1, \\ \langle \xi_{10}, \xi_{10}\rangle_\sigma &= 1, \\ \langle \eta_k^-, \eta_k^-\rangle_\sigma &= 1, \\ \langle \eta_k^+, \eta_k^+\rangle_\sigma &= 1. \end{aligned}$$

Thus, we deduce Proposition 3.3 that the characters $\rho(\xi_1+\xi_2)$, $\rho(\xi_3+\xi_4)$, $\rho(\xi_5+\xi_6)$ and $\rho(\xi_7+\xi_8)$ are sums of two irreducible characters and that $\rho(\xi_9)$, $\rho(\xi_{10})$, $\rho(\eta_k^-)$ and $\rho(\eta_k^+)$ are irreducible. Moreover, we have

$$\begin{aligned} \big\langle \mathrm{Res}_{G_2(q)}^{\tilde{G}}\, \xi_9, \theta_{12}(1)\big\rangle_{G_2(q)} &= \theta \equiv 1 \mod 2, \\ \big\langle \mathrm{Res}_{G_2(q)}^{\tilde{G}}\, \xi_{10}, \theta_{12}(-1)\big\rangle_{G_2(q)} &= \theta \equiv 1 \mod 2, \end{aligned}$$

$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}\, \eta_k^-,\, \chi_a(k) \right\rangle_{\mathrm{G}_2(q)} = 9\theta^4 + 9\theta^3 + 9\theta^2 + 6\theta + 2 \equiv 1 \mod 2,$$
$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}\, \eta_k^+,\, \chi_{ababa}(k) \right\rangle_{\mathrm{G}_2(q)} = 9\theta^4 - 9\theta^3 + 9\theta^2 - 6\theta + 2 \equiv 1 \mod 2.$$

We conclude that $\rho(\xi_9)$, $\rho(\xi_{10})$, $\rho(\eta_k^-)$ and $\rho(\eta_k^+)$ are irreducible extensions of $\theta_{12}(1)$, $\theta_{12}(-1)$, $\chi_a(k)$ and $\chi_{ababa}(k)$ respectively. Precisely, since $\rho(\xi_9)(1,\sigma) = \rho(\xi_{10})(1,\sigma) = q^3+1 > 0$, $\rho(\eta_k^-)(1,\sigma) = (q-1)(q+3\theta+1) > 0$ and $\rho(\eta_k^-)(1,\sigma) = (q-1)(q-3\theta+1) > 0$, we deduce that $\tilde{\theta}_{12}(1) = \rho(\xi_9)$, $\tilde{\theta}_{12}(-1) = \rho(\xi_{10})$, $\tilde{\chi}_a(k) = \rho(\eta_k^-)$ and $\tilde{\chi}_{ababa}(k) = \rho(\eta_k^+)$. □

Now, we decompose $\rho(\xi_1+\xi_2)$, $\rho(\xi_3+\xi_4)$ and $\rho(\xi_5+\xi_6)$

Constituents of $\rho(\xi_1+\xi_2)$:

$$\langle \xi_1+\xi_2,\, 1\rangle_{G\rtimes\langle\sigma\rangle} = 1,$$
$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_1+\xi_2),\, \theta_6 \right\rangle_{\mathrm{G}_2(q)} = 1.$$

We deduce from Proposition 3.3 that $\rho(\xi_1+\xi_2)$ is the sum of the trivial character and an extension of θ_6. The values of this extension on the outer classes are the same as the one of $\xi_1+\xi_2-1$. In particular the value on $(1,\sigma)$ is $q^2-q+1>0$, this extension is $\tilde{\theta}_6$. The values are given in Table 11.

Constituents of $\rho(\xi_3+\xi_4)$:

$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_3+\xi_4),\, \theta_5 \right\rangle_{\mathrm{G}_2(q)} = 18\theta^4 + 3\theta^2 + 2 \equiv 1 \mod 2,$$
$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_3+\xi_4),\, \theta_7 \right\rangle_{\mathrm{G}_2(q)} = 18\theta^4 + 3\theta^2 + 6 \equiv 1 \mod 2.$$

Thus $\rho(\xi_3+\xi_4)$ is the sum of an extension of θ_5 and an extension of θ_7. However, we have

$$\left\langle \xi_3+\xi_4,\, \tilde{\theta}_5 \right\rangle_{G\rtimes\langle\sigma\rangle} = 1.$$

The values of the extension of θ_7 on the outer classes are the same as $\xi_3+\xi_4-\tilde{\theta}_5$. Since the value on $(1,\sigma)$ is $q(q^2-q+1)>0$, we deduce that this extension is $\tilde{\theta}_7$. The values are given in Table 11.

Constituents of $\rho(\xi_5+\xi_6)$ *and of* $\rho(\xi_7+\xi_8)$:

$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_5+\xi_6),\, \theta_1 \right\rangle_{\mathrm{G}_2(q)} = \theta \equiv 1 \mod 2,$$
$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_5+\xi_6),\, \theta_{10} \right\rangle_{\mathrm{G}_2(q)} = \theta \equiv 1 \mod 2,$$
$$\left\langle \mathrm{Res}^{\tilde{\mathrm{G}}}_{\mathrm{G}_2(q)}(\xi_7+\xi_8),\, \theta_1 \right\rangle_{\mathrm{G}_2(q)} = \theta \equiv 1 \mod 2,$$

$$\left\langle \mathrm{Res}^{\tilde{G}}_{G_2(q)}(\xi_7+\xi_8),\, \theta_{10}\right\rangle_{G_2(q)} = \theta \equiv 1 \mod 2,$$
$$\langle \xi_5+\xi_6,\, \xi_7+\xi_8\rangle_\sigma = 0.$$

Thus, $\rho(\xi_5+\xi_6)$ and $\rho(\xi_7+\xi_8)$ are sums of an extension of θ_1 and of θ_{10} and they have a common constituent, indeed

$$\begin{aligned}\langle \rho(\xi_5+\xi_6),\, \rho(\xi_7+\xi_8)\rangle_{G\rtimes\langle\sigma\rangle} &= \frac{1}{2}\left\langle \mathrm{Res}^{\tilde{G}}_{G_2(q)}\,\rho(\xi_5+\xi_6),\, \mathrm{Res}^{\tilde{G}}_{G_2(q)}\,\rho(\xi_7+\xi_8)\right\rangle_{G_2(q)} \\ &\quad + \frac{1}{2}\langle \xi_5+\xi_6,\, \xi_7+\xi_8\rangle_\sigma \\ &= 1.\end{aligned}$$

Let ψ be the common constituent and ζ be another one. We suppose that

$$\rho(\xi_5+\xi_6) = \psi + \zeta,$$
$$\rho(\xi_7+\xi_8) = \psi + \zeta\varepsilon.$$

Thus $\rho(\xi_5+\xi_6)+\rho(\xi_7+\xi_8)$ has the same values as 2ψ on the outer classes. We obtain in this way the values of ψ on the outer classes. We then deduce the values of ζ on the outer classes. We give the values of ψ and ζ in Table 7.

We know the values of two new extensions, but we don't know if ψ is an extension of θ_1 or θ_{10}. We will show in the section 5.4 that ψ is an extension of θ_1 and ζ of θ_{10}. Before, by inducing irreducible characters of $C_{\tilde{G}}(\eta h_0, \sigma)$, we obtain the values of the extensions of θ_2.

5.3. Induction from $C_{\tilde{G}}(\eta h_0, \sigma)$. For every $t \in \mathbb{F}_q$, we set $\eta(t) = x_{a+b}(t)x_{3a+b}(-t^{3\theta})$. We have established that

$$C_{\tilde{G}}(\eta h_0, \sigma) = S_{a+b,3a+b} \times \langle (h_0, \sigma)\rangle,$$

where $S_{a+b,3a+b} = \{\eta(t) \mid t \in \mathbb{F}_q\}$.

The group $S_{a+b,3a+b}$ is abelian and isomorphic to the additive group $(\mathbb{F}_q, +)$ by $t \mapsto \eta(t)$. It follows that $C_{\tilde{G}}(\eta(1)h_0, \sigma)$ is abelian of order $4q$ and it is easy to compute these irreducible characters, which have the form $\varphi = \phi\phi'$, where ϕ is a linear character of $\mathbb{F}_q$ and ϕ' a linear character of the cyclic group of 4 elements. The elements $\eta(t)$ $(t \neq 0)$ are conjugate in $G_2(q)$ to $\eta(1)$ y $h(t) = h(t^{-1}, t^{(1-3\theta)/2}, t^{(3\theta+1)/2})$. Thus, we

Table 7. Values of ψ and ζ

	$(1,\sigma)$	$(\gamma(1),\sigma)$	$(\beta(1),\sigma)$	$(\beta(-1),\sigma)$	(Y_1,σ)	(Y_2,σ)	(Y_3,σ)	$G_1(h)$	$H_1(h)$
ψ	$(q^2-1)\theta$	$-\theta$	$-\theta$	$-\theta$	2θ	$-\theta$	$-\theta$	-1	1
ζ	0	0	$\theta^2\sqrt{-3}$	$-\theta^2\sqrt{-3}$	0	$\theta\sqrt{-3}$	$-\theta\sqrt{-3}$	0	0

deduce that the elements $\eta(t)J$ ($t \neq 0$) are conjugate to $\eta(1)J$. Using Method 3.1, we prove:

Lemma 5.1. *Let g be in $\{h_0, Jh_0\}$. The elements $(\eta(\gamma^k)g, \sigma)$ and $(\eta(1)J^k g, \sigma)$ are conjugate in $\tilde{\mathrm{G}}$, where γ is a generator of $\mathbb{F}_q^\times$.*

We deduce the distribution of the classes of $\mathrm{C}_{\tilde{\mathrm{G}}}(\eta h_0, \sigma)$ in $\tilde{\mathrm{G}}$ and the induction formula (given in Table 8).

Proposition 5.6. *Let $\phi \in \mathrm{Irr}(S_{a+b,3a+b})$ be non-trivial. Let ϕ' be the character of $\langle(h_0, \sigma)\rangle$ such that $\phi'(h_0, \sigma) = \sqrt{-1}$. Then*

$$\tilde{\theta}_2 = \rho\left(\mathrm{Ind}_{\mathrm{C}_{\tilde{\mathrm{G}}}(\eta h_0, \sigma)}^{\tilde{\mathrm{G}}} \phi\phi'\right)$$

is an extension of θ_2.

