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THE ALDER-WAINWRIGHT EFFECT FOR STATIONARY PROCESSES WITH REFLECTION POSITIVITY (II)

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1. Introduction

This paper studies two kinds of linear random difference equations

$$\Delta X(n) = -\beta_1\left(\frac{X(n)+X(n-1)}{2}\right) - \underset{\varepsilon\downarrow 0}{\text{l.i.m.}}\,(\gamma_{1,\varepsilon}*\Delta X)(n) + \alpha_1\xi(n) \quad \text{a.s. } (n\in\boldsymbol{Z}), \tag{1.1}$$

$$\Delta X(n) = -\beta_2\left(\frac{X(n)+X(n-1)}{2}\right) - \underset{\varepsilon\downarrow 0}{\text{l.i.m.}}\,(\gamma_{2,\varepsilon}*\Delta X)(n) + \alpha_2 I(n) \quad \text{a.s. } (n\in\boldsymbol{Z}) \tag{1.2}$$

with infinite delays. Here

$$\Delta X(n) = X(n) - X(n-1) \quad (n\in\boldsymbol{Z}), \tag{1.3}$$

α_j and $\beta_j (j=1, 2)$ are positive constants and, for $j=1, 2$ and $\varepsilon>0$, $\gamma_{j,\varepsilon}$ is defined by

$$\gamma_{j,\varepsilon}(n) = \chi_{[1,\infty)}(n)\int_{-1+\varepsilon}^{1-\varepsilon} t^{n-1}\,\rho_j(dt) \quad (n\in\boldsymbol{Z}) \tag{1.4}$$

with a bounded Borel measure ρ_j on $[-1, 1]$ such that

$$\rho_j(\{-1, 1\}) = 0, \int_{-1}^{1}\frac{1}{\lambda+1}\,\rho_j(d\lambda) < 1\,. \tag{1.5}$$

$\boldsymbol{\Xi}=(\xi(n); n\in\boldsymbol{Z})$ in (1.1) is a normalized Gaussian white noise or a sequence of independent Gaussian random variables with mean 0 and variance 1. $\boldsymbol{I}=(I(n); n\in\boldsymbol{Z})$ in (1.2) is a real stationary Gaussian process named the *Kubo noise* associated with the solution $\boldsymbol{X}=(X(n); n\in\boldsymbol{Z})$ of (1.2). The Kubo noise $\boldsymbol{I}$ is related to $\boldsymbol{X}$ through the following relation

$$X(n) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n} R_2(n-m)\,I(m) \quad \text{a.s.} \quad (n\in\boldsymbol{Z}) \tag{1.6}$$

(Theorem 4.1 in [11]), where R_2 is the correlation function of $\boldsymbol{X}$. For the precise definition of Kubo noise, see [11] as well as section 2. The equation (1.1) (resp. (1.2)) is a slight modification of the *first* (resp. *second*) *KMO-Langevin equation* of Okabe [10, 11]. We call (1.1) (resp. (1.2)) the *modified first* (resp. *second*) *KMO-Langevin equation.*

In [13] and [4], the Alder-Wainwright effect or the long time behavior of the correlation function was studied in the case of continuous time and it was shown that the Alder-Wainwright effect can be characterized completely in terms of the decay of the delay coefficient γ of KMO-Langevin equations. On the other hand, Okabe [10, 11, 12], as a discrete analogue of [8, 9], developed the theory of KMO-Langevin equations for a class of discrete-time real stationary Gaussian processes with reflection positivity. In particular, he showed that the time evolution of such a process $\boldsymbol{X}$ can be described in terms of two kinds of linear random difference equations named the first and the second discrete KMO-Langevin equations. Then it seems to be natural to expect a discrete analogue to the Alder-Wainwright effect for continuous KMO-Langevin equations and that is what we show in this paper.

In so doing, we encounter two problems to be considered. The first problem is that, to state the Alder-Wainwright effect, the KMO-Langevin equations of Okabe [10, 11] are a little inconvenient. Therefore we modify them as (1.1) and (1.2). It seems to be interesting to see that, for this modified second KMO-Langevin equation, the Einstein relation

$$D=\frac{R(0)}{\beta_2} \tag{1.7}$$

holds, where D is the diffusion constant of $\boldsymbol{X}$ defined by

$$D=\lim_{N\to\infty}\frac{1}{2N}E((\sum_{n=0}^{N}X(n))^2)=\sum_{n=0}^{\infty}R(n)-\frac{R(0)}{2}\,. \tag{1.8}$$

The second problem is that, in contrast with the continuous case, the correlation function $R_j(j=1, 2)$ and the delay coefficient $\gamma_j(j=1, 2)$ are not necessarily monotone in the discrete case. Since the Tauberian theorem, which is our main tool to show the Alder-Wainwright effect, needs some monotonicity, we have to suppose a condition. Let $\boldsymbol{X}_1=(X_1(n); n\in\boldsymbol{Z})$ (resp. $\boldsymbol{X}_2=(X_2(n); n\in\boldsymbol{Z})$) be the unique real stationary Gaussian solution of the equation (1.1) (resp. (1.2)) with mean 0 and covariance R_1 (resp. R_2) of the form

$$R_j(n)=\int_{-1}^{1}t^{|n|}\,\sigma_j(dt)\quad(n\in\boldsymbol{Z})\quad(j=1, 2)\,. \tag{1.9}$$

Here $\sigma_j(j=1, 2)$ is a non-zero bounded Borel measure on $[-1, 1]$ such that

$$\sigma_j(\{-1, 1\})=0,\ \int_{-1}^{1}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)\sigma_j(dt)<\infty\,. \tag{1.10}$$

The property (1.9) is called the reflection positivity. Our assumption for $\rho=\rho_j$ is

ASSUMPTION **A**.

$$\text{supp } \rho \subset [0, 1] .$$

We note that Assumption $\boldsymbol{A}$ for ρ_j does not necessarily imply

$$\text{supp } \sigma_j \subset [0, 1] \tag{1.11}$$

but we can show that σ_j is "almost" of the form (1.11). This observation is essential in our argument and once this correspondence is established, the characterization of the long time tail of R_j in terms of γ_j is proved almost in parallel with the continuous case.

Give a slowly varying function L at infinity. For $j=1, 2$, we define

$$\gamma_j(n) = \chi_{[1,\infty)}(n) \int_{-1}^{1} t^{n-1} \rho_j(dt) \quad (n \in \boldsymbol{Z}) . \tag{1.12}$$

The following two theorems are our main results.

Theorem 1.1. *Suppose that* ρ_1 *satisfies Assumption* $\boldsymbol{A}$. *Let* $0<p<\infty$. *Then the following* (1.13) *and* (1.14) *are equivalent*:

$$\gamma_1(n) \sim n^{-p} L(n) \quad (n \to \infty) , \tag{1.13}$$

$$R_1(n) \sim \frac{\alpha_1^2 p}{\beta_1^3} n^{-(1+p)} L(n) \quad (n \to \infty) . \tag{1.14}$$

Theorem 1.2. *Suppose that* ρ_2 *satisfies Assumption* $\boldsymbol{A}$. *Let* $0<p<\infty$. *Then the following* (1.15) *and* (1.16) *are equivalent*:

$$\gamma_2(n) \sim n^{-p} L(n) \quad (n \to \infty) , \tag{1.15}$$

$$R_2(n) \sim \frac{\sqrt{2\pi}\, \alpha_2 p}{\beta_2^2} n^{-(1+p)} L(n) \quad (n \to \infty) . \tag{1.16}$$

Theorems 1.1 and 1.2 show that quite a good analogue holds between the continuous and the discrete case if we consider the equations (1.1) and (1.2) under the Assumption $\boldsymbol{A}$.

This paper is organized as follows. In section 2, we rewrite the KMO-Langevin equations of Okabe [10, 11] in a form which is suitable to the Alder-Wainwright effect. Sections 3 and 4 characterize Assumption $\boldsymbol{A}$ for ρ_2 in terms of σ_2. This characterization is used when we apply a Tauberian theorem in section 6. Section 5 is a miscellaneous preliminary of the proof of Theorems 1.1 and 1.2. In section 6, we prove Theorems 1.1 and 1.2. The last proofs are parallel with the continuous case but we include them for completeness.

The problem studied here was proposed by Professor Yasunori Okabe. The author is grateful to him for his valuable advices.

2. The modified KMO-Langevin equations

The main purpose of this section is to rewrite the discrete KMO-Langevin equations of Okabe [10, 11] in a form which is suitable to state the Alder-Wainwright effect. In what follows Δ stands for the difference operator

$$\Delta f(n) = f(n) - f(n-1).$$

We define

(2.1) $\Sigma = \{\sigma; \sigma$ is a bounded Borel measure on $[-1, 1]$ such that

$$\sigma(\{-1, 1\}) = 0 \quad \text{and} \quad \int_{-1}^{1} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) \sigma(dt) < \infty\},$$

(2.2) $\mathcal{L} = \{(\alpha, \beta, \rho); \alpha > 0, \beta > 0$ and ρ is a bounded Borel measure on $[-1, 1]$ such that

$$\rho(\{-1, 1\}) = 0 \quad \text{and} \quad \int_{-1}^{1} \frac{1}{1+t} \rho(dt) < 1\}.$$

Let $\sigma \in \Sigma$ and $\boldsymbol{X} = (X(n); n \in \boldsymbol{Z})$ be a real stationary Gaussian process with mean zero and covariance function R of the form

$$R(n) = \int_{-1}^{1} t^{|n|} \sigma(dt) \quad (n \in \boldsymbol{Z}). \tag{2.3}$$

The condition $\sigma \in \Sigma$ implies that $R \in l^1(\boldsymbol{Z})$. We define a function $[R](z)$ on $\overline{U_1(0)} = \{z \in \boldsymbol{C}; |z| \leq 1\}$ by

$$[R](z) = \frac{1}{2\pi} \sum_{n=0}^{\infty} R(n) z^n = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{1-tz} \sigma(dt). \tag{2.4}$$

Theorem 2.1. *The relation*

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{1}{1-tz} \sigma(dt) = \frac{\alpha_2}{\sqrt{2\pi}} \left(\beta_2 \frac{(1+z)}{2} + 1 - z + z(1-z) \int_{-1}^{1} \frac{1}{1-tz} \rho_2(dt)\right)^{-1} \tag{2.5}$$

defines a bijection $\sigma \to \varphi_2(\sigma) = (\alpha_2, \beta_2, \rho_2)$ *from* Σ *onto* $\mathcal{L}$.

