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ASYMPTOTIC PROPERTIES OF POSTERIOR DISTRIBUTIONS IN A TRUNCATED CASE

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1. Introduction

Let $X_1, \cdots, X_n$ be independent random variables with common density $f(x-\theta)$, $-\infty<x$, $\theta<\infty$, where θ is an unknown translation parameter. We shall consider here the case that $f(x)$ is a uniformly continuous density which vanishes on the interval $(-\infty, 0]$ and is positive on the interval $(0, \infty)$ and particularly

$$f(x)\sim\alpha x \qquad \text{as} \quad x\rightarrow+0$$

with $0<\alpha<\infty$.

Let $\hat{\theta}_n=\hat{\theta}_n(X_1, \cdots, X_n)$ denote the maximum likelihood estimate of θ for the sample size n. Takeuchi [4] and Woodroofe [7] showed that $\sqrt{\frac{1}{2}\alpha n\log n}\,(\hat{\theta}_n-\theta)$ has an asymptotic standard normal distribution. The speed of convergence to the standard normal distribution has been given as $O((\log n)^{s-1})$ for every fixed $s\in(0, 1)$ by the author [2] (see Theorem 1 below). Moreover, it was shown by Takeuchi [4] and Weiss and Wolfowitz [6] that $\hat{\theta}_n$ is an asymptotically efficient estimator of θ.

Woodroofe [7] also showed that if θ is regarded as a random variable with a prior density, then the posterior probability that $\sqrt{\frac{1}{2}\alpha n\log n}\,(\theta-\hat{\theta}_n)\in J$ converges to normality $\Phi\{J\}$ in probability for every finite interval J. The purpose of the present paper is to give a refinement of his result. It is shown that the variational distance between the posterior distribution and the standard normal distribution decreases of the order $(\log n)^{-s}$ with probability $1-O((\log n)^{s-1})$ for every $s\in(0, 1)$. Similar result for minimum contrast estimates in the regular case was given by Strasser [3].

2. Conditions and the main result

We shall impose the following Condition A on $f(x)$ and Condition B on a prior distribution λ.

Condition A

(i) $f(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0]$ and is positive on $(0, \infty)$.

(ii) $f(x)$ is twice continuously differentiable on $(0, \infty)$ with derivatives $f'(x)$ and $f''(x)$. Moreover $f''(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $f'''(x)$.

(iii) For some $\alpha \in (0, \infty)$ and some $r \in (0, \infty)$

$$f'(x) = \alpha + O(x^r), \quad f''(x) = O(x^{r-1}) \quad \text{and} \quad f'''(x) = o(x^{-2}) \quad \text{as} \quad x \to +0.$$

Let $g(x) = \log f(x)$ for $x > 0$. Then the second derivative $g''(x)$ of $g(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $g''' = f'''f^{-1} - 3f'f''f^{-2} + 2(f'f^{-1})^3$. Under conditions (i) and (ii), condition (iii) is equivalent to the following condition (iii)$'$.

(iii)$'$ For some $\alpha \in (0, \infty)$ and some $r \in (0, \infty)$

$$f(x) = \alpha x + O(x^{1+r}), \quad g'(x) = x^{-1} + O(x^{r-1}), \quad g''(x) = -x^{-2} + O(x^{r-2})$$
$$\text{and} \quad g'''(x) = 2x^{-3} + o(x^{-3}) \quad \text{as} \quad x \to +0.$$

(iv) For every $t \geqq 0$

$$\int_0^\infty \{g(x+t)\}^2 f(x) dx < \infty.$$

(v) For every $a > 0$, there is a $\delta > 0$, for which

(a) $$\int_a^\infty \sup_{|u| \leqq \delta} |g'(x+u)|^3 f(x) dx < \infty,$$

(b) $$\int_a^\infty \sup_{|u| \leqq \delta} \{g''(x+u)\}^2 f(x) dx < \infty,$$

(c) $$\int_a^\infty \sup_{|u| \leqq \delta} \{g'''(x+u)\}^2 f(x) dx < \infty.$$

Let $(\boldsymbol{R}, \mathscr{B})$ be a parameter space, where $\boldsymbol{R}$ is the real line and $\mathscr{B}$ is the Borel σ-algebra of $\boldsymbol{R}$. Moreover, let λ be a prior distribution on $(\boldsymbol{R}, \mathscr{B})$. The following Condition $\boldsymbol{B}$ is owed to Strasser [3].

Condition B

(j) For every $\eta > 0$ and every compact $K \subset \boldsymbol{R}$

$$\inf_{\theta \in K} \lambda\{t \in \boldsymbol{R};\ |t-\theta| < \eta\} > 0.$$

(jj) λ has a continuous and positive density p on $\boldsymbol{R}$ with respect to the Lebesgue measure satisfying the following condition: For every compact $K \subset \boldsymbol{R}$ there exist constants $c_K > 0$ and $d_K > 0$ such that $t \in \boldsymbol{R}$, $\theta \in K$ and $|t-\theta| \leqq d_K$ imply

$$|p(t)-p(\theta)|\leqq c_K p(\theta)|t-\theta|.$$

Obviously condition (jj) implies condition (j).

