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UNISERIAL RINGS AND LIFTING PROPERTIES

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We have studied the lifting (extending) property of simple modules on direct sums of cyclic hollow (uniform) modules in [4] and [5]. We have shown there that they are closely related to generalized uniserial rings, provided that the ring is right and left artinian.

We shall extend those relations to more general rings. In the second section we shall study rings R with the lifting (extending) property of simple modules as right R-modules by making use of results in [6] and [7]. Especially, we shall show that the ring of upper (lower) tri-angular matrices (of column finite) over a division ring with countable degree have the above property. We shall stydy, in the third section, relations between right generalized uniserial (couniserial) rings and the lifting (extending) property of simple modules on direct sum of cyclic hollow (uniform) modules, when R is semi-perfect. In the final section we shall study those problems on a commutative ring. We shall determine the type of modules which have the extending property of simple modules, when R is a Dededind domain and give a characterization for a commutative ring R to have the lifting (extending) property of simple modules for any direct sum of cyclic hollow (uniform) modules. Finally we shall show that if R is noetherian, the property mentioned above is equivalent to the fact: R is a direct sum of artinian serial rings and Dededind domains.

1. Definitions

Throughout this paper we assume that a ring R contains an identity element and every R-module M is a unitary right R-module. We call M a *completely indecomposable* if $\mathrm{End}_R(M)$ is a local ring. We denote the Jacobson radical, injective hull and the socle of M by $J(M)$, $E(M)$ and $S(M)$, respectively. $M^{(J)}$ means the direct sum of J-copies of M and $|M|$, $|J|$ mean the composition length of M and the cardinal of J. We put $S_i(M)/S_{i-1}(M)=S(M/S_{i-1}(M))$ inductively, namely the *lower Loewy series*. By $\bar{M}$ we denote $M/J(M)$. If for any simple submodule A of $\bar{M}$ (resp. $S(M)$) there exists a completely indecomposable cyclic hollow (resp. uniform) direct summand N of M such that $N+J(M)/J(M)=\bar{N}=A$ (resp. A is essential in N (i.e. $A_e\subseteq N$)), we say M has *the lifting* (resp. *extending*) *property of simple modules*. See [4], [5], [6] and [7] for other definitions.

For any primitive idempotent e in R, if the set of R-submodules in eR is linear with respect to the inclusion, (namely eR is serial) and has the upper Loewy series, we say R is a *right generalized uniserial ring* [13] (we usually consider a semi-perfect ring). As the dual, if every indecomposable and injective module with non-zero socle is serial and has the lower Loewy series, we say R is *right generalized couniserial ring*.

2 Tri-angular matrix rings

We have studied a right artinian ring over which every projective (resp. injective) module has the extending (resp. lifting) property of simple modules in [8]. In this section we shall characterize some rings with above property and show that they are closely related to rings of lower (resp. upper) tri-angular matrices over a division ring with countable dgree.

Theorem 1. *Let R be a ring. Then the following conditions are equivalent:*

1) *Every projective module has the lifting property of simple modules.*
2) *R has the above property as a right R-module.*
3) *R contains a right ideal A satisfying the following conditions.*

i) $A=\sum_I \oplus e_\alpha R$ *is a locally direct summand of R and the $e_\alpha R$ are hollow* (local).

ii) $S(R/J(R))=S(A+J(R)/J(R))$.

Proof. 1)→2). This is clear.
2)→3). We may assume $S(\bar{R})\neq 0$. We note that every indecomposable and cyclic hollow projective module M_α satisfies (E-I) in [4] and if $\bar{M}_\alpha\approx\bar{M}_\beta$, $M_\alpha\approx M_\beta$. Hence, noting the above remark, we can show, by the same argument given in the proof of [6], Theorem 4 that R contains a right ideal A as in 3).
3)→1). Let P be projective. Then P is a direct summand of $R^{(J)}$ for some J. It is clear that $A^{(J)}=\sum_I \oplus e_\alpha R^{(J)}$ is a locally direct summand of $R^{(J)}$ and $S(\bar{R}^{(J)})=\bar{A}^{(J)}$. Since $\mathrm{End}_R(e_\alpha R)\to\mathrm{End}_R(\overline{e_\alpha R})\to 0$ is exact, A has the lifting property of simple modules by [4], Theorem 2. Hence, $R^{(J)}$ has the same property (cf. the proof of [6], Theorem 3). Therefore, since $J(P)=PJ(R)$, P has the lifting property of simple modules by [4], Proposition 2.

