Liess, O. and Sonobe, T.
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CURVATURE PROPERTIES OF THE SLOWNESS SURFACE OF THE SYSTEM OF CRYSTAL ACOUSTICS FOR CUBIC CRYSTALS II

OTTO LIESS and TETSUYA SONOBE

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Abstract

In this paper we study geometric properties of the slowness surface of the system of crystal acoustics for cubic crystals in the special case when the stiffness constants satisfy the condition $a = -2b$. The paper is a natural continuation of the paper [9] in which related properties were studied for general constants a and b, but assuming that we were in the nearly isotropic case, in which case $a - b$ has to be small. We also take this opportunity to correct a statement made in [9]: see Remark 1.3.

1. Introduction

In this paper we study geometric properties of the slowness surface $\tilde{S}$ of the system of crystal elasticity for cubic crystals in the special case when the stiffness constants of the crystal are related by the condition "$a = -2b$". To some extent the paper is a continuation of the paper [9] in which similar geometric properties were studied for cubic crystals in the nearly isotropic case. Information on the terminology, the notations and the condition "$a = -2b$" shall be given in a moment. The geometric properties in which we are interested are related to the curvature properties in the smooth part of $\tilde{S}$ and the structure of the singularities at the singular points. Such properties are needed in order to understand the asymptotic behavior for $|x| \to \infty$ of integrals of form

$$I(x) = \int_{\tilde{S}} e^{i\langle x,\xi\rangle} u(\xi)\, d\sigma(\xi), \tag{1.1}$$

with $d\sigma$ denoting the surface element on $\tilde{S}$. Estimates for integrals as in (1.1) in turn are an essential ingredient in establishing estimates concerning the long-time behavior of global solutions of the system of crystal elasticity for cubic crystals (See e.g., [10]).

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We recall the exact form of the system to which we refer: it is

$$
\begin{pmatrix} \partial_t^2 - a\partial_{x_1}^2 - c\Delta & -b\partial_{x_1}\partial_{x_2} & -b\partial_{x_1}\partial_{x_3} \\ -b\partial_{x_2}\partial_{x_1} & \partial_t^2 - a\partial_{x_2}^2 - c\Delta & -b\partial_{x_2}\partial_{x_3} \\ -b\partial_{x_3}\partial_{x_1} & -b\partial_{x_3}\partial_{x_2} & \partial_t^2 - a\partial_{x_3}^2 - c\Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \tag{1.2}
$$

and the “characteristic” surface associated with the system can be written in the following form (due to Kelvin):

$$
\frac{b\xi_1^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_1^2} + \frac{b\xi_2^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_2^2} + \frac{b\xi_3^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_3^2} = 1. \tag{1.3}
$$

In these equations a, b, c, are real constants which can be calculated in terms of the stiffness constants of the crystal under consideration. (Recall that for cubic crystals the number of essential stiffness constants is 3. For more details on all this, see [3] and also [6].) The fact that (1.3) defines the characteristic surface of a cubic crystal gives some restrictions on the constants a, b, c. Of these we mention that we must have $c > 0$, $a \neq 0$, $a + c > 0$, $3c - b + a > 0$ (see, e.g., [6]. Some of these restrictions simply come from the fact that the system of crystal elasticity is hyperbolic, while others may have a deeper physical interpretation: see [3], [5]). As in [3] we shall often assume that $b \geq 0$ (for a physical justification, see [10]). The surface $\{\xi \in \mathbb{R}^3;\ \xi \text{satisfies (1.3) with} \tau = 1\}$ is called the “slowness surface” of the crystal under consideration. It is essentially the intersection of the characteristic surface with the hyperplane $\tau = 1$.

In [9] the main assumption on the constants a, b, c was that $a - b$ had to be small compared with c. Since $a - b$ is a measure of the anisotropy of the crystal, this assumption says that we are in a “nearly” isotropic case. In the first main result in this paper we shall now assume that $a = -2b$, with no additional restrictions on the size of the quantity $a - b$, whereas in the second main result the restrictions on the size of the constants are quantitatively more precise than in the corresponding result in [9].

We know of no physical interpretation of the condition $a = -2b$, but from a mathematical point of view it leads to a significant simplification of the situation. Indeed, (1.3) transforms for $a = -2b$ to

$$
\sum_{j=1}^{3} \frac{b\xi_j^2}{\tau^2 - c|\xi|^2 + 3b\xi_j^2} = 1, \tag{1.4}
$$

which holds trivially when $\tau^2 = c|\xi|^2$. This shows that the factor $\tau^2 - c|\xi|^2$ must split off in the characteristic polynomial defined by (1.3) and in fact, (1.3) can in this case equivalently be written as

$$
(\tau^2 - c|\xi|^2)[(\tau^2 + (b-c)|\xi|^2)^2 - b^2(\xi_1^4 + \xi_2^4 + \xi_3^4 - \xi_1^2\xi_2^2 - \xi_1^2\xi_3^2 - \xi_2^2\xi_3^2)] = 0. \tag{1.5}
$$

(See, e.g., [13].) Since the "geometry" of the sphere $1 - c|\xi|^2 = 0$ is trivial from our point of view, we may restrict attention in the sequel to the quartic surface defined by the second factor in (1.5) when $\tau = 1$, i.e., to $S = \{\xi \in \mathbb{R}^3;\ p(\xi) = 0\}$, where

$$p(\xi) = (1 + (b - c)|\xi|^2)^2 - b^2(\xi_1^4 + \xi_2^4 + \xi_3^4 - \xi_1^2\xi_2^2 - \xi_1^2\xi_3^2 - \xi_2^2\xi_3^2) = 0. \tag{1.6}$$

Note in particular, that S already contains all the essential singularities of the full slowness surface. In fact, it is known that the full slowness surface $\tilde{S}$ for a cubic crystal has 6 singular points of "uniplanar" type which lie on the coordinate axes and 8 singular points of conical type which lie on the lines defined by the condition $|\xi_1| = |\xi_2| = |\xi_3|$. (Cf. e.g., [3], [6].) However, the uniplanar singularities are very weak in the special case $a = -2b$, in that they are just due to the fact that the sphere $|\xi| = 1/\sqrt{c}$ is tangent to S at the points of the sphere which lie on the coordinate axes, S itself being smooth at those points. As for the conically singular points, they are $(\pm 1/\sqrt{3(c-b)}, \pm 1/\sqrt{3(c-b)}, \pm 1/\sqrt{3(c-b)})$, with all combinations of signs allowed.

When $a = -2b$, the condition $a + c > 0$ mentioned above is of course equivalent to $c > 2b$. Since, except for Section 9, we shall from now on always assume that $a = -2b$, there is no need for the constant "a" in the sequel, so we shall henceforth formulate all conditions in terms of b and c. (We also mention that despite of this convention, we shall every now and then speak about the case "$a = -2b$", in order to make it clear that our results refer to this case and not to general cubic crystals.) We could in principle simplify notations further by normalizing to $c = 1$. In fact, if we replace b by $b' = b/c$, and ξ by $\eta/\sqrt{c}$, then (1.4) transforms to essentially the same equation in (τ, η) space with the constants (c, b) replaced by $(1, b/c)$. (All this amounts to a re-normalization for the speed of elastic waves in the crystal.) We shall however work with a general c for most of the calculations, since then many relations are homogeneous in the variables (b, c) and calculations are easier to check.

