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AN AFFINE PROPERTY
OF THE RECIPROCAL ASIAN OPTION PROCESS

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(Received June 2, 1999)

This note describes a property of the Asian option process, which neatly links the
process at $\mu$ with the one at $-\mu$, where $\mu$ is the drift of the geometric Brownian
motion. The proof is based on (i) a known result due to Yor, on the law of the Asian op-
tion process taken at an exponential time, and (ii) a recent result on beta and gamma
distributions.

Suppose $W$ is one-dimensional standard Brownian motion starting at the origin,
and define what this author calls the Asian option process, for want of a better name:

$$A_t^{(\mu)} = \int_0^t e^{2\mu s + 2W_s} ds, \quad t \geq 0, \quad \mu \in \mathbb{R}.$$ 

Asian options have payoffs such as $(A_t^{(\mu)} - K)_+$, and have been studied by numer-
ous authors in Finance and Mathematics; reciprocal Asian options have payoffs such as
$(K - 1/A_t^{(\mu)})_+$, and have not received much attention so far; for more details and
references, the reader is referred to [3] and [4].

**Theorem 1 ([6]).** Let $T_\lambda$ be an exponentially distributed random variable, inde-
pendent of $W$, with mean $1/\lambda$. Then

$$2A_{T_\lambda}^{(\mu)} \leq \frac{B_{1,\alpha}}{G_\beta},$$

where $B_{1,\alpha} \sim \text{Beta}(1, \alpha)$ and $G_\beta \sim \Gamma(\beta, 1)$ are independent, $\alpha = \mu/2 + \sqrt{2\lambda + \mu^2/2}$,
$\beta = \alpha - \mu$.

**Theorem 2 ([2]).** For any $a, b, c > 0$,

$$\frac{G_a}{B_{b,a+c}} + G'_c \leq \frac{G_{a+c}}{B_{b,a}},$$

where $G_a \sim \text{Gamma}(a, 1)$, $G'_c \sim \text{Gamma}(c, 1)$, $B_{b,a+c} \sim \text{Beta}(b, a + c)$, $B_{b,a} \sim \text{Beta}(b, a)$ and all variables are independent.
Theorem 3. For any \( \mu, t > 0 \),
\[
\frac{1}{2A_{t}^{(\mu)}} + G_{\mu} \equiv \frac{1}{2A_{t}^{(-\mu)}},
\]
where \( G_{\mu} \sim \Gamma(\mu, 1) \) is independent of \( W \).

The last result follows directly from the two previous ones, upon setting \( a = \beta, b = 1, c = \mu \), and then inverting the Laplace transform represented by the exponential time \( T_{\lambda} \). Theorem 3 gives an easy proof of the well-known formula in Corollary 4 (just observe that \( A_{t}^{(\mu)} = \infty \) a.s. if \( \mu \geq 0 \)).

Corollary 4. For any \( \mu > 0 \),
\[
\frac{1}{2A_{-\infty}^{(\mu)}} \sim \text{Gamma}(\mu, 1).
\]

Theorem 5. Let \( \{U_{k}; k \geq 1\} \) be independent variables with the same distribution as
\[
U \equiv \frac{B_{1}}{B_{2}}, \quad B_{1} \sim \text{Beta}(\beta, \mu), \quad B_{2} \sim \text{Beta}(1 + \beta, \mu),
\]
with \( B_{1}, B_{2} \) independent, \( \mu > 0 \), \( \beta = -\mu/2 + \sqrt{2\lambda + \mu^{2}/2} \), and let \( \{G_{\mu}^{(k)}; k \geq 0\} \) be a sequence of independent variables with a common \( \Gamma(\mu, 1) \) distribution, independent of \( \{U_{k}; k \geq 1\} \). Then

(a) \[
\frac{1}{2A_{T_{\lambda}}^{(\mu)}} \equiv U \left( \frac{1}{2A_{T_{\lambda}}^{(\mu)}} + G_{\mu} \right)
\]
(b) \[
\frac{1}{2A_{T_{\lambda}}^{(\mu)}} \equiv \sum_{k=1}^{\infty} U_{1} \cdots U_{k} G_{\mu}^{(k)}, \quad \frac{1}{2A_{T_{\lambda}}^{(-\mu)}} \equiv \sum_{k=1}^{\infty} U_{1} \cdots U_{k} G_{\mu}^{(-k)}
\]
(c) \[
\mathbb{E} e^{-s/2A_{T_{\lambda}}^{(\mu)}} = \mathbb{E} \left( \prod_{k=1}^{\infty} \frac{1}{1 + sU_{1} \cdots U_{k}} \right)^{\mu}
\]
(d) \[
\mathbb{E} e^{-s/2A_{T_{\lambda}}^{(-\mu)}} = \mathbb{E} \left( \prod_{k=1}^{\infty} \frac{1}{1 + sU_{1} \cdots U_{k}} \right)^{\mu} = \left( \frac{1}{1 + s} \right)^{\mu} \mathbb{E} e^{-s/2A_{T_{\lambda}}^{(\mu)}}.
\]

In (a), \( A_{T_{\lambda}}^{(\mu)} \) and \( G_{\mu} \) are independent; moreover, given \( G_{\mu} \) and \( U \) with the given distributions, the solution \( 1/2A_{T_{\lambda}}^{(\mu)} \) is unique (in distribution).

Part (a) follows from computing the Mellin transform \( (s \mapsto \mathbb{E}(\cdot)^{s}) \) of either side,
which from Theorem 1 is
\[
\frac{\Gamma(1 - s) \Gamma(\beta + s) \Gamma(1 + \beta + \mu)}{\Gamma(1 + \beta + \mu - s) \Gamma(\beta)} = \frac{\Gamma(\beta + s) \Gamma(\beta + \mu) \Gamma(1 + \beta - s) \Gamma(1 + \beta + \mu) \Gamma(1 - s) \Gamma(\alpha + s) \Gamma(1 + \alpha - \mu)}{\Gamma(\beta) \Gamma(\beta + \mu + s) \Gamma(1 + \beta) \Gamma(1 + \beta + \mu - s) \Gamma(1 + \alpha - \mu - s) \Gamma(\alpha)}.
\]

Uniqueness of the solution is essentially a consequence of \(E \log U < 0\), see [5], Theorem 1.5. Part (b) results from iterating (a) (see also Theorems 3 and 3 in [2]). Parts (c) and (d) result from conditioning on \(\{U_k; k \geq 1\}\) in (b).

Theorem 5 (b) is another instance of the relationship between perpetuities and the Asian option process, observed in [1]. Finally, note that Theorem 5 (a) does not say anything about \(E 1/A_t^{(\mu)}\), as \(E U = 1\) and (by (b)) \(E 1/A_t^{(\mu)} = \infty\). The expectation of \(1/A_t^{(\mu)}\) is obtained by other means in [3].

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References