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# AN AFFINE PROPERTY OF THE RECIPROCAL ASIAN OPTION PROCESS 

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This note describes a property of the Asian option process, which neatly links the process at $\mu$ with the one at $-\mu$, where $\mu$ is the drift of the geometric Brownian motion. The proof is based on (i) a known result due to Yor, on the law of the Asian option process taken at an exponential time, and (ii) a recent result on beta and gamma distributions.

Suppose $W$ is one-dimensional standard Brownian motion starting at the origin, and define what this author calls the Asian option process, for want of a better name:

$$
A_{t}^{(\mu)}=\int_{0}^{t} e^{2 \mu s+2 W_{s}} d s, \quad t \geq 0, \quad \mu \in \mathbb{R}
$$

Asian options have payoffs such as $\left(A_{t}^{(\mu)}-K\right)_{+}$, and have been studied by numerous authors in Finance and Mathematics; reciprocal Asian options have payoffs such as $\left(K-1 / A_{t}^{(\mu)}\right)_{+}$, and have not received much attention so far; for more details and references, the reader is referred to [3] and [4].

Theorem 1 ([6]). Let $T_{\lambda}$ be an exponentially distributed random variable, independent of $W$, with mean $1 / \lambda$. Then

$$
2 A_{T_{\lambda}}^{(\mu)} \stackrel{\mathcal{L}}{=} \frac{B_{1, \alpha}}{G_{\beta}}
$$

where $B_{1, \alpha} \sim \operatorname{Beta}(1, \alpha)$ and $G_{\beta} \sim \Gamma(\beta, 1)$ are independent, $\alpha=\mu / 2+\sqrt{2 \lambda+\mu^{2}} / 2$, $\beta=\alpha-\mu$.

Theorem 2 ([2]). For any $a, b, c>0$,

$$
\frac{G_{a}}{B_{b, a+c}}+G_{c}^{\prime} \stackrel{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b, a}} .
$$

where $G_{a} \sim \operatorname{Gamma}(a, 1), G_{c}^{\prime} \sim \operatorname{Gamma}(c, 1), B_{b, a+c} \sim \operatorname{Beta}(b, a+c), B_{b, a} \sim$ $\operatorname{Beta}(b, a)$ and all variables are independent.

Theorem 3. For any $\mu, t>0$,

$$
\frac{1}{2 A_{t}^{(\mu)}}+G_{\mu} \stackrel{\mathcal{L}}{=} \frac{1}{2 A_{t}^{(-\mu)}}
$$

where $G_{\mu} \sim \Gamma(\mu, 1)$ is independent of $W$.

The last result follows directly from the two previous ones, upon setting $a=\beta$, $b=1, c=\mu$, and then inverting the Laplace transform represented by the exponential time $T_{\lambda}$. Theorem 3 gives an easy proof of the well-known formula in Corollary 4 (just observe that $A_{\infty}^{(\mu)}=\infty$ a.s. if $\mu \geq 0$ ).

Corollary 4. For any $\mu>0$,

$$
\frac{1}{2 A_{\infty}^{(-\mu)}} \sim \operatorname{Gamma}(\mu, 1)
$$

Theorem 5. Let $\left\{U_{k} ; k \geq 1\right\}$ be independent variables with the same distribution as

$$
U \stackrel{\mathcal{L}}{=} \frac{B_{1}}{B_{2}}, \quad B_{1} \sim \operatorname{Beta}(\beta, \mu), \quad B_{2} \sim \operatorname{Beta}(1+\beta, \mu)
$$

