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ASYMPTOTIC PROPERTIES OF POSTERIOR DISTRIBUTIONS IN A TRUNCATED CASE

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1. Introduction

Let X_1, \dots, X_n be independent random variables with common density $f(x-\theta)$, $-\infty < x$, $\theta < \infty$, where θ is an unknown translation parameter. We shall consider here the case that $f(x)$ is a uniformly continuous density which vanishes on the interval $(-\infty, 0]$ and is positive on the interval $(0, \infty)$ and particularly

$$f(x) \sim \alpha x \quad \text{as } x \rightarrow +0$$

with $0 < \alpha < \infty$.

Let $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ denote the maximum likelihood estimate of θ for the sample size n . Takeuchi [4] and Woodroffe [7] showed that $\sqrt{\frac{1}{2}\alpha n \log n}(\hat{\theta}_n - \theta)$ has an asymptotic standard normal distribution. The speed of convergence to the standard normal distribution has been given as $O((\log n)^{s-1})$ for every fixed $s \in (0, 1)$ by the author [2] (see Theorem 1 below). Moreover, it was shown by Takeuchi [4] and Weiss and Wolfowitz [6] that $\hat{\theta}_n$ is an asymptotically efficient estimator of θ .

Woodroffe [7] also showed that if θ is regarded as a random variable with a prior density, then the posterior probability that $\sqrt{\frac{1}{2}\alpha n \log n}(\theta - \hat{\theta}_n) \in J$ converges to normality $\Phi\{J\}$ in probability for every finite interval J . The purpose of the present paper is to give a refinement of his result. It is shown that the variational distance between the posterior distribution and the standard normal distribution decreases of the order $(\log n)^{-s}$ with probability $1 - O((\log n)^{s-1})$ for every $s \in (0, 1)$. Similar result for minimum contrast estimates in the regular case was given by Strasser [3].

2. Conditions and the main result

We shall impose the following Condition A on $f(x)$ and Condition B on a prior distribution λ .

Condition A

(i) $f(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0]$ and is positive on $(0, \infty)$.

(ii) $f(x)$ is twice continuously differentiable on $(0, \infty)$ with derivatives $f'(x)$ and $f''(x)$. Moreover $f''(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $f'''(x)$.

(iii) For some $\alpha \in (0, \infty)$ and some $r \in (0, \infty)$

$$f'(x) = \alpha + O(x^r), \quad f''(x) = O(x^{r-1}) \quad \text{and} \quad f'''(x) = o(x^{-2}) \quad \text{as} \quad x \rightarrow +0.$$

Let $g(x) = \log f(x)$ for $x > 0$. Then the second derivative $g''(x)$ of $g(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $g''' = f'''f^{-1} - 3f'f''f^{-2} + 2(f'f^{-1})^3$. Under conditions (i) and (ii), condition (iii) is equivalent to the following condition (iii)'.

(iii)' For some $\alpha \in (0, \infty)$ and some $r \in (0, \infty)$

$$f(x) = \alpha x + O(x^{1+r}), \quad g'(x) = x^{-1} + O(x^{r-1}), \quad g''(x) = -x^{-2} + O(x^{r-2})$$

$$\text{and} \quad g'''(x) = 2x^{-3} + o(x^{-3}) \quad \text{as} \quad x \rightarrow +0.$$

(iv) For every $t \geq 0$

$$\int_0^\infty \{g(x+t)\}^2 f(x) dx < \infty.$$

(v) For every $a > 0$, there is a $\delta > 0$, for which

(a)
$$\int_a^\infty \sup_{|u| \leq \delta} |g'(x+u)|^3 f(x) dx < \infty,$$

(b)
$$\int_a^\infty \sup_{|u| \leq \delta} \{g''(x+u)\}^2 f(x) dx < \infty,$$

(c)
$$\int_a^\infty \sup_{|u| \leq \delta} \{g'''(x+u)\}^2 f(x) dx < \infty.$$

Let $(\mathbf{R}, \mathcal{B})$ be a parameter space, where \mathbf{R} is the real line and \mathcal{B} is the Borel σ -algebra of \mathbf{R} . Moreover, let λ be a prior distribution on $(\mathbf{R}, \mathcal{B})$. The following Condition **B** is owed to Strasser [3].

Condition B

(j) For every $\eta > 0$ and every compact $K \subset \mathbf{R}$

$$\inf_{\theta \in K} \lambda \{t \in \mathbf{R}; |t - \theta| < \eta\} > 0.$$

(jj) λ has a continuous and positive density p on \mathbf{R} with respect to the Lebesgue measure satisfying the following condition: For every compact $K \subset \mathbf{R}$ there exist constants $c_K > 0$ and $d_K > 0$ such that $t \in \mathbf{R}$, $\theta \in K$ and $|t - \theta| \leq d_K$ imply

$$|p(t) - p(\theta)| \leq c_K p(\theta) |t - \theta|.$$

Obviously condition (jj) implies condition (j).