Our first remark on the geometry of S is that it is a two-sheeted bounded surface with an inner and an outer sheet which we shall call S_o, respectively S_i. By this we mean that we can write S in the form $S = S_i \cup S_o$ where both surfaces are bounded and are such that S_i lies in the closure of the bounded component of $\mathbb{R}^3 \setminus S_o$. These surfaces are clearly symmetric under a permutation of the variables and under reflection with respect to the coordinate planes $\xi_i = 0$. The ξ_3-coordinates of the points on the positive ξ_3 semi-axis are $1/\sqrt{c - 2b}$ and $1/\sqrt{c}$. The two surfaces have exactly 8 points in common, which are the conically singular points mentioned above. It follows indeed from the fact that the system of crystal elasticity is hyperbolic that the full slowness surface $\tilde{S}$ is a three sheeted surface and S is obtained from the full slowness surface by removing the smooth sheet $\{\xi;\ |\xi|^2 = 1/c\}$. The inner sheet S_i is easily seen to be strictly convex: if we could find a line which intersected S_i in 3 or more points, then this line would intersect S in at least five points. (For this to be true when one or two of these points is a singular point, we have to take into account multiplicities: when our line passes through a singular point, we count the intersection "twice".) Since S

is a quartic this is impossible, so any line which intersects S_i has only two points of intersection with S_i.

The main results of the paper are

Theorem 1.1. *Assume $a = -2b$, $b > 0$. Then there are no non-trivial curves embedded in S with a common tangent plane.*

Theorem 1.2. *Still under the assumption $a = -2b$, there is a calculable constant $\delta > 0$ such that for $0 < b \leq \delta c$ the surface S_o has no points where the Gaussian and the mean curvature vanish simultaneously.*

REMARK 1.3. In the nearly isotropic case (and for "generic" constants (b, c)) similar results have already been established in [9]. However in that paper it was wrongly stated that "the mean curvature of the slowness surface S does not vanish in the nearly isotropic case". The correct statement there should be that "in the nearly isotropic case the mean curvature and the total curvature can not vanish simultaneously". Our aim in this paper is to remove the restriction to the nearly isotropic case in the special case when $a = -2b$ in the case of Theorem 1.1 and to make the restrictions on the size of what nearly isotropic means more precise in the case of Theorem 1.2. (The argument in [9] does not allow for this.)

The reason why the Theorems 1.1, 1.2, are interesting for estimates of integrals as in (1.1) is explained (e.g.) in Proposition 1.2 in [9]. On the way to proving them we shall also establish other results on the geometric nature of S and shall study S near the singular points. While we do so in this paper as preliminary results for the proof of Theorem 1.2 we should mention that the results referring to singular points are exactly in the form in which they are needed to establish decay estimates for the associated system of crystal elasticity.

REMARK 1.4. We have not made any serious attempt to calculate an optimal value for δ. The reason is that while most partial results are sharp, we hope that Theorem 1.2 itself can be improved by arguing in a different way.

REMARK 1.5. We recall here that Theorem 1.1 is in sharp contrast with what happens for Fresnel's surface in crystal optics for biaxial crystals: it was discovered by R.W. Hamilton that there are four circles embedded in Fresnel's surface such that the points on each circle admit a common tangent plane. (Cf. e.g., [2].) This is intimately related to "conical refraction" in crystal optics. For some comments on this and on the fact that there is still some kind of conical refraction in crystal acoustics for cubic crystals, see, e.g., [6].

As for Theorem 1.2, we restrict our attention to S_o, since S_i is strictly convex and it should be possible to prove much better results with other methods.

We shall show in Section 2 that for $b > 0$, S lies completely in the cube $\{\xi\,;\,|\xi_j| \leq 1/\sqrt{c-2b},\ j = 1, 2, 3\}$. Since the point $(0, 0, 1/\sqrt{c-2b})$ (and similar ones on the other axes) lies in S, this comes as no surprise, but a look at the only somewhat more general case of the slowness surface for general cubic crystals may convince us that some argument is needed to check the statement.

We also mention that when we keep c fixed and let b tend increasingly towards $c/2$, then the points on the outer sheet on the axes will tend to infinity whereas on the inner sheet the non-zero component will have constant value $\pm 1/\sqrt{c}$. Since the two sheets are glued together at the conically singular points, which remain in the bounded region $\sup_j |\xi_j| \leq 1/\sqrt{3c}$, we see that with $b \nearrow (c/2)$, the surface will look wilder and wilder.

There is also another information about the surfaces S which is quite easy to obtain. We assume for simplicity, here (but also often in the sequel when this leads to simplifications) that $b > 0$: it follows from (1.4) that

$$|\xi| \geq \frac{1}{\sqrt{c}} \quad \text{when} \quad \xi \in S. \tag{1.7}$$

Indeed, if $|\xi| < 1/\sqrt{c}$ then $|b\xi_j|^2/(1 - c|\xi|^2 + 3b\xi_j^2) < 1/3$, for every j, so (1.4) cannot hold.

Acknowledgments written by the first author. The present paper owes its existence to a discussion the two authors had in September 2003 in Kyoto. Both of us were then interested in the long-time behavior of global solutions to the system of crystal acoustics in the cubic case, the first author for general nearly isotropic crystals, the second for the case when $a = -2b$. I myself was in that discussion more interested in theorems of type Theorem 1.1, T. Sonobe in how the singularities of the slowness surface bear on loss of decay of solutions for large times. None of us had a strategy to prove theorems of type Theorem 1.1, but T. Sonobe, at the time a Ph.D. student at Osaka University, with professor M. Sugimoto as an advisor, expressed his strong belief, which for him was perhaps based on graphical evidence and preliminary calculations, that his insights were true and calculable. Soon after this discussion, T. Sonobe fell seriously ill and passed away in the Summer of 2004. At that time, I still had no idea of how to prove Theorem 1.1, neither in the nearly isotropic general case, nor in the case $a = -2b$. It was only much later that I understood what was going on for the nearly isotropic case and then also how the special case $a = -2b$ simplifies things enough to study crystals away from the nearly isotropic case if we assume $a = -2b$. Since there were no further contacts between T. Sonobe and me after September 2003, I am to blame alone for any shortcomings of the paper.