Corollary. *Let R be a ring such that $R/J(R)$ is artinian. Then R has the lifting property of simple modules as a right or left R-module if and only if R is semi-perfect.*

Proof. Since $R/J(R)$ is artinian, $R=A$ if and only if R has the lifting property of simple modules as a right R-module. The concept of semi-perfect ring is left and right symmetric.

As the dual to Theorem 1

Theorem 2. *Let R be a ring. We assume that every uniform direct summand M with $S(M)\neq 0$ of R is artinian. Then the following conditions are equivalent:*

1) *Every projective module has the extending poperty of simple modules.*
2) *$R\oplus R$ has the above property as a right R-module.*
3) *R contains a right ideal A satisfying the following conditions.*

i) *$A=\sum_{\alpha\in I}\sum_{\beta\in I(\alpha)}\oplus e_{\alpha\beta}R$ is a locally direct summand of R and the $e_{\alpha\beta}R$ are uniform and $S(e_{\alpha\beta}R)\neq 0$.*

ii) *$S(R)=S(A)$.*

iii) *$S(e_{\alpha 1}R)\approx S(e_{\alpha\beta}R)$ and $S(e_{\alpha 1}R)\not\approx S(e_{\alpha' 1}R)$* if $\alpha\neq\alpha'$.

iv) *$I(\alpha)$ is a well ordered set,* $\mathrm{Hom}_R(S(e_{\alpha\beta}R), S(e_{\alpha\beta'}R))$ *is extendible to* $\mathrm{Hom}_R(e_{\alpha\beta}R, e_{\alpha\beta'}R)$ (and so $e_{\alpha\beta}R$ is isomorphic to a submodule of $e_{\alpha\beta'}R$) *for* $\beta\leqslant\beta'$ (cf. [8]).

Proposition 1. *Let R be a right artinian ring. Then R has the extending property of simple modules as a right R-module if and only if $R=\sum_{i=1}^{n}\oplus e_iR$ satisfies the following two conditions.*

1) *Each e_iR is uniform.*

2) *Any element f in* $\mathrm{Hom}_R(S(e_iR), S(e_jR))$ *is extendible to an element in* $\mathrm{Hom}_R(e_iR, e_jR)$ *or f^{-1} is extendible to an element in* $\mathrm{Hom}_R(e_jR, e_iR)$ *for $i\neq j$.*

Proof. This is clear from [7], Corollary 1 to Proposition 1 and [5], Corollary 8.

Proposition 2. *Let R be a self-injective ring as a right R-module. Then we have the following.*

1) *Every projective module has the extending propoerty of simple modules.*
2) *R contains a right ideal A satisfying the following conditions.*

i) *$A=\sum_{I}\oplus e_{\alpha}R$ is a locally direct summand of R and the $e_{\alpha}R$ are indecomoposable and injective.*

ii) *$S(R)=S(A)$.*

Proof. Put $S(R)=\sum_{I}\oplus S_{\alpha}$: the S_{α} are simple. Then $A=\sum_{I}\oplus E(S_{\alpha})$ is a desired right ideal. Hence, every direct sum $R^{(J)}$ has the extending property of simple modules (cf. [7], the proof of Themore 6). Therefore, every projective module has the same property by [7], Proposition 1.

If R is the full ring K_{∞} of linear transformations of a vector space over a division ring K, the above A is the socle of R.

Let T be the ring of lower tri-angular matrices (of column finite) over K with countable dsgree. Let $\{e_{ij}\}$ the set of matrix units in T. We put $e_i=e_{ii}$. Let B be a two-sided ideal of T not containing any e_i. We put $\tilde{T}=T/B$. Then

$e_nT+B/B\approx\tilde{e}_n\tilde{T}$ and $\sum_{n=1}^{\infty}\oplus\tilde{e}_n\tilde{T}$ is a two-sided ideal of $\tilde{T}$. Let R be an intermediate ring containing $1_{\tilde{T}}$ between $\sum_{n=1}^{\infty}\oplus\tilde{e}_n\tilde{T}$ and $\tilde{T}$. Then $\tilde{e}_nR=\tilde{e}_n(\sum\oplus\tilde{e}_m\tilde{T})$ $=\tilde{e}_n\tilde{T}$ is serial and of finite length. Further $S(R)=S(\sum_{n=1}^{\infty}\oplus\tilde{e}_nR)$. Let e be any idempotent in R. Then $e=\sum_{i=1}^{n-1}\tilde{e}_{ni}x_i+\tilde{e}_n+\sum_{m>n,m\geqslant m'}\tilde{e}_{mm'}x_{mm'}$ for some n and $g=e\tilde{e}_n$ is idempotent and $eR\supseteq gR\supseteq\tilde{e}_nR$. Hence, R satisfies all conditions of 3) in Theorem 2, replacing A by $\sum_{n=1}^{\infty}\oplus\tilde{e}_nR$. More generally, we may replace some diagonal parts in T by K_∞ and modify the forms of T (cf. [1]). Then we have the same structure of such a generalized ring as one of R. Thus, we have

