



Title	An affine property of the reciprocal Asian option process
Author(s)	Dufresne, Daniel
Citation	Osaka Journal of Mathematics. 2001, 38(2), p. 379-381
Version Type	VoR
URL	https://doi.org/10.18910/11747
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

AN AFFINE PROPERTY OF THE RECIPROCAL ASIAN OPTION PROCESS

DANIEL DUFRESNE

(Received June 2, 1999)

This note describes a property of the Asian option process, which neatly links the process at μ with the one at $-\mu$, where μ is the drift of the geometric Brownian motion. The proof is based on (i) a known result due to Yor, on the law of the Asian option process taken at an exponential time, and (ii) a recent result on beta and gamma distributions.

Suppose W is one-dimensional standard Brownian motion starting at the origin, and define what this author calls the *Asian option process*, for want of a better name:

$$A_t^{(\mu)} = \int_0^t e^{2\mu s + 2W_s} ds, \quad t \geq 0, \quad \mu \in \mathbb{R}.$$

Asian options have payoffs such as $(A_t^{(\mu)} - K)_+$, and have been studied by numerous authors in Finance and Mathematics; reciprocal Asian options have payoffs such as $(K - 1/A_t^{(\mu)})_+$, and have not received much attention so far; for more details and references, the reader is referred to [3] and [4].

Theorem 1 ([6]). *Let T_λ be an exponentially distributed random variable, independent of W , with mean $1/\lambda$. Then*

$$2A_{T_\lambda}^{(\mu)} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta},$$

where $B_{1,\alpha} \sim \text{Beta}(1, \alpha)$ and $G_\beta \sim \Gamma(\beta, 1)$ are independent, $\alpha = \mu/2 + \sqrt{2\lambda + \mu^2}/2$, $\beta = \alpha - \mu$.

Theorem 2 ([2]). *For any $a, b, c > 0$,*

$$\frac{G_a}{B_{b,a+c}} + G'_c \stackrel{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b,a}}.$$

where $G_a \sim \text{Gamma}(a, 1)$, $G'_c \sim \text{Gamma}(c, 1)$, $B_{b,a+c} \sim \text{Beta}(b, a + c)$, $B_{b,a} \sim \text{Beta}(b, a)$ and all variables are independent.

Theorem 3. For any $\mu, t > 0$,

$$\frac{1}{2A_t^{(\mu)}} + G_\mu \stackrel{\mathcal{L}}{=} \frac{1}{2A_t^{(-\mu)}},$$

where $G_\mu \sim \Gamma(\mu, 1)$ is independent of W .

The last result follows directly from the two previous ones, upon setting $a = \beta$, $b = 1$, $c = \mu$, and then inverting the Laplace transform represented by the exponential time T_λ . Theorem 3 gives an easy proof of the well-known formula in Corollary 4 (just observe that $A_\infty^{(\mu)} = \infty$ a.s. if $\mu \geq 0$).

Corollary 4. For any $\mu > 0$,

$$\frac{1}{2A_\infty^{(-\mu)}} \sim \text{Gamma}(\mu, 1).$$

Theorem 5. Let $\{U_k; k \geq 1\}$ be independent variables with the same distribution as

$$U \stackrel{\mathcal{L}}{=} \frac{B_1}{B_2}, \quad B_1 \sim \text{Beta}(\beta, \mu), \quad B_2 \sim \text{Beta}(1 + \beta, \mu),$$

with B_1, B_2 independent, $\mu > 0$, $\beta = -\mu/2 + \sqrt{2\lambda + \mu^2}/2$, and let $\{G_\mu^{(k)}; k \geq 0\}$ be a sequence of independent variables with a common $\text{Gamma}(\mu, 1)$ distribution, independent of $\{U_k; k \geq 1\}$. Then