2. Preliminary remarks

1. For later purpose we now introduce the notations $\xi' = (\xi_1, \xi_2)$,

$$
\begin{aligned}
&d_0(b, c) = (b-c)^2 - b^2 = c(c-2b),\\
&d_2(\xi', b, c) = 2(1+(b-c)|\xi'|^2)(b-c) + b^2|\xi'|^2,\\
&d_4(\xi', b, c) = (1+(b-c)|\xi'|^2)^2 - b^2(\xi_1^4 + \xi_2^4 - \xi_1^2\xi_2^2),
\end{aligned}
\tag{2.1}
$$

and

$$
p(\xi, b, c) = d_0(b, c)\xi_3^4 + d_2(\xi' b, c)\xi_3^2 + d_4(\xi', b, c). \tag{2.2}
$$

In particular, S is then given by the condition $\{\xi\,;\ p(\xi, b, c) = 0\}$. The coefficient d_0 is strictly positive, since by assumption $c > 2b$, $c > 0$.

We now first study the curvature of S at the point $P_+ = (0, 0, 1/\sqrt{c-2b})$.

For this purpose we parameterize the outer sheet of S by ξ', i.e., we write this sheet locally near $(0, 0, 1/\sqrt{c-2b})$ as the graph of some function $\xi' \to g(\xi')$ which satisfies

$$
d_0(b, c)g^4(\xi') + d_2(\xi', b, c)g^2(\xi') + d_4(\xi', b, c) \equiv 0, \quad g(0) = \frac{1}{\sqrt{c-2b}}.
$$

(For notations, see (2.2).) Thus g also depends on b and c, but since we shall use the notation g only in the present argument, we shall not make this dependence explicit in the notation.

If we denote by $H_{\xi'\xi'}$ the Hessian in the variables ξ' we must then have (since it is clear by symmetry that $\nabla_{\xi'} g(0) = 0$) $4d_0(b,c)g^3(0)H_{\xi'\xi'}g(0) + 2d_2(0,b,c)g(0)H_{\xi'\xi'}g(0) + g^2(0)H_{\xi'\xi'}d_2(0, b, c) + H_{\xi'\xi'}d_4(0, b, c) = 0$. This gives

$$
H_{\xi'\xi'}g(0) = -\sqrt{c-2b}\,\frac{(c-2b)^{-1}H_{\xi'\xi'}d_2(0, b, c) + H_{\xi'\xi'}d_4(0, b, c)}{4d_0(b, c)/(c-2b) + d_2(0, b, c)}. \tag{2.3}
$$

Here $4d_0(b, c)/(c-2b) + d_2(0, b, c) = 4c + 2(b-c) = 2(c+b)$ is strictly positive, so the Hessian will have a sign if $(c-2b)^{-1}H_{\xi'\xi'}d_2(0, b, c) + H_{\xi'\xi'}d_4(0, b, c)$ has one. A trivial computation shows that

$$
\begin{aligned}
&\frac{1}{c-2b}H_{\xi'\xi'}d_2(0, b, c) + H_{\xi'\xi'}d_4(0, b, c)\\
&= \frac{2}{c-2b}\begin{pmatrix} 2(b-c)^2 + b^2 & 0 \\ 0 & 2(b-c)^2 + b^2 \end{pmatrix} + \begin{pmatrix} 4b-4c & 0 \\ 0 & 4b-4c \end{pmatrix}
\end{aligned}
\tag{2.4}
$$

and this is positive definite if $0 < 2b < c$. We see therefore that in the neighborhood of the poles the surface S has non-vanishing Gaussian curvature. Moreover, it is also clear from these calculations that the value of ξ_3 on S has locally a maximum at $\xi' = 0$. (We shall see later on that the maximum is global.)

2. In the engineering literature it is customary to study the "principal sections" of S, i.e., the curves which appear as intersections of S with the planes $\xi_j = 0,\ j = 1,2,3$. Such a study may be regarded as a substitute for a full geometric study of surfaces of type S, which may be out of reach.

In this section we shall obtain, somewhat more generally, some preliminary information on the sections $\Gamma_\mu = S \cap \{\xi \in \mathbb{R}^3;\ \xi_3 = \mu\}$ where μ is a real number, which we may assume nonnegative. With the notations

$$\begin{aligned} &a_0 = c(c-2b), \quad a_1 = 2(b-c)^2 + b^2, \quad a_2 = 2(1+(b-c)\mu^2)(b-c) + b^2\mu^2, \\ &a_3 = (1+(b-c)\mu^2)^2 - b^2\mu^4, \end{aligned} \tag{2.5}$$

(which are a variant of notations considered above) the point (ξ', μ) will lie on Γ_μ precisely when

$$Q_1(\xi_1, \xi_2) = a_0(\xi_1^4 + \xi_2^4) + a_1\xi_1^2\xi_2^2 + a_2(\xi_1^2 + \xi_2^2) + a_3 = 0. \tag{2.6}$$

Thus in particular, $a_0 > 0$ for $c > 2b$, and for $b > 0$ we have $a_3 \le 0$ precisely when $1/\sqrt{c} \le \mu \le 1/\sqrt{c-2b}$, whereas for $b < 0$ we shall have $a_3 \le 0$ precisely when $1/\sqrt{c-2b} \le \mu \le 1/\sqrt{c}$.

To gain a first insight into the shape of S_o and S_i, we start with some elementary calculations referring to the curves Γ_μ. Q_1 is a fourth order polynomial equation of a form which is (easily seen to be) precisely of the type which has been studied in section 6 in [7]. We explicitly want to insist on the fact that in what follows we are interested only in real points of the quartics which we consider.

As a preparation we mention the following trivial:

REMARK 2.1. Let

$$t \to \tilde{Q}(t) = b_0t^4 + b_1t^2 + b_2 \tag{2.7}$$

be a fourth order polynomial with real coefficients b_i. Assume that $b_0 > 0$. Then $\tilde{Q}(t) = 0$ admits exactly two real roots (one by necessity positive, the other negative) if and only if $b_2 \le 0$.

In the sequel of this section we shall assume (unless specified otherwise) that $b > 0$. Let us also consider the polynomials

$$\tilde{Q}(t) = a_0(\alpha^4t^4 + \beta^4t^4) + a_1\alpha^2\beta^2t^4 + a_2(\alpha^2+\beta^2)t^2 + a_3 = 0,$$

which are obtained by restricting the polynomial $Q(\xi')$ to the lines $\xi_1 = \alpha t,\ \xi_2 = \beta t$, $\alpha^2 + \beta^2 = 1$ passing through the origin. These polynomials are of the form in (2.7) with $b_0 = a_0(\alpha^4 + \beta^4) + a_1\alpha^2\beta^2$ and $b_2 = a_3$. In particular, $b_0 > 0$ for $2b < c$. (In fact, both a_0 and a_1 are then positive.) It follows that as long as $1/\sqrt{c} \le \mu \le 1/\sqrt{c-2b}$ we must have exactly two points of intersection of the line $t \to (\alpha t, \beta t)$ with $S_\mu = \{\xi'; (\xi', \mu) \in S\}$, whatever value of (α, β) we consider. It is then clear that Γ_μ is a simple curve which, when regarded as a curve in the ξ'-plane, surrounds the origin once and is symmetric with respect to the origin. Also when $|\mu| < 1/\sqrt{c}$, the lines considered a moment ago will intersect Γ_μ, but the number of intersection points must then be four. (By symmetry the number of points of intersection must be even, and it cannot be two, since two points of intersection occur only in the region $1/\sqrt{c} \le |\mu| \le 1/\sqrt{c-2b}$.) Γ_μ must therefore consist of two connected curves which surround the origin. The inner curve must be strictly convex, since it is the intersection of S_i with the plane $\xi_3 = \mu$. (Similar remarks hold when $b < 0$, but now of course the region with exactly two points of intersection is $1/\sqrt{c-2b} \le |\mu| \le 1/\sqrt{c}$.)