Let P_θ denote the conditional probability of $(X_1, \cdots, X_n)$ given θ and define

$$\Phi\{B\}=\int_B \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\,,\qquad B\in\mathcal{B}\,.$$

The following theorem is often needed in the sequel.

Theorem 1 (Matsuda [2]). *Suppose that Condition A holds. Then for every* $s\in(0,1)$ *there exists a positive constant c such that for all* θ, $t\in\boldsymbol{R}$ *and* $n\geqq 1$

$$|P_\theta\{a_n(\hat{\theta}_n-\theta)\leqq t\}-\Phi\{(-\infty, t]\}|\leqq c(\log n)^{s-1}\,,$$

where $2a_n^2=\alpha n(\log n+\log\log n)$ *and the constant c tends to infinity as* $s\to 0$.

It is remarked that the upper bound $(\log n)^{s-1}$ in Theorem 1 is replaced by a better bound $(\log n)^{-1}$, provided t is restricted to $(-\infty, M)$ with $0<M<\infty$. But, using $\sqrt{\frac{1}{2}\alpha n\log n}$ instead of a_n, the upper bound in Theorem 1 becomes $(\log\log n)(\log n)^{-1}$ which is worse than the order $(\log n)^{-1}$. Thus we use a_n rather than $\sqrt{\frac{1}{2}\alpha n\log n}$.

Let R_n denote the conditional distribution of θ given $X_1, \cdots, X_n$ and define a probability measure Q_n by

$$Q_n\{B\}=R_n\{\theta\in\boldsymbol{R};\ a_n(\theta-\hat{\theta}_n)\in B\}\,,\qquad B\in\mathcal{B}\,.$$

Theorem 2. *Suppose that Condition A and condition* (jj) *hold. Then for every* $s\in(0,1)$ *and every compact* $K\subset\boldsymbol{R}$ *there exist constants* $c_1>0$ *and* $c_2>0$ *such that for all* $n\geqq 1$

$$\sup_{\theta\in K}P_\theta\{\|Q_n-\Phi\|\geqq c_1(\log n)^{-s}\}\leqq c_2(\log n)^{s-1}\,,$$

where $\|\cdot\|$ *means the totally variation of a measure.*

For the proof of Theorem 2 we need several lemmas and propositions.

3. Auxiliary results

In this section, $\theta=0$ will be chosen for simplicity and write P instead of P_0. Let E be the expectation with respect to P. The following Lemma 1 and Lemma 2 are closely related to Lemma 1 and Lemma 2 in Strasser [3], respectively.

Lemma 1. *Let conditions* (i) *and* (iv) *be satisfied. Then for every* $\varepsilon>0$

there exists $d>0$ *such that*

$$P\{\sup_{t\leqq -\varepsilon} n^{-1}\sum_{i=1}^{n} g(X_i-t)\geqq E\{g(X)\}-d\} = O(n^{-1}).$$

Proof. Let M be a positive number chosen such that

$$E\{\sup_{t<-M} g(X-t)\} < E\{g(X)\}.$$

For every $t\in[-M, -\varepsilon]$ there exists an open neighborhood U_t of t such that

$$E\{\sup_{u\in U_t} g(X-u)\} < E\{g(X)\}.$$

The existence of such a positive number M and that of such a U_t follow from Wald [5] (see Woodroofe [7] and also [2]). As $\{U_t;\, t\in[-M, -\varepsilon]\}$ covers the compact set $[-M, -\varepsilon]$, there exists a finite subcover of this set $[-M, -\varepsilon]$ determined by $t_j\in[-M, -\varepsilon]$, $j=1, \cdots, m$. For notational convenience, let $U_0=(-\infty, -M)$ and $U_j=U_{t_j}$, $j=1, \cdots, m$. Write

$$d_j = E\{g(X)\} - E\{\sup_{t\in U_j} g(X-t)\} > 0, \qquad j = 0, \cdots, m$$

and let $2d=\min\{d_j;\, j=0, \cdots, m\}>0$. Then

$$\sup_{t\leqq -\varepsilon} n^{-1}\sum_{i=1}^{n} g(X_i-t)\geqq E\{g(X)\}-d$$

implies

$$n^{-1}\sum_{i=1}^{n}\sup_{t\in U_j} g(X_i-t) - E\{\sup_{t\in U_j} g(X-t)\}\geqq d$$

for some $j\in\{0, \cdots, m\}$. Hence we have

$$\begin{aligned}
&P\{\sup_{t\leqq -\varepsilon} n^{-1}\sum_{i=1}^{n} g(X_i-t)\geqq E\{g(X)\}-d\}\\
&\quad\leqq \sum_{i=0}^{m} P\{|n^{-1}\sum_{j=1}^{n}\sup_{t\in U_j} g(X_i-t) - E\{\sup_{t\in U_j} g(X-t)\}|\geqq d\}.
\end{aligned}$$

Now the assertion of Lemma 1 follows from Chebyshev's inequality because of conditions (i) and (iv).