Proof. By Theorem 3.1 in [11], the relation

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{1}{1-tz} \sigma(dt) = \frac{\tilde{\alpha}_2}{\sqrt{2\pi}} \left(\tilde{\beta}_2(1+z) + 1 - z + (1-z^2) \int_{-1}^{1} \frac{1}{1-tz} \tilde{\rho}_2(dt)\right)^{-1}$$

defines a bijection $\sigma \rightarrow \tilde{\phi}_2(\sigma) = (\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\rho}_2)$ from Σ onto $\tilde{\mathcal{L}}$, where $z \in U_1(0)$ and

$$\begin{aligned}(2.6) \qquad \tilde{\mathcal{L}} = \{(\tilde{\alpha}, \tilde{\beta}, \tilde{\rho}); \tilde{\alpha}>0, \tilde{\beta}>0 \quad &\text{and} \quad \tilde{\rho} \text{ is a bounded} \\ &\text{Borel measure on } [-1, 1] \text{ such that } \tilde{\rho}(\{-1, 1\}) = 0\} \,.\end{aligned}$$

Then the theorem follows easily if we consider the bijection $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\rho}_2) \rightarrow (\alpha_2, \beta_2, \rho_2)$ from $\tilde{\mathcal{L}}$ onto $\mathcal{L}$ defined by

$$(2.7) \quad \alpha_2 = \frac{\tilde{\alpha}_2}{1+\tilde{\rho}_2([-1, 1])}, \beta_2 = \frac{2\tilde{\beta}_2}{1+\tilde{\rho}_2([-1, 1])}, \tilde{\rho}_2 = \frac{1+t}{1+\tilde{\rho}_2([-1, 1])} \tilde{\rho}_2 \,. \quad \blacksquare$$

We recall the definition of Kubo noise according to [10, 11]. Let Δ, h and E be the spectral density, outer function and canonical representation kernel of $\boldsymbol{X}$ respectively:

$$(2.8) \qquad R(n) = \int_{-\pi}^{\pi} e^{-in\theta} \Delta(\theta) \, d\theta \quad (n \in \boldsymbol{Z}) \,,$$

$$(2.9) \qquad h(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \Delta(\theta) \, d\theta\right) \quad (z \in U_1(0)) \,,$$

$$(2.10) \qquad h(e^{i\theta}) = \lim_{r \uparrow 1} h(re^{i\theta}) \quad \text{a.e. } \theta \in (-\pi, \pi) \,,$$

$$(2.11) \qquad E(n) = \hat{h}(n) = \int_{-\pi}^{\pi} e^{-in\theta} h(e^{i\theta}) \, d\theta \quad (n \in \boldsymbol{Z}) \,.$$

Then we know that $E \in l^2(\boldsymbol{Z})$, $E(n)=0$ $(n=-1, -2, \cdots)$ and there exists a normalized Gaussian white noise $\boldsymbol{\Xi}=(\xi(n); n \in \boldsymbol{Z})$ such that

$$(2.12) \qquad X(n) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{n} E(n-m) \, \xi(m) \quad \text{in} \quad L^2(\Omega, \mathcal{F}, P) \,,$$

$$(2.13) \qquad \sigma(X(m); m \leq n) = \sigma(\xi(m); m \leq n) \quad (n \in \boldsymbol{Z}) \,.$$

The Kubo noise $\boldsymbol{I}=(I(n); n \in \boldsymbol{Z})$ associated with $\boldsymbol{X}$ is the stationary Gaussian process defined by

$$(2.14) \qquad I(n) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{n} E_I(n-m) \, \xi(m) \quad \text{a.s.} \quad (n \in \boldsymbol{Z}) \,,$$

where

$$(2.15) \qquad E_I(n) = \widehat{h_I}(z) = \int_{-\pi}^{\pi} e^{-in\theta} h_I(e^{i\theta}) \, d\theta \quad (n \in \boldsymbol{Z}) \,,$$

$$(2.16) \qquad h_I(z) = \frac{1}{\sqrt{2\pi}} \frac{h(z)}{[R](z)} \quad (z \in \overline{U_1(0)}) \,.$$

By Theorem 4.1 in [11] we see that $\boldsymbol{I}$ has the following properties:

$$(2.17)\qquad X(n)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{n}R(n-m)\,I(m)\quad \text{a.s.}\quad (n\in \boldsymbol{Z}),$$

$$(2.18)\qquad \sigma(X(m);\, m\leq n)=\sigma(I(m);\, m\leq n)\quad (n\in \boldsymbol{Z}).$$

Now for any $\varepsilon\geq 0$ we define

$$(2.19)\qquad \gamma_{2,\varepsilon}(n)=\chi_{[1\ \infty)}(n)\int_{-1+\varepsilon}^{1-\varepsilon}t^{n-1}\,\rho_2(dt)\quad (n\in \boldsymbol{Z}),$$

$$(2.20)\qquad \gamma_2(n)=\gamma_{2,0}(n)\quad (n\in \boldsymbol{Z}).$$

By using the Kubo noise $\boldsymbol{I}$ and the triple $(\alpha_2, \beta_2, \rho_2)$, the time evolution of $\boldsymbol{X}$ is described as follows.

Theorem 2.2.

(2.21)

$$\Delta X(n)=-\beta_2\left(\frac{X(n)+X(n-1)}{2}\right)-\underset{\varepsilon\downarrow 0}{\text{l.i.m.}}\,(\gamma_{2,\varepsilon}*\Delta X)(n)+\alpha_2 I(n)\quad \text{a.s.}\ (n\in \boldsymbol{Z})$$

Proof. Let $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\rho}_2)$ be an element of $\mathcal{L}$ defined by (2.7). We put for any $\varepsilon\geq 0$,

$$\tilde{\gamma}_{2,\varepsilon}(n)=\begin{cases}0\quad (n=-1,-2,\cdots),\\ \displaystyle\int_{-1+\varepsilon}^{1-\varepsilon}t^n\,\tilde{\rho}_2(dt)\quad (n=0,1),\\ \displaystyle\int_{-1+\varepsilon}^{1-\varepsilon}(t^n-t^{n-2})\,\tilde{\rho}_2(dt)\quad (n=2,3,\cdots),\end{cases}\quad ,\quad \tilde{\gamma}_2(n)=\tilde{\gamma}_{2,0}(n)\quad (n\in \boldsymbol{Z}).$$

Then we have

$$(2.22)\qquad \begin{aligned}&\tilde{\gamma}_{2,\varepsilon}(0)\,X(n)+\tilde{\gamma}_{2,\varepsilon}(1)\,X(n-1)\\ &\quad=(1+\tilde{\gamma}_2(0))\,\Delta\gamma_{2,\varepsilon}(1)\,X(n-1)+\tilde{\gamma}_{2,\varepsilon}(0)\,\Delta X(n),\end{aligned}$$

$$(2.23)\qquad \tilde{\gamma}_{2,\varepsilon}(l)=(1+\tilde{\gamma}_2(0))\,\Delta\gamma_{2,\varepsilon}(l)\quad (l=2,3,\cdots).$$

On the other hand, by Theorem 5.1 in [11], we know that the time evolution of $\boldsymbol{X}$ is governed by the following second KMO-Langevin equation

$$(2.24)\qquad \Delta X(n)=-\tilde{\beta}_2(X(n)+X(n-1))-(\tilde{\gamma}_2*X)(n)+\tilde{\alpha}_2 I(n)\quad \text{a.s.}\ (n\in \boldsymbol{Z}).$$

Since we have for any $\varepsilon\geq 0$,

$$|\tilde{\gamma}_{2,\varepsilon}(l)|\leq\int_{-1}^{1}|t|^{l-2}(1-t^2)\,\tilde{\rho}_2(dt)\quad (l=2,3,\cdots)$$

and

$$\sum_{l=2}^{\infty}\int_{-1}^{1}|t|^{l-2}(1-t^2)\,\tilde{\rho}_2(dt)=\int_{-1}^{1}(1+|t|)\,\tilde{\rho}_2(dt)<\infty,$$

it follows that

$$(2.25)\quad E[|(\tilde{\gamma}_2*X)(n)-(\tilde{\gamma}_{2,\varepsilon}*X)(n)|^2]\leq R(0)\Big(\sum_{l=0}^{\infty}|\tilde{\gamma}_2(l)-\tilde{\gamma}_{2,\varepsilon}(l)|\Big)^2\to 0\quad(\varepsilon\downarrow 0).$$

From (2.22), (2.23) and (2.25) we have

$$(\tilde{\gamma}_2*X)(n)=(1+\tilde{\gamma}_2(0))\lim_{\varepsilon\downarrow 0}(\gamma_{2,\varepsilon}*\Delta X)(n)+\tilde{\gamma}_2(0)\,\Delta X(n).$$

Then the theorem follows from (2.24) and the above. ■

DEFINITION 2.3. We call the stochastic difference equation (2.21) the *modified second KMO-Langevin equation* and the triple $(\alpha_2, \beta_2, \rho_2)$ or $(\alpha_2, \beta_2, \gamma_2)$ the *modified second KMO-Langevin data.*

In the next theorem we express α_2 and β_2 in terms of $\sigma=\varphi_2^{-1}((\alpha_2, \beta_2, \rho_2))$.