$$\begin{aligned} \text{(a)} \quad & \frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} U \left(\frac{1}{2A_{T_\lambda}^{(\mu)}} + G_\mu \right) \\ \text{(b)} \quad & \frac{1}{2A_{T_\lambda}^{(\mu)}} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\infty} U_1 \cdots U_k G_\mu^{(k)}; \quad \frac{1}{2A_{T_\lambda}^{(-\mu)}} \stackrel{\mathcal{L}}{=} G_\mu^{(0)} + \sum_{k=1}^{\infty} U_1 \cdots U_k G_\mu^{(k)} \\ \text{(c)} \quad & \mathbf{E} e^{-s/2A_{T_\lambda}^{(\mu)}} = \mathbf{E} \left(\prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k} \right)^\mu \\ \text{(d)} \quad & \mathbf{E} e^{-s/2A_{T_\lambda}^{(-\mu)}} = \mathbf{E} \left(\frac{1}{1+s} \prod_{k=1}^{\infty} \frac{1}{1 + sU_1 \cdots U_k} \right)^\mu = \left(\frac{1}{1+s} \right)^\mu \mathbf{E} e^{-s/2A_{T_\lambda}^{(\mu)}}. \end{aligned}$$

In (a), $A_{T_\lambda}^{(\mu)}$ and G_μ are independent; moreover, given G_μ and U with the given distributions, the solution $1/2A_{T_\lambda}^{(\mu)}$ is unique (in distribution).

Part (a) follows from computing the Mellin transform ($s \mapsto \mathbf{E}(\cdot)^s$) of either side,

which from Theorem 1 is

$$\begin{aligned} & \frac{\Gamma(1-s)\Gamma(\beta+s)\Gamma(1+\beta+\mu)}{\Gamma(1+\beta+\mu-s)\Gamma(\beta)} \\ &= \frac{\Gamma(\beta+s)\Gamma(\beta+\mu)}{\Gamma(\beta)\Gamma(\beta+\mu+s)} \frac{\Gamma(1+\beta-s)\Gamma(1+\beta+\mu)}{\Gamma(1+\beta)\Gamma(1+\beta+\mu-s)} \frac{\Gamma(1-s)\Gamma(\alpha+s)\Gamma(1+\alpha-\mu)}{\Gamma(1+\alpha-\mu-s)\Gamma(\alpha)}. \end{aligned}$$

Uniqueness of the solution is essentially a consequence of $\mathbf{E} \log U < 0$, see [5], Theorem 1.5. Part (b) results from iterating (a) (see also Theorems 3 and 3 in [2]). Parts (c) and (d) result from conditioning on $\{U_k; k \geq 1\}$ in (b).

Theorem 5 (b) is another instance of the relationship between perpetuities and the Asian option process, observed in [1]. Finally, note that Theorem 5 (a) does not say anything about $\mathbf{E} 1/A_t^{(\mu)}$, as $\mathbf{E} U = 1$ and (by (b)) $\mathbf{E} 1/A_{T_\lambda}^{(\mu)} = \infty$. The expectation of $1/A_t^{(\mu)}$ is obtained by other means in [3].

ACKNOWLEDGEMENTS. Support from the Australian Research Council and from the Natural Science and Engineering Research Council of Canada is gratefully acknowledged.

References

- [1] Dufresne, D.: *The distribution of a perpetuity, with applications to risk theory and pension funding*. Scand. Actuarial. J. (1990), 39–79.
- [2] Dufresne, D.: *Algebraic properties of beta and gamma distributions, and applications*. Adv. Appl. Math. **20** (1998), 285–299.
- [3] Dufresne, D.: *Laguerre series for Asian and other options*. To appear in Mathematical Finance.
- [4] Geman, H. and Yor, M.: *Bessel processes, Asian options and perpetuities*. Mathematical Finance **3** (1993), 349–375.
- [5] Vervaat, W.: *On a stochastic difference equation and a representation of non-negative infinitely divisible random variables*. Adv. Appl. Prob. **11** (1979), 750–783.
- [6] Yor, M.: *Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires*. C. R. Acad. Sci. Paris Série I **314** (1992), 951–956.

Department of Mathematics and Statistics
 University of Montreal
 PO Box 6128, Downtown Station
 Montreal, Quebec
 Canada H3C 3J7
 e-mail: dufresne@dms.umontreal.ca