Lemma 2. *Let conditions* (i)–(iv) *and* (v) (a) *be satisfied. Then for every* $d>0$ *there exists* $\eta>0$ *such that*

$$P\{\inf_{-\eta<t<0} n^{-1}\sum_{i=1}^{n} g(X_i-t)\leqq E\{g(X)\}-d\} = O(n^{-1}).$$

Proof. Let $a>0$ be so small that $g'(x)>0$ for $0<x<2a$. Next choose

$\delta>0$ to satisfy condition (v) (a). Then for $\eta<\delta$ we have

$$n^{-1}\sum_{i=1}^{n} g(X_i-t) = n^{-1}\sum_{i=1}^{n} g(X_i)-n^{-1}t\sum_{i=1}^{n} g'(X_i-t^*)$$
$$\geqq n^{-1}\sum_{i=1}^{n} g(X_i)+n^{-1}t\sum\nolimits_a^\infty \sup_{|u|\leqq\delta} |g'(X_i+u)|$$

for some $t^*\in(-\eta, 0)$. Here and in what follows, $\sum_u^v$ denotes summation over $i\leqq n$ for which $u\leqq X_i<v$. Hence

$$|n^{-1}\sum_{i=1}^{n} g(X_i)-E\{g(X)\}|<\frac{d}{3}$$

and

$$|n^{-1}\sum\nolimits_a^\infty \sup_{|u|\leqq\delta} |g'(X_i+u)|-\int_a^\infty \sup_{|u|\leqq\delta} |g'(x+u)|f(x)dx|<\frac{d}{3}$$

imply

$$n^{-1}\sum_{i=1}^{n} g(X_i-t)\geqq E\{g(X)\}-\frac{d}{3}+t\left\{\frac{d}{3}+\int_a^\infty \sup_{|u|\leqq\delta} |g'(x+u)|f(x)dx\right\}.$$

Choosing $\eta<\min\left\{1, \delta, \frac{d}{3}\left[\int_a^\infty \sup_{|u|\leqq\delta} |g'(x+u)|f(x)dx\right]^{-1}\right\}$, we obtain

$$\inf_{-\eta<t<0} n^{-1}\sum_{i=1}^{n} g(X_i-t)>E\{g(X)\}-d.$$

Lemma 2 follows from Chebyshev's inequality because of conditions (iv) and (v)(a).

Lemma 3. *Let conditions* (i)–(iii) *and* (v)(*b*) *be satisfied. Then for every* $s\in(0, 1)$

$$P\{|a_n^{-2}\sum_{i=1}^{n} g''(X_i)+1|\geqq(\log n)^{-s}\} = O((\log n)^{s-1}).$$

Proof. According to condition (iii)′ choose $a>0$ and $c>0$ such that $|f(x)-\alpha x|\leqq cx^{1+r}$ and $|g''(x)+x^{-2}|\leqq cx^{r-2}$ for $0<x<a$. For $i\leqq n$ let

$$\begin{aligned} Y_{ni} &= g''(X_i), && \text{if } b_n\leqq X_i<a, \\ &= 0, && \text{if } X_i<b_n \text{ or } a\leqq X_i, \end{aligned}$$

where $b_n=a_n^{-1}(\log n)^{s/2}$. Since $E\{Y_{n1}^2\}=O(b_n^{-2})=O(n(\log n)^{1-s})$, it follows from Chebyshev's inequality that

$$P\{|a_n^{-2}\sum_{i=1}^{n}(Y_{ni}-E\{Y_{ni}\})|\geqq\frac{1}{4}(\log n)^{-s}\} = O((\log n)^{s-1}).$$

Considering $E\{Y_{n1}\}=-\alpha\log a_n+O(\log\log n)$, this leads to

$$P\{|a_n^{-2}\sum_{i=1}^{n} Y_{ni}+1|\geqq\frac{1}{2}(\log n)^{-s}\} = O((\log n)^{s-1}).$$

Moreover, using $P\{\sum_{i=1}^{n} Y_{ni}\neq\sum_0^a g''(X_i)\}=O((\log n)^{s-1})$, we obtain

$$P\{|a_n^{-2}\sum_0^a g''(X_i)+1|\geqq\frac{1}{2}(\log n)^{-s}\} = O((\log n)^{s-1}).$$

Since also

$$P\{|a_n^{-2}\sum_a^\infty g''(X_i)|\geqq\frac{1}{2}(\log n)^{-s}\} = O(n^{-1})$$

by Chebyshev's inequality, the proof is completed.