Proposition 3. *Let R be a ring as above. Then every projective R-module has the extending property of simple modules.*

Let T' be the ring of upper tri-angular matrices over K with countable degree and let B' be a two-sided ideal in T' not containing any e_i. Put $\tilde{T}'=T'/B'$ and $J'(\tilde{T}')=\{(a_{ij})\,|\,a_{ii}=0$ for all $i\}$. We have a meaning of $\sum_{n=1}^{\infty}L^n$ for any L in $J'(\tilde{T}')$. Hence, $J'(\tilde{T}')$ is the Jacobson radical of $\tilde{T}'$ (see [12]) and $S(\tilde{T}'/J(\tilde{T}'))=\sum_{n=1}^{\infty}\oplus\tilde{e}_n\tilde{T}'/\tilde{e}_nJ(\tilde{T}')$. Let R' be an intermediate ring containing $1_{\tilde{T}}$ between $\tilde{T}'$ and $\sum_{n=1}^{\infty}\oplus\tilde{e}_n\tilde{T}'$. Then $S(R'/J(R'))=\sum\oplus\tilde{e}_nR'/J(R')$. Hence, R' satisfies the conditions in 3) of Theorem 2. Similary R satisfies the same condition. Thus, we have

Proposition 4. *Let R and R' be as above. Then every projective R-(R'-) module has the lifting property of simple modules.*

Let T be the ring before Proposition 3 with matrix units h_{ij}.

Theorem 3. *Let R be a ring. Then R is isomorphic to an intermediate ring containing 1_T between T and $\sum_{n=1}^{\infty}\oplus h_{nn}T$ if and only if R contains a two-sided ideal A satisfying the following conditions.*

1) i)$\sim$iv) *in Theorem 2 are satisfied, namely every projective module has the extending property of simple modules.*

2) *A is a faithful left R-module.*

3) $|I|=1$.

4) *$e_\alpha R\not\approx e_\beta R$ if $\alpha\neq\beta$ ($e_\alpha=e_{1\alpha}$).*

5) *Each $e_\alpha R$ is of finite length and every submodule is projective.*

Proof. Since A has the extending property by [5], Corollary 8, $\{e_\alpha\}_{I(\alpha)}$ is a countable and linearly ordered set with respect to $<^*$ by 3)$\sim$5) and [5], Corollary 8. Hence, we may assume $e_1R<^*e_2R<^*\cdots<^*e_nR<^*\cdots$. We note that every element in $\mathrm{Hom}_R(S(e_nR), S(e_mR))$ is uniquely extendible to $\mathrm{Hom}_R(e_nR, e_mR)$ for $n\leqslant m$ by 5). We fix an element $f_{n,n-1}$ in $\mathrm{Hom}_R(S(e_{n-1}R), S(e_nR))$ and denote its unique extension in $\mathrm{Hom}_R(e_{n-1}R, e_nR)$ by $g_{n,n-1}$ and the identity mapping of e_nR by $g_{n,n}$. We put $g_{m,n}=g_{m,m-1}g_{m-1,m-2}\cdots g_{n+1,n}$ for $m>n$ (cf. [1]). It is clear that $\mathrm{Hom}_R(e_nR, e_mR)=g_{m,n}\mathrm{End}_R(e_nR)=\mathrm{End}_R(e_mR)g_{m,n}$ and $\mathrm{Hom}_R(e_nR, e_nR)\approx\mathrm{Hom}_R(e_1R, e_1R)$ via $g_{n,1}=g_{n,n-1}\cdots g_{?,1}$. Since $\mathrm{Hom}_R(e_mR, e_nR)=0$ for $m>n$ by 5), $\mathrm{End}_R(A)$ is isomorphic to T: the ring of lower tri-angular matirces of column finite over a division ring $\mathrm{End}_R(e_1R)$ with countable degree. Since A is a faithful left R-module and $\mathrm{Hom}_R(e_nR, e_mR)=e_mRe_n\subseteq A$, R is an intermediate ring between T and $\sum\oplus g_{n,n}T$. The converse is clear.