with $B_{1}, B_{2}$ independent, $\mu>0, \beta=-\mu / 2+\sqrt{2 \lambda+\mu^{2}} / 2$, and let $\left\{G_{\mu}^{(k)} ; k \geq 0\right\}$ be a sequence of independent variables with a common $\operatorname{Gamma}(\mu, 1)$ distribution, independent of $\left\{U_{k} ; k \geq 1\right\}$. Then
(a) $\quad \frac{1}{2 A_{T_{\lambda}}^{(\mu)}} \stackrel{\mathcal{L}}{=} U\left(\frac{1}{2 A_{T_{\lambda}}^{(\mu)}}+G_{\mu}\right)$
(b) $\quad \frac{1}{2 A_{T_{\lambda}}^{(\mu)}} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\infty} U_{1} \cdots U_{k} G_{\mu}^{(k)}$; $\frac{1}{2 A_{T_{\lambda}}^{(-\mu)}} \stackrel{\mathcal{L}}{=} G_{\mu}^{(0)}+\sum_{k=1}^{\infty} U_{1} \cdots U_{k} G_{\mu}^{(k)}$
(c) $\quad \mathrm{E} e^{-s / 2 A_{T_{\lambda}}^{(\mu)}}=\mathrm{E}\left(\prod_{k=1}^{\infty} \frac{1}{1+s U_{1} \cdots U_{k}}\right)^{\mu}$
(d) $\mathrm{E} e^{-s / 2 A_{T_{\lambda}}^{(-\mu)}}=\mathrm{E}\left(\frac{1}{1+s} \prod_{k=1}^{\infty} \frac{1}{1+s U_{1} \cdots U_{k}}\right)^{\mu}=\left(\frac{1}{1+s}\right)^{\mu} \mathrm{E} e^{-s / 2 A_{T_{\lambda}}^{(\mu)}}$.

In (a), $A_{T_{\lambda}}^{(\mu)}$ and $G_{\mu}$ are independent; moreover, given $G_{\mu}$ and $U$ with the given distributions, the solution $1 / 2 A_{T_{\lambda}}^{(\mu)}$ is unique (in distribution).

Part (a) follows from computing the Mellin transform $\left(s \mapsto \mathrm{E}(\cdot)^{s}\right)$ of either side,
which from Theorem 1 is

$$
\begin{aligned}
& \frac{\Gamma(1-s) \Gamma(\beta+s) \Gamma(1+\beta+\mu)}{\Gamma(1+\beta+\mu-s) \Gamma(\beta)} \\
& \quad=\frac{\Gamma(\beta+s) \Gamma(\beta+\mu)}{\Gamma(\beta) \Gamma(\beta+\mu+s)} \frac{\Gamma(1+\beta-s) \Gamma(1+\beta+\mu)}{\Gamma(1+\beta) \Gamma(1+\beta+\mu-s)} \frac{\Gamma(1-s) \Gamma(\alpha+s) \Gamma(1+\alpha-\mu)}{\Gamma(1+\alpha-\mu-s) \Gamma(\alpha)} .
\end{aligned}
$$

Uniqueness of the solution is essentially a consequence of $\mathrm{E} \log U<0$, see [5], Theorem 1.5. Part (b) results from iterating (a) (see also Theorems 3 and 3 in [2]). Parts (c) and (d) result from conditioning on $\left\{U_{k} ; k \geq 1\right\}$ in (b).

Theorem 5 (b) is another instance of the relationship between perpetuities and the Asian option process, observed in [1]. Finally, note that Theorem 5 (a) does not say anything about $\mathrm{E} 1 / A_{t}^{(\mu)}$, as $\mathrm{E} U=1$ and (by (b)) $\mathrm{E} 1 / A_{T_{\lambda}}^{(\mu)}=\infty$. The expectation of $1 / A_{t}^{(\mu)}$ is obtained by other means in [3].

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## References

[1] Dufresne, D.: The distribution of a perpetuity, with applications to risk theory and pension funding. Scand. Actuarial. J. (1990), 39-79.
[2] Dufresne, D.: Algebraic properties of beta and gamma distributions, and applications. Adv. Appl. Math. 20 (1998), 285-299.
[3] Dufresne, D.: Laguerre series for Asian and other options. To appear in Mathematical Finance.
[4] Geman, H. and Yor, M.: Bessel processes, Asian options and perpetuities. Mathematical Finance 3 (1993), 349-375.
[5] Vervaat, W.: On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob. 11 (1979), 750-783.
[6] Yor, M.: Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires. C. R. Acad. Sci. Paris Série I 314 (1992), 951-956.

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