The only μ for which the two components have singularities are when $\mu = \pm 1/\sqrt{3(c-b)}$. In this case in fact the two curves have the points $(\pm 1/\sqrt{3(c-b)}, \pm 1/\sqrt{3(c-b)}, \mu)$ in common.

We can sum up the results which we have obtained so far in the following lemma:

Lemma 2.2. *Assume $b \ge 0$. $\xi \in S$ then implies $|\mu| \le 1/\sqrt{c-2b}$. In particular, $\Gamma_\mu \ne \emptyset$ precisely when $|\mu| \le 1/\sqrt{c-2b}$. When $1/\sqrt{c} < |\mu| < 1/\sqrt{c-2b}$ then Γ_μ consists of a simple smooth closed curve surrounding the origin. When $0 \le |\mu| < 1/\sqrt{c}$ and $|\mu| \ne 1/\sqrt{3c-3b}$ then Γ_μ consists of two connected components, both of which are simple smooth curves which surround the origin.*

A similar result is true by the same argument for the case $b < 0$, in that then Γ_μ shall have a single component when $1/\sqrt{c-2b} < |\mu| < 1/\sqrt{c}$ and two components for $|\mu| \le 1/\sqrt{c-2b}$.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is by elementary arguments. Since the inner sheet is strictly convex, it will suffice to show that there is no plane tangent to the outer sheet along entire curves. We shall argue by contradiction. We shall thus assume that there is a plane, which we shall denote by Σ, which is tangent to S along an entire curve and shall show that this leads to a contradiction. We recall here for completeness some arguments which are probably quite common when one studies curvature for low degree algebraic surfaces. (Also see [2] for a related argument.)

Let thus Σ be a plane in $\mathbb{R}^3$ and assume that Σ is tangent to S along some nontrivial curve $G \subset S$. We also denote $\Sigma \cap S$ by $\tilde{G}$. $\tilde{G}$ is a bounded plane quartic and we have $G \subset \tilde{G}$. (When we speak about G or $\tilde{G}$, we shall sometimes regard these curves as curves in $\mathbb{R}^3$ and sometimes as plane curves in the plane Σ.) The first thing to show is:

Proposition 3.1. *Under the above assumptions G is an ellipse and $G = \tilde{G}$.*

Proof. After an orthogonal change of coordinates and a translation we may assume that $\Sigma = \{\xi \in \mathbb{R}^3;\ \xi_3 = 0\}$. Denote by Q the polynomial p restricted to Σ in these new coordinates. We regard Q as a polynomial in the variables (ξ_1, ξ_2), i.e., we define Q by $Q(\xi_1,\xi_2) = p(\xi_1,\xi_2,0)$, keeping in mind that the notation of variables refers to the new coordinates. We claim that Q vanishes of order two on G. In fact, since $\{\xi\,;\ \xi_3 = 0\}$ is tangent to S along G, we must have that $(\partial/\partial\xi_i)p(P) = 0$, $i = 1, 2$, for every $P \in G$. This shows that the gradient of Q in the variables (ξ_1, ξ_2) must vanish on G. We conclude with the aid of Proposition 8.6 in [9] that Q must be reducible, i.e., that we can write Q as Q_1Q_2 for two nontrivial polynomials Q_1, Q_2. It is also clear that Q cannot have factors of degree one, since in the opposite case $\tilde{G}$ would be unbounded. Thus, both Q_1 and Q_2 are of degree two and both must vanish on G, since otherwise the gradient of Q could not vanish on the curve G. It follows that G is a parabola, hyperbola or ellipse, but since it is bounded it must be an ellipse. But then, G is the "complete" set of real zeros, both for Q_1 and for Q_2, and the two polynomials must be proportional since they vanish on the same ellipse. □

REMARK 3.2. Let G be the ellipse along which Σ is tangent to S. Also assume that coordinates have been chosen as above. In particular, Σ is of form $\Sigma = \{\xi \in \mathbb{R}^3;\ \xi_3 = 0\}$. Then for every point $P \in G$ there is a neighborhood $\tilde{W}$ in Σ so that the projection of S_o to Σ contains $\tilde{W}$. Moreover, if we choose $\tilde{W}$ small enough, then we can find $\delta > 0$ with the property that when $(\xi_1, \xi_2, 0) \in \tilde{W}$, then there is exactly one ξ_3 with $|\xi_3| < \delta$ such that $(\xi_1, \xi_2, \xi_3) \in S_o$. In fact, P must lie in the smooth part of S, so $\operatorname{grad} p(P) \neq 0$. Since Σ is tangent to S, it follows that we must have $(\partial/\partial\xi_3)p(P) \neq 0$. The statement follows therefore from the implicit function theorem. It follows from this that the orthogonal projection of S_o onto the plane Σ contains a neighborhood W of G. Moreover, if we choose W small enough, then we can find $\delta > 0$ so that for every $(\xi_1, \xi_2, 0) \in W$, there is exactly one ξ_3 with $|\xi_3| < \delta$ such that $(\xi_1, \xi_2, \xi_3) \in S_o$.

We consider next a point P in the smooth part of S and let Σ_o be the tangent plane to S at P. To simplify notations we shall make again an affine change of variables such that P in the new coordinates is $0 \in \mathbb{R}^3$ and the tangent plane is $\Sigma = \{\xi \in \mathbb{R}^3;\ \xi_3 = 0\}$. Locally near P we can write S_o as the graph of some smooth function $\xi' \to h(\xi')$, i.e., if Ω is a small neighborhood of the origin in $\mathbb{R}^3$ then $S \cap \Omega = \{\xi \in \Omega;\ \xi_3 = h(\xi')\}$. The fact that Σ is tangent to S at 0 means that $\nabla_{\xi'}h(0) = 0$. We also consider some line L in the plane Σ. After a further linear change of coordinates we may assume that L has the form $L = \{(\xi_1, 0, 0);\ \xi_1 \in \mathbb{R}\}$. The fact that Σ is tangent to S at P implies that the order of contact of any line L in the plane Σ which passes through P with S is at least two.