Proof. We have $\phi'(J) = -1$ and $\phi'(Jh_0, \sigma) = -\sqrt{-1}$. By using Table 8, we induce to $\tilde{\mathrm{G}}$ the character $\phi\phi'$, which has values zero except on the classes $(\eta h_0, \sigma)$ and $(\eta^{-1}h_0, \sigma)$. We set $\phi_0 = \mathrm{Ind}\,\phi\phi'$. We have

$$\phi_0(\eta h_0, \sigma) = -\phi_0(\eta^{-1}h_0, \sigma) = \sqrt{-1}\left(\sum_{t \text{ square}} \phi(\eta(t)) - \sum_{t \text{ not square}} \phi(\eta(t))\right).$$

Table 8. Induction formula of $\mathrm{C}_{\tilde{\mathrm{G}}}(h_0, \sigma)$ to $\tilde{\mathrm{G}}$

x	$\mathrm{Ind}_{\mathrm{C}_{\tilde{\mathrm{G}}}(\eta h_0, \sigma)}^{\tilde{\mathrm{G}}} \varphi(x)$
1	$\dfrac{q^5(q^6-1)(q^2-1)}{2}$
$\eta(1)$	$q^3 \sum_{t \neq 0} \phi(\eta(t))$
J	$\dfrac{q^3(q^2-1)^2}{2}\phi'(J)$
$\eta(1)J$	$q \sum_{t \neq 0} \phi(\eta(t))\phi'(J)$
(h_0, σ)	$\dfrac{q(q^2-1)}{2}(\phi'(h_0, \sigma) + \phi'(h_0 J, \sigma))$
$(\eta(1)h_0, \sigma)$	$\sum_{t \text{ square}} \phi(\eta(t))\phi'(h_0, \sigma) + \sum_{t \text{ not square}} \phi(\eta(t))\phi'(h_0 J, \sigma)$
$(\eta(1)h_0 J, \sigma)$	$\sum_{t \text{ square}} \phi(\eta(t))\phi'(Jh_0, \sigma) + \sum_{t \text{ not square}} \phi(\eta(t))\phi'(h_0, \sigma)$

Using the known formulas for Gauss sums, we obtain that

$$\sum_{t \text{ square}} \phi(\eta(t)) - \sum_{t \text{ not square}} \phi(\eta(t)) = \theta\sqrt{-3}.$$

Moreover, we have

$$\langle \phi_0, \phi_0 \rangle_\sigma = 1 \quad \text{and} \quad \left\langle \mathrm{Res}^{\tilde{G}}_{G_2(q)} \phi_0, \theta_2 \right\rangle_{G_2(q)} = \frac{1}{4}(q^4 + q^3 + q - 3).$$

To determine the parity of the integer $(1/4)(q^4 + q^3 + q - 3)$, it is sufficient to see that the numerator is divisible by 8. However, $q \equiv 3 \mod 8$. Then $(q^4 + q^3 + q - 3) \equiv 4 \mod 8$, and $\left\langle \mathrm{Res}^{\tilde{G}}_{G_2(q)} \phi_0, \theta_2 \right\rangle_{G_2(q)}$ is odd. By Proposition 3.3, we obtain that $\rho(\phi_0)$ is an irreducible extension of θ_2. Since the value on $(1, \sigma)$ is zero, we can fix the notation such that

$$\tilde{\theta}_2 = \rho(\phi_0). \qquad \square$$

5.4. Modular methods. We recall the definition of p-blocks and their orthogonality relation. For more details, we refer to [11] or [12]. Let G be a finite group and $R \subseteq \mathbb{C}$ the ring of algebraic integers of $\mathbb{C}$. Let p be a prime number. We define an equivalence relation on $\mathrm{Irr}(G)$ by

$$\chi_1 \sim \chi_2 \iff \frac{|C|\chi_1(C)}{\chi_1(1)} \equiv \frac{|C|\chi_2(C)}{\chi_2(1)} \mod pR, \tag{10}$$

for all p-regular classes C of G. The equivalence classes are called the p-blocks of G. The p-block containing the trivial character is called the principal p-block of G and is denoted by $B_0(G)$.

An element g in G is called p-singular if its order is divisible by p. Let B be a p-block of G, x a p-singular element and y a p-regular element. Then, we have

$$\sum_{\chi \in B} \chi(x)\overline{\chi(y)} = 0. \tag{11}$$

To finish, let χ be in $\mathrm{Irr}(G)$. Since $|G|/\chi(1)$ is an integer, we define the defect $d(\chi)$ of χ by the p-adic valuation of $|G|/\chi(1)$. Let B be a p-block of G, then we define the defect of B by $d(B) = \max\{d(\chi) \mid \chi \in B\}$. We have a characterization of the p-blocks with defect 0: a p-block B has defect 0 if and only if its cardinal is 1, that is there exists $\chi \in \mathrm{Irr}(G)$ such that $B = \{\chi\}$. Moreover, an irreducible character χ is in a p-block with defect 0 if and only if it vanishes on the p-singular elements.

5.4.1. Extension of θ_1 and θ_{10}. In 5.2, we have seen that the character ψ is an extension of θ_1 or θ_{10} (and ζ is the extension for the other one).

Proposition 5.7. *The character ψ (resp. ζ) is $\tilde{\theta}_1$ (resp. $\tilde{\theta}_{10}$).*

Proof. The degree of θ_1 (resp. θ_{10}) is $(1/6)q(q+1)^2(q^2+q+1)$ (resp. $(1/6)q(q-1)^2(q^2-q+1)$). For every $n \geq 0$, we have $q^2-q+1>1$. Fix p a prime divisor of (q^2-q+1). Since (q^2-q+1) and $2q^6(q^2-1)(q^3-1)(q+1)$ are coprime, it follows that θ_{10} belongs to a p-block of defect 0 and θ_1 lies in a p-block of maximal defect. By using the characterization of the p-blocks with defect zero, we conclude that the extension of θ_{10} is ζ and that of θ_1 is ψ. □

5.4.2. Extension of θ_{11}. We must compute the extensions of the $(1/6)(q-3)\sigma$-stable characters of degree $(q-1)^2(q^2-q+1)(q^3-1)$ and of θ_{11}. The strategy to obtain the extension of θ_{11} is to find a prime number p, such that the extension of θ_{11} is the only unknown character in his p-block. Thus, by using the orthogonality relation (11) we determine the values of the extension. We have $q^2-q+1=(q-3\theta+1)(q+3\theta+1)$, with $q-3\theta+1$ and $q+3\theta+1$ relative prime. By using relation (10), we show:

Proposition 5.8. *Let p be a prime number dividing q^2-q+1.*

- *If $q-3\theta+1=p^d\alpha$ with α prime to p, there exists an extension ψ_0 of θ_{11} such that*

$$B_0\tilde{\mathrm{G}} = \left\{1,\, \psi_0,\, \tilde{\theta}_1,\, \varepsilon\tilde{\theta}_5,\, \varepsilon\tilde{\theta}_{12}(-1),\, \varepsilon\tilde{\theta}_{12}(1),\, \varepsilon\,\tilde{\chi}_{a,k\alpha}\right\}.$$

- *If $q+3\theta+1=p^d\alpha$ with α prime to p, there exists an extension ψ_0 of θ_{11} such that*

$$B_0\tilde{\mathrm{G}} = \left\{1,\, \psi_0,\, \varepsilon\tilde{\theta}_1,\, \varepsilon\tilde{\theta}_5,\, \tilde{\theta}_{12}(1),\, \tilde{\theta}_{12}(-1),\, \varepsilon\,\tilde{\chi}_{ababa,k\alpha}\right\}.$$

We fix p a prime divisor of $q+3\theta+1$, and we denote by ψ the extension of θ_{11} contained in the principal p-block of $\tilde{\mathrm{G}}$.

Proposition 5.9. *The values of ψ on the outer classes are*:

	$(1,\sigma)$	$(\gamma(1),\sigma)$	(h_0,σ)	$(\eta(1)h_0,\sigma)$	$(\eta(1)h_0J,\sigma)$	$(h_{aba}\epsilon,\sigma)$	(h_a,σ)	(h_{ababa},σ)
ψ	$q(1-q)$	q	$q-1$	-1	-1	-2	1	1

Proof. To determine the values of ψ, we use the orthogonality relation of the principal p-block of $\tilde{\mathrm{G}}$. Let x be an element of a σ-stable class $E_6(i)$ (see Table 14) of $\mathrm{G}_2(q)$ whose order is not prime to p. We have

$$\theta_1(x)=-1, \quad \theta_5(x)=1, \quad \theta_{11}(x)=1, \quad \theta_{12}(-1)=-1, \quad \theta_{12}(1)=1.$$

Applying the relation (11) with the p-regular element $x_a(1)x_b(1)$. We compute:

$$\sum_k \chi_{ababa,\alpha k}(x) = -1.$$

Since the class of x in $G_2(q)$ is σ-stable, it follows that the extension to $\tilde{G}$ of the character χ of $G_2(q)$ and χ have the same values on x. Let y be an outer element that does not belong to the class $H_1(h)$ ($h \in L_{ababa}$). Thus, y is p-regular. If we set $\beta_y = \psi(y)$ and apply the relation (11) in the principal p-block, we obtain a linear equation with one variable β_y which is easy to solve. We compute:

$$\beta_\sigma = q(1-q), \quad \beta_{(\gamma(1),\sigma)} = q, \quad \beta_{(h_0,\sigma)} = q-1, \quad \beta_{(\eta(1)h_0,\sigma)} = \beta_{(\eta(1)h_0 J,\sigma)} = -1,$$
$$\beta_{(h_{aba}\epsilon,\sigma)} = -2, \quad \beta_{(h_a,\sigma)} = 1,$$

and

$$\beta_{(\beta(1),\sigma)} = \beta_{(\beta(-1),\sigma)} = \beta_{(Y_1,\sigma)} = \beta_{(Y_2,\sigma)} = \beta_{(Y_3,\sigma)} = \beta_{(h_1,\sigma)} = 0.$$

Now, we must determine the values on the class $H_1(h)$.

Suppose $q > 3$. Then $q - 3\theta + 1 > 1$. Let p' be a prime divisor of this number. By Proposition 5.8, the principal p'-block of $\tilde{G}$ contains an extension ψ' of θ_{11}. By using relation (11) in the principal p'-block, we obtain that the value of ψ' on $(1,\sigma)$ is $q(1-q)$. This proves that $\psi = \psi'$. However, the elements in the classes $H_1(h)$ ($h \in L_{ababa}$) are p'-regular and relation (11) shows that $\psi(H_1(h)) = 1$, for all $h \in L_{ababa}$.