Let T' be the ring before Proposition 4.

Theorem 4. *Let R' be a ring. Then R' is isomorphic to an intermediate ring containing 1_T, between T' and $\sum_{n=1}^{\infty}\oplus h_{n,n}T'$ if and only if R' contains a two-sided ideal A' satisfying the following conditioiors.*

1) i) *of Theorem* 1 *is astisfied, namely every projective module has the lifting property of simple modules.*

2) *A' is a faithful left R-module.*

3) *$\bar{e}_\alpha\bar{R}'\not\approx\bar{e}_\beta\bar{R}'$ for all $\alpha\neq\beta$.*

4) *Each submodule $e_\alpha C\neq 0$ of $e_\alpha R'$ is projective and $e_\alpha R'/e_\alpha C$ is of finite length.*

Proof. This is dual to Theorem 3.

REMARKS. 1. If we omit 4) (resp. 3)) in Theorem 3 (resp. 4), we may replace T (resp. T') by the general form as before Proposition 3.

2. Let K be a field with automorphism σ such that $K\neq\sigma(K)$ and let $\{e_{ij}\}$ be a set of matrix units. We put $R=Ke_{11}\oplus Ke_{22}\oplus Ke_{12}$. Then R is a right artinian ring by setting $ke_{11}k'e_{21}=k^{\sigma}k'e_{12}$ and $ke_{12}k'e_{22}=kk'e_{12}$ for $k, k'\in K$ and R has the extending property of simple modules as a right R-module. However $\mathrm{End}_R(S(e_{11}R))$ is not extendible to $\mathrm{End}_R(e_{11}R)$ and R does not have the extending property of simple modules as a left R-module (cf. [8]). $R\oplus R$ does not have the extending property of simple modules as a right R-module.

3. Generalized uniserial rings

Let R be an algebra over a field K of finite dimension (more generally R is an artinian algebra with duality).

Proposition 5. *Let R be as above. Then the following conditions are equivalent:*

1) *Every direct sum of hollow right R-modules has the lifting property of simple modules.*

2) *Every direct sum of uniform left R-modules has the extending property of simple modules.*

3) *R is a right generalized uniserial ring.*

4) *R is a left generalized couniserial ring.*

Proof. Since R has the duality, the proposition is clear from [4], Theorem 4.

Theorem 5. *Let R be a semi-perfect ring. Then the following conditions are equivalent*:

1) *For any primitive idempotent* e, $eR\supset eJ\supset eJ^2\supset\cdots\supset eJ^n\supset\cdots\supset e(\bigcap_n J^n)$ *is a unique composition series of right ideals* eA *with* $|eR/eA|<\infty$ *in* eR.

2) *Every direct sum of cyclic hollow modules with finite length has the lifting property of simple modules.*

Proof. Since R is semi-perfect, we have a complete set of mutually orthogonal primitive idempotents $\{e_i\}$ with $1=\sum e_j$. 1)$\rightarrow$2). Let M be a cyclic hollow module with finite length and $M\approx e_iR/e_iA$ for some e_i in $\{e_j\}$ and a right ideal A. Then $e_iA=e_iJ^n$ for some integer n. Hence, e_iA is a character submodule of e_iR. Accordingly, we have 2) by [4], Theorem 2.
2)$\rightarrow$1). We can show by induction on n and the same argument in [4], Theorem 4 that e_iJ^n/e_iJ^{n+1} is simple or zero. Hence, if $|eR/eA|<\infty$, $eA=eJ^t$ for some t. Let $|eR/eA|=\infty$. Since $\overline{eR/eA}\approx\overline{eR/eJ^n}$ for any n, there exists an epimorphism $g_n\colon eR/eA\rightarrow eR/eJ^n$. Hence, $(eA\subseteq)$ $\ker g_n=eJ^n/eA$ by the above. Accordingly, $eA\subseteq\bigcap_n eJ^n=e(\bigcap_n J^n)$. Hence, every right ideal in eR is contained in $e(\bigcap_n J^n)$ or equal to eJ^n.

Corollary. *Let R be a semi-perfect ring with* $\bigcap_n J^n=0$. *Then the following conditions are equivalent*:

1) *R is a right generalized uniserial ring.*

2) *Every direct sum of cyclic hollow modules has the lifting property of simple modules.*

Proof. Since $\bigcap_n J^n=0$, every hollow module is isomorphic to e_iR/e_iJ^n or e_iR if 1) is satisfied. Hence, we have 2) by [4], Theorem 2.