Theorem 2.6.

$$\text{(i)}\qquad \alpha_2=\sqrt{\frac{2}{\pi}}\,\sigma([-1,1])\int_{-1}^{1}\frac{1}{1-t}\,\sigma(dt)\Big(\int_{-1}^{1}\frac{1+t}{1-t}\,\sigma(dt)\Big)^{-1}.$$

$$\text{(ii)}\qquad \beta_2=2\sigma([-1,1])\Big(\int_{-1}^{1}\frac{1+t}{1-t}\,\sigma(dt)\Big)^{-1}.$$

Proof. By substituting $z=0$ into (2.5) we have

$$\frac{1}{2\pi}\,\sigma([-1,1])=\alpha_2\Big(\sqrt{2\pi}\,(\frac{\beta_2}{2}+1)\Big)^{-1}$$

and making $z\uparrow 1$ in (2.5) we obtain

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{1-t}\,\sigma(dt)=\alpha_2(\sqrt{2\pi}\,\beta_2)^{-1}.$$

The theorem follows from these two equalities. ■

Let D be the diffusion constant of $\boldsymbol{X}$ defined by (1.8). By Theorem 6.1 (iii) in [11] and (2.7), we see that the Einstein relation

$$(2.26)\qquad D=\frac{R(0)}{\beta_2}$$

holds between D and β_2. We remark that for the second KMO-Langevin data of Okabe [11] there is a deviation from the Einstein relation.

Next, according to the above modification of the second KMO-Langevin equation, we rewrite the first KMO-Langevin equation of Okabe [10]. Let $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\rho}_1)\in\tilde{\mathcal{L}}$ be the first KMO-Langevin data of [10] characterized by the relation

$$h(z)=\frac{\tilde{\alpha}_1}{\sqrt{2\pi}}\Big(\tilde{\beta}_1(1+z)+1-z+(1-z^2)\int_{-1}^{1}\frac{1}{1-tz}\,\tilde{\rho}_1(dt)\Big)^{-1}\quad(z\in U_1(0)),$$

where h is the outer function of $\boldsymbol{X}$. In the same way as Theorem 2.1, by applying a similar substitution

$$\alpha_1=\frac{\tilde{\alpha}_1}{1+\tilde{\rho}_1([-1,1])},\ \beta_1=\frac{2\tilde{\beta}_1}{1+\tilde{\rho}_1([-1,1])},\ \rho_1=\frac{1+t}{1+\tilde{\rho}_1([-1,1])}\tilde{\rho}_1, \tag{2.27}$$

to $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\rho}_1)\in\tilde{\mathcal{L}}$, we obtain from Theorem 4.1 in [10] the following theorem.

Theorem 2.5. *The relation*

$$h(z)=\frac{\alpha_1}{\sqrt{2\pi}}\left(\beta_1\frac{(1+z)}{2}+1-z+z(1-z)\int_{-1}^{1}\frac{1}{1-tz}\rho_1(dt)\right)^{-1} \tag{2.28}$$

defines a bijection $\sigma\to\varphi_1(\sigma)=(\alpha_1,\beta_1,\rho_1)$ from Σ onto $\mathcal{L}$.

For any $\varepsilon\geq 0$ we define

$$\gamma_{1,\varepsilon}(n)=\chi_{[1,\infty)}(n)\int_{-1+\varepsilon}^{1-\varepsilon}t^{n-1}\rho_1(dt)\quad(n\in\boldsymbol{Z}), \tag{2.29}$$

$$\gamma_1(n)=\gamma_{1,0}(n)\quad(n\in\boldsymbol{Z}). \tag{2.30}$$

Then the time evolution of $\boldsymbol{X}$ is described by using the triple $(\alpha_1,\beta_1,\rho_1)$ and the white noise $\boldsymbol{\Xi}$ in (2.14) and (2.15) as follows.

Theorem 2.6.

(2.31)
$$\Delta X(n)=-\beta_1\left(\frac{X(n)+X(n-1)}{2}\right)-\underset{\varepsilon\downarrow 0}{\text{l.i.m.}}\,(\gamma_{1,\varepsilon}*\Delta X)(n)+\alpha_1\,\xi(n)\quad\text{a.s. }(n\in\boldsymbol{Z}).$$

Since Theorem 2.6 follows from Theorem 6.1 in Okabe [10] almost in the same way with Theorem 2.2, we omit the proof.

DEFINITION 2.7. We call the stochastic difference equation (2.31) the *modified first KMO-Langevin equation* and the triple $(\alpha_1,\beta_1,\rho_1)$ or $(\alpha_1,\beta_1,\gamma_1)$ the *modified first KMO-Langevin data.*

EXAMPLE 2.8. For any $p>0$, we put

$$\tilde{\rho}=\chi_{(0,1)}(t)\frac{1}{\Gamma(p)}\frac{1}{1+t}(-\log t)^{p-1}dt, \tag{2.32}$$

where dt is the Lebesgue measure on [0,1]. We define

$$\rho=\frac{1+t}{1+\tilde{\rho}([0,1])}\tilde{\rho}. \tag{2.33}$$

Then for any $\alpha>0$ and $\beta>0$, we see that $(\alpha,\beta,\rho)\in\mathcal{L}$. By a direct calculation, we obtain

$$\gamma(n)=\chi_{[1,\infty)}(n)\int_0^t t^{n-1}\rho(dt)=\chi_{[1\ \infty)}(n)\frac{1}{1+\tilde{\rho}([0,1])}\frac{1}{n^p}. \tag{2.34}$$

EXAMPLE 2.9. For any $p>0$, we put

$$\tilde{\rho} = \chi_{(0,1)}(t)\frac{1}{\Gamma(p)}\frac{t}{t+1}(-\log t)^{p-1}\,dt\,, \tag{2.35}$$

and define ρ by (2.33). Then we have

$$\gamma(n) = \chi_{[1,\infty)}(n)\int_0^1 t^{n-1}\rho(dt) = \chi_{[1,\infty)}(n)\frac{1}{1+\tilde{\rho}([0,1])}\frac{1}{(n+1)^p}\,. \tag{2.36}$$

3. Relation between the supports of measures (1)

In this and next sections we characterize Assumption $\boldsymbol{A}$ in section 1 for ρ in terms of $\sigma=\varphi_2^{-1}((\alpha,\beta,\rho))$. The main results are Theorem 3.1 and Corollary 3.2 below.

We define

$$\mathcal{L}^+ = \{(\alpha,\beta,\rho)\in\mathcal{L};\ \mathrm{supp}\,\rho\subset[0,1]\}\,, \tag{3.1}$$

$$\mathcal{L}_1^+ = \{(\alpha,\beta,\rho)\in\mathcal{L}^+;\ \rho(\{0\})=0\,,\quad -\infty<\frac{\beta}{2}-1+\int_0^1\frac{1}{t}\rho(dt)<0\}\,, \tag{3.2}$$

$$\mathcal{L}_2^+ = \{(\alpha,\beta,\rho)\in\mathcal{L}^+;\ \rho(\{0\})=0\,,\quad \frac{\beta}{2}-1+\int_0^1\frac{1}{t}\rho(dt)=0\}\,, \tag{3.3}$$

$$\mathcal{L}_3^+ = \{(\alpha,\beta,\rho)\in\mathcal{L}^+;\ \rho(\{0\})=0\,,\quad 0<\frac{\beta}{2}-1+\int_0^1\frac{1}{t}\rho(dt)<\infty\}\,, \tag{3.4}$$

$$\mathcal{L}_4^+ = \{(\alpha,\beta,\rho)\in\mathcal{L}_+;\ \int_{[0,1]}\frac{1}{t}\rho(dt)=+\infty\}\,. \tag{3.5}$$

We note that $\mathcal{L}^+=\mathcal{L}_1^+\cup\mathcal{L}_2^+\cup\mathcal{L}_3^+\cup\mathcal{L}_4^+$ (disjoint union). Next we define

$$\Sigma_1^+ = \{\sigma\in\Sigma;\ \mathrm{supp}\,\sigma\subset[0,1]\,,\quad \sigma(\{0\})=0\,,\quad \int_0^1\frac{1}{t}\sigma(dt)<\infty\}\,, \tag{3.6}$$

$$\Sigma_2^+ = \{\sigma\in\Sigma;\ \mathrm{supp}\,\sigma\subset[0,1]\,,\quad \int_{[0,1]}\frac{1}{t}\sigma(dt)=+\infty\}\,, \tag{3.7}$$

$$\Sigma_0^- = \{\sigma\in\Sigma;\ \sigma=\sigma_0\,\delta_{p_0}\,,\quad \sigma_0>0\,,\quad -1<p_0<0\}\,, \tag{3.8}$$

$$\Sigma_0^{+-} = \{\sigma\in\Sigma;\ \sigma=\sigma_-+\sigma_+\,,\quad \sigma_-\in\Sigma_0^-\,,\quad \sigma_+\in\Sigma_1^+\}\,, \tag{3.9}$$

$$\Sigma_3^+ = \{\sigma\in\Sigma_0^{+-};\ -\infty<\int_{-1}^1\frac{1}{t}\sigma(dt)<0\}\,, \tag{3.10}$$

$$\Sigma_4^+ = \{\sigma\in\Sigma_0^{+-};\ \int_{-1}^1\frac{1}{t}\sigma(dt)=0\}\,, \tag{3.11}$$

$$\Sigma^+ = \Sigma_1^+\cup\Sigma_2^+\cup\Sigma_3^+\cup\Sigma_4^+\,, \tag{3.12}$$

$$\Sigma^{+-} = \{\sigma\in\Sigma;\ \sigma=\sigma_-+\sigma_+\,,\quad \sigma_-=\sigma_0\,\delta_{p_0}\,,\quad \sigma_0\geq 0\,,\quad -1<p_0<0 \ \text{ and } \ \sigma_+\in\Sigma_1^+\cup\Sigma_2^+\}\,. \tag{3.13}$$

Note that $\Sigma^{+}\subset\Sigma^{+-}$. In (3.5) and (3.7), the integrand t^{-1} is assumed to be $+\infty$ at 0.

Theorem 3.1.

(i) $\varphi_2(\Sigma_1^+)=\mathcal{L}_1^+$, $\varphi_2(\Sigma_2^+)=\mathcal{L}_2^+$, $\varphi_2(\Sigma_3^+)=\mathcal{L}_3^+$, $\varphi_2(\Sigma_4^+)=\mathcal{L}_4^+$.