In the case where $q = 3$, there is only one class $H_1(h)$. The scalar product relation shows that the value of ψ on this class is 1. □

5.5. Induced characters of $H' \rtimes \langle\sigma\rangle$. The maximal σ-stable torus H' of order $(q+1)^2$ is described in §4.2. We will induce the irreducible characters of $\tilde{H}' = H' \rtimes \langle\sigma\rangle$ to $\tilde{G}$. We recall that

$$H' = \langle\tau\rangle \times \langle\tau'\rangle \times \langle g_{a+b}\rangle \times \langle g_{3a+b}\rangle,$$

with τ and τ' of order 4 and g_{a+b} and g_{3a+b} of order $(q+1)/4$. We know how σ is acting on H'. Precisely, we recall that $\sigma(\tau) = J\tau$, $\sigma(\tau') = J\tau^2\tau'$, $\sigma(g_{a+b}) = g_{3a+b}$. Moreover, we have $\tau'^2 = J$.

The group H' is abelian and has $(q+1)$ σ-stable classes. We deduce from Proposition 3.1 that $\tilde{H}'$ has $(q+1)$ outer classes with system of representatives (h,σ), $(h\tau,\sigma)$, $(h\tau',\sigma)$ and $(h\tau\tau',\sigma)$, where $h \in H'_\sigma$. The centralizer of these elements in $\tilde{G}$ has order $2(q+1)$. The irreducible characters of H' are described by

$$\chi_{k_1,k_2,k_3,k_4}(\tau^{i_1}\tau'^{i_2}g_{a+b}^{i_3}g_{3a+b}^{i_4}) = \tau_0^{((q+1)/4)(i_1k_1+i_2k_2)+4(i_3k_3+i_4k_4)},$$

where k_1, $k_2 \in \mathbb{Z}/4\mathbb{Z}$ and k_3, $k_4 \in \mathbb{Z}/(1/4)(q+1)\mathbb{Z}$. The σ-stable characters of H' are:

$$\phi_{k_1,k_2,k_3} = \chi_{2k_1,2k_2,k_3,k_3},$$

where k_1, k_2 belong to $\{0, 1\}$. The σ-stable element of H' are:

$$\tau^{2i_1}J^{i_2}(g_{a+b}g_{3a+b})^{i_3}.$$

Proposition 5.10. *Let h be in L_{aba}, we have*:

$$\begin{aligned}
\mathrm{Cl}(1,\sigma)\cap\tilde{H}' &= \mathrm{Cl}(1,\sigma),\\
\mathrm{Cl}(h_0,\sigma)\cap\tilde{H}' &= \mathrm{Cl}(\tau,\sigma)\cup\mathrm{Cl}(\tau',\sigma)\cup\mathrm{Cl}(\tau\tau',\sigma),\\
\mathrm{Cl}(h,\sigma)\cap\tilde{H}' &= \mathrm{Cl}(h^{\pm1},\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}+1)/2},\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}-1)/2},\sigma),\\
\mathrm{Cl}(h\tau,\sigma)\cap\tilde{H}' &= \mathrm{Cl}(h^{\pm1}\tau,\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}+1)/2}\tau\tau',\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}-1)/2}\tau',\sigma),\\
\mathrm{Cl}(h\tau',\sigma)\cap\tilde{H}' &= \mathrm{Cl}(h^{\pm1}\tau',\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}+1)/2}\tau,\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}-1)/2}\tau\tau',\sigma),\\
\mathrm{Cl}(h\tau\tau',\sigma)\cap\tilde{H}' &= \mathrm{Cl}(h^{\pm1}\tau\tau',\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}+1)/2}\tau',\sigma)\cup\mathrm{Cl}(h^{\pm(3^{n+1}-1)/2}\tau,\sigma).
\end{aligned}$$

Proof. To determine the distribution of the classes of $\tilde{H}'$ in $\tilde{\mathrm{G}}$, we use Lemma 3.1. The subgroup H'_σ which consists of σ-stable elements of H' with odd order is conjugate in $\mathbf{G}$ to:

$$\{h(z^2, z^{-(3^{n+1}+1)}, z^{(3^{n+1}-1)}) \mid z^{(q+1)/2}=1\}.$$

Let $x\in H'_\sigma$ be a non trivial element. Then x is conjugate in $\mathbf{G}$ to $h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)$ for some $z_0\in\overline{\mathbb{F}}_3$ such that $z_0^{(q+1)/2}=1$. The elements of H'_σ that are conjugate in $\mathrm{G}_2(q)$ to x are exactly the elements of H'_σ that are conjugate in $\mathbf{G}$ to one of the 6 elements:

$$\begin{aligned}
&h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big), \quad h\big(z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}, z_0^2\big),\\
&h\big(z_0^{(3^{n+1}-1)}, z_0^2, z_0^{-(3^{n+1}+1)}\big), \quad h\big(z_0^{-2}, z_0^{(3^{n+1}+1)}, z_0^{-(3^{n+1}-1)}\big),\\
&h\big(z_0^{(3^{n+1}+1)}, z_0^{-(3^{n+1}-1)}, z_0^{-2}\big) \quad\text{and}\quad h\big(z_0^{-(3^{n+1}-1)}, z_0^{-2}, z_0^{(3^{n+1}+1)}\big).
\end{aligned}$$

We remark that

$$\begin{aligned}
h\big(z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}, z_0^2\big) &= h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)^{-(3^{n+1}+1)/2},\\
h\big(z_0^{(3^{n+1}-1)}, z_0^2, z_0^{-(3^{n+1}+1)}\big) &= h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)^{(3^{n+1}-1)/2},\\
h\big(z_0^{-2}, z_0^{(3^{n+1}+1)}, z_0^{-(3^{n+1}-1)}\big) &= h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)^{-1},\\
h\big(z_0^{(3^{n+1}+1)}, z_0^{-(3^{n+1}-1)}, z_0^{-2}\big) &= h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)^{(3^{n+1}+1)/2},\\
h\big(z_0^{-(3^{n+1}-1)}, z_0^{-2}, z_0^{(3^{n+1}+1)}\big) &= h\big(z_0^2, z_0^{-(3^{n+1}+1)}, z_0^{(3^{n+1}-1)}\big)^{-(3^{n+1}-1)/2}.
\end{aligned}$$

We deduce that the elements of H'_σ conjugate in $\mathrm{G}_2(q)$ to $(g_{a+b}g_{3a+b})^i$ are:

$$(g_{a+b}g_{3a+b})^{\pm i}, \quad (g_{a+b}g_{3a+b})^{\pm(3^{n+1}+1)i/2}, \quad (g_{a+b}g_{3a+b})^{\pm(3^{n+1}-1)i/2}.$$

This is due to the fact that if two elements x_0 and y_0 are conjugate, then x_0^i is conjugate to y_0^i. In [15], it is proven that there exists an element t_0 in $\mathrm{R}(q)$ with order 3 such that

$$t_0^{-1}(g_{a+b}g_{3a+b})^i t_0 = (g_{a+b}g_{3a+b})^{-(3^{n+1}+1)i/2} \quad \text{and} \quad t_0^{-1}Jt_0 = \tau^2.$$

It follows that

$$t_0^{-2}(g_{a+b}g_{3a+b})^i t_0^2 = (g_{a+b}g_{3a+b})^{(3^{n+1}-1)i/2} \quad \text{and} \quad t_0^{-2}Jt_0^2 = J\tau^2.$$

We set $s_0 = h_0\tau \in \mathrm{R}(q)$. We then have

$$s_0(g_{a+b}g_{3a+b})^i s_0^{-1} = (g_{a+b}g_{3a+b})^{-i}.$$

To know the distribution of (hx, σ), where $x \in \{1, \tau, \tau', \tau\tau'\}$, we must determine the 2-unipotent classes in $\mathrm{C}_{\tilde{\mathrm{G}}}(h) = \tilde{H}'$ of $t_0(x,\sigma)t_0^{-1}$ and $s_0(x,\sigma)s_0^{-1}$. We have $s_0\tau s_0^{-1} = \tau^3$, $s_0\tau' s_0^{-1} = \tau'^3$. This shows that $s_0(x,\sigma)s_0^{-1}$ is conjugate to (x,σ) in $\tilde{H}'$. Since $t_0 \in \mathrm{R}(q)$, we have $t_0(x,\sigma)t_0^{-1} = (t_0xt_0^{-1}, \sigma)$. We denote by $P = \langle \tau, \tau' \rangle$ a 2-Sylow of H'. We have $t_0\tau t_0^{-1} \in P$ and $t_0\tau' t_0^{-1} \in P$. Indeed,

$$t_0(g_{a+b}g_{3a+b})^{-(3^{n+1}+1)i/2}\tau t_0^{-1} = (g_{a+b}g_{3a+b})^i t_0\tau t_0^{-1}$$

is a 2-decomposition of Jordan. Since $\mathrm{C}_{G_2(q)}((g_{a+b}g_{3a+b})^i) = H'_\sigma$, it follows that $t_0\tau t_0^{-1} \in H'$. Moreover the order of $t_0\tau t_0^{-1}$ is a power of 2. Since H' has a unique 2-Sylow (because H' is abelian), we deduce $t_0\tau t_0^{-1} \in P$. Similarly, $t_0\tau' t_0^{-1} \in P$. However τ and τ' generate P. Thus t_0 is acting by conjugation on P. Now, we show that:

$$t_0\{\tau, J\tau, \tau^3, J\tau^3\}t_0^{-1} = \{\tau', J\tau', J\tau'\tau^2, \tau'\tau^2\},$$
$$t_0\{\tau', J\tau', J\tau'\tau^2, \tau'\tau^2\}t_0^{-1} = \{\tau\tau', J\tau'\tau^3, \tau'\tau^3, J\tau\tau'\},$$
$$t_0\{\tau\tau', J\tau'\tau^3, \tau'\tau^3, J\tau\tau'\}t_0^{-1} = \{\tau, J\tau, \tau^3, J\tau^3\}.$$

Indeed, since t_0 acts on P by conjugation, it follows that t_0 acts on the cyclic subgroups with order 4 of P. The group P has 6 cyclic subgroup with order 4. Each subgroup has a unique element of order 2. Precisely, we have:

4-cyclic subgroup		element of order 2
$\langle\tau\rangle$, $\langle J\tau\rangle$	$\rightsquigarrow$	τ^2
$\langle\tau'\rangle$, $\langle J\tau'\rangle$	$\rightsquigarrow$	J
$\langle\tau\tau'\rangle$, $\langle J\tau\tau'\rangle$	$\rightsquigarrow$	$J\tau^2$