Theorem 6. *Let R be any ring. Then the following conditions are equivalent*:

1) *Every direct sum of uniform modules of finite length has the extending property of simple module.*

2) *For every indecomposable injective module* E, $0\subseteq S_1(E)\subseteq S_2(E)\subseteq\cdots\subseteq S_n(E)\subseteq\cdots$ *is a unique composition series of submodules of finite length.*

Proof. 1)$\rightarrow$2). Put $T=\mathrm{End}_R(E)$ and $J'=J(T)$. Then $\mathrm{End}_R(S_1(E))=T/J'$

if $S_1(E)\neq 0$. We assume $S_i/S_{i-1}(E)$ is simple for all $i\leqslant n$. If $|B|\leqslant n$ for a submodule B of E, $B\subseteq S_n(E)$. Let $A_i(\supseteq S_n(E))$ be a submodule of E such that $A_i/S_n(E)$ is simple for $i=1, 2$. Then taking the identity in T, we obtain some f in J' such that $(1+f)(A_i)\subseteq A_{i'}$, $(i, i')=(1, 2)$ by [5], Corollary 8. Since $|f(A_i)|<|A_i|$, $f(A_i)\subseteq S_n(E)$. Hence, $A_i\subseteq A_{i'}$ and so $A_i=A_{i'}$. Therefore, $S_{n+1}(E)/S_n(E)$ is simple or zero.
2)→1). Let M be a uniform module of finite length. Then $E(M)$ has a non-zero socle. Hence, we have 1) by [5], Corollary 8.

Corollary. *Let R be a ring such that every indecomposable injective module E with* $S(E)\neq 0$ *is equal to* $\cup S_i(E)$ (e.g. R is a semi-primary or commutative noetherian ring). *Then the following conditions are equivalent*:

1) *Every direct sum of uniform modules with non-zero socles has the extending property of simple modules.*

2) *R is a right generalized couniserial ring.*

Proof. If R is right generalized couniserial, then for every uniform module M with $S(M)\neq 0$, $M=E(M)$ or $M=S_i(E)$ and so M is completely indecomposable.

REMARKS. 1. If R is a semi-primary, EJ^i/EJ^{i+1} is semi-simple and $EJ^n\subseteq S_1(E)$ if $EJ^{n+1}=0$.

2. If R is a commutative noetherian ring, then $E=\cup S_i(E)$ by [11].

Theorem 7. *Let R be a right artinian and right genralized uniserial ring. Then an R-module M has the lifting property of simple modules if and only if M is a direct sum of cyclic hollow submodules.*

Proof. This is clear from Corollary to Theorem 5 and [6], Corollary to Theorem 4.

We shall give the dual result to Theorem 7.

Lemma 1. *Let R be a right noetherian ring and M an R-module. We assume that every uniform direct summand of M is artinian and $M\,_e\!\supseteq S(M)$. If M has the extending property of simple modules, then M is a direct sum of uniform submodules.*

Proof. From the assumption and [7], Theorem 6, M contains a submodule M' such that $M'=\sum_I\oplus M_\alpha$ is a locally direct summand of M and $S(M)=S(M')$, where the M_α are uniform. We shall show $M=M'$. If $M\neq M'$, there exist x in M-M' and a non-zero right ideal A such that $xA\subseteq M'$ for $M\,_e\!\supseteq S(M)=S(M')$. Since R is right noetherian, we can find a maximal one B among A above, i.e. $m\in M-M'$ and $mB\subseteq M'$. Now, mB is finitely generated and so $mB\subseteq\sum_J\oplus M_{\alpha_i}(=M_J)$ for some finite subset J of I. M' being a locally direct summand of

M, $M=M_J\oplus M^*$. Let $m=m_1+m_2$; $m_1\in M_J$, $m_2\in M^*$. Then $m_2B\subseteq mB+m_1B\subseteq M_J$ and $m_2B\subseteq M_J\cap M^*=0$. Since $m_2\notin M'$ and B is maximal, $m_2R\approx R/B$. Let $S(m_2R)\approx C/B$. Then $m_2C=S(m_2R)\subseteq S(M)=S(M')\subseteq M'$, which is a contradiction. Therefore, $M=M'$.

If R is a right noetherian and generalized couniserial ring, every injective module is a direct sum of injective hull of simple modules. Hence, R is right artinian by [14], Theorem 4.5 in p. 85.

Theorem 8. *Let R be a right artinian and generalized couniserial ring. Then an R-module M has the extending property of simple modules if and only if M is a direct sum of uniform submodules.*

Proof. Every uniform module is artinian if R is couniserial, and $M_e\supseteq S(M)$ if R is right artinian. Hence, we have the theorem by Lemma 1 and Corollary to Theorem 6.