(ii) *Let* $\sigma\in\Sigma_1^+\cup\Sigma_3^+$ *and* $(\alpha,\beta,\rho)=\varphi_2(\sigma)$. *Then*

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{t}\,\sigma(dt)=-\frac{\alpha}{\sqrt{2\pi}}\left(\frac{\beta}{2}-1+\int_0^1\frac{1}{t}\,\rho(dt)\right)^{-1}. \tag{3.14}$$

Corollary 3.2.

$$\varphi_2(\Sigma^+)=\mathcal{L}^+. \tag{3.15}$$

Before going into the details, we make several comments on the method of the proof. We show Theorem 3.1 first for discrete measures and in the general case by approximation by discrete measures. Originally this approximation method was used in Okabe [7] for the continuous case. The results of [7] were applied to the discrete case in Okabe [10,11]. It should be noted that though we use some results in [10, 11], which were proved by reducing to the continuous case, we can also prove them by this approximation method. It means that if we wish, we can develope the theory of the discete case independently of the continuous case.

Theorem 3.1 is proved finally in the next section. In this section, as a first step, we prove it for discrete measures. We define

$$\mathcal{L}_{i,d}^+=\{(\alpha,\beta,\rho)\in\mathcal{L}_i^+;\ \rho=\sum_{n=1}^{N}\rho_n\,\delta_{q_n}\quad\text{for some}\quad N\in\boldsymbol{N},\ \text{where}\quad \rho_n>0(n=1,\cdots,N),\ 0<q_1<q_2<\cdots<q_N<1\}\quad(i=1\text{ or }3), \tag{3.16}$$

$$\Sigma_{1,d}^+=\{\sigma\in\Sigma_1^+;\ \sigma=\sum_{n=0}^{N}\sigma_n\,\delta_{p_n}\quad\text{for some}\quad N\in\boldsymbol{N},\quad\text{where}\quad \sigma_n>0\ (n=0,\cdots,N)\quad\text{and}\quad 0<p_0<p_2<\cdots<p_N<1\}, \tag{3.17}$$

$$\Sigma_{3,d}^+=\{\sigma\in\Sigma_3^+;\ \sigma=\sum_{n=0}^{N}\sigma_n\,\delta_{p_n}\quad\text{for some}\quad N\in\boldsymbol{N},\quad\text{where}\quad \sigma_n>0\ (n=0,\cdots,N),\ -1<p_0<0<p_1<\cdots<p_N<1\}. \tag{3.18}$$

Lemma 3.2.

(i) $\varphi_2(\Sigma_{1,d}^+)=\mathcal{L}_{1,d}^+$, $\varphi_2(\Sigma_{3,d}^+)=\mathcal{L}_{3,d}^+$.

(ii) *Let* $\sigma\in\Sigma_{1,d}^+\cup\Sigma_{3,d}^+$ *and* $(\alpha,\beta,\rho)=\varphi_2(\sigma)$. *Then*

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{t}\,\sigma(dt)=-\frac{\alpha}{\sqrt{2\pi}}\left(\frac{\beta}{2}-1+\int_0^1\frac{1}{t}\,\rho(dt)\right)^{-1}. \tag{3.19}$$

Proof. Step 1. First we assume that $(\alpha, \beta, \rho) \in \mathcal{L}_{1,d}^+ \cup \mathcal{L}_{3,d}^+$. Then ρ is of the form $\sum_{n=1}^N \rho_n \delta_{q_n}$. By Theorem 2.1 we see that

$$[R](z) = \frac{\alpha \prod_{n=1}^N (1-q_n z)}{\sqrt{2\pi}\, g(z)},$$

$$\begin{aligned} g(z) &= \left(\beta \frac{(1+z)}{2} + 1 - z\right) \prod_{n=1}^N (1-q_n z) + z(1-z) \sum_{n=1}^N \rho_n \prod_{k \neq n} (1-q_k z) \\ &= (-1)^N \left(\prod_{n=1}^N q_n\right) \left(\frac{\beta}{2} - 1 + \int_0^1 \frac{1}{t} \rho(dt)\right) z^{N+1} + \cdots \end{aligned}$$

and so

$$\text{(3.20)} \qquad \begin{cases} \operatorname{sgn}(g(-\infty)) = -\operatorname{sgn}\left(\frac{\beta}{2} - 1 + \int_0^1 \frac{1}{t} \rho(dt)\right), \ \operatorname{sgn}(g(-1)) = 1, \\ \operatorname{sgn}(g(1)) = 1, \ \operatorname{sgn}(g(q_n^{-1})) = (-1)^{N-n+1} \quad (n = 1, \cdots, N), \\ \operatorname{sgn}(g(+\infty)) = (-1)^N \operatorname{sgn}\left(\frac{\beta}{2} - 1 + \int_0^1 \frac{1}{t} \rho(dt)\right). \end{cases}$$

In view of (3.20) there are $N+1$ numbers $p_n (n = 0, \cdots, N)$ and a positive constant c_1 such that

$$g(z) = c_1 \prod_{n=0}^N (1-p_n z),$$

$$\begin{cases} 0 < p_0 < q_1 < p_1 < \cdots < q_N < p_N < 1 & \text{if} \quad (\alpha, \beta, \rho) \in \mathcal{L}_{1,d}^+, \\ -1 < p_0 < 0 < q_1 < p_1 < \cdots < q_N < p_N < 1 & \text{if} \quad (\alpha, \beta, \rho) \in \mathcal{L}_{3,d}^+. \end{cases}$$

Therefore we can expand $[R](z)$ as

$$\text{(3.21)} \qquad [R](z) = \frac{1}{2\pi} \sum_{n=0}^N \sigma_n \frac{1}{1-p_n z},$$

$$\text{(3.22)} \qquad \sigma_n = \frac{\sqrt{2\pi}\, \alpha \prod_{k=1}^N (1-q_k p_n^{-1})}{c_1 \prod_{k \neq n} (1-p_k p_n^{-1})} > 0 \quad (n = 0, \cdots, N).$$

(3.21) and (3.22) show that $\sigma = \sum_{n=0}^{N+1} \sigma_n \delta_{p_n} = \varphi_2^{-1}((\alpha, \beta, \rho))$. From (2.5) we have

$$\text{(3.23)} \quad \frac{1}{2\pi} \sum_{n=0}^N \frac{\sigma_n}{z^{-1} - p_n} = \frac{\alpha}{\sqrt{2\pi}} \left(\beta \frac{z^{-1}+1}{2} + z^{-1} - 1 + \sum_{n=1}^N \rho_n \frac{z^{-1}-1}{z^{-1}-q_n}\right)^{-1}$$

and letting $z \to \infty$ we obtain (3.19). Thus $\mathcal{L}_{1,d}^+ \subset \varphi_2(\Sigma_{1,d}^+)$ and $\mathcal{L}_{3,d}^+ \subset \varphi_2(\Sigma_{3,d}^+)$.

Step 2. Conversely we assume $\sigma = \sum_{n=0}^N \sigma_n \delta_{p_n} \in \Sigma_{1,d}^+ \cup \Sigma_{3,d}^+$. By (2.4) we have

$$[R](z) = f(z) / \prod_{n=0}^N (1-p_n z),$$

$$f(z) = \frac{1}{2\pi} \sum_{n=0}^N \sigma_n \prod_{k \neq n} (1-p_k z) = \frac{(-1)^N}{2\pi} \left(\prod_{n=0}^N p_n\right) \left(\int_{-1}^1 \frac{1}{t} \sigma(dt)\right) z^N + \cdots$$

and so

$$\operatorname{sgn}(f(p_n^{-1})) = (-1)^{N-n} \quad (n = 0, \cdots, N) \qquad \text{if} \quad \sigma \in \Sigma_{1,d}^{+} \tag{3.24}$$

$$\begin{cases} \operatorname{sgn}(f(p_n^{-1})) = (-1)^{N-n} \quad (n = 1, \cdots, N) \\ \operatorname{sgn}(f(+\infty)) = (-1)^{N} \end{cases} \qquad \text{if} \quad \sigma \in \Sigma_{3,d}^{+}. \tag{3.25}$$

By (3.24) and (3.25) we see that there are N numbers $q_n (n=1, \cdots, N)$ and a positive constant c_2 such that

$$f(z) = c_2 \prod_{n=1}^{N} (1-q_n z),$$

$$\begin{cases} p_0 < q_1 < p_1 < \cdots < q_N < p_N & \text{if} \quad \sigma \in \Sigma_{1,d}^{+}, \\ 0 < q_1 < p_1 < q_2 < \cdots < q_N < p_N & \text{if} \quad \sigma \in \Sigma_{3,d}^{+}. \end{cases}$$

We put

$$\alpha = \sqrt{2\pi} \left(\frac{1}{[R](0)} - \frac{1}{2[R](1)} \right)^{-1}, \tag{3.26}$$

$$\beta = \frac{\alpha}{\sqrt{2\pi}\,[R](1)}, \tag{3.27}$$

$$a(z) = \frac{1}{[R](z)} - \frac{\sqrt{2\pi}}{\alpha} \left\{ \frac{\beta(1+z)}{2} + 1 - z \right\}. \tag{3.28}$$

Since $[R](0) = (2\pi)^{-1} \sigma([-1, 1]) > 0$, $[R](1) = (2\pi)^{-1} \int_{-1}^{1} (1-t)^{-1} \sigma(dt) > 0$ and $2[R](1) - [R](0) = (2\pi)^{-1} \int_{-1}^{1} (1+t)(1-t)^{-1} \sigma(dt) > 0$, we see that $\alpha > 0$ and $\beta > 0$. Furthermore since $a(0) = a(1) = 0$, there is a polynomial $b(z)$ of degree $\leq N-1$ such that

$$a(z) = \frac{\sqrt{2\pi}\, z(1-z)\, b(z)}{\alpha \prod_{n=1}^{N} (1-q_n z)}. \tag{3.29}$$

Substituting (3.29) into (3.28), we obtain

$$b(z) = \frac{\alpha \prod_{n=0}^{N} (1-p_n z)}{c_2 \sqrt{2\pi}\, z(1-z)} - \left\{ \frac{\beta(1+z)}{2z(1-z)} + \frac{1}{z} \right\} \prod_{n=1}^{N} (1-q_n z). \tag{3.30}$$