Since $t_0\tau^2t_0^{-1} = J$, $t_0Jt_0^{-1} = J\tau^2$ and $t_0J\tau^2t_0^{-1} = \tau^2$ and the order is preserved by conjugacy, we also obtain the result. We recall that the σ-classes of P are

$$\{1, J, \tau^2, J\tau, J\tau^2\},$$
$$\{\tau, J\tau, \tau^3, J\tau^3\},$$
$$\{\tau', J\tau', J\tau'\tau^2, \tau'\tau^2\},$$
$$\{\tau\tau', J\tau'\tau^3, \tau'\tau^3, J\tau\tau'\}.$$

Then, we conclude that $(t_0\tau t_0^{-1},\sigma)$ (resp. $(t_0\tau' t_0^{-1},\sigma)$ and $(t_0\tau\tau' t_0^{-1},\sigma)$) is conjugate in $\tilde{H}'$ to (τ',σ) (resp. $(\tau\tau',\sigma)$ and (τ,σ)). Finally, we deduce that:

	conjugate to
$(h^{-(3^{n+1}+1)/2}, \sigma)$	(h, σ)
$(h^{-(3^{n+1}+1)/2}\tau, \sigma)$	$(h\tau', \sigma)$
$(h^{-(3^{n+1}+1)/2}\tau', \sigma)$	$(h\tau\tau', \sigma)$
$(h^{-(3^{n+1}+1)/2}\tau\tau', \sigma)$	$(h\tau, \sigma)$
$(h^{(3^{n+1}-1)/2}, \sigma)$	(h, σ)
$(h^{(3^{n+1}-1)/2}\tau, \sigma)$	$(h\tau\tau', \sigma)$
$(h^{(3^{n+1}-1)/2}\tau', \sigma)$	$(h\tau, \sigma)$
$(h^{(3^{n+1}-1)/2}\tau\tau', \sigma)$	$(h\tau', \sigma)$

This completes the proof. □

We immediately deduce the induction formula, given in Table 9. By Lemma 3.5, we construct the outer characters of $\tilde{H}'$. Let ϕ_{k_1,k_2,k_3} be a σ-stable character of H'. To simplify, we again denote $\mathrm{Ind}_{\tilde{H}'}^{\tilde{\mathrm{G}}}\left(\tilde{\phi}_{k_1,k_2,k_3}\right)$ by ϕ_{k_1,k_2,k_3} the induction from $\tilde{H}'$ to $\tilde{\mathrm{G}}$ of

Table 9. Induction from $\tilde{H}'$ to $\tilde{\mathrm{G}}$

x	$\mathrm{Ind}_{\tilde{H}'}^{\tilde{\mathrm{G}}}\,\phi(x)$
$(1, \sigma)$	$q^3(q-1)(q^2-q+1)\phi(1,\sigma)$
(h_0, σ)	$q(q-1)(\phi(\tau,\sigma)+\phi(\tau',\sigma)+\phi(\tau\tau',\sigma))$
(h, σ)	$\phi(h^{\pm 1},\sigma)+\phi(h^{\pm(3^{n+1}+1)/2},\sigma)+\phi(h^{\pm(3^{n+1}-1)/2},\sigma)$
$(h\tau, \sigma)$	$\phi(h^{\pm 1}\tau,\sigma)+\phi(h^{\pm(3^{n+1}+1)/2}\tau\tau',\sigma)+\phi(h^{\pm(3^{n+1}-1)/2}\tau',\sigma)$
$(h\tau', \sigma)$	$\phi(h^{\pm 1}\tau',\sigma)+\phi(h^{\pm(3^{n+1}+1)/2}\tau,\sigma)+\phi(h^{\pm(3^{n+1}-1)/2}\tau\tau',\sigma)$
$(h\tau\tau', \sigma)$	$\phi(h^{\pm 1}\tau\tau',\sigma)+\phi(h^{\pm(3^{n+1}+1)/2}\tau',\sigma)+\phi(h^{\pm(3^{n+1}-1)/2}\tau,\sigma)$

$\tilde{\phi}_{k_1,k_2,k_3}$. We remark that $\phi_{1,0,0} = \phi_{0,1,0} = \phi_{1,1,0}$ and we set

$$\psi_0 = \phi_{0,0,0} \quad \text{and} \quad \psi_1 = \phi_{1,0,0}.$$

We set:

$$\begin{aligned}\varphi_{0,0,k} &= \psi_0 + \varepsilon\phi_{0,0,k},\\ \varphi_{1,0,k} &= \psi_1 + \varepsilon\phi_{1,0,k},\\ \varphi_{0,1,k} &= \psi_1 + \varepsilon\phi_{0,1,k},\\ \varphi_{1,1,k} &= \psi_1 + \varepsilon\phi_{1,1,k}.\end{aligned}$$

There are $(1/24)(q-3)$ distinct characters in each family. Moreover, the characters $\varphi_{0,0,k}$ have σ-norm 7 and $\varphi_{1,0,k}$, $\varphi_{0,1,k}$ are $\varphi_{1,1,k}$ have σ-norm 3.

THE CASE OF $\varphi_{0,0,k}$: we show that

$$\langle\varphi_{0,0,k}, 1_{\tilde{G}}\rangle_\sigma = 1, \quad \langle\varphi_{0,0,k}, \tilde{\theta}_5\rangle_\sigma = -1, \quad \langle\varphi_{0,0,k}, \tilde{\theta}_{11}\rangle_\sigma = 2,$$

and the other outer scalar products are zero. The class function $\varphi_{0,0,k} - 1_{\tilde{G}} - \varepsilon\tilde{\theta}_5 - 2\tilde{\theta}_{11}$ is a character and:

$$\langle\rho(\varphi_{0,0,k} - 1_{\tilde{G}} - \varepsilon\tilde{\theta}_5 - 2\tilde{\theta}_{11}), \rho(\varphi_{0,0,k} - 1_{\tilde{G}} - \varepsilon\tilde{\theta}_5 - 2\tilde{\theta}_{11})\rangle_\sigma = 1.$$

It follows from Proposition 3.3 that $\rho(\varphi_{0,0,k} - 1_{\tilde{G}} - \varepsilon\tilde{\theta}_5 - 2\tilde{\theta}_{11})$ is an irreducible character.

THE CASE OF $\varphi_{1,0,k}$, $\varphi_{0,1,k}$, AND $\varphi_{1,1,k}$: let $\epsilon, \epsilon' \in \{0, 1\}$ be such that $\epsilon \neq 0$ or $\epsilon' \neq 0$. We have

$$\langle\varphi_{\epsilon,\epsilon',k}, \tilde{\theta}_6\rangle_\sigma = 1 \quad \text{and} \quad \langle\varphi_{\epsilon,\epsilon',k}, \tilde{\theta}_7\rangle_\sigma = -1.$$

The class function $\varphi_{\epsilon,\epsilon',k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7$ is a character and

$$\langle\rho(\varphi_{\epsilon,\epsilon',k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7), \rho(\varphi_{\epsilon,\epsilon',k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7)\rangle_\sigma = 1.$$

Then $\rho(\varphi_{\epsilon,\epsilon',k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7)$ is an irreducible character by Proposition 3.3.

The irreducible characters obtained are distinct. We obtain $(1/6)(q-3)$ new outer irreducible characters of $\tilde{G}$. We set

$$\begin{aligned}\chi_{aba,0,0}(k) &= \mathrm{Res}^{\tilde{G}}_{G_2(q)}\, \rho(\varphi_{0,0,k} - 1_{\tilde{G}} - \varepsilon\tilde{\theta}_5 - 2\tilde{\theta}_{11}),\\ \chi_{aba,1,0}(k) &= \mathrm{Res}^{\tilde{G}}_{G_2(q)}\, \rho(\varphi_{1,0,k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7),\\ \chi_{aba,0,1}(k) &= \mathrm{Res}^{\tilde{G}}_{G_2(q)}\, \rho(\varphi_{0,1,k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7),\\ \chi_{aba,1,1}(k) &= \mathrm{Res}^{\tilde{G}}_{G_2(q)}\, \rho(\varphi_{1,1,k} - \tilde{\theta}_6 - \varepsilon\tilde{\theta}_7).\end{aligned} \tag{12}$$

These characters are the $(1/6)(q-3)$ irreducible σ-stable characters of $G_2(q)$ with degree $(q-1)(q^2-q+1)(q^3-1)$.

5.6. Character Table of $\tilde{\mathrm{G}}$. Let γ_0 (resp. τ_0 and σ_0) be a $(q-1)$-th (resp. $(q+1)$-th and (q^2-q+1)-th) primitive complex root of 1. We give in Table 10 the notations and the values of the outer characters of $\tilde{\mathrm{G}}$ on the outer classes in Table 11.

6. Perfect isometry

First, we recall generalities: let G and H be finite groups and $(\mathrm{K}, \mathcal{O}, p)$ be a p-modular system large enough for both G and H. Let μ in $\mathbb{Z}\,\mathrm{Irr}(G \times H)$. We define $I_\mu : \mathbb{Z}\,\mathrm{Irr}(H) \to \mathbb{Z}\,\mathrm{Irr}(G)$ by

$$\forall \xi \in \mathbb{Z}\,\mathrm{Irr}(H),\ g \in G, \quad I_\mu(\xi)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\xi(h).$$

Its adjoint relative to the usual scalar products on G and H is $R_\mu \colon \mathbb{Z}\,\mathrm{Irr}(G) \to \mathbb{Z}\,\mathrm{Irr}(H)$ such that

$$\forall \chi \in \mathbb{Z}\,\mathrm{Irr}(G),\ h \in H, \quad R_\mu(\chi)(h) = \frac{1}{|G|} \sum_{g \in G} \mu(g^{-1}, h)\chi(g).$$

The virtual character μ of $G \times H$ is called perfect if

- For all $g \in G$ and $h \in H$, we have $\mu(g, h)/|\mathrm{C}_G(g)| \in \mathcal{O}$ and $\mu(g, h)/|\mathrm{C}_G(g)| \in \mathcal{O}$.
- If $\mu(g, h) \neq 0$, g has order prime to p if and only if h has order prime to p.

Let e (resp. f) be a central idempotent of $\mathcal{O}G$ (resp. $\mathcal{O}H$) and $A = \mathcal{O}Ge$ (resp. $B = \mathcal{O}Hf$) a block of G (resp. of H). The blocks A and B are said to be perfectly isometric if there exists a perfect character μ of $\mathbb{Z}\,\mathrm{Irr}(G \times H)$ such that I_μ induces by restriction a bijective isometry between $\mathbb{Z}\,\mathrm{Irr}(KB)$ and $\mathbb{Z}\,\mathrm{Irr}(KA)$.