Let $M_n=\min(X_1, \cdots, X_n)$ and let $b_n=a_n^{-1}(\log n)^{s/2}$ with $s\in(0, 1)$ as in the proof of Lemma 3.

Lemma 4. *Let conditions* (i), (ii) *and* (iii) *be satisfied. Then for every* $s\in(0, 1)$ *and sufficiently small* $a>0$

$$P\{|a_n^{-3}\sum_0^a (X_i-2b_n)^{-3}|\geqq(\log n)^{-(3/2)s}, M_n>2b_n\}=O((\log n)^{s-1}).$$

Proof. Let $a>0$ be so small that $f(x)<2\alpha x$ for $0<x<a$. Then define $\{Y_{ni}; i=1, \cdots, n\}$ by

$$\begin{aligned} Y_{ni} &= (X_i-2b_n)^{-3}, && \text{if } 3b_n\leqq X_i<a, \\ &= 0, && \text{if } X_i<3b_n \text{ or } a\leqq X_i. \end{aligned}$$

Since $E\{Y_{n1}^2\}=O(b_n^{-4})$, it follows from Chebyshev's inequality that

$$P\{|a_n^{-3}\sum_{i=1}^{n}(Y_{ni}-E\{Y_{ni}\})|\geqq\frac{1}{2}(\log n)^{-(3/2)s}\} = O((\log n)^{s-1}).$$

Moreover, using $a_n^{-3}\sum_{i=1}^{n} E\{Y_{ni}\}=O((\log n)^{-1-s/2})$ we obtain

$$P\{|a_n^{-3}\sum_{i=1}^{n} Y_{ni}|\geqq(\log n)^{-(3/2)s}\} = O((\log n)^{s-1}),$$

which leads to the desired result.

For notational convenience define

$$\begin{aligned} G_n(t) &= \sum_{i=1}^{n} g(X_i-t), && \text{if } t<M_n, \\ &= -\infty, && \text{if } t\geqq M_n. \end{aligned}$$

The following Lemma 5 and Lemma 6 refine Lemma 3.4 and Lemma 4.1 in Woodroofe [7], respectively.

Lemma 5. *Let conditions* (i)–(iii), (v)(b) *and* (v)(c) *be satisfied. Then for every* $s \in (0, 1)$ *there exists* $c > 0$ *such that*

$$P\{\sup_{|t|\leqq 2b_n} |a_n^{-2}G_n''(t)+1| \geqq c(\log n)^{-s}\} = O((\log n)^{s-1}).$$

Proof. Since $P\{M_n \leqq 2b_n\} = O((\log n)^{s-1})$, we can assume that $M_n > 2b_n$. Then $G_n''(t) = \sum_{i=1}^n g''(X_i - t)$ for $|t| \leqq 2b_n$. Using the equality

$$a_n^{-2}\sum_{i=1}^n g''(X_i - t) = a_n^{-2}\sum_{i=1}^n g''(X_i) - a_n^{-2}\sum_{i=1}^n \int_0^t g'''(X_i - u)du$$

we have

$$\sup_{|t|\leqq 2b_n} |a_n^{-2}G_n''(t)+1| \leqq |a_n^{-2}\sum_{i=1}^n g''(X_i)+1| + 6a_n^{-2}b_n \textstyle\sum_0^a (X_i - 2b_n)^{-3}$$
$$+2a_n^{-2}b_n \textstyle\sum_a^\infty \sup_{|u|\leqq 2b_n} |g'''(X_i+u)| .$$

Here we used the fact that $|g'''(x)| \leqq 3x^{-3}$ for $0 < x < 2a$ with sufficiently small $a > 0$. Now the assertion follows from Lemma 3 and Lemma 4.

Lemma 5, together with Theorem 1, yields the following lemma.

Lemma 6. *Let Condition A be satisfied. Then for every* $s \in (0, 1)$ *there exists* $c > 0$ *such that*

$$P\{\sup_{|t|\leqq b_n} |a_n^{-2}G_n''(\hat{\theta}_n + t)+1| \geqq c(\log n)^{-s}\} = O((\log n)^{s-1}),$$

where $b_n = a_n^{-1}(\log n)^{s/2}$.

Lemma 7 (Lemma 2 in [2]). *Let conditions* (i)–(iii) *and* (iv) *be satisfied. Then for every* $\varepsilon > 0$

$$P\{|\hat{\theta}_n| \geqq \varepsilon\} = O(n^{-1}).$$

Lemma 8 (Lemma 1 in [2]). *Let conditions* (i)–(iii) *and* (v) (b) *be satisfied. Then for sufficiently small* $\varepsilon > 0$, *there are events* D_n, $n \geqq 1$, *for which* $P\{D_n^c\} = O(n^{-1})$ *and* D_n *implies* $\sup_{-\varepsilon \leqq t < M_n} n^{-1}G_n''(t) < -1$.