By (3.30) we see that if we define positive constants $\rho_k (k=1, \cdots, N)$ by

$$\rho_k = \frac{b(q_k^{-1})}{\prod_{n \neq k} (1-q_n q_k^{-1})} = \frac{\alpha \prod_{n=0}^{N} (1-p_n q_k^{-1})}{c_2 \sqrt{2\pi}\, q_k^{-1} (1-q_k^{-1}) \prod_{n \neq k} (1-q_n q_k^{-1})},$$

then the following polynomial

$$\left(\prod_{n=1}^{N} (1-q_n z) \right) \sum_{n=1}^{N} \rho_n \frac{1}{1-q_n z}$$

of degree $\leq N-1$ coincides with $b(z)$ at each value $q_k^{-1}(k=1, \cdots, N)$. Therefore we have

$$b(z) \equiv \left(\prod_{n=1}^{N}(1-q_n z)\right)\sum_{n=1}^{N}\rho_n\frac{1}{1-q_n z}. \tag{3.31}$$

By (3.28), (3.29) and (3.31), we obtain

$$[R](z)=\frac{\alpha}{2\pi}\left(\frac{\beta(1+z)}{2}+1-z+z(1-z)\int_0^1\frac{1}{1-tz}\rho(dt)\right)^{-1}, \rho=\sum_{n=1}^{N}\rho_n\delta_{q_n},$$

which shows that $\varphi_2(\sigma)=(\alpha, \beta, \rho)\in\mathcal{L}^+$. Again by (3.23) we have (3.19). Thus $\varphi_2(\sigma)\in\mathcal{L}^+_{1,d}$ if $\sigma\in\Sigma^+_{1,d}$ and $\varphi_2(\sigma)\in\mathcal{L}^+_{3,d}$ if $\sigma\in\Sigma^+_{3,d}$. This completes the proof of the lemma. ■

4. Relation between the supports of measures (2)

In this section we complete the proof of Theorem 3.1. First we prepare two lemmas.

Lemma 4.1. *Let* $(\alpha, \beta, \rho)\in\mathcal{L}^+$. *Then* $\rho([0, 1])<2$.

Proof. The lemma follows from the estimate

$$\rho([0, 1])\leq\int_0^1\frac{2}{t+1}\rho(dt)<2.$$

■

Lemma 4.2 *Let* $\sigma\in\Sigma$. *Let* $\alpha>0, \beta>0$ *and* ρ *be a bounded Borel measure on* $[-1, 1]$ *such that* supp $\rho\subset[0, 1]$, $\rho(\{1\})=0$ *and* (2.5) *hold. Then* $\int_{-1}^1(1+t)^{-1}\rho(dt)<1$ *and so* $\mathcal{L}^+\ni(\alpha, \beta, \rho)=\varphi_2(\sigma)$.

Proof. The lemma follows easily by letting $z\downarrow -1$ in (2.5). ■

Proof of Theorem 3.1. Step 1. Let $\sigma\in\Sigma_3^+$. Then σ is of the form

$$\sigma=\sigma_-+\sigma_+, \quad \sigma_-=\sigma_0\,\delta_{p_0}\in\Sigma_0^-. \quad \sigma_-\in\Sigma_1^+. \tag{4.1}$$

We put

$$\tilde{\sigma}_+=\frac{1}{t(1-t)}\sigma_+. \tag{4.2}$$

Since $\tilde{\sigma}_+([0, 1])=\int_{[0,1]}(t^{-1}+(1-t)^{-1})\,\sigma_+(dt)<\infty$, $\tilde{\sigma}_+$ is a bounded Borel measure on $[0, 1]$. We choose $\tilde{\sigma}_+^{(n)}\in\Sigma^+_{1,d}(n=1, 2, \cdots)$ so that w-$\lim_{n\to\infty}\tilde{\sigma}_+^{(n)}=\tilde{\sigma}_+$ on $[0, 1]$. We put $\sigma_+^{(n)}=t(1-t)\,\tilde{\sigma}_+^{(n)}$, $\sigma^{(n)}=\sigma_-+\sigma_+^{(n)}$ and $(\alpha^{(n)}, \beta^{(n)}, \rho^{(n)})=\varphi_2(\sigma^{(n)})$. Since

$$\lim_{n\to\infty}\int_{[-1,1]}\frac{1}{t}\sigma^{(n)}(dt)=\int_{[-1,1]}\frac{1}{t}\sigma(dt)<0, \tag{4.3}$$

we may assume from the beginning that $\int_{[-1,1]} t^{-1}\,\sigma^{(n)}(dt)<0$ for all n. Thus $\sigma^{(n)}\in\Sigma_{3,d}^{+}$ and, by Lemma 3.2, $(\alpha^{(n)}, \beta^{(n)}, \rho^{(n)})\in\mathcal{L}_{3,d}^{+}$. By Lemma 4.1, we have $\sup_n \rho^{(n)}([0, 1])\leq 2$. Therefore there exist a subsequence $n_1<n_2<\cdots$ and a bounded Borel measure ρ' on $[0, 1]$ such that $\text{w-lim}_{k\to\infty} \rho^{(n_k)}=\rho'$ on $[0, 1]$. By Theorem 2.4, $\alpha^{(n_k)}$ and $\beta^{(n_k)}$ converge to α and β as $k\to\infty$ respectively, where α and β are positive constants given by (i) and (ii) in Theorem 2.3. Then it follows from Theorem 2.1 that for any $z\in U_1(0)$,

(4.4)

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{1-tz}\,\sigma(dt)=\frac{\alpha}{\sqrt{2\pi}}\Big(\beta\,\frac{(1+z)}{2}+1-z+z(1-z)\int_{[0,1]}\frac{1}{1-tz}\,\rho'(dt)\Big)^{-1}.$$

If we put $c=1-\rho'(\{1\})/2$ and define a bounded Borel measure ρ'' on $[0, 1]$ by $\rho''(A)=\rho'(A\cap[0, 1))$, then the right hand side of (4.4) is equal to

$$\frac{(\alpha/c)}{\sqrt{2\pi}}\Big(\frac{(\beta+\rho'(\{1\}))}{c}\,\frac{(1+z)}{2}+1-z+z(1-z)\int_{[0,1]}\frac{1}{1-tz}\,\frac{1}{c}\,\rho''(dt)\Big)^{-1}$$

and therefore Lemma 4.2 yields $\varphi_2(\sigma)=(\alpha/c, (\beta+\rho'(\{1\}))/c, \rho''/c)\in\mathcal{L}^{+}$. Here we note that, by Theorem 2.4, $c=1$ and $\rho'(\{1\})=0$. Anyway we obtain $\varphi_2(\sigma)\in\mathcal{L}^{+}$; so that $\varphi_2(\Sigma_3^{+})\subset\mathcal{L}^{+}$.

In the same way, by approximating any element $\sigma\in\Sigma_1^{+}$ by elements of $\Sigma_{1,d}^{+}$, we can show that $\varphi_2(\Sigma_1^{+})\subset\mathcal{L}^{+}$.

Step 2. Let $\sigma\in\Sigma_2^{+}$. We put for any $n=1, 2, \cdots$,

$$\sigma^{(n)}=\chi_{[1/n,1]}(t)\,\sigma+\sigma(\{0\})\,\delta_{1/n}\,,\quad(\alpha^{(n)}, \beta^{(n)}, \rho^{(n)})=\varphi_2(\sigma^{(n)})\,.$$

Since $\int_{[0,1]} t^{-1}\,\sigma^{(n)}(dt)<\infty$, it follows from Step 2 that $(\alpha^{(n)}, \beta^{(n)}, \rho^{(n)})\in\mathcal{L}^{+}$. Then Theorem 2.3 and the monotone convergence theorem yield $\lim_{n\to\infty}\alpha^{(n)}=\alpha$ and $\lim_{n\to\infty}\beta^{(n)}=\beta$, where α and β are positive constants given by (i) and (ii) in Theorem 2.4 respectively. On the other hand, by the Lebesgue's convergence theorem, we have $\lim_{n\to\infty}\int_{[0,1]}(1-tz)^{-1}\,\sigma^{(n)}(dt)=\int_{[0,1]}(1-tz)^{-1}\,\sigma(dt)$. Then in the same way as Step 1, we obtain $\varphi_2(\sigma)\in\mathcal{L}^{+}$ and so $\varphi_2(\Sigma_2^{+})\subset\mathcal{L}^{+}$.

Step 3. Let $\sigma\in\Sigma_4^{+}$. Then σ is of the form (4.1). We put

$$\sigma_{-}^{(n)}=(\sigma_0+\frac{1}{n})\,\delta_{p_0}\,,\quad\sigma^{(n)}=\sigma_{-}^{(n)}+\sigma_{+}\quad(n=1, 2, \cdots)\,.$$

Since $\int_{[-1,1]} t^{-1}\,\sigma^{(n)}(dt)<0$, we have $\sigma^{(n)}\in\Sigma_3^{+}$ and so, by Step 1, $\varphi_2(\sigma^{(n)})\in\mathcal{L}^{+}$. Then in the same way as Step 1, we obtain $\varphi(\sigma)\in\mathcal{L}^{+}$; so that $\varphi(\Sigma_4^{+})\subset\mathcal{L}^{+}$.

By Steps 1–3, we conclude that $\varphi_2(\Sigma^{+})\subset\mathcal{L}^{+}$.