Let P_θ denote the conditional probability of (X_1, \dots, X_n) given θ and define

$$\Phi\{B\} = \int_B \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx, \quad B \in \mathcal{B}.$$

The following theorem is often needed in the sequel.

Theorem 1 (Matsuda [2]). *Suppose that Condition A holds. Then for every $s \in (0, 1)$ there exists a positive constant c such that for all $\theta, t \in \mathbf{R}$ and $n \geq 1$*

$$|P_\theta\{a_n(\hat{\theta}_n - \theta) \leq t\} - \Phi\{(-\infty, t]\}| \leq c(\log n)^{s-1},$$

where $2a_n^2 = \alpha n(\log n + \log \log n)$ and the constant c tends to infinity as $s \rightarrow 0$.

It is remarked that the upper bound $(\log n)^{s-1}$ in Theorem 1 is replaced by a better bound $(\log n)^{-1}$, provided t is restricted to $(-\infty, M)$ with $0 < M < \infty$.

But, using $\sqrt{\frac{1}{2}\alpha n \log n}$ instead of a_n , the upper bound in Theorem 1 becomes $(\log \log n)(\log n)^{-1}$ which is worse than the order $(\log n)^{-1}$. Thus we use a_n rather than $\sqrt{\frac{1}{2}\alpha n \log n}$.

Let R_n denote the conditional distribution of θ given X_1, \dots, X_n and define a probability measure Q_n by

$$Q_n\{B\} = R_n\{\theta \in \mathbf{R}; a_n(\theta - \hat{\theta}_n) \in B\}, \quad B \in \mathcal{B}.$$

Theorem 2. *Suppose that Condition A and condition (jj) hold. Then for every $s \in (0, 1)$ and every compact $K \subset \mathbf{R}$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $n \geq 1$*

$$\sup_{\theta \in K} P_\theta\{\|Q_n - \Phi\| \geq c_1(\log n)^{-s}\} \leq c_2(\log n)^{s-1},$$

where $\|\cdot\|$ means the totally variation of a measure.

For the proof of Theorem 2 we need several lemmas and propositions.

3. Auxiliary results

In this section, $\theta = 0$ will be chosen for simplicity and write P instead of P_0 . Let E be the expectation with respect to P . The following Lemma 1 and Lemma 2 are closely related to Lemma 1 and Lemma 2 in Strasser [3], respectively.

Lemma 1. *Let conditions (i) and (iv) be satisfied. Then for every $\varepsilon > 0$*

there exists $d > 0$ such that

$$P\left\{\sup_{t \leq -\varepsilon} n^{-1} \sum_{i=1}^n g(X_i - t) \geq E\{g(X)\} - d\right\} = O(n^{-1}).$$

Proof. Let M be a positive number chosen such that

$$E\left\{\sup_{t < -M} g(X - t)\right\} < E\{g(X)\}.$$

For every $t \in [-M, -\varepsilon]$ there exists an open neighborhood U_t of t such that

$$E\left\{\sup_{u \in U_t} g(X - u)\right\} < E\{g(X)\}.$$

The existence of such a positive number M and that of such a U_t follow from Wald [5] (see Woodroffe [7] and also [2]). As $\{U_t; t \in [-M, -\varepsilon]\}$ covers the compact set $[-M, -\varepsilon]$, there exists a finite subcover of this set $[-M, -\varepsilon]$ determined by $t_j \in [-M, -\varepsilon]$, $j = 1, \dots, m$. For notational convenience, let $U_0 = (-\infty, -M)$ and $U_j = U_{t_j}$, $j = 1, \dots, m$. Write

$$d_j = E\{g(X)\} - E\left\{\sup_{t \in U_j} g(X - t)\right\} > 0, \quad j = 0, \dots, m$$

and let $2d = \min\{d_j; j = 0, \dots, m\} > 0$. Then

$$\sup_{t \leq -\varepsilon} n^{-1} \sum_{i=1}^n g(X_i - t) \geq E\{g(X)\} - d$$

implies

$$n^{-1} \sum_{i=1}^n \sup_{t \in U_j} g(X_i - t) - E\left\{\sup_{t \in U_j} g(X - t)\right\} \geq d$$

for some $j \in \{0, \dots, m\}$. Hence we have

$$\begin{aligned} & P\left\{\sup_{t \leq -\varepsilon} n^{-1} \sum_{i=1}^n g(X_i - t) \geq E\{g(X)\} - d\right\} \\ & \leq \sum_{j=0}^m P\left\{|n^{-1} \sum_{i=1}^n \sup_{t \in U_j} g(X_i - t) - E\left\{\sup_{t \in U_j} g(X - t)\right\}| \geq d\right\}. \end{aligned}$$

Now the assertion of Lemma 1 follows from Chebyshev's inequality because of conditions (i) and (iv).