The following lemma also may be proved analogously to Lemma 8.

Lemma 9. *Let conditions* (i)–(iii) *and* (v)(c) *be satisfied. Then for sufficiently small* $\varepsilon > 0$, *there are events* F_n, $n \geqq 1$, *for which* $P\{F_n^c\} = O(n^{-1})$ *and* F_n *implies* $\sup_{-\varepsilon \leqq t < M_n} n^{-1}G_n'''(t) < -1$.

Lemma 10. *Let conditions* (i), (ii) *and* (iii) *be satisfied. Then for every*

$s\in(0, 1)$, *every* $b>0$ *and sufficiently small* $a>0$

$$P\{|a_n^{-2}\sum_0^a(X_i+2bd_n)^{-2}-1|\geqq(\log n)^{-(1+s)/2}\} = O((\log n)^{s-1}),$$

where $d_n=a_n^{-1}(\log n)^{1/2}$.

We shall omit the proof since Lemma 10 may be proved analogously to Lemma 4.

4. Estimation of the speed of convergence

For each $n\geqq 1$ and each $s\in(0, 1)$, let $H_n(s)=[-(\log n)^{s/2}, (\log n)^{s/2}]$. In this section, we shall estimate the speed with which $Q_n\{H_n(s)^c\}$ converges to 0. For the convenience of calculation, we shall divide $H_n(s)^c$ into five parts as follows:

$$\begin{aligned}
I_n(\varepsilon) &= (-\infty, -a_n\varepsilon],\\
I_n(\varepsilon, b) &= (-a_n\varepsilon, -b(\log n)^{1/2}],\\
J_n(b, s) &= (-b(\log n)^{1/2}, -(\log n)^{s/2}),\\
J_n(s) &= ((\log n)^{s/2}, \log n)\\
\text{and}\qquad J_n &= [\log n, \infty)
\end{aligned}$$

with $\varepsilon>0$ and $b>0$. We first show the following proposition which is similar to Theorem 1 in Strasser [3].

Proposition 1. *Let conditions* (i)–(v)(*a*) *and* (j) *be satisfied. Then for every* $\varepsilon>0$ *there exists* $c>0$ *such that for every compact* $K\subset\boldsymbol{R}$

$$\sup_{\theta\in K}P_\theta\{R_n\{t\in\boldsymbol{R};\ |t-\theta|\geqq\varepsilon\}>\exp(-cn)\} = O(n^{-1}).$$

Proof. Since θ is a translation parameter, it is easily seen that $\sup_{\theta\in\boldsymbol{R}}P_\theta\{M_n-\theta\geqq\varepsilon\}=P\{M_n\geqq\varepsilon\}=o(n^{-1})$. Therefore, we shall assume that $M_n-\theta<\varepsilon$. Then we have

$$\begin{aligned}
R_n\{|t-\theta|\geqq\varepsilon\} &= \frac{\int_{|t-\theta|\geqq\varepsilon}\exp\{G_n(t)\}\lambda(dt)}{\int_{\boldsymbol{R}}\exp\{G_n(t)\}\lambda(dt)}\\
&\leqq \frac{\int_{t\leqq\theta-\varepsilon}\exp\{G_n(t)\}\lambda(dt)}{\int_{\theta-\eta<t<\theta}\exp\{G_n(t)\}\lambda(dt)}\\
&\leqq \exp\{-n[\inf_{-\eta<t<0}n^{-1}G_n(\theta+t)-\sup_{t\leqq-\varepsilon}n^{-1}G_n(\theta+t)\\
&\qquad +n^{-1}\log\lambda\{-\eta<t-\theta<0\}]\}
\end{aligned}$$

for $\eta>0$. By Lemma 1 there exists $d>0$ (depending on ε) such that

$$\sup_{t\leqq -\varepsilon} n^{-1}G_n(\theta+t)<E_\theta\{g(X-\theta)\}-d$$

with probability $1-O(n^{-1})$, where $O(n^{-1})$ is uniform in θ for $\theta\in\boldsymbol{R}$. Also, by Lemma 2 there exists $\eta>0$ (depending on ε) such that

$$\inf_{-\eta<t<0} n^{-1}G_n(\theta+t)>E_\theta\{g(X-\theta)\}-\frac{d}{4}$$

with probability $1-O(n^{-1})$ as just stated. Since $-\infty<\beta\equiv\inf_{\theta\in K}\log\lambda\{-\eta<t-\theta<0\}\leqq 0$ by condition (j), for any $0<c<\frac{d}{2}$ we have

$$\inf_{-\eta<t<0} n^{-1}G_n(\theta+t)-\sup_{t\leqq-\varepsilon} n^{-1}G_n(\theta+t)+n^{-1}\beta>c$$

for all sufficiently large n. This completes the proof of Proposition 1.