Theorem 9. *Let R be a right and left artinian ring. We assume R is self-dual* (e.g. algebra over a field of finite dimension).[1] *Then the following conditions are equivalent*:

1) *R is left and right generalized uniserial.*
2) *Every right R-module has the lifting property of simple modules.*
3) *Every right R-module has the extending property of simple modules.*

Proof. 1)→2) and 3). This is clear from Theorems 6 and 7 and [13].
2)→1). R is right generalized uniserial by Corollary to Theorem 5. Let e be a primitive idempotent in R. Since Re is hollow, the dual $(Re/Ae)^*$ of Re/Ae is uniform for any left ideal Ae of Re. Further $(Re/Ae)^*$ is hollow by 2). Hence, Re/Ae is uniform. Therefore, Re is serial.
3)→1). R is right couniserial by Corollary to Theorem 6. Since eR/eA is indecomposable for any right ideal eA, eR/eA is uniform by 3).

4. Commutative rings

In this section we assume that R is a commutative ring. Then every cyclic hollow module M is isomorphic to R/A; the ideal A is contained in a unique maximal ideal $P=P(M)$. Hence, R/A is a local ring. Conversely, if R is a local ring, then R/A is a cyclic hollow module for any ideal A. Let $M\approx R/A$ and $P=P(M)$. Since R/A is a local ring, R/A is an R_P-module and every R-submodule of R/A is an R_P-module. Hence, $R/A=R/A\otimes R_P=R_P/A_P$.

Lemma 2. *Let R be a commutative ring and P prime ideal. Let A be an ideal in R_P. Then every R-submodule of R_P/A is an R_P-module if and only if $R_P=A+\tilde{R}$ as R-modules, where $\tilde{R}=\rho(R)$ and ρ is the natural homomorphism of R into R_P.*

1) We shall show in the forthcoming paper that the assumption is superfluous.

If there exists such an ideal $A \neq R_P$, then P is maximal.

Proof. We assume $R_P = A + \tilde{R}$. Let B be an intermediate R-submodule between R_P and A, and let $x = a + \tilde{r}$ in R_P ($a \in A$, $r \in R$). Then $Bx \subseteq Ba + Br \subseteq B$. Hence, B/A is an R_P-module. Conversely, we assume that every R-submodule of R_P/A is an R_P-module. Then $A + \tilde{R}/A$ is an R_P-module. Hence, $A + \tilde{R} = R_P$. Since $A \subseteq P_P$, $R_P = P_P + \tilde{R}$, which implies $R/P \approx R_P/P_P$ and hence, P is maximal.

Proposition 6. *Let R be as above. Let P be a maximal ideal and A an ideal contained in P. Then the following conditions are equivalent:*

1) *P is a unique maximal ideal containing $B = \rho^{-1}(A_P)$.*
2) *R_P/A_P is a cyclic hollow R-module.*
3) *$R_P = A_P + \tilde{R}$ as R-modules.*

Proof. 1)→2). Since $B_P = A_P$, we obtain 2) by the beginning.
2)→3) and 1). R_P/A_P is a cyclic hollow R-module, $R_P/A_P \approx R/C$ for some ideal C with $Q = P(R/C)$. Hence, $(\bar{R}_Q =) R_Q/C_Q \overset{f}{\approx} R_P/A_P (= \bar{R}_P)$ as R-modules. If $P \neq Q$, $\bar{1}_Q P = \bar{1}_Q P R_Q = \bar{R}_Q$. Hence, $f(\bar{1}_Q)P = \bar{R}_P$, a contradiction. Accordingly, $Q = P$ and $R_P/A_P \approx R_P/C_P$ as R_P-modules. Therefore, $A_P = C_P$ and $A_P + \tilde{R} = R_P$ by Lemma 2 and P is a unique maximal ideal containing $\rho^{-1}(A_P) = \rho^{-1}(C_P)$.
3)→2). If $R_P = A_P + \tilde{R}$, R_P/A_P is a cyclic hollow module by Lemma 2.