Step 4. Let $(\alpha, \beta, \rho)\in\mathcal{L}_3^{+}$. With $\rho(\{0\})=0$ in mind, we put $\tilde{\rho}=t^{-1}\rho$. Then $\tilde{\rho}$ is a bounded Borel measure on $[0, 1]$. We choose $\tilde{\rho}^{(n)}\in\Sigma_{1,d}^{+}(n=1, 2, \cdots)$ so that $\text{w-lim}_{n\to\infty}\tilde{\rho}^{(n)}=\tilde{\rho}$ on $[0, 1]$. We put $\rho^{(n)}=t\tilde{\rho}^{(n)}$. Since

$$\lim_{n\to\infty}\Big(\frac{\beta}{2}-1+\int_{[0,1]}\frac{1}{t}\,\rho^{(n)}(dt)\Big)=\frac{\beta}{2}-1+\int_{[0,1]}\frac{1}{t}\,\rho(dt)>0\,,$$

we may assume from the beginning that $(\alpha, \beta, \rho^{(n)}) \in \mathcal{L}_{3,d}^{+} (n=1, 2, \cdots)$. We put $\sigma^{(n)} = \varphi_2^{-1}((\alpha, \beta, \rho^{(n)}))$. By Lemma 3.2, $\sigma^{(n)}$ is of the form

$$\sigma^{(n)} = \sigma_{-}^{(n)} + \sigma_{+}^{(n)},\ \sigma_{+}^{(n)} \in \Sigma_1^{+},\ \sigma_{-}^{(n)} = s^{(n)}\, \delta_{p^{(n)}},\ 0 < s^{(n)} < \infty,\ -1 < p^{(n)} < 0 .$$

By (2.5), it holds that

$$\sigma_{+}^{(n)}([0, 1]),\ s^{(n)} \leq \sigma^{(n)}([0, 1]) = \sqrt{2\pi}\, \alpha(\beta/2+1)^{-1} .$$

Therefore there exist a subsequence $n_1 < n_2 < \cdots$, a bounded Borel measure σ_{+} on $[0, 1]$, $p_0 \in [-1, 0]$ and $s_0 \in [0, \sqrt{2\pi}\, \alpha(\beta/2+1)^{-1}]$ such that

$$\text{w-}\lim_{k\to\infty} \sigma_{+}^{(n_k)} = \sigma_{+} \quad \text{on} \quad [0, 1],\ \lim_{k\to\infty} p^{(n_k)} = p_0,\ \lim_{k\to\infty} s^{(n_k)} = s_0 . \tag{4.5}$$

Then it follows from (2.5) that for any $z \in U_1(0)$,

$$\begin{aligned} &\frac{1}{2\pi} \frac{s_0}{1-p_0 z} + \frac{1}{2\pi} \int_{[0,1]} \frac{1}{1-tz}\, \sigma_{+}(dt) \\ &= \frac{\alpha}{\sqrt{\pi 2}} \left(\frac{\beta}{2}(1+z) + 1 - z + z(1-z) \int_{[0,1]} \frac{1}{1-tz}\, \rho(dt) \right)^{-1} . \end{aligned} \tag{4.6}$$

We note that

$$\frac{1-z}{1-tz} = \left(\frac{1-t}{1-z} + t \right)^{-1} \downarrow 0 \quad (z \uparrow 1) . \tag{4.7}$$

Hence, by letting $z \uparrow 1$ in (4.6), we have $\sigma_{+}(\{1\}) = 0$ and $\int_{[0,1]} (1-t)^{-1}\, \sigma_{+}(dt) < \infty$, which show that $\sigma_{+} \in \Sigma_1^{+} \cup \Sigma_2^{+}$. If $s_0 \neq 0$, by letting $z \downarrow -1$ in (4.6), we see that $p_0 \neq -1$. Thus we obtain $\varphi_2^{-1}((\alpha, \beta, \rho)) = s_0\, \delta_{p_0} + \sigma_{+} \in \Sigma^{+-}$; so that $\varphi_2^{-1}(\mathcal{L}_3^{+}) \subset \Sigma^{+-}$.

Almost in the same way, we can show that $\varphi_2^{-1}(\mathcal{L}_1^{+}) \subset \Sigma_1^{+} \cup \Sigma_2^{+}$.

Step 5. Let $(\alpha, \beta, \rho) \in \mathcal{L}_2^{+}$. We put

$$\beta^{(n)} = \beta\left(1 - \frac{1}{n}\right),\ \sigma^{(n)} = \varphi_2^{-1}((\alpha, \beta^{(n)}, \rho)) \quad (n = 2, 3, \cdots) .$$

Since $\beta^{(n)}/2 - 1 + \int_{[0,1]} t^{-1}\, \rho(dt) < 0$, we have, by Step 4, $\sigma^{(n)} \in \Sigma_1^{+} \cup \Sigma_2^{+}$ Then, by using these sequences, we obtain $\varphi_2^{-1}((\alpha, \beta, \rho)) \in \Sigma_1^{+} \cup \Sigma_2^{+}$ in the same way as Step 4; so that $\varphi_2^{-1}(\mathcal{L}_2^{+}) \subset \Sigma_1^{+} \cup \Sigma_2^{+}$.

Step 6. Let $(\alpha, \beta, \rho) \in \mathcal{L}_4^{+}$. We put

$$\rho^{(n)} = \chi_{[1/n, 1]}(t)\, \rho + \sigma(\{0\})\, \delta_{1/n} .$$

By the monotone convergence theorem, we have $\lim_{n\to\infty} \int_0^1 t^{-1}\, \rho^{(n)}(dt) = +\infty$. Hence we see that, for sufficiently large n, $(\alpha, \beta, \rho^{(n)}) \in \mathcal{L}_3^{+}$ and so, by Step 4, $\varphi_2^{-1}((\alpha, \beta, \rho^{(n)})) \in \Sigma^{+-}$. Then in the same way as Step 4, we see that $\varphi_2^{-1}((\alpha, \beta, \rho)) \in \Sigma^{+-}$; so that $\varphi_2^{-1}(\mathcal{L}_4^{+}) \subset \Sigma^{+-}$.

Step 7. Let $(\alpha, \beta, \rho)\in\mathcal{L}^{+}$ and $\sigma=\varphi_2^{-1}((\alpha, \beta, \rho))$. Then by Steps 4–6, $\sigma\in\Sigma^{+-}$.

By substituting z in (2.5) by w^{-1}, we obtain

$$f(w)g(w) = -\frac{\alpha}{2\pi} \quad (|w|>1)\,, \tag{4.8}$$

where

$$f(w) = \frac{1}{2\pi}\int_{-1}^{1}\frac{1}{t-w}\,\sigma(dt)\,,$$

$$g(w) = \beta\left(\frac{w+1}{2}\right)+w-1+(1+w)\int_{0}^{1}\frac{1}{t-w}\,\rho(dt)\,.$$

Since $f(w)$ and $g(w)$ are both holomorphic on the domain $D=\boldsymbol{C}\setminus(\{p_0\}\cup[0, 1])$, (4.8) holds also on D. Then by letting $w\uparrow 0$ in (4.8), we see that

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{t}\,\sigma(dt) = \begin{cases} -\dfrac{\alpha}{\sqrt{2\pi}}\left(\dfrac{\beta}{2}-1+\displaystyle\int_{0}^{1}\frac{1}{t}\,\rho(dt)\right)^{-1} & \text{if } \in(\alpha, \beta, \rho)\in\mathcal{L}_1^{+}\cup\mathcal{L}_3^{+}\,, \\ +\infty \quad \text{if} \quad (\alpha, \beta, \rho)\in\mathcal{L}_2^{+}\,, \\ 0 \quad \text{if} \quad (\alpha, \beta, \rho)\in\mathcal{L}_4^{+}\,. \end{cases}$$

Thus we obtain

$$\varphi_2^{-1}(\mathcal{L}_1^{+})\subset\Sigma_1^{+},\ \varphi_2^{-1}(\mathcal{L}_2^{+})\subset\Sigma_2^{+},\ \varphi_2^{-1}(\mathcal{L}_3^{+})\subset\Sigma_3^{+},\ \varphi_2^{-1}(\mathcal{L}_4^{+})\subset\Sigma_4^{+}\,.$$

Since $\varphi_2(\Sigma^{+})\subset\mathcal{L}^{+}$, this completes the proof of Theorem 3.1. ■

EXAMPLE 4.3. We consider the triple (α, β, ρ) in Example 2.8. Since

$$\int_{0}^{1}\frac{1}{t}\,\rho(dt) = \frac{1}{1+\tilde{\rho}([0, 1])}\frac{1}{\Gamma(p)}\int_{0}^{\infty}x^{p-1}\,dx = +\infty\,,$$

we have $(\alpha, \beta, \rho)\in\mathcal{L}_4^{+}$.

EXAMPLE 4.4. We consider the triple (α, β, ρ) in Example 2.9. With $\int_{0}^{1}t^{-1}\,\rho(dt)<1$ in mind, we define

$$\beta_c = 2\left(1-\int_{0}^{1}\frac{1}{t}\,\rho(dt)\right)>0\,. \tag{4.9}$$

Then we have

$$(\alpha, \beta, \rho)\in\begin{cases} \mathcal{L}_1^{+} & \text{if} \quad 0<\beta<\beta_d\,, \\ \mathcal{L}_2^{+} & \text{if} \quad \beta=\beta_c\,, \\ \mathcal{L}_3^{+} & \text{if} \quad \beta_c<\beta<\infty\,. \end{cases}$$

5. Preliminaries

In this section, we prepare some facts which are used when we prove Theorems 1.1 and 1.2 in the next section.

Lemma 5.1. *Suppose that a real sequence* $(\gamma(n))_{n=0}^{\infty}$ *converges to* 0. *Then*

$$\lim_{\eta \downarrow 0} (1-e^{-\eta}) \sum_{n=0}^{\infty} e^{-\eta n} \gamma(n) = 0 . \tag{5.1}$$

The proof of Lemma 5.1 is pallarel to Lemma 3.4 in [4], so that we omit it.

Let $N \in \boldsymbol{Z}$. A real sequence $(a(n))_{n=N}^{\infty}$ is called *slowly increasing* if it holds that

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sup_{n \leq m \leq \lambda n} (a(m)-a(n)) \leq 0 \quad (\text{hence} = 0) \tag{5.2}$$

(see [3]). We need the following Tauberian condition.

(5.3) $(a(n))$ is eventually positive and $(\log a(n))$ is slowly increasing.

Lemma 5.2. *Let* $(a(n))$ *be a positive non-increasing sequence. Let* $\rho > 0$. *Then* $(n^{\rho} a(n))$ *staisfies the Tauberian condition* (5.3).

The proof of Lemma 5.2 is parallel to Lemma 3.6 in [4], so that we omit it.

The following Karamata's Tauberian Theorem for sequences plays a crucial role in our proof of Theorems 1.1 and 1.2 in the next section.