The following result immediately follows from Proposition 1 and Lemma 7.

Proposition 2. *Let conditions* (i)–(v)(a) *and* (j) *be satisfied. Then for every $\varepsilon>0$ there exists $c>0$ such that for every compact $K\subset\boldsymbol{R}$*

$$\sup_{\theta\in K} P_\theta\{Q_n\{I_n(\varepsilon)\}>\exp(-cn)\}=O(n^{-1})\,.$$

Easy computations show that condition (jj) and Lemma 7 imply that for every compact $K\subset\boldsymbol{R}$ there exist c_1, c_2, $0<c_1<c_2<\infty$, and $c_3>0$ such that

$$\inf_{\theta\in K} P_\theta\{c_1\eta_n\leqq\lambda\{|t-\hat{\theta}_n|\leqq\eta_n\}\leqq c_2\eta_n\}\geqq 1-c_3n^{-1} \tag{4.1}$$

for all $n\geqq 1$ and for every positive sequence $\{\eta_n\}$ with $\eta_n\to 0$ as $n\to\infty$.

Proposition 3. *Let Condition A and condition* (jj) *be satisfied. Then for every $s\in(0, 1)$, every $b>0$, every $k>0$ and every compact $K\subset\boldsymbol{R}$*

$$\sup_{\theta\in K} P_\theta\{Q_n\{J_n(b, s)\}\geqq(\log n)^{-k}\}=O((\log n)^{s-1})\,.$$

Proof. Lemma 8 implies that, with probability $1-O(n^{-1})$, $G_n(t)$ is a concave function in $t\in[\theta-2\varepsilon, M_n)$, if $\varepsilon>0$ is a sufficiently small number. Using Lemma 7 we can assume that $|\hat{\theta}_n-\theta|<\varepsilon$. Hence for all sufficiently large n we have

$$\begin{aligned}\sup\{G_n(t);\ \hat{\theta}_n-ba_n^{-1}(\log n)^{1/2}<t<\hat{\theta}_n-b_n\}&\leqq G_n(\hat{\theta}_n-b_n)\\ &\leqq G_n(\hat{\theta}_n)+\frac{b_n^2}{2}\sup_{|t|\leqq b_n} G_n''(\hat{\theta}_n+t)\\ &\leqq G_n(\hat{\theta}_n)-\frac{1}{4}(\log n)^s\,.\end{aligned}$$

The last inequality follows from Lemma 6. A similar argument will show that

$$\inf \{G_n(t);\ |t-\hat{\theta}_n| \leqq a_n^{-1}\} \geqq \min \{G_n(\hat{\theta}_n-a_n^{-1}),\ G_n(\hat{\theta}_n+a_n^{-1})\}$$
$$\geqq G_n(\hat{\theta}_n)+\frac{a_n^{-2}}{2} \inf_{|t|\leqq a_n^{-1}} G_n''(\hat{\theta}_n+t)$$
$$\geqq G_n(\hat{\theta}_n)-\frac{3}{4}\,.$$

Therefore, for $\theta \in K$

$$Q_n\{J_n(b, s)\} \leqq \frac{\int_{\hat{\theta}_n-ba_n^{-1}(\log n)^{1/2}}^{\hat{\theta}_n-b_n} \exp \{G_n(t)\}\lambda(dt)}{\int_{\hat{\theta}_n-a_n^{-1}}^{\hat{\theta}_n+a_n^{-1}} \exp \{G_n(t)\}\lambda(dt)}$$
$$\leqq \frac{\exp \{G_n(\hat{\theta}_n)-\frac{1}{4}(\log n)^s\}\lambda\{|t-\hat{\theta}_n| \leqq ba_n^{-1}(\log n)^{/2}\}}{\exp \left\{G_n(\hat{\theta}_n)-\frac{3}{4}\right\}\lambda\{|t-\hat{\theta}_n| \leqq a_n^{-1}\}}\,.$$

Taking account of (4.1), we obtain

$$Q_n\{J_n(b, s)\} \leqq cb(\log n)^{1/2} \exp \{-\frac{1}{4}(\log n)^s\} < (\log n)^{-k}$$

for all sufficiently large n, where c is a real number depending on K. Thus the proof is completed.

The following Proposition 4 may be proved similarly to Proposition 3, and so the proof will be omitted here.