Theorem 10. *Let R be a commutative ring. Then the following conditions are equivalent:*

1) *Every direct sum of cyclic hollow modules has the lifting property of simple modules.*

2) *For each maximal ideal P, the set of ideals A such that P is a unique maximal ideal containing A is linearly ordered with respect to inclusion.*

Proof. 2)→1). Let $M_i \approx R/A_i$ be a cyclic hollow module for $i = 1, 2$. If $\bar{M}_1 \approx \bar{M}_2$, $P(M_1) = P(M_2)$ as above. Hence, $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$ by 2). Therefore, we have 1) by [4], Theorem 2.
1)→2). Let A_1, A_2 be ideals in R as in 2). Since R/A_i is a cyclic R-hollow module and $\overline{R/A_1} \approx \overline{R/A_2}$, we obtain an element x in $R - P$ such that $xA_1 \subseteq A_2$ or $xA_2 \subseteq A_1$ by [4], Theorem 2. Let Q be a maximal ideal $\neq P$. Then $A_{iQ} = R_Q$ for $i = 1, 2$. Hence, $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Therefore, the set of those ideals A_i is a linearly ordered set.

Corollary. *Let R be a commutative local ring. Then* 2) *in the above is equivalent to*

2′) *R is a generalized uniserial ring* (namely a valuation ring).

The dual to Theorem 10 is

Theorem 11. *Let R be a commutative ring. Then the following conditions are equivalent:*

1) *Every direct sum of uniform modules with finite length has the extending property of simple modules.*

2) *For each maximal ideal P in R, the set of submodules with finite length of $E(R/P)$ is serial.*

Proof. Each uniform module with finite length is isomorphic to a submodule of $E(R/P)$ for some maximal ideal P in R. Hence, the theorem is clear from Theorem 6.

Finally we shall assume that R is a commutative noetherian ring. Then every indecomposable and injective module E has of the form $E(R/P)$, where P is a prime ideal and $E=\bigcup_i E_i$; $E_i=\{x\in E \mid \mathrm{Ann}(x)=P^i\}$ [11]. If $S(E)\neq 0$, P is a maximal ideal in R. Then $E(R/P)$ is artinian and every R-submodule is quasi-injective by [3], Lemma 2 and Theorem 3.

Theorem 12. *Let R be a commutative noetherian ring. Let M be an R-module with $M\,{}_e\!\supseteq S(M)$. Then M has the extending property of simple modules if and only if $M\approx\sum_P\sum_{I(P)}\oplus M_{P\alpha}$, which satisfies the following conditions.*

1) *$M_{P\alpha}$ is a submodule of $E(R/P)$ for each P.*

2) *$\{M_{P\alpha}\}_\alpha$ is linear with respect to inclusion for each P,*

where P runs through all maximal ideals in R.

Proof. If $M\approx\sum_P\sum_I\oplus M_{P\alpha}$, M has the extending property of simple modules by 1), 2) and [5], Corollary 8, since $M_{P\alpha}$ is quasi-injective. Conversely, we assume that M has the extending property of simple modules. Let N be a uniform submodule of M. Since $M\,{}_e\!\supseteq S(M)$, $S(N)\neq 0$. Hence, N is a submodule of some $E(R/P)$ and N is artinian. Therefore, M is isomorphic to a submodule $M'=\sum_{P,\alpha}\oplus M_{P\alpha}$ satisfying 1) and 2) by Lemma 1 and [7], Theorem 6.

Especially, let R be a Dededind domain and M an R-module. Then $M=M(t)\oplus M_0$: $M(t)$ is the torsion part and $S(M_0)=0$. Hence, M has the extending property of simple modules if and only if so does $M(t)$. We note that $M\,{}_e\!\supseteq S(M)$ if and only if M is torsion.

Corollary. *Let R be a Dedekind domain and M a torsion R-module. Then M has the extending propetty of simple modules if and only if M is a direct sum of an injective module and cyclic P-groups R/P^n, where P runs through all non-zero primes in R.*

Proposition 7. *Let R be a commutative and noetherian ring and M an R-module with $S(M)\,{}_e\!\subseteq M$. Then M has the extending property of direct decompositions of $S(M)$ if and only if M is a quasi-injective. In this case M has the extending* *property of direct sums of uniform modules.*

Proof. If M has the extending property above, $M=\sum_{P,\alpha}\oplus M_{P\alpha}$ as in Theorem 12 by the definition. Furthermore, $M_{P\alpha}=M_{P\beta}$ by [5], Corollary 20. Since R is noetherian, $\sum_{\alpha}\oplus M_{P\alpha}$ is quasi-injective by [10], Theorem 1.1. Hence, M is also quasi-injective. The converse is clear by [5], Proposition 25.