Theorem 5.3. (A discrete version of Theorem 1.7.6 in [3]) *Assume that* $(a(n))_{n=0}^{\infty}$ *is eventually positive,* $\rho > -1$, L *is slowly varying at infinity and*

$$\hat{a}(s) := s \sum_{n=0}^{\infty} e^{-sn} a(n) \tag{5.4}$$

is convergent for $s > 0$. *Then*

$$a(n) \sim n^{\rho} L(n)/\Gamma(1+\rho) \quad (n \to \infty) \tag{5.5}$$

implies

$$\hat{a}(s) \sim s^{-\rho} L(1/s) \quad (s \downarrow 0) . \tag{5.6}$$

Conversely, (5.6) *implies* (5.5) *if* $(a(n))$ *satisfies the Tauberian condition* (5.3).

Theorem 5.3 is reduced to Theorem 1.7.6 in [3] (or Theorem 3.7 in [4]) by the following lemma.

Lemma 5.4. *Let* $(a(n))$ *be a slowly increasing sequence. We define* $f(x) = a([x])$, *where* $[x]$ *is the greatest integer not greatet than* x. *Then* f *is a slowly increasing function.*

Proof. For any $\lambda > 1$, choose $M > 0$ so large that $n > M$ implies $(n+1)/n <$

$\sqrt{\lambda}$. Let $x>M+1$, $\lambda'\in(1,\sqrt{\lambda})$ and $t\in[1,\lambda']$. Then since

$$[tx]\le tx<\lambda'([x]+1)<\lambda[x],$$

we have

$$f(tx)-f(x)\le \sup_{[x]\le m\le\lambda[x]}(a(m)-a([x])).$$

Therefore it follows that

$$\begin{aligned}\lim_{\lambda'\downarrow 1}\limsup_{x\to\infty}\sup_{t\in[1,\lambda']}(f(tx)-f(x))&\le\limsup_{x\to\infty}\sup_{[x]\le m\le\lambda[x]}(a(m)-a([x]))\\ &=\limsup_{n\to\infty}\sup_{n\le m\le\lambda n}(a(m)-a(n)).\end{aligned}$$

Since λ is arbitrary, we obtain the lemma. ∎

Lemma 5.5. *Let $p\in\boldsymbol{R}$, $\varepsilon>0$, $q\ge 0$ and L be a slowly varying function at infinity. We assume $q-p-\varepsilon<-1$ and the sequence $(a(n))_{n=0}^{\infty}$ satisfies*

$$a(n)\sim n^{-p}L(n)\quad(n\to\infty). \tag{5.7}$$

Then

$$\lim_{\eta\downarrow 0}\eta^{\varepsilon}\sum_{n=0}^{\infty}e^{-\eta n}n^{q}a(n)=0. \tag{5.8}$$

The proof of Lemma 5.5 is almost the same with Lemma 3.5 in [4], so that we omit it.

Lemma 5.6. *Let σ be a bounded Borel measure on $[0, 1]$ such that* sup $\{t; t\in\operatorname{supp}\sigma\}=1$. *We put*

$$U(n)=\int_0^1 t^n\,\sigma(dt)\quad(n=0,1,\cdots). \tag{5.9}$$

Then for any $M\in\boldsymbol{N}$,

$$\lim_{n\to\infty}\frac{U(n+M)}{U(n)}=1. \tag{5.10}$$

Proof. First we note that $U(n+M)\le U(n)$ $(n=1,2,\cdots)$.
Let $0<\varepsilon<1$. Since $\sigma((1-\varepsilon,1])>0$, we have

$$\begin{aligned}0&\le\frac{\int_{[0,1-\varepsilon]}t^n\,\sigma(dt)}{\int_{(1-\varepsilon,1]}t^n\,\sigma(dt)}\\ &\le\sigma([0,1-\varepsilon])\left(\int_{(1-\varepsilon,1]}\left(\frac{t}{1-\varepsilon}\right)^n\sigma(dt)\right)^{-1}\to 0\quad(n\to\infty).\end{aligned} \tag{5.11}$$

Furthermore we have

$$U(n+M)\ge(1-\varepsilon)^M\int_{(1-\varepsilon,1]}t^n\,\sigma(dt). \tag{5.12}$$

It follows from (5.11) and (5.12) that

$$\liminf_{n\to\infty} \frac{U(n+M)}{U(n)} \geq \lim_{n\to\infty} (1-\varepsilon)^M \left(1+\frac{\int_{[0,1-\varepsilon]} t^n \sigma(dt)}{\int_{(1-\varepsilon,1]} t^n \sigma(dt)}\right)^{-1} = (1-\varepsilon)^M .$$

Since $\varepsilon \in (0, 1)$ is arbitrary, we have

$$\lim_{n\to\infty} \frac{U(n+M)}{U(n)} = 1 .$$

This completes the proof. ∎

Lemma 5.7. *Let $(E(n))_{n=0}^{\infty}$ be a real summable sequence and $(U(n))_{n=0}^{\infty}$ be a real positive non-increasing sequence. Assume that $(U(n))$ satisfies* (5.10). *We put*

(5.13) $$R(n) = \sum_{m=0}^{\infty} U(n+m) E(m) \quad (n = 0, 1, \cdots),$$

(5.14) $$c = \sum_{m=0}^{\infty} E(m) .$$

Then

(5.15) $$R(n) \sim cU(n) \quad (n \to \infty) .$$

Proof. Without loss of generality, we may assume that $E(n) \geq 0 (n=0, 1, \cdots)$. Since U is non-increasing, it follows that for any $n=0, 1, \cdots$ and $M \in \boldsymbol{N}$,

(5.16) $$R(n) \leq cU(n)$$

(5.17) $$R(n) \geq \left(\sum_{m=0}^{M} E(m)\right) U(n+M) .$$

By (5.10), (5.16) and (5.17), we have

$$\sum_{m=0}^{M} E(m) \leq \liminf_{n\to\infty} \frac{R(n)}{U(n)} \leq \limsup_{n\to\infty} \frac{R(n)}{U(n)} \leq c$$

and making $M \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{R(n)}{U(n)} = c .$$

This completes the proof. ∎

6. Proof of Theorems 1.1 and 1.2

In this section, we complete the proof of Theorems 1.1 and 1.2.

Theorem 6.1. *Let $\alpha > 0, \beta > 0, p > 0$ and L be a slowly varying function at infinity. Let $(\gamma(n))_{n=1}^{\infty}$ be a positive non-increasing sequence which tends to zero as $n \to \infty$. Let $(U(n))_{n=0}^{\infty}$ be a sequence of the form*

$$(6.1) \qquad U(n)=U_+(n)+c_0 p_0^n \quad (n=0, 1, \cdots),$$

where $c_0 \geq 0$, $-1<p_0<0$ *and* $(U_+(n))_{n=0}^{\infty}$ *is a positive, non-increasing and summable sequence. We assume that, for any* $\eta>0$, U *and* γ *satisfy*

$$(6.2) \quad \sum_{m=0}^{\infty} e^{-\eta n} U(n) = \sqrt{2\pi}\,\alpha \frac{1}{\beta\dfrac{(1+e^{-\eta})}{2}+1-e^{-\eta}+(1-e^{-\eta})\sum_{n=1}^{\infty} e^{-\eta n}\gamma(n)}.$$

Then

$$(6.3) \qquad \gamma(n) \sim n^{-p} L(n) \quad (n \to \infty)$$

if and only if

$$(6.4) \qquad U(n) \sim \frac{\sqrt{2\pi}\,\alpha p}{\beta^2} n^{-(1+p)} L(n) \quad (n \to \infty).$$

Proof. We put $d=[p]$, where $[p]$ is the greatest integer not greater than p. We divide the proof into two steps.

Step 1. We first assume (6.3). The fundamental idea of our proof is to differentiate both sides of (6.2) $d+1$ times with respect to η so that we can apply Theorem 5.3. By Lemma 3.2 in [4], we obtain

$$(6.5) \quad \sum_{n=0}^{\infty} e^{-\eta n} n^{d+1} U_+(n) = (-1)^{d+1}\sqrt{2\pi}\,\alpha\left\{-\frac{f^{(d+1)}(\eta)}{f(\eta)^2}+\frac{F_{d+1}(\eta)}{f(\eta)^{d+2}}\right\}-R^{(d+1)}(\eta),$$

where

$$R(\eta)=\frac{c_0}{1-p_0 e^{-\eta}},$$
$$f(\eta)=\beta\frac{(1+e^{-\eta})}{2}+1-e^{-\eta}+(1-e^{-\eta})\sum_{n=1}^{\infty} e^{-\eta n}\gamma(n)$$

and $F_{d+1}(\eta)$ is a polynomial in $\{f^{(l)}(\eta); l=0, 1, \cdots, d\}$. On the right hand side of (6.5), the first term turns out to be the main term. For each $l=1, 2, \cdots$, the l-th derivative of $f(\eta)$ is given by

$$f^{(l)}(\eta)=A_l(\eta)+B_l(\eta)+C_l(\eta)+D_l(\eta),$$

where

$$A_l(\eta)=(-1)^l\frac{\beta}{2}e^{-\eta}-(-1)^l e^{-\eta},$$
$$B_l(\eta)=(-1)^l(1-e^{-\eta})\sum_{n=1}^{\infty} e^{-\eta n} n^l \gamma(n),$$
$$C_l(\eta)=(-1)^{l-1} l e^{-\eta}\sum_{n=1}^{\infty} e^{-\eta n} n^{l-1}\gamma(n),$$
$$D_l(\eta)=(-1)^{l+1}\sum_{m=2}^{l}\binom{l}{m} e^{-\eta}\sum_{n=1}^{\infty} e^{-\eta n} n^{l-m}\gamma(n).$$

Since $d+1-p>0$ and

$$n^{d+1}\gamma(n)\sim n^{d+1-p}L(n)\,, \quad n^d\gamma(n)\sim n^{d-p}L(n) \quad (n\to\infty)\,,$$

Theorem 5.3 yields

$$B_{d+1}(\eta)+C_{d+1}(\eta)\sim(-1)^d\,\Gamma(d+1-p)\,p\eta^{-(d+1-p)}L(1/\eta) \quad (\eta\downarrow 0)\,. \tag{6.6}$$