Table 10. Notations

$$\begin{aligned}
\gamma_i &= \gamma_0^{4.3^{n+1}i} + \gamma_0^{-4.3^{n+1}i},\\
\pi_3 &= \sigma_0^{(q+3\theta+1)^2},\\
\pi_4 &= \sigma_0^{(q-3\theta+1)^2},\\
\delta_i &= \pi_3^{i} + \pi_3^{-i} + \pi_3^{qi} + \pi_3^{-qi} + \pi_3^{q^2 i} + \pi_3^{-q^2 i},\\
\delta'_i &= \pi_4^{i} + \pi_4^{-i} + \pi_4^{qi} + \pi_4^{-qi} + \pi_4^{q^2 i} + \pi_4^{-q^2 i},\\
\beta_{i,0} &= \tau_0^{8i} + \tau_0^{-8i} + \tau_0^{4(3^{n+1}+1)i} + \tau_0^{-4(3^{n+1}+1)i} + \tau_0^{4(3^{n+1}-1)i} + \tau_0^{-4(3^{n+1}-1)i},\\
\beta_{i,1} &= \tau_0^{8i} + \tau_0^{-8i} - \tau_0^{4(3^{n+1}+1)i} - \tau_0^{-4(3^{n+1}+1)i} - \tau_0^{4(3^{n+1}-1)i} - \tau_0^{-4(3^{n+1}-1)i},\\
\beta_{i,2} &= -\tau_0^{8i} - \tau_0^{-8i} + \tau_0^{4(3^{n+1}+1)i} + \tau_0^{-4(3^{n+1}+1)i} - \tau_0^{4(3^{n+1}-1)i} - \tau_0^{-4(3^{n+1}-1)i},\\
\beta_{i,3} &= -\tau_0^{8i} - \tau_0^{-8i} - \tau_0^{4(3^{n+1}+1)i} - \tau_0^{-4(3^{n+1}+1)i} + \tau_0^{4(3^{n+1}-1)i} + \tau_0^{-4(3^{n+1}-1)i}.
\end{aligned}$$

Table 11. (Continued) Outer characters of $G_2(q) \rtimes \langle\sigma\rangle$

	$(1,\sigma)$	(X,σ)	(T,σ)	(T^{-1},σ)	(Y_1,σ)	(Y_2,σ)	(Y_3,σ)	(h_0,σ)	$(\eta h_0,\sigma)$	$(\eta^{-1}h_0,\sigma)$
$\tilde{\theta}_1$	$\theta(q^2-1)$	$-\theta$	$-\theta$	$-\theta$	2θ	$-\theta$	$-\theta$	0	0	0
$\tilde{\theta}_2$	0	0	0	0	0	0	0	0	$\sqrt{q}$	$-\sqrt{q}$
$\tilde{\theta}_5$	q^3	0	0	0	0	0	0	q	0	0
$\tilde{\theta}_6$	q^2-q+1	$1-q$	1	1	1	1	1	-1	-1	-1
$\tilde{\theta}_7$	$q(q^2-q+1)$	q	0	0	0	0	0	$-q$	0	0
$\tilde{\theta}_{10}$	0	0	$\theta^2\sqrt{-3}$	$-\theta^2\sqrt{-3}$	0	$\theta\sqrt{-3}$	$-\theta\sqrt{-3}$	0	0	0
$\tilde{\theta}_{11}$	$q(q-1)$	$-q$	0	0	0	0	0	$1-q$	1	1
$\tilde{\theta}_{12}(1)$	$\theta(q^2-1)$	$-\theta$	$-\theta+\theta^2\sqrt{-3}$	$-\theta-\theta^2\sqrt{-3}$	$-\theta$	$\dfrac{\theta+\theta\sqrt{-3}}{2}$	$\dfrac{\theta-\theta\sqrt{-3}}{2}$	0	0	0
$\tilde{\theta}_{12}(-1)$	$\theta(q^2-1)$	$-\theta$	$-\theta-\theta^2\sqrt{-3}$	$-\theta+\theta^2\sqrt{-3}$	$-\theta$	$\dfrac{\theta-\theta\sqrt{-3}}{2}$	$\dfrac{\theta+\theta\sqrt{-3}}{2}$	0	0	0
$\tilde{\chi}_1(k)$	q^3+1	1	1	1	1	1	1	$(-1)^k(q+1)$	$(-1)^k$	$(-1)^k$
$\tilde{\chi}_{aba,0,0}(k)$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$3(q-1)$	-3	-3
$\tilde{\chi}_{aba,1,0}(k)$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$	1	1
$\tilde{\chi}_{aba,0,1}(k)$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$	1	1
$\tilde{\chi}_{aba,1,1}(k)$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$	1	1
$\tilde{\chi}_a(k)$	$(q^2-1)(q+3\theta+1)$	$-q-1-3\theta$	$-3\theta-1$	$-3\theta-1$	-1	-1	-1	0	0	0
$\tilde{\chi}_{ababa}(k)$	$(q^2-1)(q-3\theta+1)$	$-q-1+3\theta$	$3\theta-1$	$3\theta-1$	-1	-1	-1	0	0	0

Table 11. Outer characters of $G_2(q) \rtimes \langle\sigma\rangle$

	(h_1, σ)	$(h_1 J, \sigma)$	(h_{aba}, σ)	$(h_{aba}\tau, \sigma)$	$(h_{aba}\tau', \sigma)$	$(h_{aba}\tau\tau', \sigma)$	(h_a, σ)	(h_{ababa}, σ)
$\tilde{\theta}_1$	0	0	0	0	0	0	-1	1
$\tilde{\theta}_2$	0	0	0	0	0	0	0	0
$\tilde{\theta}_5$	1	1	-1	-1	-1	-1	-1	-1
$\tilde{\theta}_6$	1	-1	3	-1	-1	-1	0	0
$\tilde{\theta}_7$	1	-1	-3	1	1	1	0	0
$\tilde{\theta}_{10}$	0	0	0	0	0	0	0	0
$\tilde{\theta}_{11}$	0	0	2	2	2	2	-1	-1
$\tilde{\theta}_{12}(1)$	0	0	0	0	0	0	-1	1
$\tilde{\theta}_{12}(-1)$	0	0	0	0	0	0	-1	1
$\tilde{\chi}_1(k)$	γ_{ik}	$(-1)^k\gamma_{ik}$	0	0	0	0	0	0
$\tilde{\chi}_{aba,0,0}(k)$	0	0	$-\beta_{kj,0}$	$-\beta_{kj,0}$	$-\beta_{kj,0}$	$-\beta_{kj,0}$	0	0
$\tilde{\chi}_{aba,1,0}(k)$	0	0	$-\beta_{kj,0}$	$-\beta_{kj,3}$	$-\beta_{kj,1}$	$-\beta_{kj,2}$	0	0
$\tilde{\chi}_{aba,0,1}(k)$	0	0	$-\beta_{kj,0}$	$-\beta_{kj,1}$	$-\beta_{kj,2}$	$-\beta_{kj,3}$	0	0
$\tilde{\chi}_{aba,1,1}(k)$	0	0	$-\beta_{kj,0}$	$-\beta_{kj,2}$	$-\beta_{kj,3}$	$-\beta_{kj,1}$	0	0
$\tilde{\chi}_a(k)$	0	0	0	0	0	0	$-\delta_{kj}$	0
$\tilde{\chi}_{ababa}(k)$	0	0	0	0	0	0	0	$-\delta'_{kj}$

For generalities on perfect isometries, we refer to [1]. We recall two results used to establish a perfect isometry between the principal blocks of $\mathrm{R}(q)$ and $\tilde{\mathrm{G}}$. The proofs are in [1]:

Proposition 6.1. *Any homomorphism* $J\colon \mathbb{Z}\,\mathrm{Irr}(H) \to \mathbb{Z}\,\mathrm{Irr}(G)$ *is of the form* I_μ, *where* μ *is a virtual character of* $G \times H$.

We recall that $\mathrm{C_K}(G, e)$ is the K-vector space of class functions $\alpha\colon G \to \mathrm{K}$ such that for every $g \in G$, $\alpha(eg) = \alpha(g)$, and $\mathrm{C_K}(G, e, p)$ is the subspace of class functions which vanish on p-singular elements of G. We define in the same way $\mathrm{C}_{\mathcal{O}}(G, e)$ and $\mathrm{C}_{\mathcal{O}}(H, e, p)$. We have:

Proposition 6.2. *The character* μ *is perfect if and only if*

1. I_μ *maps* $\mathrm{C}_{\mathcal{O}}(H, f)$ *into* $\mathrm{C}_{\mathcal{O}}(G, e)$ *and* R_μ *maps* $\mathrm{C}_{\mathcal{O}}(G, e)$ *into* $\mathrm{C}_{\mathcal{O}}(H, f)$.
2. I_μ *maps* $\mathrm{C_K}(H, f, p)$ *into* $\mathrm{C_K}(G, e, p)$ *and* R_μ *maps* $\mathrm{C_K}(G, e, p)$ *into* $\mathrm{C_K}(H, f, p)$.

Now, we will establish a perfect isometry between the principal p-blocks of the Ree group and $\tilde{\mathrm{G}}$. Since a perfect isometry preserves the order of defect groups, we only consider the prime divisors p of $|\tilde{\mathrm{G}}|$ such that p is prime to the index of $\mathrm{R}(q)$ in $\tilde{\mathrm{G}}$ (that is, p is a prime divisor of $q^2 - q + 1$). We can now prove Theorem 1.2:

Proof of Theorem 1.2. We prove the result only in the case where p is a prime divisor of $(q - 3\theta + 1)$. The other case is similar. The principal p-block of $B_0(\tilde{\mathrm{G}})$ is given in Propositions 5.8 and 5.9. By using relation (10), we obtain the principal p-block of $\mathrm{R}(q)$. In the following, we put $A_0 = B_0(\mathrm{R}(q)) = \mathcal{O}\mathrm{R}(q)e_0$ and $B_0 = B_0(\tilde{\mathrm{G}}) = \mathcal{O}\tilde{\mathrm{G}} f_0$. We define I and R as in the statement of Theorem 1.2. Since I is a homomorphism between $\mathbb{Z}\,\mathrm{Irr}(\mathrm{R}(q))$ and $\mathbb{Z}\,\mathrm{Irr}(\tilde{\mathrm{G}})$, it follows by Proposition 6.1 that I has the form I_μ, where μ is virtual character of $\tilde{\mathrm{G}} \times \mathrm{R}(q)$. To prove that μ is perfect, we use Proposition 6.2.