Lemma 2. *Let conditions (i)–(iv) and (v) (a) be satisfied. Then for every $d > 0$ there exists $\eta > 0$ such that*

$$P\left\{\inf_{-\eta < t < 0} n^{-1} \sum_{i=1}^n g(X_i - t) \leq E\{g(X)\} - d\right\} = O(n^{-1}).$$

Proof. Let $a > 0$ be so small that $g'(x) > 0$ for $0 < x < 2a$. Next choose

$\delta > 0$ to satisfy condition (v) (a). Then for $\eta < \delta$ we have

$$\begin{aligned} n^{-1} \sum_{i=1}^n g(X_i - t) &= n^{-1} \sum_{i=1}^n g(X_i) - n^{-1} t \sum_{i=1}^n g'(X_i - t^*) \\ &\geq n^{-1} \sum_{i=1}^n g(X_i) + n^{-1} t \sum_a^\infty \sup_{|u| \leq \delta} |g'(X_i + u)| \end{aligned}$$

for some $t^* \in (-\eta, 0)$. Here and in what follows, \sum_u^v denotes summation over $i \leq n$ for which $u \leq X_i < v$. Hence

$$|n^{-1} \sum_{i=1}^n g(X_i) - E\{g(X)\}| < \frac{d}{3}$$

and

$$|n^{-1} \sum_a^\infty \sup_{|u| \leq \delta} |g'(X_i + u)| - \int_a^\infty \sup_{|u| \leq \delta} |g'(x + u)| f(x) dx| < \frac{d}{3}$$

imply

$$n^{-1} \sum_{i=1}^n g(X_i - t) \geq E\{g(X)\} - \frac{d}{3} + t \left\{ \frac{d}{3} + \int_a^\infty \sup_{|u| \leq \delta} |g'(x + u)| f(x) dx \right\}.$$

Choosing $\eta < \min \left\{ 1, \delta, \frac{d}{3} \left[\int_a^\infty \sup_{|u| \leq \delta} |g'(x + u)| f(x) dx \right]^{-1} \right\}$, we obtain

$$\inf_{-\eta < t < 0} n^{-1} \sum_{i=1}^n g(X_i - t) > E\{g(X)\} - d.$$

Lemma 2 follows from Chebyshev's inequality because of conditions (iv) and (v)(a).

Lemma 3. *Let conditions (i)–(iii) and (v)(b) be satisfied. Then for every $s \in (0, 1)$*

$$P\{|a_n^{-2} \sum_{i=1}^n g''(X_i) + 1| \geq (\log n)^{-s}\} = O((\log n)^{s-1}).$$

Proof. According to condition (iii)' choose $a > 0$ and $c > 0$ such that $|f(x) - \alpha x| \leq cx^{1+r}$ and $|g''(x) + x^{-2}| \leq cx^{r-2}$ for $0 < x < a$. For $i \leq n$ let

$$\begin{aligned} Y_{ni} &= g''(X_i), & \text{if } b_n \leq X_i < a, \\ &= 0, & \text{if } X_i < b_n \text{ or } a \leq X_i, \end{aligned}$$

where $b_n = a_n^{-1}(\log n)^{s/2}$. Since $E\{Y_{ni}^2\} = O(b_n^{-2}) = O(n(\log n)^{1-s})$, it follows from Chebyshev's inequality that

$$P\{|a_n^{-2} \sum_{i=1}^n (Y_{ni} - E\{Y_{ni}\})| \geq \frac{1}{4} (\log n)^{-s}\} = O((\log n)^{s-1}).$$

Considering $E\{Y_{ni}\} = -\alpha \log a_n + O(\log \log n)$, this leads to

$$P\left\{\left|a_n^{-2} \sum_{i=1}^n Y_{ni} + 1\right| \geq \frac{1}{2} (\log n)^{-s}\right\} = O((\log n)^{s-1}).$$

Moreover, using $P\left\{\sum_{i=1}^n Y_{ni} \neq \sum_0^a g''(X_i)\right\} = O((\log n)^{s-1})$, we obtain

$$P\left\{\left|a_n^{-2} \sum_0^a g''(X_i) + 1\right| \geq \frac{1}{2} (\log n)^{-s}\right\} = O((\log n)^{s-1}).$$

Since also

$$P\left\{\left|a_n^{-2} \sum_a^\infty g''(X_i)\right| \geq \frac{1}{2} (\log n)^{-s}\right\} = O(n^{-1})$$

by Chebyshev's inequality, the proof is completed.