Proposition 4. *Let Condition A and condition* (jj) *be satisfied. Then for every* $s \in (0, 1)$, *every* $k>0$ *and every compact* $K \subset \boldsymbol{R}$

$$\sup_{\theta \in K} P_\theta\{Q_n\{J_n(s)\} \geqq (\log n)^{-k}\} = O((\log n)^{s-1})\,.$$

Proposition 5. *Let Condition A be satisfied. Then for every* $s \in (0, 1)$

$$\sup_{\theta \in \boldsymbol{R}} P_\theta\{Q_n\{J_n\} > 0\} = O((\log n)^{s-1})\,.$$

Proof. It is easily seen that $\sup_{\theta \in \boldsymbol{R}} P_\theta\{M_n-\theta \geqq \frac{1}{2}a_n^{-1} \log n\} = O(n^{-c})$ for some $c>0$. Theorem 1 implies that

$$\sup_{\theta \in \boldsymbol{R}} P_\theta\{|\hat{\theta}_n-\theta| \geqq b_n\} = O((\log n)^{s-1})\,.$$

Therefore, we may assume that

$$M_n-\theta<\frac{1}{2}a_n^{-1}\log n \quad\text{and}\quad |\hat{\theta}_n-\theta|<b_n\,.$$

Then $t\geqq\hat{\theta}_n+a_n^{-1}\log n$ implies $t>M_n$ for sufficiently large n. Since $R_n\{t>M_n\}=0$, the assertion of the proposition holds.

Proposition 6. *Let Condition A and condition* (jj) *be satisfied. Then for every* $s\in(0, 1)$, *every* $k>0$, *every compact* $K\subset\boldsymbol{R}$ *and sufficiently small* $\varepsilon>0$ *there exists* $b>0$ *such that*

$$\sup_{\theta\in K}P_\theta\{Q_n\{I_n(\varepsilon, b)\}\geqq n^{-k}\} = O((\log n)^{s-1})\,.$$

Proof. By Theorem 1 we can assume that $|\hat{\theta}_n-\theta|<bd_n$ where $d_n=a_n^{-1}(\log n)^{1/2}$. Since $G_n(t)$ is concave on $[\theta-2\varepsilon, M_n)$ with sufficiently small $\varepsilon>0$, Lemma 9 implies

$$\begin{aligned}\sup\{G_n(t);\ -\varepsilon<t-\hat{\theta}_n<-bd_n\}&\leqq G_n(\hat{\theta}_n-bd_n)\\ &\leqq G_n(\hat{\theta}_n)+\frac{b^2d_n^2}{2}G_n''(\hat{\theta}_n-bd_n)\end{aligned}$$

for all sufficiently large n.

Let $a>0$ be so small that $g''(x)<-\frac{1}{2}x^{-2}$ for $0<x<2a$ and choose $\delta>0$ to satisfy condition (v)(b). Then, it follows from Lemma 10 that

$$\begin{aligned}\textstyle\sum_\theta^{\theta+a}g''(X_i-\hat{\theta}_n+bd_n)&\leqq-\frac{1}{2}\textstyle\sum_\theta^{\theta+a}(X_i-\hat{\theta}_n+bd_n)^{-2}\\ &\leqq-\frac{1}{2}\textstyle\sum_\theta^{\theta+a}(X_i-\theta+2bd_n)^{-2}\\ &\leqq-\frac{1}{4}a_n^2\,.\end{aligned}$$

Since $|\sum_{\theta+a}^{\infty}g''(X_i-\hat{\theta}_n+bd_n)|\leqq\sum_{\theta+a}^{\infty}\sup_{|u|\leqq\delta}|g''(X_i-\theta+u)|$ for all sufficiently large n, we have $\sum_{\theta+a}^{\infty}g''(X_i-\hat{\theta}_n+bd_n)=O(n)$ from Chebyshev's inequality. Hence, there is $L>0$ such that

$$\sup\{G_n(t);\ -\varepsilon<t-\hat{\theta}_n<-bd_n\}\leqq G_n(\hat{\theta}_n)-\frac{b^2}{8}\log n+L$$

for all sufficiently large n. Thus it follows from (4.1) that

$$\begin{aligned}Q_n\{I_n(\varepsilon, b)\}&\leqq\frac{\exp\left\{G_n(\hat{\theta}_n)-\frac{b^2}{8}\log n+L\right\}}{\exp\left\{G_n(\hat{\theta}_n)-\frac{3}{4}\right\}\lambda\{|t-\hat{\theta}_n|\leqq a_n^{-1}\}}\\ &\leqq ca_nn^{-b^2/8}\,,\end{aligned}$$

where c is a real number depending on K. Choosing $b^2=8(1+k)$, it can be easily seen that $Q_n\{I_n(\varepsilon, b)\}<n^{-k}$. This completes the proof.

Now we are able to estimate the speed of convergence in the following proposition.