(1) Let C be a class of $\mathrm{R}(q)$ (resp. $\tilde{\mathrm{G}}$). We associate the map α_C (resp. β_C) in $\mathrm{C}_{\mathcal{O}}(\mathrm{R}(q), e_0)$ (resp. $\mathrm{C}_{\mathcal{O}}(\tilde{\mathrm{G}}, f_0)$) defined by $\alpha_C(h) = 1_{Ce_0}(he_0)$ (resp. $\beta_C(g) = 1_{Cf_0}(gf_0)$). We define on $\mathrm{Cl}(\mathrm{R}(q))$ (resp. $\tilde{\mathrm{G}}$) an equivalence relation by C_1 being in relation with C_2 if $\alpha_{C_1} = \alpha_{C_2}$ (resp. $\beta_{C_1} = \beta_{C_2}$). Let X (resp. Y) the set of classes for this relation. The family $\{\alpha_C\}_{C\in X}$ (resp. $\beta_{C\in Y}$) is a $\mathcal{O}$-basis of $\mathrm{C}_{\mathcal{O}}(\mathrm{R}(q), e_0)$ (resp. $\mathrm{C}_{\mathcal{O}}(\tilde{\mathrm{G}}, f_0)$). Let $\mathrm{Irr}(A_0)$ (resp. $\mathrm{Irr}(B_0)$) be the set of irreducible characters in A_0 (resp. B_0), which is a K-basis of $\mathrm{C}_K(\mathrm{R}(q), e_0)$ (resp. $\mathrm{C}_K(\tilde{\mathrm{G}}, f_0)$). We decompose α_C in the K-basis $\mathrm{Irr}(A_0)$. We obtain

$$\alpha_C = \frac{|C|}{|\mathrm{R}(q)|} \sum_{\phi\in\mathrm{Irr}(A_0)} \phi(C)\phi.$$

We now prove that $I_\mu(\alpha_C)$ has values in $\mathcal{O}$.

If p^d divides $|C|$, since the irreducible characters of $\mathrm{R}(q)$ and $\tilde{\mathrm{G}}$ have values in $\mathcal{O}$, it follows that $I_\mu(\alpha_C) \in \mathrm{C}_{\mathcal{O}}(\tilde{\mathrm{G}}, f_0)$.

Suppose p^d does not divide $|C|$ (that is p does not divide $|C|$). We will show that $\sum_{\phi\in\mathrm{Irr}(A_0)} \phi(C)I_\mu(\phi)$ has values in $\mathbb{Z}$, and is divisible by p^d. Let δ_1 be a p^d-th root of 1. Let E_d be the non-zero classes of $\mathbb{Z}/p^d\mathbb{Z}$ for the relation $\sim$ defined as in Theorem 1.2. We set $\delta_i = \delta_1^i + \delta_1^{-i} + \delta_1^{iq} + \delta_1^{-iq} + \delta_1^{iq^2} + \delta_1^{-iq^2}$. We remark that

$$\sum_{i\in E_d} \delta_i = -1 \quad \text{and} \quad \sum_{i,j\in E_d} \delta_i\delta_j = \begin{cases} -6 & \text{if} \quad i \neq j \\ q^d - 6 & \text{if} \quad i = j \end{cases}. \tag{13}$$

By reducing modulo p the values of the characters of $\mathrm{Irr}(A_0)$ on the classes C (where $|C|$ prime to p) in Table 15, and by using relation (13), it is sufficient to show that the map

$$\psi_a = 1 + \varepsilon\tilde{\theta}_5 - \tilde{\theta}_1 + \varepsilon\tilde{\theta}_{11} + \varepsilon\tilde{\theta}_{12}(-1) + \varepsilon\tilde{\theta}_{12}(1) + 6\sum_{k\in\mathfrak{R}_0} \varepsilon\,\tilde{\chi}_a(\alpha k),$$

has integer values that are divisible by p^d. This can be checked without difficulty by using the character table of $\tilde{\mathrm{G}}$ (see Table 11) and we have

$$I(\mathrm{C}_{\mathcal{O}}(\mathrm{R}(q), e_0)) \subseteq \mathrm{C}_{\mathcal{O}}(\tilde{\mathrm{G}}, f_0).$$

By using relation (13) and Table 12, it is sufficient to prove that the map

$$\Psi_a = 1 - \xi_3 - \xi_5 - \xi_7 - \xi_9 - \xi_{10} - 6\sum_{k\in\mathfrak{R}_0} \eta^-_{\alpha k}$$

has integer values divisible by p^d. We verify this with the table of $\mathrm{R}(q)$ in Table 15. We have proved

$$R\big(\mathrm{C}_{\mathcal{O}}\big(\tilde{\mathrm{G}}, f_0\big)\big) \subseteq \mathrm{C}_{\mathcal{O}}(\mathrm{R}(q), e_0).$$

Table 12. Values modulo p^d of $\mathrm{Irr}(B_0)$

	1	$E_6(i)$	$(1,\sigma)$	$G_1(h)$
1	1	1	1	1
$\varepsilon\tilde{\theta}_1$	-1	-1	-1	-1
$\varepsilon\tilde{\theta}_5$	1	1	1	1
$\tilde{\theta}_{11}$	1	1	1	1
$\tilde{\theta}_{12}(\pm 1)$	1	1	1	1
$\varepsilon\,\tilde{\chi}_a(\alpha k)$	6	$\delta_1^{\pm i} + \delta_1^{\pm iq} + \delta_1^{\pm iq^2}$	6	$\delta_1^{\pm i} + \delta_1^{\pm iq} + \delta_1^{\pm iq^2}$

(2) Recall that if G is a finite group and ϕ a class function of G, we denote by $\hat{\phi}$ the restriction of ϕ to the p-regular values of G. If A is a p-block, we denote by $\widehat{\mathrm{Irr}}_p(A)$ the set of $\hat{\chi}$, where χ belongs to A. The set $\widehat{\mathrm{Irr}}_p(A)$ generates $\mathrm{C_K}(G, e_0, p)$.

In our problem, we see that it is sufficient to prove that

$$I_\mu\big(\widehat{\mathrm{Irr}}(A_0)\big) \subseteq \mathrm{C_K}(\tilde{\mathrm{G}},\ f_0,\ p) \quad \text{and} \quad R_\mu\big(\widehat{\mathrm{Irr}}(B_0)\big) \subseteq \mathrm{C_K}(\mathrm{R}(q), e_0,\ p).$$

Let $\psi \in \mathrm{Irr}(A_0)$. Since $\hat{\psi}$ belongs to the K-space with K-basis $\mathrm{Irr}(A_0)$, we have

$$\hat{\psi} = \sum_{\phi \in \mathrm{Irr}(A_0)} a_\phi \phi.$$

Let g be a p-singular element of $\tilde{\mathrm{G}}$. Thus it belongs to either the class $E_6(i)$ (see the beginning of §4.4) or to $G_1(h)$, where $h \in L_a$ (see Table 2).

First suppose that g is in $E_6(i)$, then there exists a p^d-th complex root of 1 and α_1 a p-singular element of $\mathrm{R}(q)$ such that

$$\chi_a(\alpha)(g) = \delta_1^{\pm 1} + \delta_1^{\pm q} + \delta_1^{\pm q^2} = -\eta_\alpha^-(\alpha_1).$$

It follows that

$$\forall k, \quad \chi_a(\alpha k)(g) = \delta_1^{\pm k} + \delta_1^{\pm qk} + \delta_1^{\pm q^2 k} = -\eta_{\alpha k}^-(\alpha_1).$$

Moreover, we have

$$1_{\tilde{\mathrm{G}}}(g) = 1_{\mathrm{R}(q)}(\alpha_1), \quad \tilde{\theta}_5\varepsilon(g) = -\xi_3(\alpha_1), \quad \tilde{\theta}_1 = \xi_5(\alpha_1)$$

and

$$\tilde{\theta}_{11}\varepsilon = -\xi_7(\alpha_1), \quad \tilde{\theta}_{12}(-1)\varepsilon = -\xi_9(\alpha_1), \quad \tilde{\theta}_{12}(1)\varepsilon = -\xi_{10}(\alpha_1).$$

Thus, we deduce that for every $\phi \in \mathrm{Irr}(A_0)$, we have $I_\mu(\phi)(g) = \phi(\alpha_1)$. It follows that

$$I_\mu(\hat{\psi})(g) = \sum_{\phi \in \mathrm{Irr}(A_0)} a_\phi \phi(\alpha_1) = \hat{\psi}(\alpha_1) = 0.$$

Suppose $g \in G_1(h)$ for some p-singular element h in L_a, then

$$I_\mu(\hat{\psi})(g) = \sum_{\phi \in \mathrm{Irr}(A_0)} a_\phi I_\mu(\phi)(g) = \hat{\psi}(h) = 0.$$

Thus, $\forall \psi \in \mathrm{Irr}(A_0)$, we have $I_\mu(\hat{\psi}) \in \mathrm{C_K}(\tilde{\mathrm{G}}, f_0, p)$. We proceed similarly to show that $R_\mu\big(\widehat{\mathrm{Irr}}(B_0)\big) \subseteq \mathrm{C_K}(\mathrm{R}(q), e_0, p)$. We have shown:

$$I_\mu(\mathrm{C_K}(\mathrm{R}(q), e_0, p)) \subseteq \mathrm{C_K}(\tilde{\mathrm{G}},\ f_0,\ p),$$
$$R_\mu(\mathrm{C_K}(\tilde{\mathrm{G}},\ f_0,\ p)) \subseteq \mathrm{C_K}(\mathrm{R}(q), e_0,\ p).$$

It follows that μ is a perfect character and the result is proven. □

7. Appendix

7.1. The Group $\mathbf{G_2(\mathbb{F}_{3^{2n+1}})}$. Let $n \in \mathbb{N}$. We set $q = 3^{2n+1}$. Conjugacy classes and character Table of $G_2(q)$ are described in [8]. Let $E = \{t^3 - t \mid t \in \mathbb{F}_q\}$ and π_E be the canonical map from $\mathbb{F}_q$ to $\mathbb{F}_q/E$. We denote by $\xi \in \mathbb{F}_q$ an element such that $\pi_E(\xi) = 1$. Let κ (resp. κ_0) be a primitive $(q^6 - 1)$-th root of unity in $\overline{\mathbb{F}}_q$ (resp. in $\mathbb{C}$). Let ω be a cubic complex root. We set:

$$\tilde{\sigma} = \kappa^{(q+1)(q^3-1)},$$
$$\tilde{\tau} = \kappa^{(q-1)(q^3+1)},$$
$$\tilde{\theta} = \kappa^{q^4+q^2+1},$$
$$\tilde{\eta} = \tilde{\theta}^{q-1},$$
$$\tilde{\gamma} = \tilde{\theta}^{q+1}.$$