Let $M_n = \min(X_1, \dots, X_n)$ and let $b_n = a_n^{-1}(\log n)^{s/2}$ with $s \in (0, 1)$ as in the proof of Lemma 3.

Lemma 4. *Let conditions (i), (ii) and (iii) be satisfied. Then for every $s \in (0, 1)$ and sufficiently small $a > 0$*

$$P\left\{\left|a_n^{-3} \sum_0^a (X_i - 2b_n)^{-3}\right| \geq (\log n)^{-(3/2)s}, M_n > 2b_n\right\} = O((\log n)^{s-1}).$$

Proof. Let $a > 0$ be so small that $f(x) < 2\alpha x$ for $0 < x < a$. Then define $\{Y_{ni}; i=1, \dots, n\}$ by

$$Y_{ni} = \begin{cases} (X_i - 2b_n)^{-3}, & \text{if } 3b_n \leq X_i < a, \\ 0, & \text{if } X_i < 3b_n \text{ or } a \leq X_i. \end{cases}$$

Since $E\{Y_{n1}^2\} = O(b_n^{-4})$, it follows from Chebyshev's inequality that

$$P\left\{\left|a_n^{-3} \sum_{i=1}^n (Y_{ni} - E\{Y_{ni}\})\right| \geq \frac{1}{2} (\log n)^{-(3/2)s}\right\} = O((\log n)^{s-1}).$$

Moreover, using $a_n^{-3} \sum_{i=1}^n E\{Y_{ni}\} = O((\log n)^{-1-s/2})$ we obtain

$$P\left\{\left|a_n^{-3} \sum_{i=1}^n Y_{ni}\right| \geq (\log n)^{-(3/2)s}\right\} = O((\log n)^{s-1}),$$

which leads to the desired result.

For notational convenience define

$$G_n(t) = \begin{cases} \sum_{i=1}^n g(X_i - t), & \text{if } t < M_n, \\ -\infty, & \text{if } t \geq M_n. \end{cases}$$

The following Lemma 5 and Lemma 6 refine Lemma 3.4 and Lemma 4.1 in Woodroffe [7], respectively.

Lemma 5. *Let conditions (i)–(iii), (v)(b) and (v)(c) be satisfied. Then for every $s \in (0, 1)$ there exists $c > 0$ such that*

$$P \left\{ \sup_{|t| \leq 2b_n} |a_n^{-2} G_n''(t) + 1| \geq c(\log n)^{-s} \right\} = O((\log n)^{s-1}).$$

Proof. Since $P\{M_n \leq 2b_n\} = O((\log n)^{s-1})$, we can assume that $M_n > 2b_n$. Then $G_n''(t) = \sum_{i=1}^n g''(X_i - t)$ for $|t| \leq 2b_n$. Using the equality

$$a_n^{-2} \sum_{i=1}^n g''(X_i - t) = a_n^{-2} \sum_{i=1}^n g''(X_i) - a_n^{-2} \sum_{i=1}^n \int_0^t g'''(X_i - u) du$$

we have

$$\begin{aligned} \sup_{|t| \leq 2b_n} |a_n^{-2} G_n''(t) + 1| &\leq |a_n^{-2} \sum_{i=1}^n g''(X_i) + 1| + 6a_n^{-2} b_n \sum_0^a (X_i - 2b_n)^{-3} \\ &\quad + 2a_n^{-2} b_n \sum_0^a \sup_{|u| \leq 2b_n} |g'''(X_i + u)|. \end{aligned}$$

Here we used the fact that $|g'''(x)| \leq 3x^{-3}$ for $0 < x < 2a$ with sufficiently small $a > 0$. Now the assertion follows from Lemma 3 and Lemma 4.

Lemma 5, together with Theorem 1, yields the following lemma.

Lemma 6. *Let Condition A be satisfied. Then for every $s \in (0, 1)$ there exists $c > 0$ such that*

$$P \left\{ \sup_{|t| \leq b_n} |a_n^{-2} G_n''(\hat{\theta}_n + t) + 1| \geq c(\log n)^{-s} \right\} = O((\log n)^{s-1}),$$

where $b_n = a_n^{-1}(\log n)^{s/2}$.

Lemma 7 (Lemma 2 in [2]). *Let conditions (i)–(iii) and (iv) be satisfied. Then for every $\varepsilon > 0$*

$$P\{|\hat{\theta}_n| \geq \varepsilon\} = O(n^{-1}).$$

Lemma 8 (Lemma 1 in [2]). *Let conditions (i)–(iii) and (v)(b) be satisfied. Then for sufficiently small $\varepsilon > 0$, there are events $D_n, n \geq 1$, for which $P\{D_n^c\} = O(n^{-1})$ and D_n implies $\sup_{-e \leq t < M_n} n^{-1} G_n'''(t) < -1$.*

The following lemma also may be proved analogously to Lemma 8.