We also observe that if L intersects S in the points P^i, $i = 1, \ldots, s$, then the sum of the orders of contact at these points with S can be at most four. In fact, if we look at the intersection of S with the plane $\xi_2 = 0$, then we obtain a quartic T in the variables ξ_1, ξ_3. If we denote by $\tilde{q}$ the defining equation of this quartic, then a point $P^i = (\xi_{1,i}, 0) \in T$ which has contact of order k_i, will give a root of multiplicity k_i of $\xi_1 \to \tilde{q}(\xi_1, 0)$ at $\xi_{1,i}$. It remains then to note that since $\xi_1 \to \tilde{q}(\xi_1, 0)$ is a fourth order polynomial, we can at most have 4 roots when multiplicities are taken into account.

As a further preparation for the proof of Theorem 1.1 we prove one more "if"-result. (But a particularity of our argument is that in the end we shall see that in the situations of interest to us, no planes as in the statement of the following proposition can exist!)

Proposition 3.3. *If Σ is a plane which is tangent to S along a non-trivial curve Γ, then S must lie on one side of Σ.*

Proof. Since we already know that we may restrict attention to the outer sheet, it will suffice to show that if some plane Σ is tangent to the outer sheet of S along an entire curve G, then S must lie on one side of Σ.

We choose affine coordinates in the way done before such that $\Sigma = \{\xi\,;\, \xi_3 = 0\}$ and recall that under the assumptions of the proposition, $S \cap \{\xi\,;\, \xi_3 = 0\}$ is an ellipse. We also pick some open neighborhood W in the (ξ_1, ξ_2)-plane of this ellipse as in Remark 3.2 so that every point in W is the orthogonal projection onto Σ of a uniquely defined point (given in Remark 3.2), in S_o. Let further $U, V \subset \Sigma$ be the sets $U = \{\xi' \in W;\, \exists \xi_3 > 0 \text{ s.t. } (\xi', \xi_3) \in S_o\}$, $V = \{\xi' \in W;\, \exists \xi_3 < 0 \text{ s.t. } (\xi', \xi_3) \in S_o\}$, where in both cases the point ξ_3 is given by Remark 3.2. By the definition of W, we have then that $W = U \cup V \cup \Gamma$. If we assume that both sets U and V are non-void (and this is the exact meaning of the statement that S_o lies on both sides of Σ), then $\partial U \cap \partial V \cap W$ is a closed curve in W contained in the quartic $\{\xi' \in \mathbb{R}^2;\, p(\xi', 0) = 0\}$. (Here p is written in the new coordinates and, with notations introduced above, $p(\xi', 0) = Q(\xi')$.) We have seen above that as a curve the latter is the ellipse G. Since G consists of only one connected component, we must have $G = \partial U \cap \partial V \cap W$. This means that locally near any point of G we must have U on one side of $G \cap \Sigma$ and V on the other. ("Sides" are now taken inside Σ, whereas before they were "with respect" to Σ. The argument is here as follows. We write W as $W_+ \cup W_- \cup G$, where W_+ is the intersection of the unbounded component of $\Sigma \setminus G$ with W and W_- is the intersection of the bounded component of $\Sigma \setminus G$ with W. If now for example both $W_+ \cap U$ and $W_+ \cap V$ were non-void, then $\partial U \cap \partial V$ would have points in $W_+ \setminus G$, which is not true.) If we consider a line L which passes through some point $P \in G$ and is transversal to G, and if we consider a function h defined in a neighborhood Ω in $\mathbb{R}^2$ of P such that S is near P the graph of h, then h must change sign along L when we pass through P.

Now we change coordinates still further to have that $L = \{(\xi_1, 0, 0);\, \xi_1 \in \mathbb{R}\}$, $S \cap U = \{\xi \in U;\, \xi_3 = h(\xi')\}$, $h(\xi_1, 0) < 0$ for $\xi_1 < 0$, $|\xi_1|$ small, $h(\xi_1, 0) > 0$ for $\xi_1 > 0$,

ξ_1 small. We can also assume, by rotating L a little bit (if necessary) inside Σ around the projection of P to Σ, that L intersects G in a second point. We now claim that $(\partial^2 h/\partial\xi_1^2)(0) = 0$, thus proving that L has a contact of order 3 with S at P. In fact, we know already that $h(0) = (\partial h/\partial\xi_1)(0) = 0$, so a change of sign of $\xi_1 \to h(\xi_1, 0)$ is only possible when $(\partial^2 h/\partial\xi_1^2)(0) = 0$. This means that at 0 L has a contact of order at least 3 with S, which is not possible since we also have another point on $L \cap S$ with contact of order at least two. We conclude that one of the sets U or V must be void and therefore S lies on one side of Σ. □

We shall now use the exact form of our surface in a more direct way, but we shall still argue along traditional lines, by intersecting S with some specially chosen planes. For suitable choices of these planes we shall see that the fact that there are tangent planes to S along curves, imposes some strong restrictions. These restrictions will in the end be too strong to hold simultaneously.

The first family of planes with which we shall intersect is $\{\xi\,;\,\xi_3 = \mu\}$, $\mu \in \mathbb{R}$, thus obtaining the curves Γ_μ introduced above. We have seen that these curves are nontrivial precisely when $|\mu| < 1/\sqrt{c-2b}$.

We also observe that the equation (2.6) can be solved explicitly for ξ_1 or for ξ_2 and that it is symmetric in ξ_1, ξ_2. Elementary considerations, together with the results established in [7] give, under the assumption $b > 0$, the following statements:

the outer component is not necessarily convex. However, if it has inflection points, the number of inflection points is precisely 8, two in each quadrant of the plane. In any given quadrant, the two inflection points are symmetric with respect to the diagonal of the respective quadrant. We do not show how this statement can be obtained from the results in [7], since we shall prove a more precise statement in Section 4 below.

In particular, lines which are tangent to the outer curve in at least 2 distinct points and for which the curve remains on one side of the line are perpendicular to the diagonal in the first and third quadrant and perpendicular to the anti-diagonal in the second and fourth quadrant. (See Fig. 1.)

We now return to our main goal, which is the proof of Theorem 1.1. Let us then assume that Σ is tangent to S along an entire curve G. We assume that this curve has a nontrivial portion in the first octant in $\mathbb{R}^3$. If we intersect with $\xi_3 = \mu$ with $\mu > 0$ chosen such that $\Gamma_\mu \cap G$ is nontrivial, then $\Sigma \cap \{\xi_3 = \mu\}$ is a curve which is tangent to $S \cap \{\xi_3 = \mu\}$ in at least two points.