We denote by π_κ the isomorphism between $\langle\kappa\rangle$ and $\langle\kappa_0\rangle$ such that $\kappa_0 = \pi_\kappa(\kappa)$. For every $\zeta \in \langle\kappa\rangle$, we set

$$\zeta_0 = \pi_\kappa(\zeta).$$

Proposition 7.1. *In* Table 13, *we give the conjugacy classes of* $G_2(q)$. *We write*

$$\alpha_i = \gamma_0^i + \gamma_0^{-i} \quad \textit{and} \quad \beta_i = \tau_0^i + \tau_0^{-i}.$$

Moreover, we set

$$\alpha_{i,j,k,l} = \alpha_{ik+jl} + \alpha_{il+jk} + \alpha_{i(k-l)-jl} + \alpha_{il-j(k-l)} + \alpha_{i(k-l)+jk} + \alpha_{ik+j(k-l)},$$
$$\beta_{i,j,k,l} = \beta_{ik+jl} + \beta_{il+jk} + \beta_{i(k-l)-jl}\beta_{il-j(k-l)} + \beta_{i(k-l)+jl} + \beta_{ik+j(k-l)},$$
$$\sigma_i = \sigma_0^i + \sigma_0^{-i} + \sigma_0^{iq} + \sigma_0^{-iq} + \sigma_0^{iq^2} + \sigma_0^{-iq^2},$$
$$\alpha'_{i,j,k,l} = \alpha_{i(k+l)} + \alpha_{i(k-2l)} + \alpha_{i(l-2k)},$$
$$\alpha''_{i,k,l} = \alpha_{ik} + \alpha_{il} + \alpha_{i(k-l)},$$
$$\beta'_{i,k,l} = \beta_{i(k+l)} + \beta_{i(k-2l)} + \beta_{i(l-2k)},$$
$$\beta''_{i,j,k,l} = (\beta_{ik} + \beta_{il} + \beta_{i(k-l)}),$$
$$\epsilon_{k,l} = (-1)^k + (-1)^l + (-1)^{k+l}.$$

We give the values of Irr(G) *that are interessting for us in this work in* Table 14.

Table 13. Conjugacy classes of $G_2(q)$

Notation	Representative	Number	Order of Centralizer
A_1	$h(1, 1, 1)$	1	$q^6(q^2-1)(q^6-1)$
A_2	$x_{3a+2b}(1)$	1	$q^6(q^2-1)$
A_{31}	$x_{2a+b}(1)$	1	$q^6(q^2-1)$
A_{32}	$x_{2a+b}(1)x_{3a+2b}(1)$	1	q^6
A_{41}	$x_{a+b}(1)x_{3a+b}(-1)$	1	$2q^4$
A_{42}	$x_{a+b}(1)x_{3a+b}(1)$	1	$2q^4$
A_{51}	$x_a(1)x_b(1)$	1	$3q^2$
A_{52}	$x_a(1)x_b(1)x_{3a+b}(\xi)$	1	$3q^2$
A_{53}	$x_a(1)x_b(1)x_{3a+b}(-\xi)$	1	$3q^2$
B_1	$h(1, -1, -1)$	1	$q^2(q^2-1)^2$
B_2	$h(1, -1, -1)x_{3a+b}(1)$	1	$q^2(q^2-1)$
B_3	$h(1, -1, -1)x_{a+b}(1)$	1	$q^2(q^2-1)$
B_4	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(-1)$	1	$2q^2$
B_5	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(1)$	1	$2q^2$
$C_{11}(i)$	$h_{\tilde{\gamma}}(i, -2i, i)$	$\frac{1}{2}(q-3)$	$q(q-1)(q^2-1)$
$C_{12}(i)$	$h_{\tilde{\gamma}}(i, -2i, i)x_{3a+2b}(1)$	$\frac{1}{2}(q-3)$	$q(q-1)$
$C_{21}(i)$	$h_{\tilde{\gamma}}(i, -i, 0)$	$\frac{1}{2}(q-3)$	$q(q-1)(q^2-1)$
$C_{22}(i)$	$h_{\tilde{\gamma}}(i, -i, 0)x_{2a+b}(1)$	$\frac{1}{2}(q-3)$	$q(q-1)$
$D_{11}(i)$	$h_{\tilde{\eta}}(i, -2i, i)$	$\frac{1}{2}(q-1)$	$q(q+1)(q^2-1)$
$D_{12}(i)$	$h_{\tilde{\eta}}(i, -2i, i)x_{3a+2b}(1)$	$\frac{1}{2}(q-1)$	$q(q+1)$
$D_{21}(i)$	$h_{\tilde{\eta}}(i, -i, 0)$	$\frac{1}{2}(q-1)$	$q(q+1)(q^2-1)$
$D_{22}(i)$	$h_{\tilde{\eta}}(i, -i, 0)x_{2a+b}(1)$	$\frac{1}{2}(q-1)$	$q(q+1)$
$E_1(i, j)$	$h_{\tilde{\gamma}}(i, j, -i-j)$	$\frac{1}{12}(q-3)(q-5)$	$(q-1)^2$
$E_2(i, j)$	$h_{\tilde{\theta}}(i, (q-1)i, -qi)$	$\frac{1}{4}(q-1)^2$	q^2-1
$E_3(i)$	$h_{\tilde{\theta}}(i, qj, -(q+1)i)$	$\frac{1}{4}(q-1)^2$	q^2-1
$E_4(i, j)$	$h_{\tilde{\eta}}(i, j, -i-j)$	$\frac{1}{12}(q-1)(q-3)$	$(q+1)^2$
$E_5(i)$	$h_{\tilde{\tau}}(i, qi, q^2i)$	$\frac{1}{6}q(q+1)$	q^2+q+1
$E_6(i)$	$h_{\tilde{\sigma}}(i, -qi, q^2i)$	$\frac{1}{6}q(q-1)$	q^2-q+1

	A_1	A_2	A_{31}	A_{32}	A_{41}	A_{42}	A_{51}	A_{52}	A_{53}
θ_1	$\frac{1}{6}q(q+1)^2(q^2+q+1)$	$\frac{1}{6}q(q+1)(2q+1)$	$\frac{1}{6}q(q+1)(2q+1)$	$\frac{1}{6}q(3q+1)$	$\frac{1}{6}q(q+1)$	$-\frac{1}{6}q(q-1)$	$\frac{2}{3}q$	$-\frac{1}{3}q$	$-\frac{1}{3}q$
θ_2	$\frac{1}{2}q(q+1)(q^3+1)$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	0	0	0
θ_5	q^6	0	0	0	0	0	0	0	0
θ_6	q^4+q^2+1	q^2+1	q^2+1	q^2+1	1	1	1	1	1
θ_7	$q^2(q^4+q^2+1)$	q^2	q^2	q^2	0	0	0	0	0
θ_{10}	$\frac{1}{6}q(q-1)^2(q^2-q-1)$	$\frac{1}{6}q(q-1)(2q-1)$	$\frac{1}{6}q(q-1)(2q-1)$	$-\frac{1}{6}q(3q-1)$	$\frac{1}{6}q(q+1)$	$-\frac{1}{6}q(q-1)$	$\frac{2}{3}q$	$-\frac{1}{3}q$	$-\frac{1}{3}q$
θ_{11}	$\frac{1}{2}q(q-1)(q^3-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	0	0	0
$\theta_{12}(1)$	$\frac{1}{3}q(q^2-1)^2$	$-\frac{1}{3}q(q^2-1)$	$-\frac{1}{3}q(q^2-1)$	$\frac{1}{3}q$	$\frac{1}{3}q(q+1)$	$-\frac{1}{3}q(q-1)$	$\frac{1}{3}q$	$\frac{1}{3}q+q\omega$	$\frac{1}{3}q+q\omega^{-1}$
$\theta_{12}(-1)$	$\frac{1}{3}q(q^2-1)^2$	$-\frac{1}{3}q(q^2-1)$	$-\frac{1}{3}q(q^2-1)$	$\frac{1}{3}q$	$\frac{1}{3}q(q+1)$	$-\frac{1}{3}q(q-1)$	$\frac{1}{3}q$	$\frac{1}{3}q+q\omega^{-1}$	$\frac{1}{3}q+q\omega$
$\chi_9(k,l)$	$(q+1)(q^2+q+1)(q^3+1)$	$(q+1)(q^2+q+1)$	$(q+1)(q^2+q+1)$	$2q^2+2q+1$	$2q+1$	$2q+1$	1	1	1
$\chi_{12}(k,l)$	$(q-1)(q^2-q+1)(q^3-1)$	$-(q-1)(q^2-q+1)$	$-(q-1)(q^2-q+1)$	$2q^2-2q+1$	$-(2q-1)$	$-(2q-1)$	1	1	1
$\chi_{14}(k)$	$(q+1)(q^2-1)(q^3-1)$	$-(q+1)(q^3-1)$	$-(q+1)(q^3-1)$	$-(q^2-q+1)$	$q+1$	$q+1$	1	1	1

	B_1	B_2	B_3	B_4	B_5	$C_{11}(i)$	$C_{12}(i)$	$C_{21}(i)$	$C_{22}(i)$
θ_1	$\frac{1}{2}(q+1)^2$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$-\frac{1}{2}(q-1)$	$q+1$	1	$q+1$	1
θ_2	$\frac{1}{2}(q+1)^2$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$-\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$	$q+1$	1	$q+1$	1
θ_5	q^2	0	0	0	0	q	0	q	0
θ_6	$1-2q$	$1-q$	$1-q$	1	1	$1+(q+1)(-1)^i$	$1+(-1)^i$	$1+(q+1)(-1)^i$	$1+(-1)^i$
θ_7	q^2-2q	$-q$	$-q$	0	0	$q+(q+1)(-1)^i$	$(-1)^i$	$q+(q+1)(-1)^i$	$(-1)^i$
θ_{10}	$-\frac{1}{3}(q-1)^2$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	0	0	0	0
θ_{11}	$-\frac{1}{2}(q-1)^2$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	0	0	0	0
$\theta_{12}(1)$	0	0	0	0	0	0	0	0	0
$\theta_{12}(-1)$	0	0	0	0	0	0	0	0	0
$\chi_9(k,l)$	$(q+1)^2\epsilon_{k,l}$	$(q+1)\epsilon_{k,l}$	$(q+1)\epsilon_{k,l}$	$\epsilon_{k,l}$	$\epsilon_{k,l}$	$(q+1)\alpha'_{i,k,l}$	$\alpha'_{i,k,l}$	$(q+1)\alpha''_{i,k,l}$	$\alpha''_{i,k,l}$
$\chi_{12}(k,l)$	$(q-1)^2\epsilon_{k,l}$	$-(q-1)\epsilon_{k,l}$	$-(q-1)\epsilon_{k,l}$	$\epsilon_{k,l}$	$\epsilon_{k,l}$	0	0	0	0
$\chi_{14}(k)$	0	0	0	0	0	0	0	0	0