Lemma 9. *Let conditions (i)–(iii) and (v)(c) be satisfied. Then for sufficiently small $\varepsilon > 0$, there are events $F_n, n \geq 1$, for which $P\{F_n^c\} = O(n^{-1})$ and F_n implies $\sup_{-e \leq t < M_n} n^{-1} G_n'''(t) < -1$.*

Lemma 10. *Let conditions (i), (ii) and (iii) be satisfied. Then for every*

$s \in (0, 1)$, every $b > 0$ and sufficiently small $a > 0$

$$P\{|a_n^{-2} \sum_0^a (X_i + 2bd_n)^{-2} - 1| \geq (\log n)^{-(1+s)/2}\} = O((\log n)^{s-1}),$$

where $d_n = a_n^{-1}(\log n)^{1/2}$.

We shall omit the proof since Lemma 10 may be proved analogously to Lemma 4.

4. Estimation of the speed of convergence

For each $n \geq 1$ and each $s \in (0, 1)$, let $H_n(s) = [-(\log n)^{s/2}, (\log n)^{s/2}]$. In this section, we shall estimate the speed with which $Q_n\{H_n(s)^c\}$ converges to 0. For the convenience of calculation, we shall divide $H_n(s)^c$ into five parts as follows:

$$\begin{aligned} I_n(\varepsilon) &= (-\infty, -a_n \varepsilon], \\ I_n(\varepsilon, b) &= (-a_n \varepsilon, -b(\log n)^{1/2}), \\ J_n(b, s) &= (-b(\log n)^{1/2}, -(\log n)^{s/2}), \\ J_n(s) &= ((\log n)^{s/2}, \log n) \end{aligned}$$

and

$$J_n = [\log n, \infty)$$

with $\varepsilon > 0$ and $b > 0$. We first show the following proposition which is similar to Theorem 1 in Strasser [3].

Proposition 1. *Let conditions (i)–(v)(a) and (j) be satisfied. Then for every $\varepsilon > 0$ there exists $c > 0$ such that for every compact $K \subset \mathbf{R}$*

$$\sup_{\theta \in K} P_\theta\{R_n\{t \in \mathbf{R}; |t - \theta| \geq \varepsilon\} > \exp(-cn)\} = O(n^{-1}).$$

Proof. Since θ is a translation parameter, it is easily seen that $\sup_{\theta \in \mathbf{R}} P_\theta\{M_n - \theta \geq \varepsilon\} = P\{M_n \geq \varepsilon\} = o(n^{-1})$. Therefore, we shall assume that $M_n - \theta < \varepsilon$. Then we have

$$\begin{aligned} R_n\{|t - \theta| \geq \varepsilon\} &= \frac{\int_{|t - \theta| \geq \varepsilon} \exp\{G_n(t)\} \lambda(dt)}{\int_{\mathbf{R}} \exp\{G_n(t)\} \lambda(dt)} \\ &\leq \frac{\int_{t \leq \theta - \varepsilon} \exp\{G_n(t)\} \lambda(dt)}{\int_{\theta - \eta < t < \theta} \exp\{G_n(t)\} \lambda(dt)} \\ &\leq \exp\{-n[\inf_{-\eta < t < \theta} n^{-1}G_n(\theta + t) - \sup_{t \leq \theta - \varepsilon} n^{-1}G_n(\theta + t) \\ &\quad + n^{-1} \log \lambda\{-\eta < t - \theta < 0\}]\} \end{aligned}$$

for $\eta > 0$. By Lemma 1 there exists $d > 0$ (depending on ε) such that

$$\sup_{t \leq -\varepsilon} n^{-1} G_n(\theta + t) < E_\theta \{g(X - \theta)\} - d$$

with probability $1 - O(n^{-1})$, where $O(n^{-1})$ is uniform in θ for $\theta \in \mathbf{R}$. Also, by Lemma 2 there exists $\eta > 0$ (depending on ε) such that

$$\inf_{-\eta < t < 0} n^{-1} G_n(\theta + t) > E_\theta \{g(X - \theta)\} - \frac{d}{4}$$

with probability $1 - O(n^{-1})$ as just stated. Since $-\infty < \beta \equiv \inf_{\theta \in K} \log \lambda \{-\eta < t - \theta < 0\} \leq 0$ by condition (j), for any $0 < c < \frac{d}{2}$ we have

$$\inf_{-\eta < t < 0} n^{-1} G_n(\theta + t) - \sup_{t \leq -\varepsilon} n^{-1} G_n(\theta + t) + n^{-1} \beta > c$$

for all sufficiently large n . This completes the proof of Proposition 1.