Proposition 8 *Let R be a commutative and perfect ring and M an R-module. Then M has the lifting property of direct decompositions of $M/J(M)$ if and only if M is quasi-projective.*

Proof. If M has the above property, $M=\sum_{I}\oplus M_{\alpha}$ with M_{α} cyclic hollow. Since R is commutative and perfect, $M_{\alpha}\approx eR/eA_{\alpha}$ is quasi-projective by [9], Proposition 2.1, where e is a primitive idempotent and A_{α} is an ideal. Furthermore, if $eR/eA_{\alpha}\approx eR/eA_{\beta}$ for $\alpha, \beta\in I$, $eA_{\alpha}=eA_{\beta}$ by [4], Theorem 3. Hence, M is quasi-projective. The converse is clear by [4], Corollary 3.

Proposition 9. *Let R be a commutative and noetherian ring. Then R has the extending property of simple modules as an R-module if and only if R is a direct sum of Frobenius rings and a ring with zero socle. In this case every projective module has the extending property of simple modules.*

Proof. We assume that R has the extending property of simple modules. We may assume by [7], Proposition 1 that R is directly indecomposable. If $S(R)\neq 0$, R is uniform and R is artinian by [3], Theorem 3. Hence, R is a Frobenius ring. The remaining parts are clear from Theorem 2.

Theorem 13. *Let R be a commutative and noetherian ring. Then the following conditions are equivalent*:

1) *Every direct sum of cyclic hollow modules has the lifting property of simple modules.*

2) *Every direct sum of uniform modules with non-zero socles has the extending property of simple modules.*

3) *R_P is a serial ring for every maximal ideal P.*

4) *R is a direct sum of artinian uniserial rings and Dedekind domains.*

Proof. 1)$\to$3). Let P be a maximal ideal. Then all P^n are contained in a unique maximal ideal P. Hence, P^n/P^{n+1} is simple by Theorem 10. Therefore, $R_P\supset P_P\supset P_P^2\supset\cdots$ is a unique series of ideals, since $\bigcap_n P_P^n=0$.

3)$\to$1). This is clear by Theorem 10, [4], Theorem 2 and taking localization with respect to maximal ideals.

2)$\to$3). Let P be a maximal ideal in R and put $E=E(R/P)$. Then $E_i=S_i(E)$.

P_P^n/P_P^{n+1} is simple or equal to zero by Corollary to Theorem 6 and [11], Theorem 3.10.

3)→2). If R_P is a serial ring for a maximal ideal P, E_i/E_{i-1} is simple by [11], Theorem 3.10. Hence, $E=\cup E_i$ is serial. Therefore we have 2) by Corollary to Theorem 6.

4)→3). This is clear.

1), 3)→4). We note that every factor ring of R satisfies 1). Since R is noetherian, we may assume that R is an irreducible ring as an R-module. We know Krull dim $R\leqslant 1$ by 3). Let $0=A_1\cap\cdots\cap A_k\cap B_1\cap\cdots\cap B_m\cap C_1\cap\cdots\cap C_n$ be a irredundant representation by primary ideals A_i, B_j and C_q such that the associate prime ideal $P(A_i)$ of A_i is maximal and minimal, $P(B_j)$ is of height 1 and $P(C_q)$ is minimal and not maximal ideal in R (we may assume all $P(\ )$'s are distinct). Let P be a maximal ideal in R. We note that P does not contain two distinct minimal ideals by 3). Since $P(A_i)$ is a maximal and minimal, A_i is relatively prime to another ideals A_j $(j\neq i)$, B_j and C_q. Hence, $A_1=0$ or $0=B_1\cap\cdots\cap B_m\cap C_1\cap\cdots\cap C_n$, since R is irreducible. We assume the latter. Since $P(B_i)$ contains a unique minimal ideal $P(C_{\pi(i)})$ from the above, we have $0=(B_1\cap C_{\pi(1)})\cap(B_2\cap C_{\pi(2)})\cap\cdots\cap(B_m\cap C_{\pi(m)})\cap C_{i1}\cap\cdots\cap C_{in-m}$. We know again from the above $0=B_1\cap C_{\pi(1)}$ or $C_{i1}=0$. In the former case, $P(C_{\pi(1)})=p$ is a unique minimal prime ideal in R. Let P be any maximal ideal in R. Since $p\subseteq P$ and R_P is serial, $p_P=0$. Hence, $p=0$, which is a contradiction. Thus, we have shown that R is a primary ring with a unique minimal ideal q. If R is artinian, R is local and so $R=R_P$ is a uniserial ring by 3). If R is not artinian, R_P is a Dedekind domain for any maximal ideal P. Hence, $q_P=0$. Accordingly, R is a domain and so R is a Dedekind domain by 3).

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