By the above, this is only possible if the line of tangency, when regarded in the ξ'-plane, is perpendicular to the diagonal and has exactly two points of tangency which are symmetric with respect to the diagonal. If we now move μ upwards, then we will have for a while at any moment two points of intersection, until a moment when we have only one point of intersection (remember that the curve of tangency is an ellipse, so the geometry of the situation is simple), which for symmetry reasons must be of form (ξ_1, ξ_2, μ) with $\xi_1 = \xi_2$. (In every other situation we will have a pair of points.) We conclude that G has a nontrivial intersection with the plane $\xi_1 = \xi_2$

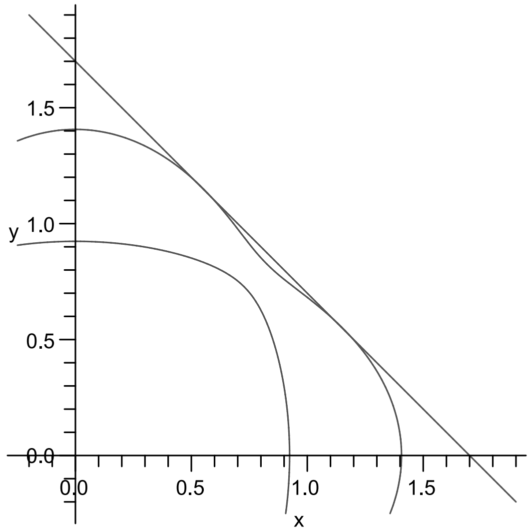

Fig. 1. Line is tangent to outer curve at two points. The two inflection points must lie between the two points of contact of the tangent with the outer curve.

and by symmetry then also with the planes $\xi_1 = \xi_3$, $\xi_2 = \xi_3$. If we now argue by symmetry then we can see that the normal to the tangent plane in question must be $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. (For additional details see [9].) On the other hand, the curve $S \cap \{\xi \in \mathbb{R}^3;\, \xi_1 = \xi_2\}$ is quite simple, since (1.6) reduces for $\xi_1 = \xi_2$ to

$$(1 + (b-c)(2\xi_1^2 + \xi_3^2))^2 - b^2(\xi_1^2 - \xi_3^2)^2 = 0. \tag{3.1}$$

This can also be written as

$$\begin{aligned} &[(1 + (b-c)(2\xi_1^2 + \xi_3^2)) - b(\xi_1^2 - \xi_3^2)] \\ &\times [(1 + (b-c)(2\xi_1^2 + \xi_3^2)) + b(\xi_1^2 - \xi_3^2)] = 0, \end{aligned} \tag{3.2}$$

which shows that the curve $\Lambda = S \cap \{\xi_1 = \xi_2\}$ is the union of the two ellipses

$$\begin{aligned} &\{(b-2c)\xi_1^2 + (2b-c)\xi_3^2 + 1 = 0,\ \xi_1 = \xi_2\}, \\ &\{(3b-2c)\xi_1^2 - c\xi_3^2 + 1 = 0,\ \xi_1 = \xi_2\}. \end{aligned} \tag{3.3}$$

We denote by Λ', respectively by Λ'', the image of these ellipses under the projection $(\xi_1, \xi_2, \xi_3) \to (\xi_1, \xi_3)$. Thus Λ is a curve in $\mathbb{R}^3$, whereas the Λ', Λ'' are ellipses in the (ξ_1, ξ_3)-plane. The two ellipses intersect in the points $\xi_1^2 = \xi_3^2 = 3^{-1}(c-b)^{-1}$. If we now denote by L the projection into the (ξ_1, ξ_3)-plane of the line $\Sigma \cap \{\xi\,;\, \xi_1 = \xi_2\}$, then L should be a common tangent to these two ellipses and should have normal direction

$v = (1, -1/2)/\|(1, -1/2)\|$. It is however not difficult to find the common tangent of two explicitly given ellipses and to show that for our two ellipses the common tangent cannot have the direction v, thus concluding the proof. (If the "common tangent" is to have the given direction, then it must be of form $\xi_1 = -(1/2)\xi_3 + n$ for some constant n. We need therefore show that there are no points $P_+ = (\tilde{\xi}_1, \tilde{\xi}_3)$, respectively $P_- = (\tilde{\eta}_1, \tilde{\eta}_3)$, on this line at which the line is tangent to Λ' at P_+ and tangent to Λ'' at P_-. The fact that P_+, P_- are on the line gives the conditions

$$\tilde{\xi}_1 = -\frac{1}{2}\tilde{\xi}_3 + n, \quad \tilde{\eta}_1 = -\frac{1}{2}\tilde{\eta}_3 + n, \tag{3.4}$$

whereas the fact that they are on the ellipses gives

$$(b - 2c)\tilde{\xi}^2 + (2b - c)\tilde{\xi}_3{}^2 + 1 = 0, \quad (3b - 2c)\tilde{\eta}_1{}^2 - c\tilde{\eta}_3{}^2 + 1 = 0. \tag{3.5}$$

In addition we have two more conditions, which say that the line is tangent to the ellipses at P_+, respectively P_-. We can express the latter by requiring for example that the gradient of the equation for the ellipses calculated at the corresponding points is proportional to the vector $(1, -1/2)$. This gives us a mildly nonlinear system of 6 equations for the 5 unknowns $n, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\eta}_1, \tilde{\eta}_2$: there are only five unknowns since we have already inserted the information that we know what the normal to the tangent line must be. It is not difficult to see, by explicitly studying the system that it is not compatible. For further details on how this is done, see [9], pp. 206–207.)

4. Inflection points of the curves Γ_μ

In this section we study the existence of inflection points on the curves Γ_μ, assuming $b > 0$. Part of the results of this section do not have the same level of interest for applications to decay estimates in crystal theory than have those in the other sections, but Proposition 4.6 (which requires roughly speaking half of the efforts we make in the section) is needed in the proof of Theorem 1.2, and we have also referred to this section in the proof of Theorem 1.1. Another possible merit of the calculations here is that we obtain a very explicit characterization of the shape of the curves Γ_μ, a problem which has been studied in the engineering literature some time ago. (Cf. e.g., [11], [12].) As a preliminary remark we recall that for $|\mu| < 1/\sqrt{c - 2b}$ the curves Γ_μ are nontrivial and consist of one or two closed connected components. More precisely, when $1/\sqrt{c} < |\mu| < 1/\sqrt{c - 2b}$ then we have one closed connected curve and when $0 \leq |\mu| < \sqrt{c}$, then we can write Γ_μ as $\Gamma_\mu = \Gamma_{\mu,i} \cup \Gamma_{\mu,o}$ where the $\Gamma_{\mu,i} \cup \Gamma_{\mu,o}$ are closed connected curves which we shall call the "inner", respectively the "outer" component. The terminology refers to the fact that if we regard Γ_μ as a curve in $\mathbb{R}^2$, by identifying (ξ', μ) with ξ', then $\Gamma_{\mu,i}$ and $\Gamma_{\mu,o}$ can be chosen so that $\Gamma_{\mu,i}$ lies completely in the bounded connected component of $\mathbb{R}^2 \setminus \Gamma_{\mu,o}$: see Fig. 2 for an example.