The following result immediately follows from Proposition 1 and Lemma 7.

Proposition 2. *Let conditions (i)–(v)(a) and (j) be satisfied. Then for every $\varepsilon > 0$ there exists $c > 0$ such that for every compact $K \subset \mathbf{R}$*

$$\sup_{\theta \in K} P_\theta \{Q_n \{I_n(\varepsilon)\} > \exp(-cn)\} = O(n^{-1}).$$

Easy computations show that condition (jj) and Lemma 7 imply that for every compact $K \subset \mathbf{R}$ there exist $c_1, c_2, 0 < c_1 < c_2 < \infty$, and $c_3 > 0$ such that

$$(4.1) \quad \inf_{\theta \in K} P_\theta \{c_1 \eta_n \leq \lambda \{|t - \hat{\theta}_n| \leq \eta_n\} \leq c_2 \eta_n\} \geq 1 - c_3 n^{-1}$$

for all $n \geq 1$ and for every positive sequence $\{\eta_n\}$ with $\eta_n \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 3. *Let Condition A and condition (jj) be satisfied. Then for every $s \in (0, 1)$, every $b > 0$, every $k > 0$ and every compact $K \subset \mathbf{R}$*

$$\sup_{\theta \in K} P_\theta \{Q_n \{J_n(b, s)\} \geq (\log n)^{-k}\} = O((\log n)^{s-1}).$$

Proof. Lemma 8 implies that, with probability $1 - O(n^{-1})$, $G_n(t)$ is a concave function in $t \in [\theta - 2\varepsilon, M_n]$, if $\varepsilon > 0$ is a sufficiently small number. Using Lemma 7 we can assume that $|\hat{\theta}_n - \theta| < \varepsilon$. Hence for all sufficiently large n we have

$$\begin{aligned} \sup \{G_n(t); \hat{\theta}_n - ba_n^{-1}(\log n)^{1/2} < t < \hat{\theta}_n - b_n\} &\leq G_n(\hat{\theta}_n - b_n) \\ &\leq G_n(\hat{\theta}_n) + \frac{b_n^2}{2} \sup_{|t| \leq b_n} G_n''(\hat{\theta}_n + t) \\ &\leq G_n(\hat{\theta}_n) - \frac{1}{4} (\log n)^s. \end{aligned}$$

The last inequality follows from Lemma 6. A similar argument will show that

$$\begin{aligned} \inf \{G_n(t); |t-\hat{\theta}_n| \leq a_n^{-1}\} &\geq \min \{G_n(\hat{\theta}_n - a_n^{-1}), G_n(\hat{\theta}_n + a_n^{-1})\} \\ &\geq G_n(\hat{\theta}_n) + \frac{a_n^{-2}}{2} \inf_{|t| \leq a_n^{-1}} G_n''(\hat{\theta}_n + t) \\ &\geq G_n(\hat{\theta}_n) - \frac{3}{4}. \end{aligned}$$

Therefore, for $\theta \in K$

$$\begin{aligned} Q_n \{J_n(b, s)\} &\leq \frac{\int_{\hat{\theta}_n - ba_n^{-1}(\log n)^{1/2}}^{\hat{\theta}_n - b_n} \exp \{G_n(t)\} \lambda(dt)}{\int_{\hat{\theta}_n - a_n^{-1}}^{\hat{\theta}_n + a_n^{-1}} \exp \{G_n(t)\} \lambda(dt)} \\ &\leq \frac{\exp \{G_n(\hat{\theta}_n) - \frac{1}{4}(\log n)^s\} \lambda \{|t - \hat{\theta}_n| \leq ba_n^{-1}(\log n)^{1/2}\}}{\exp \{G_n(\hat{\theta}_n) - \frac{3}{4}\} \lambda \{|t - \hat{\theta}_n| \leq a_n^{-1}\}}. \end{aligned}$$

Taking account of (4.1), we obtain

$$Q_n \{J_n(b, s)\} \leq cb(\log n)^{1/2} \exp \left\{ -\frac{1}{4}(\log n)^s \right\} < (\log n)^{-k}$$

for all sufficiently large n , where c is a real number depending on K . Thus the proof is completed.

The following Proposition 4 may be proved similarly to Proposition 3, and so the proof will be omitted here.