Furthermore, since $(d+1-m)-p-2^{-1}(d+1-p)<-1(m=2,\cdots,d+1)$, it follows from Lemma 5.5 that

$$\lim_{\eta\downarrow 0}\eta^{1/2\cdot(d+1-p)}D_{d+1}(\eta)=0\,.$$

Hence we have

$$\lim_{\eta\downarrow 0}\frac{A_{d+1}(\eta)+D_{d+1}(\eta)}{\eta^{-(d+1-p)}L(1/\eta)}=\lim_{\eta\downarrow 0}\frac{\eta^{1/2\cdot(d+1-p)}(A_{d+1}(\eta)+D_{d+1}(\eta))}{\eta^{-1/2\cdot(d+1-p)}L(1/\eta)}=0\,. \tag{6.7}$$

By (6.6) and (6.7), we see that $f^{(d+1)}(\eta)$ decays as

$$f^{(d+1)}(\eta)\sim(-1)^d\,\Gamma(d+1-p)\,p\eta^{-(d+1-p)}L(1/\eta) \quad (\eta\downarrow 0)\,. \tag{6.8}$$

On the other hand, it follows from Lemma 5.5 that for any $\varepsilon>0$,

$$\lim_{\eta\downarrow 0}\eta^{\varepsilon}f^{(l)}(\eta)=0 \quad (l=1,\cdots,d)\,,$$

and hence

$$\lim_{\eta\downarrow 0}\eta^{\varepsilon}F_{d+1}(\eta)=0\,. \tag{6.9}$$

Furthermore, by Lemma 5.1, we have

$$\lim_{\eta\downarrow 0}f(\eta)=\beta\,. \tag{6.10}$$

It is clear that for any $\varepsilon>0$,

$$\lim_{\eta\downarrow 0}\eta^{\varepsilon}R^{(d+1)}(\eta)=0\,. \tag{6.11}$$

Now we return to (6.5). From (6.8), (6.9), (6.10) and (6.11) we conclude

$$\eta\sum_{n=0}^{\infty}e^{-\eta n}n^{d+1}U_+(n)\sim\frac{\sqrt{2\pi}\,\alpha\Gamma(d+1-p)\,p}{\beta^2}\eta^{-(d-p)}L(1/\eta) \quad (\eta\downarrow 0)\,. \tag{6.12}$$

Noting $d-p>-1$ we apply Lemma 5.2 and Theorem 5.3 to (6.12) and obtain

$$n^{d+1}U_+(n)\sim\frac{\sqrt{2\pi}\,\alpha p}{\beta^2}n^{d-p}L(n) \quad (n\to\infty)\,.$$

Thus (6.4) follows.

Step 2. Next, to prove another half, we assume (6.4). By (6.2) and (6.10), we have

$$\sum_{n=0}^{\infty} U(n) = \frac{\sqrt{2\pi}\,\alpha}{\beta}\,. \tag{6.13}$$

Using (6.13), we transform (6.2) as

$$\sum_{n=1}^{\infty} e^{-\eta n}\,\gamma(n) = \frac{\beta \sum_{n=0}^{\infty} e^{-\eta n} \sum_{k=n}^{\infty} U(k)}{\sum_{n=0}^{\infty} e^{-\eta n}\, U(n)} - \frac{\beta}{2} - 1\,. \tag{6.14}$$

The fundamental idea of the proof below is similar to Step 1. This time we differentiate both sides of (6.14) d times with respect to η. Then, by Lemma 3.3 in [4], we obtain

$$\sum_{n=0}^{\infty} e^{-\eta t}\, n^d\, \gamma(n) = (-1)^d \left\{\beta \frac{h^{(d)}(\eta)}{g(\eta)} + \frac{G_d(\eta)}{g(\eta)^{2^d}} - \left(\frac{\beta}{2}+1\right)\delta_{0d}\right\}, \tag{6.15}$$

where

$$g(\eta) = \sum_{n=0}^{\infty} e^{-\eta n}\, U(n)\,, \quad h(\eta) = \sum_{n=0}^{\infty} e^{-\eta n} \sum_{k=n}^{\infty} U(k)$$

and G_d is a polynomial in $\{h^{(l)}(\eta); l=0, 1, \cdots, d-1\}$ and $\{g^{(l)}(\eta); l=0, 1, \cdots, d\}$. On the right hand side of (6.15), the first term turns out to be the main term.

(6.13) shows that

$$\lim_{\eta \downarrow 0} g(\eta) = \frac{\sqrt{2\pi}\,\alpha}{\beta}\,. \tag{6.16}$$

For each $l=0, 1, \cdots$, the l-th derivatives of $f(\eta)$ and $g(\eta)$ are gievn by

$$g^{(l)}(\eta) = (-1)^l \sum_{n=0}^{\infty} e^{-\eta n}\, n^l\, U(n)\,, \quad h^{(l)}(\eta) = (-1)^l \sum_{n=0}^{\infty} e^{-\eta n}\, n^l \sum_{k=n}^{\infty} U(k)\,.$$

From the equality

$$\sum_{k=n}^{\infty} U(k) = \frac{c_0 p_0^n}{1-p_0} + \sum_{k=n}^{\infty} U_+(k)$$

and the monotone density theorem (c.f Theorem 1.7.2 in [3] and its remark), we have

$$\sum_{k=n}^{\infty} U(k) \sim \frac{\sqrt{2\pi}\,\alpha}{\beta^2}\, n^{-p}\, L(n) \quad (n \to \infty)\,. \tag{6.17}$$

Since $d-p>-1$ and $U(n)$ is eventually positive, it follows from (6.17) and Theorem 5.3 that

$$h^{(d)}(\eta) \sim (-1)^d \frac{\sqrt{2\pi}\,\alpha\,\Gamma(d-p+1)}{\beta^2}\, \eta^{p-d-1}\, L(1/\eta) \quad (\eta \downarrow 0)\,. \tag{6.18}$$

On the other hand, for any $\varepsilon>0$ and $l=0, 1, \cdots, d-1$, Lemma 5.5 yields

$$\lim_{\eta \downarrow 0} \eta^{\varepsilon} h^{(l)}(\eta) = 0 . \tag{6.19}$$

In the same way, for any $\varepsilon>0$ and $l=0, 1, \cdots, d$, we have

$$\lim_{\eta \downarrow 0} \eta^{\varepsilon} g^{(l)}(\eta) = 0 . \tag{6.20}$$

By (6.19) and (6.20), we obtain for any $\varepsilon>0$,

$$\lim_{\eta \downarrow 0} \eta^{\varepsilon} G_d(\eta) = 0 . \tag{6.21}$$

Now we return to (6.15). In view of (6.16), (6.18) and (6.21), we derive

$$\eta \sum_{n=0}^{\infty} e^{-\eta n} n^d \gamma(n) \sim \Gamma(d-p+1)\, \eta^{p-d} L(1/\eta) \quad (\eta \downarrow 0) .$$

Then, by Lemma 5.2 and Theorem 5.3, we obtain

$$n^d \gamma(n) \sim n^{d-p} L(n) \quad (n \to \infty) .$$

This shows (6.3) and completes the proof of Theorem 6.1. ■

Proof of Theorem 1.2. Let $(\alpha_2, \beta_2, \rho_2) \in \mathcal{L}^+$ and let $\sigma_2 = \varphi_2^{-1}((\alpha_2, \beta_2, \rho_2))$. We consider the correlation function R_2 of the form (1.9) and the delay coefficient γ_2 of the form (1.13). By Corollary 3.2, we have $\sigma_2 \in \Sigma^+$, so that R_2 is of the form (6.1). Furthermore, by (2.5), R_2 and $(\alpha_2, \beta_2, \gamma_2)$ satisfy for any $\eta>0$,

$$\sum_{n=0}^{\infty} e^{-\eta n} R_2(n) = \sqrt{2\pi}\, \alpha_2 \left\{ \beta_2 \frac{(1+e^{-\eta})}{2} + 1 - e^{-\eta} + (1-e^{-\eta}) \sum_{n=1}^{\infty} e^{-\eta n} \gamma_2(n) \right\}^{-1} .$$

Then Theorem 1.2 follows from Theorem 6.1 immediately. ■

Proof of Theorem 1.1. Let $(\alpha_1, \beta_1, \rho_1) \in \mathcal{L}^+$ and let $\sigma_1 = \varphi_1^{-1}((\alpha_1, \beta_1, \rho_1))$. We consider the correlation function R_1 of the form (1.9) and the delay coefficient γ_1 of the form (1.13). Let $\boldsymbol{X}$ be the solution of (1.1). Let E_1 be the canonical representation kernel of $\boldsymbol{X}$ and let $h(z)$ be the outer function of $\boldsymbol{X}$. We put $\nu = \varphi_2^{-1}((\alpha_1, \beta_1, \rho_1))$. Then, by (2.34), we have

$$h(z) = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{1-tz} \nu(dt) \quad (z \in U_1(0)) ,$$

and hence, by Theorems 4.2 and 4.3 in [10],

$$E_1(n) = \chi_{[0,\infty)}(n) \int_{-1}^{1} t^n \nu(dt) \quad (n \in \boldsymbol{Z}) .$$

Therefore, by Theorem 1.2, (1.13) is equivalent to

$$E_1(n) \sim \frac{\sqrt{2\pi}\, \alpha_1 p}{\beta_1^2} n^{-(1+p)} L(n) \quad (n \to \infty) . \tag{6.22}$$

On the other hand, we note that since $\nu\in\Sigma^+$, E_1 is of the form

$$E_1(n)=E_+(n)+c_0p_0^n \quad (n=0,1,\cdots), \tag{6.23}$$

where $c_0\geq 0$, $1-<p_0<0$ and $(E_+(n))_{n=0}^{\infty}$ is a positive, non-increasing and summable sequence. In the same way with (6.13), we can show

$$\sum_{n=1}^{\infty}E_1(n)=\frac{\sqrt{2\pi}\,\alpha_1}{\beta_1}.$$

In view of (2.13) in [10] and (6.23), we see that

$$R_1(n)=\frac{1}{2\pi}\sum_{m=0}^{\infty}E_+(n+m)\,E_1(m)+\frac{c_0}{2\pi}\left(\sum_{m=0}^{\infty}p_0^m\,E_1(m)\right)p_0^n\,.$$

Then, it follows from Lemma 5.7 that (6.22) is equivalent to (1.14). This completes the proof of Theorem 1.1. ■

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