As seen above, $\Gamma_{\mu,i}$ must be strictly convex. For this reason our arguments will refer almost exclusively to $\Gamma_{\mu,o}$. (For simplicity we shall speak about the outer component

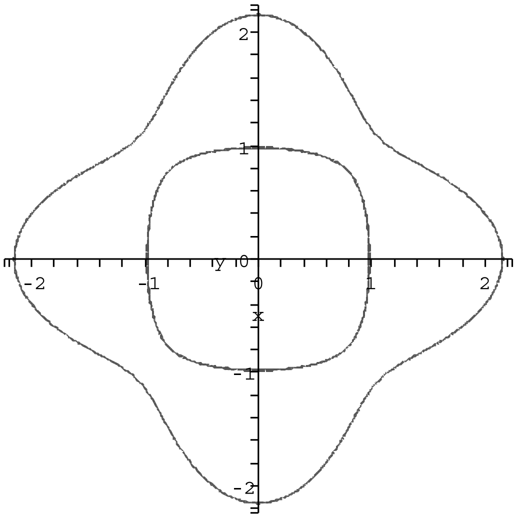

Fig. 2. Γ_μ consists of an inner and an outer component. It has a number of symmetries.

also in cases when Γ_μ consists of only one connected component. In this case the "outer" component is Γ_μ itself.) Inflection points are interesting in that they can not appear if the Gaussian curvature of S is strictly positive at the corresponding point on S. Since the Gaussian curvature of S is strictly positive at $(0, 0, 1/\sqrt{c-2b})$, the curves Γ_μ have no inflection points for μ close to $1/\sqrt{c-2b}$. Geometric considerations show that the Γ_μ will have inflection points when μ is near the conically singular values, which are $\pm 1/\sqrt{3(c-b)}$. It is also interesting to note that for μ close to zero, the situation will depend on the value of b: for small values of b the curves Γ_μ have no inflection points, but for moderately large values, they will (as we shall see later on in this section) have. We shall now make a quantitatively more precise analysis of these statements.

Theorem 4.1. *There is a calculable constant $\tilde{c}$ such that the plane sections Γ_μ can have inflection points for $\mu > 0$ only when $|\mu - 1/\sqrt{3c-3b}| \leq \tilde{c}b$.*

REMARK 4.2. The value of $\tilde{c}$ can be calculated noticing that we shall have inflection points on Γ_μ precisely when the sign of the quantity "D" introduced later on in this section is positive. For details when this is the case see Lemma 4.8 below.

Proof of Theorem 4.1 is based on the following theorem of H.G. Zeuthen:

Theorem 4.3 (1877. Cf. [14]. Also see [4].). *A nonsingular quartic in the real plane has at most eight real inflection points.*

This theorem has been used in a similar context already in [7] and actually the argument below shall be based on an idea used in that paper. We also mention that we could use that paper in a more direct way. However, calculations would remain tedious also if we did so, and we prefer a direct study of Γ_μ in order not to obscure the simple ideas which are underlying our results. Moreover, in order to obtain the results in Lemma 4.8 later on we would need a considerable part of the calculations we shall perform in exactly the form in which we shall do them and could not refer to any corresponding part in [7].)

REMARK 4.4. The curve Γ_μ is singular exactly when $\mu^2 = (3c - 3b)^{-1}$. Since we want to apply Zeuthen's theorem, we shall therefore exclude this case in the following considerations.

We start with some calculations related to the north-pole in the (ξ_1, ξ_2)-plane of Γ_μ. By this we mean the point on $\Gamma_{\mu,o}$ with $\xi_1 = 0$, $\xi_2 > 0$. (The value of ξ_2 will be calculated in a moment.)

We shall now parameterize $\Gamma_{\mu,o}$ locally near the pole by $\xi_1 \to (\xi_1, h(\xi_1))$, for some suitable function h. (Also see (4.1) below.)

As a preparation we consider the following trivial abstract result:

REMARK 4.5. a) Let $T(t)$ be a fourth degree polynomial with real coefficients of form

$$T(t) = \alpha_0 t^4 + \alpha_2 t^2 + \alpha_4,$$

with $\alpha_0 > 0$ such that $T(t) = 0$ has a positive root τ. Denote by $\tilde{\tau} = \max \tau_i$, where the maximum is over the real roots τ_i of T. (There are at least two real roots.) Then $2\alpha_0\tilde{\tau}^2 + \alpha_2 > 0$.

b) If the $\alpha_i$are functions of t and h is a $\mathcal{C}^2$-function defined in a neighborhood of 0 such that $h'(0) = 0$, then the coefficient of $h''(0)$ in $(d/dt)^2(\alpha_0 h^4(t) + \alpha_2(t)h^2(t) + \alpha_4)$ at $t = 0$ is $2h(0)(2\alpha_0(0)h^2(0) + \alpha_2(0))$.

In fact, denote by $\tilde{T}(\sigma)$ the polynomial $\alpha_0\sigma^2 + \alpha_2\sigma + \alpha_4$. If $\tau$is a real root of T, then $\sigma = \tau^2$ is a positive root of $\tilde{T}(\sigma)$. But then $2\alpha_0\tilde{\tau}^2 + \alpha_2 = 2\alpha_0\sigma + \alpha_2 = \sqrt{\Delta} > 0$ where $\Delta = \alpha_2^2 - 4\alpha_0\alpha_4 = \alpha_0^2(\sigma_1 - \sigma_2)^2$ is the discriminant of $\tilde{T}$ and σ_i are the two roots of $\tilde{T}$. (We have to take the positive square root of Δ, since $\tilde{\tau}^2$ is the bigger of the two roots of $\tilde{T}$.) Note that $\tilde{T}$ has two real roots, since its coefficients are real and it has one positive root by assumption. This gives a). b) is trivial.

The first result we want to prove is

Proposition 4.6. *$h''(0)$ is negative for every admissible value of (b, c, μ), i.e., when $b < c/2$, $\mu < 1/\sqrt{c - 2b}$.*

The proof of this result is by completely straightforward calculations. The reason why we give it with details is that calculations are somewhat tedious. The value of $h''(0)$ can be calculated by derivating the equation

$$(4.1)\qquad \begin{aligned}&(1+(b-c)(\xi_1^2+h^2(\xi_1)+\mu^2))^2\\ &-b^2(\xi_1^4+h^4(\xi_1)+\mu^4-\xi_1^2h^2(\xi_1)-\xi_1^2\mu^2-h^2(\xi_1)\mu^2)=0\end{aligned}$$

twice in ξ_1 and putting $\xi_1=0$ in the end. The calculation is simplified by the fact that we know beforehand (since h is even) that $h'(0)=0$ and that we need only the values at $\xi_1=0$. We then obtain (using (2.1), with the roles of ξ_2 and ξ_3 inverted) the equation

$$(4.2)\qquad Ah(0)h''(0)+B=0,$$

where

$$(4.3)\qquad \begin{aligned}A&=(-8bc+4c^2)h^2(0)+6b^2\mu^2-8\mu^2bc+4b-4c+4\mu^2c^2,\\ B&=4(1+(b-c)(h^2(0)+\mu^2))(b-c)+2b^2(h^2(0)+\mu^2).\end{aligned}$$

Here μ and $h(0)$ are related by the fact that we must have $(1+(b-c)(h(0)^2+\mu^2))^2-b^2(h^4(0)+\mu^4-h^2(0)\mu^2)=0$, which gives for $h^2(0)$, as a first guess, the values

$$(4.4)\qquad \frac{-2b+2c-(2c^2+3b^2-4bc)\mu^2\pm b\sqrt{4+(12c^2-24bc+9b^2)\mu^4+12(b-c)\mu^2}}{2(c^2-2bc)}.$$