Proposition 4. *Let Condition A and condition (jj) be satisfied. Then for every $s \in (0, 1)$, every $k > 0$ and every compact $K \subset \mathbf{R}$*

$$\sup_{\theta \in K} P_\theta \{Q_n \{J_n(s)\} \geq (\log n)^{-k}\} = O((\log n)^{s-1}).$$

Proposition 5. *Let Condition A be satisfied. Then for every $s \in (0, 1)$*

$$\sup_{\theta \in \mathbf{R}} P_\theta \{Q_n \{J_n\} > 0\} = O((\log n)^{s-1}).$$

Proof. It is easily seen that $\sup_{\theta \in \mathbf{R}} P_\theta \{M_n - \theta \geq \frac{1}{2}a_n^{-1} \log n\} = O(n^{-c})$ for some $c > 0$. Theorem 1 implies that

$$\sup_{\theta \in \mathbf{R}} P_\theta \{|\hat{\theta}_n - \theta| \geq b_n\} = O((\log n)^{s-1}).$$

Therefore, we may assume that

$$M_n - \theta < \frac{1}{2} a_n^{-1} \log n \quad \text{and} \quad |\hat{\theta}_n - \theta| < b_n.$$

Then $t \geq \hat{\theta}_n + a_n^{-1} \log n$ implies $t > M_n$ for sufficiently large n . Since $R_n\{t > M_n\} = 0$, the assertion of the proposition holds.

Proposition 6. *Let Condition A and condition (jj) be satisfied. Then for every $s \in (0, 1)$, every $k > 0$, every compact $K \subset \mathbf{R}$ and sufficiently small $\varepsilon > 0$ there exists $b > 0$ such that*

$$\sup_{\theta \in K} P_\theta \{Q_n\{I_n(\varepsilon, b)\} \geq n^{-k}\} = O((\log n)^{s-1}).$$

Proof. By Theorem 1 we can assume that $|\hat{\theta}_n - \theta| < b d_n$ where $d_n = a_n^{-1}(\log n)^{1/2}$. Since $G_n(t)$ is concave on $[\theta - 2\varepsilon, M_n)$ with sufficiently small $\varepsilon > 0$, Lemma 9 implies

$$\begin{aligned} \sup \{G_n(t); -\varepsilon < t - \hat{\theta}_n < -b d_n\} &\leq G_n(\hat{\theta}_n - b d_n) \\ &\leq G_n(\hat{\theta}_n) + \frac{b^2 d_n^2}{2} G_n''(\hat{\theta}_n - b d_n) \end{aligned}$$

for all sufficiently large n .

Let $a > 0$ be so small that $g''(x) < -\frac{1}{2}x^{-2}$ for $0 < x < 2a$ and choose $\delta > 0$ to satisfy condition (v)(b). Then, it follows from Lemma 10 that

$$\begin{aligned} \sum_{\theta}^{\theta+a} g''(X_i - \hat{\theta}_n + b d_n) &\leq -\frac{1}{2} \sum_{\theta}^{\theta+a} (X_i - \hat{\theta}_n + b d_n)^{-2} \\ &\leq -\frac{1}{2} \sum_{\theta}^{\theta+a} (X_i - \theta + 2b d_n)^{-2} \\ &\leq -\frac{1}{4} a_n^2. \end{aligned}$$

Since $|\sum_{\theta+a}^{\infty} g''(X_i - \hat{\theta}_n + b d_n)| \leq \sum_{\theta+a}^{\infty} \sup_{|u| \leq \delta} |g''(X_i - \theta + u)|$ for all sufficiently large n , we have $\sum_{\theta+a}^{\infty} g''(X_i - \hat{\theta}_n + b d_n) = O(n)$ from Chebyshev's inequality. Hence, there is $L > 0$ such that

$$\sup \{G_n(t); -\varepsilon < t - \hat{\theta}_n < -b d_n\} \leq G_n(\hat{\theta}_n) - \frac{b^2}{8} \log n + L$$

for all sufficiently large n . Thus it follows from (4.1) that

$$\begin{aligned} Q_n\{I_n(\varepsilon, b)\} &\leq \frac{\exp \left\{ G_n(\hat{\theta}_n) - \frac{b^2}{8} \log n + L \right\}}{\exp \left\{ G_n(\hat{\theta}_n) - \frac{3}{4} \right\} \lambda \{ |t - \hat{\theta}_n| \leq a_n^{-1} \}} \\ &\leq c a_n n^{-b^2/8}, \end{aligned}$$

where c is a real number depending on K . Choosing $b^2=8(1+k)$, it can be easily seen that $Q_n\{I_n(\varepsilon, b)\} < n^{-k}$. This completes the proof.

Now we are able to estimate the speed of convergence in the following proposition.