Table 14. Character table of $G_2(q)$

	$D_{11}(i)$	$D_{12}(i)$	$D_{21}(i)$	$D_{22}(i)$	$E_1(i,j)$	$E_2(i)$	$E_3(i)$	$E_4(i,j)$	$E_5(i)$	$E_6(i)$
θ_1	0	0	0	0	2	0	0	0	0	-1
θ_2	0	0	0	0	2	0	0	0	-1	0
θ_5	$-q$	0	$-q$	0	1	-1	-1	1	1	1
θ_6	$1-(q-1)(-1)^i$	$1+(-1)^i$	$1-(q-1)(-1)^i$	$1+(-1)^i$	$\epsilon_{k,l}$	$(-1)^i$	$(-1)^i$	$\epsilon_{k,l}$	0	0
θ_7	$-q-(q-1)(-1)^i$	$(-1)^i$	$-q-(q-1)(-1)^i$	$(-1)^i$	$\epsilon_{k,l}$	$-(-1)^i$	$-(-1)^i$	$\epsilon_{k,l}$	0	0
θ_{10}	$q-1$	-1	$q-1$	-1	0	0	0	-2	1	0
θ_{11}	$q-1$	-1	$q-1$	-1	0	0	0	-2	0	1
$\theta_{12}(1)$	0	0	0	0	0	0	0	0	-1	1
$\theta_{12}(-1)$	0	0	0	0	0	0	0	0	-1	1
$\chi_9(k,l)$	0	0	0	0	$\alpha_{i,j,k,l}$	0	0	0	0	0
$\chi_{12}(k,l)$	$-(q-1)\beta'_{i,k,l}$	$\beta'_{i,k,l}$	$-(q-1)\beta''_{i,k,l}$	$\beta''_{i,k,l}$	0	0	0	$\beta_{i,j,k,l}$	0	0
$\chi_{14}(k)$	0	0	0	0	0	0	0	0	0	σ_i

7.2. The Ree group of type G_2. We set $\theta = 3^n$. The Ree group $R(q)$ is described in [14] and its character table is computed in [15]. We recall that the conjugacy classes of $R(q)$ are given in Theorem 4.1 and that we have:

Proposition 7.2. *Let $x \in \{1, a, aba, ababa\}$. We denote by h_x representatives of odd order elements of type w_x. Let $\gamma_i = \gamma_0^{4.3\theta i} + \gamma_0^{-4.3\theta i}$ and*

$$\delta_i = \pi_3^i + \pi_3^{-i} + \pi_3^{qi} + \pi_3^{-qi} + \pi_3^{q^2 i} + \pi_3^{-q^2 i} \quad \text{and} \quad \delta_i' = \pi_4^i + \pi_4^{-i} + \pi_4^{qi} + \pi_4^{-qi} + \pi_4^{q^2 i} + \pi_4^{-q^2 i},$$

with $\pi_3 = \sigma_0^{(q+3\theta+1)^2}$ and $\pi_4 = \sigma_0^{(q-3\theta+1)^2}$. Finally, we set

$$\beta_{i,0} = \tau_0^{8i} + \tau_0^{-8i} + \tau_0^{4(3\theta+1)i} + \tau_0^{-4(3\theta+1)i} + \tau_0^{4(3\theta-1)i} + \tau_0^{-4(3\theta-1)i},$$
$$\beta_{i,1} = \tau_0^{8i} + \tau_0^{-8i} - \tau_0^{4(3\theta+1)i} - \tau_0^{-4(3\theta+1)i} - \tau_0^{4(3\theta-1)i} - \tau_0^{-4(3\theta-1)i},$$
$$\beta_{i,2} = -\tau_0^{8i} - \tau_0^{-8i} + \tau_0^{4(3\theta+1)i} + \tau_0^{-4(3\theta+1)i} - \tau_0^{4(3\theta-1)i} - \tau_0^{-4(3\theta-1)i},$$
$$\beta_{i,3} = -\tau_0^{8i} - \tau_0^{-8i} - \tau_0^{4(3\theta+1)i} - \tau_0^{-4(3\theta+1)i} + \tau_0^{4(3\theta-1)i} + \tau_0^{-4(3\theta-1)i}.$$

Then the character table of $R(q)$ *is given in* Table 15.

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Table 15. Character table of R(q)

	1	X	T	T^{-1}	Y_1	Y_2	Y_3	J
ξ_2	q^2-q+1	$1-q$	1	1	1	1	1	-1
ξ_3	q^3	0	0	0	0	0	0	q
ξ_4	$q(q^2-q+1)$	q	0	0	0	0	0	$-q$
ξ_5	$\frac{1}{2}(q-1)(q+3\theta+1)$	$-\frac{1}{2}(q+\theta)$	$\frac{1}{2}(-\theta+\theta^2\sqrt{-3})$	$\frac{1}{2}(-\theta-\theta^2\sqrt{-3})$	θ	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	$-\frac{1}{2}(q-1)$
ξ_6	$\frac{1}{2}(q-1)(q-3\theta+1)$	$\frac{1}{2}(q-\theta)$	$\frac{1}{2}(-\theta+\theta^2\sqrt{-3})$	$\frac{1}{2}(-\theta-\theta^2\sqrt{-3})$	θ	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	$\frac{1}{2}(q-1)$
ξ_7	$\frac{1}{2}(q-1)(q+3\theta+1)$	$-\frac{1}{2}(q+\theta)$	$\frac{1}{2}(-\theta-\theta^2\sqrt{-3})$	$\frac{1}{2}(-\theta+\theta^2\sqrt{-3})$	θ	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	$-\frac{1}{2}(q-1)$
ξ_8	$\frac{1}{2}(q-1)(q-3\theta+1)$	$\frac{1}{2}(q-\theta)$	$\frac{1}{2}(-\theta-\theta^2\sqrt{-3})$	$\frac{1}{2}(-\theta+\theta^2\sqrt{-3})$	θ	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	$\frac{1}{2}(q-1)$
ξ_9	$\theta(q^2-1)$	$-\theta$	$-\theta+\theta^2\sqrt{-3}$	$-\theta-\theta^2\sqrt{-3}$	$-\theta$	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	0
ξ_{10}	$\theta(q^2-1)$	$-\theta$	$-\theta-\theta^2\sqrt{-3}$	$-\theta+\theta^2\sqrt{-3}$	$-\theta$	$\frac{1}{2}(-\theta-\theta\sqrt{-3})$	$\frac{1}{2}(-\theta+\theta\sqrt{-3})$	0
η_r	q^3+1	1	1	1	1	1	1	$(q+1)$
η_r'	q^3+1	1	1	1	1	1	1	$-(q+1)$
η_t	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$3(q-1)$
$\eta_{t,1}'$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$
$\eta_{t,2}'$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$
$\eta_{t,3}'$	$(q-1)(q^2-q+1)$	$2q-1$	-1	-1	-1	-1	-1	$1-q$
η_k^-	$(q^2-1)(q+3\theta+1)$	$-q-1-3\theta$	$-3\theta-1$	$-3\theta-1$	-1	-1	-1	0
η_k^+	$(q^2-1)(q-3\theta+1)$	$-q-1+3\theta$	$3\theta-1$	$3\theta-1$	-1	-1	-1	0

Table 15. Character table of R(q)

	JT	JT^{-1}	h_1	h_1J	h_{aba}	$h_{aba}\tau^2$	$h_{aba}J$	$h_{aba}J\tau^2$	h_a	h_{ababa}
ξ_2	-1	-1	1	-1	1	-1	-1	-1	0	0
ξ_3	0	0	1	1	3	-1	-1	-1	-1	-1
ξ_4	0	0	1	-1	-3	1	1	1	0	0
ξ_5	$\frac{1}{2}(1-\theta\sqrt{-3})$	$\frac{1}{2}(1+\theta\sqrt{-3})$	0	0	1	1	1	1	-1	0
ξ_6	$\frac{1}{2}(-1+\theta\sqrt{-3})$	$\frac{1}{2}(-1-\theta\sqrt{-3})$	0	0	-1	-1	-1	-1	0	1
ξ_7	$\frac{1}{2}(1+\theta\sqrt{-3})$	$\frac{1}{2}(1-\theta\sqrt{-3})$	0	0	1	1	1	1	-1	0
ξ_8	$\frac{1}{2}(-1-\theta\sqrt{-3})$	$\frac{1}{2}(-1+\theta\sqrt{-3})$	0	0	-1	-1	-1	-1	0	1
ξ_9	0	0	0	0	0	0	0	0	-1	1
ξ_{10}	0	0	0	0	0	0	0	0	-1	1
η_r	1	1	γ_{ir}	γ_{ir}	0	0	0	0	0	0
η'_r	-1	-1	γ_{ir}	$-\gamma_{ir}$	0	0	0	0	0	0
η_t	-3	-3	0	0	$-\beta_{tj,0}$	$-\beta_{tj,0}$	$-\beta_{tj,0}$	$-\beta_{tj,0}$	0	0
$\eta'_{t,1}$	1	1	0	0	$-\beta_{tj,0}$	$-\beta_{tj,3}$	$-\beta_{tj,1}$	$-\beta_{tj,2}$	0	0
$\eta'_{t,2}$	1	1	0	0	$-\beta_{tj,0}$	$-\beta_{tj,1}$	$-\beta_{tj,2}$	$-\beta_{tj,3}$	0	0
$\eta'_{t,3}$	1	1	0	0	$-\beta_{tj,0}$	$-\beta_{tj,2}$	$-\beta_{tj,3}$	$-\beta_{tj,1}$	0	0
η_k^-	0	0	0	0	0	0	0	0	$-\delta_{kj}$	0
η_k^+	0	0	0	0	0	0	0	0	0	$-\delta'_{kj}$

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