Proposition 7. *Let Condition A and condition* (jj) *be satisfied. Then for every* $s\in(0,1)$,*every* $k>0$ *and every compact* $K\subset \boldsymbol{R}$ *there exists* $c>0$ *such that*

$$\sup_{\theta\in K} P_\theta\{Q_n\{H_n(s)^c\}\geqq c(\log n)^{-k}\} = O((\log n)^{s-1})\,.$$

5. Proof of Theorem 2

According to Proposition 7, it is enough to see that for every $s\in(0,1)$ and every compact $K\subset\boldsymbol{R}$ there exists $c>0$ such that

$$\sup_{\theta\in K} P_\theta\{\sup_{B\in\mathcal{B}} |Q_n\{B\cap H_n(s)\}-\Phi\{B\}|\geqq c(\log n)^{-s}\} = O((\log n)^{s-1})\,.$$

This implies that we need only to show

$$\sup_{\theta\in K} P_\theta\{\sup_{B\in\mathcal{B}} |\tilde{Q}_n\{B\}-\Phi\{B\}|\geqq c(\log n)^{-s}\} = O((\log n)^{s-1})\,,$$

where

$$\tilde{Q}_n\{B\} = \frac{Q_n\{B\cap H_n(s)\}}{Q_n\{H_n(s)\}}\,, \qquad B\in\mathcal{B}\,.$$

Since $\sup_{\theta\in R} P_\theta\{|\hat{\theta}_n-\theta|\geqq 1\}=O(n^{-1})$ by Lemma 7, we shall assume that $|\hat{\theta}_n-\theta|<1$. Let $\tilde{K}=\{t;\ \inf_{v\in K}|t-v|\leqq 1\}$. Then $\theta\in K$ implies $\hat{\theta}_n\in\tilde{K}$. Applying condition (jj) to $\tilde{K}$, we have

$$|p(\hat{\theta}_n+a_n^{-1}u)-p(\hat{\theta}_n)|\leqq n^{-1/2}p(\hat{\theta}_n)$$

for $u\in H_n(s)$ and all sufficiently large n. From Lemma 6 we obtain

$$-\frac{u^2}{2}(1+L_1(\log n)^{-s})\leqq G_n(\hat{\theta}_n+a_n^{-1}u)-G_n(\hat{\theta}_n)\leqq -\frac{u^2}{2}(1-L_1(\log n)^{-s})$$

for all $u\in H_n(s)$, where L_1 is a positive real number. Hence, for all sufficiently large n, we have the upper bound of $\tilde{Q}_n\{B\}$ as follows:

$$\begin{aligned}\tilde{Q}_n\{B\} &= \frac{\int_{B\cap H_n(s)}\exp\ \{G_n(\hat{\theta}_n+a_n^{-1}u)\}\,p(\hat{\theta}_n+a_n^{-1}u)du}{\int_{H_n(s)}\exp\ \{G_n(\hat{\theta}_n+a_n^{-1}u)\}\,p(\hat{\theta}_n+a_n^{-1}u)du}\\ &\leqq (1+3n^{-1/2})\frac{\int_{B\cap H_n(s)}\exp\ \{-\frac{u^2}{2}(1-L_1\,(\log n)^{-s})\}\,du}{\int_{H_n(s)}\exp\ \{-\frac{u^2}{2}(1+L_1\,(\log n)^{-s})\}\,du}\end{aligned}$$

$$\leqq \frac{(1+3n^{-1/2})\left[\int_B \exp\left(-\frac{u^2}{2}\right)du+L_2(\log n)^{-s}\right]}{\sqrt{2\pi}-L_3(\log n)^{-s}}$$
$$\leqq \Phi\{B\}+L_4(\log n)^{-s},$$

where $L_2 \sim L_4$ are positive constants. A similar argument shows that the lower bound of $\tilde{Q}_n\{B\}$ is $\Phi\{B\}-L_5(\log n)^{-s}$. This completes the proof of Theorem 2.

REMARK. Easy computations show that the distribution of $\{n^{-1}\sum_0^a X_i^{-2}-\frac{\alpha}{2}\log n\}$ converges weakly to a stable law $V(x)$ with characteristic exponent 1. It is well known that

$$\lim_{x\to\infty} x\{1-V(x)+V(-x)\} = c,$$

where c is a positive constant (see Gnedenko and Kolmogorov [1]). If the distribution of $\{n^{-1}\sum_0^a X_i^{-2}-\frac{\alpha}{2}\log n\}$ is replaced by the limiting distribution $V(x)$, then we obtain

$$P\{|a_n^{-2}\sum_0^a X_i^{-2}-1|\geqq(\log n)^{-s}\}$$
$$\geqq P\{|n^{-1}\sum_0^a X_i^{-2}-\frac{\alpha}{2}\log n|\geqq\alpha(\log n)^{1-s}\}$$
$$\geqq\frac{c}{2\alpha}(\log n)^{s-1}$$

for sufficiently large n. Thus it seems to be impossible to improve Lemma 3 and consequently Theorem 2.

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