Proposition 7. *Let Condition A and condition (jj) be satisfied. Then for every $s \in (0, 1)$, every $k > 0$ and every compact $K \subset \mathbf{R}$ there exists $c > 0$ such that*

$$\sup_{\theta \in K} P_\theta \{Q_n\{H_n(s)^2\} \geq c(\log n)^{-k}\} = O((\log n)^{s-1}).$$

5. Proof of Theorem 2

According to Proposition 7, it is enough to see that for every $s \in (0, 1)$ and every compact $K \subset \mathbf{R}$ there exists $c > 0$ such that

$$\sup_{\theta \in K} P_\theta \{ \sup_{B \in \mathcal{B}} |Q_n\{B \cap H_n(s)\} - \Phi\{B\}| \geq c(\log n)^{-s} \} = O((\log n)^{s-1}).$$

This implies that we need only to show

$$\sup_{\theta \in K} P_\theta \{ \sup_{B \in \mathcal{B}} |\tilde{Q}_n\{B\} - \Phi\{B\}| \geq c(\log n)^{-s} \} = O((\log n)^{s-1}),$$

where

$$\tilde{Q}_n\{B\} = \frac{Q_n\{B \cap H_n(s)\}}{Q_n\{H_n(s)\}}, \quad B \in \mathcal{B}.$$

Since $\sup_{\theta \in \mathbf{R}} P_\theta \{|\hat{\theta}_n - \theta| \geq 1\} = O(n^{-1})$ by Lemma 7, we shall assume that $|\hat{\theta}_n - \theta| < 1$. Let $\tilde{K} = \{t; \inf_{v \in K} |t - v| \leq 1\}$. Then $\theta \in K$ implies $\hat{\theta}_n \in \tilde{K}$. Applying condition (jj) to \tilde{K} , we have

$$|p(\hat{\theta}_n + a_n^{-1}u) - p(\hat{\theta}_n)| \leq n^{-1/2} p(\hat{\theta}_n)$$

for $u \in H_n(s)$ and all sufficiently large n . From Lemma 6 we obtain

$$-\frac{u^2}{2}(1 + L_1(\log n)^{-s}) \leq G_n(\hat{\theta}_n + a_n^{-1}u) - G_n(\hat{\theta}_n) \leq -\frac{u^2}{2}(1 - L_1(\log n)^{-s})$$

for all $u \in H_n(s)$, where L_1 is a positive real number. Hence, for all sufficiently large n , we have the upper bound of $\tilde{Q}_n\{B\}$ as follows:

$$\begin{aligned} \tilde{Q}_n\{B\} &= \frac{\int_{B \cap H_n(s)} \exp\{G_n(\hat{\theta}_n + a_n^{-1}u)\} p(\hat{\theta}_n + a_n^{-1}u) du}{\int_{H_n(s)} \exp\{G_n(\hat{\theta}_n + a_n^{-1}u)\} p(\hat{\theta}_n + a_n^{-1}u) du} \\ &\leq (1 + 3n^{-1/2}) \frac{\int_{B \cap H_n(s)} \exp\{-\frac{u^2}{2}(1 - L_1(\log n)^{-s})\} du}{\int_{H_n(s)} \exp\{-\frac{u^2}{2}(1 + L_1(\log n)^{-s})\} du} \end{aligned}$$

$$\begin{aligned} &\leq \frac{(1+3n^{-1/2})\left[\int_B \exp\left(-\frac{u^2}{2}\right)du + L_2(\log n)^{-s}\right]}{\sqrt{2\pi} - L_3(\log n)^{-s}} \\ &\leq \Phi\{B\} + L_4(\log n)^{-s}, \end{aligned}$$

where $L_2 \sim L_4$ are positive constants. A similar argument shows that the lower bound of $\tilde{Q}_n\{B\}$ is $\Phi\{B\} - L_5(\log n)^{-s}$. This completes the proof of Theorem 2.

REMARK. Easy computations show that the distribution of $\{n^{-1} \sum_0^a X_i^{-2} - \frac{\alpha}{2} \log n\}$ converges weakly to a stable law $V(x)$ with characteristic exponent 1. It is well known that

$$\lim_{x \rightarrow \infty} x\{1 - V(x) + V(-x)\} = c,$$

where c is a positive constant (see Gnedenko and Kolmogorov [1]). If the distribution of $\{n^{-1} \sum_0^a X_i^{-2} - \frac{\alpha}{2} \log n\}$ is replaced by the limiting distribution $V(x)$, then we obtain

$$\begin{aligned} &P\{|a_n^{-2} \sum_0^a X_i^{-2} - 1| \geq (\log n)^{-s}\} \\ &\geq P\{|n^{-1} \sum_0^a X_i^{-2} - \frac{\alpha}{2} \log n| \geq \alpha(\log n)^{1-s}\} \\ &\geq \frac{c}{2\alpha} (\log n)^{s-1} \end{aligned}$$

for sufficiently large n . Thus it seems to be impossible to improve Lemma 3 and consequently Theorem 2.

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