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AN AFFINE PROPERTY OF THE RECIPROCAL ASIAN OPTION PROCESS

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This note describes a property of the Asian option process, which neatly links the process at μ with the one at $-\mu$, where μ is the drift of the geometric Brownian motion. The proof is based on (i) a known result due to Yor, on the law of the Asian option process taken at an exponential time, and (ii) a recent result on beta and gamma distributions.

Suppose W is one-dimensional standard Brownian motion starting at the origin, and define what this author calls the *Asian option process*, for want of a better name:

$$A_t^{(\mu)} = \int_0^t e^{2\mu s + 2W_s} ds, \qquad t \geq 0, \quad \mu \in \mathbb{R}.$$

Asian options have payoffs such as $(A_t^{(\mu)} - K)_+$, and have been studied by numerous authors in Finance and Mathematics; reciprocal Asian options have payoffs such as $(K - 1/A_t^{(\mu)})_+$, and have not received much attention so far; for more details and references, the reader is referred to [3] and [4].

Theorem 1 ([6]). *Let T_λ be an exponentially distributed random variable, independent of W, with mean $1/\lambda$. Then*

$$2A_{T_\lambda}^{(\mu)} \overset{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta},$$

where $B_{1,\alpha} \sim \mathrm{Beta}(1, \alpha)$ and $G_\beta \sim \Gamma(\beta, 1)$ are independent, $\alpha = \mu/2 + \sqrt{2\lambda + \mu^2}/2$, $\beta = \alpha - \mu$.

Theorem 2 ([2]). *For any $a, b, c > 0$,*

$$\frac{G_a}{B_{b,a+c}} + G'_c \overset{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b,a}}.$$

where $G_a \sim \mathrm{Gamma}(a, 1)$, $G'_c \sim \mathrm{Gamma}(c, 1)$, $B_{b,a+c} \sim \mathrm{Beta}(b, a + c)$, $B_{b,a} \sim \mathrm{Beta}(b, a)$ and all variables are independent.

Theorem 3. *For any $\mu, t > 0$,*

$$\frac{1}{2A_t^{(\mu)}} + G_\mu \stackrel{\mathcal{L}}{=} \frac{1}{2A_t^{(-\mu)}},$$

where $G_\mu \sim \Gamma(\mu, 1)$ is independent of W.

The last result follows directly from the two previous ones, upon setting $a = \beta$, $b = 1$, $c = \mu$, and then inverting the Laplace transform represented by the exponential time T_λ. Theorem 3 gives an easy proof of the well-known formula in Corollary 4 (just observe that $A_\infty^{(\mu)} = \infty$ a.s. if $\mu \geq 0$).

Corollary 4. *For any $\mu > 0$,*

$$\frac{1}{2A_\infty^{(-\mu)}} \sim \mathrm{Gamma}(\mu, 1).$$

Theorem 5. *Let $\{U_k;\, k \geq 1\}$ be independent variables with the same distribution as*

$$U \stackrel{\mathcal{L}}{=} \frac{B_1}{B_2}, \qquad B_1 \sim \mathrm{Beta}(\beta, \mu), \qquad B_2 \sim \mathrm{Beta}(1+\beta, \mu),$$

with B_1, B_2 independent, $\mu > 0$, $\beta = -\mu/2 + \sqrt{2\lambda + \mu^2}/2$, and let $\{G_\mu^{(k)};\, k \geq 0\}$ be a sequence of independent variables with a common Gamma$(\mu, 1)$ *distribution, independent of $\{U_k;\, k \geq 1\}$. Then*

(a) $$\frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} U\left(\frac{1}{2A_{T_\lambda}^{(\mu)}} + G_\mu\right)$$

(b) $$\frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\infty} U_1 \cdots U_k G_\mu^{(k)}; \qquad \frac{1}{2A_{T_\lambda}^{(-\mu)}} \stackrel{\mathcal{L}}{=} G_\mu^{(0)} + \sum_{k=1}^{\infty} U_1 \cdots U_k G_\mu^{(k)}$$

(c) $$\mathsf{E}\, e^{-s/2A_{T_\lambda}^{(\mu)}} = \mathsf{E}\left(\prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k}\right)^{\mu}$$

(d) $$\mathsf{E}\, e^{-s/2A_{T_\lambda}^{(-\mu)}} = \mathsf{E}\left(\frac{1}{1+s}\prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k}\right)^{\mu} = \left(\frac{1}{1+s}\right)^{\mu} \mathsf{E}\, e^{-s/2A_{T_\lambda}^{(\mu)}}.$$

In (a), *$A_{T_\lambda}^{(\mu)}$ and G_μ are independent*; *moreover*, *given G_μ and U with the given distributions, the solution $1/2A_{T_\lambda}^{(\mu)}$ is unique* (*in distribution*).

Part (a) follows from computing the Mellin transform ($s \mapsto \mathsf{E}\,(\cdot)^s$) of either side, which from Theorem 1 is

$$\frac{\Gamma(1-s)\Gamma(\beta+s)\Gamma(1+\beta+\mu)}{\Gamma(1+\beta+\mu-s)\Gamma(\beta)}$$
$$= \frac{\Gamma(\beta+s)\Gamma(\beta+\mu)}{\Gamma(\beta)\Gamma(\beta+\mu+s)}\frac{\Gamma(1+\beta-s)\Gamma(1+\beta+\mu)}{\Gamma(1+\beta)\Gamma(1+\beta+\mu-s)}\frac{\Gamma(1-s)\Gamma(\alpha+s)\Gamma(1+\alpha-\mu)}{\Gamma(1+\alpha-\mu-s)\Gamma(\alpha)}.$$

Uniqueness of the solution is essentially a consequence of $\mathsf{E}\log U < 0$, see [5], Theorem 1.5. Part (b) results from iterating (a) (see also Theorems 3 and 3 in [2]). Parts (c) and (d) result from conditioning on $\{U_k;\, k \geq 1\}$ in (b).

Theorem 5 (b) is another instance of the relationship between perpetuities and the Asian option process, observed in [1]. Finally, note that Theorem 5 (a) does not say anything about $\mathsf{E}\,1/A_t^{(\mu)}$, as $\mathsf{E}\,U = 1$ and (by (b)) $\mathsf{E}\,1/A_{T_\lambda}^{(\mu)} = \infty$. The expectation of $1/A_t^{(\mu)}$ is obtained by other means in [3].

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