Of course we are interested only in those triples $(b,c,\mu)$for which $h^2(0)$ is a positive number, which means in particular that the expression

$$L(b,c,\mu)=4+(12c^2-24bc+9b^2)\mu^4+12(b-c)\mu^2$$

must be non-negative for (b,c,μ) satisfying $\mu^2\le 1/(c-2b)$. This is clear in that we know from Section 2 that for $|\mu|\le 1/\sqrt{c-2b}$ the equation (4.1) admits for every $\xi_1=0$ a positive solution $h(0)$. We also mention for later use that

$$(4.5)\qquad L\left(b,1,\frac{1}{\sqrt{1-2b}}\right)=\frac{(b-2)^2}{(1-2b)^2},\quad L\left(b,1,\frac{1}{\sqrt{2-3b}}\right)=1.$$

We further observe that (4.4) gives us two values, one for the “plus”-sign and one for the “minus”-sign. However since the denominator $2(c^2-2bc)$ in (4.4) is positive for the relevant b,c, and since we restrict attention to $b\ge 0$, the bigger of the two values defined by (4.4), which is the one which corresponds to the north pole, is when the sign is “plus”. We shall work therefore from now on with this value of $h^2(0)$.

We now return to the proof of Proposition 4.6. Since $h(0)>0$, it is clear that the proposition follows from the following lemma.

Lemma 4.7. *The expression A defined in* (4.3) *is always positive, and so is B.*

Proof. The fact that A is always positive follows from Remark 4.5. To establish the sign of B, we have to evaluate $4(1+(b-c)(h^2(0)+\mu^2))(b-c)+2b^2(h^2(0)+\mu^2)$ when $h^2(0)$ is given by (4.4), the sign being again "+". We can also write B as $4(b-c)+(6b^2-8bc+4c^2)(h^2(0)+\mu^2)$. When we insert the value of $h^2(0)$ into this, we obtain

$$\begin{aligned}&(6b^2+4c^2-8bc)\\&\times\Bigg(\frac{1}{2(c^2-2bc)}\big(2c-2b+4bc\mu^2-2c^2\mu^2-3b^2\mu^2\\&\qquad+\sqrt{4b^2+12b^3\mu^2+9b^4\mu^4-12b^2c\mu^2-24b^3c\mu^4+12b^2c^2\mu^4}\big)+\mu^2\Bigg)\\&+4b-4c.\end{aligned}$$

We have to show that this quantity is positive. Our task will be (a little bit) simplified if we replace μ^2 by t and study the quantity

$$\begin{aligned}&N(b,c,t)\\&=(6b^2+4c^2-8bc)\\&\quad\times\Bigg(\frac{1}{2(c^2-2bc)}\big(2c-2b+4bct-2c^2t-3b^2t\\&\qquad+\sqrt{4b^2+12b^3t+9b^4t^2-12b^2ct-24b^3ct^2+12b^2c^2t^2}\big)+t\Bigg)\\&\quad+4b-4c\end{aligned}\tag{4.6}$$

instead. The idea is to fix b, c and study the values of the functions $t\to N(b,c,t)$ for the appropriate t. The situation becomes notationally simpler if we assume $c=1$. The value of N for $t=0$ is $N(b,1,0)=2b(2-b)/(1-2b)$, which is positive when $0\le b<1/2$. As for the case when $t=1/(1-2b)$, which is the biggest value t can have for some given b, we obtain, if we also take into account that by (4.5) $L(b,1,1/\sqrt{1-2b})=(2-b)^2/(1-2b)^2$, $N(b,1,(1-2b)^{-1})=(6b^2-8b+4)[(2-4b)^{-1}(2-2b+4b/(1-2b)-2/(1-2b)-3b^2/(1-2b)+b(-b+2)/(1-2b))+1/(1-2b)]+4b-4$. After some calculations this gives (again) that $N(b,1,(1-2b)^{-1})=2b(2-b)/(1-2b)$.

The proof will now come to an end if we show that the function $t\to N(b,1,t)$ is positive when $(d/dt)N(b,1,t)=0$. The zeros of the t-derivative of $N(b,1,t)$ are the same with the zeros of the t-derivative of the function $(2(1-2b))^{-1}[4bt-2t-3b^2t+(4b^2+12b^3t+9b^4t^2-12b^2t-24b^3t^2+12b^2t^2)^{1/2}]+t$ and are also equal to the zeros of t-derivative of the function

$$P(t)=-3b^2t+\sqrt{b^2(4+12bt+9b^2t^2-12t-24bt^2+12t^2)}.$$

Clearly $\partial_t P(t) = 0$ comes to

$$3b\sqrt{4 + 12bt + 9b^2t^2 - 12t - 24bt^2 + 12t^2} = 6b + 9b^2t - 6 - 24bt + 12t. \tag{4.7}$$

We divide both sides by 3 and square the result to obtain

$$\begin{aligned}&(16 - 64b + 9b^4 + 88b^2 - 48b^3)t^2 + (12b^3 - 44b^2 + 48b - 16)t + 4 - 8b + 4b^2\\ &= b^2(4 + 12bt + 9b^2t^2 - 12t - 24bt^2 + 12t^2).\end{aligned}$$

This is a second order polynomial equation which has the solutions $t_1 = 1/(2 - 3b)$, $t_2 = 1/(2 - b)$. Of these, only t_1 is however also a solution of the equation (4.7), and t_2 is due to the fact that we squared both sides of the equality there. We have thus found the point where $\partial_t N(b, 1, t)$ vanishes, namely at $t = 1/(2 - 3b)$. It remains to calculate the value of $N(b, 1, t)$ at $t = 1/(2 - 3b)$ and to check that it is positive. We obtain in fact that $N(b, 1, 1/(2 - 3b)) = 4b(1 - 2b)/((1 - 2b)(2 - 3b)) = 4b/(2 - 3b)$. (The value of $L(b, 1, 1/(2 - 3b))$ is 1 by (4.5).) □

As a next step in our geometric study of the curves Γ_μ we now study the curvatures at the points of Γ_μ lying on the diagonal, respectively on the anti-diagonal in $\mathbb{R}^2$. We shall work out details for the diagonal $\{\xi' \in \mathbb{R}^2;\ \xi_1 = \xi_2\}$. We have already used above that $S \cap \{\xi \in \mathbb{R}^3;\ \xi_1 = \xi_2\}$ is quite simple and also in the present calculations this fact can be brought (in some partial calculations) to fruition. (Recall that $|\mu| \neq 1/\sqrt{3c - 3b}$.) This seems easiest if we change variables setting $\xi_1 = s + t$, $\xi_2 = t - s$. (The diagonal corresponds then to $s = 0$.) The equation (2.6) transforms to

$$\begin{aligned}&(1 + (b - c)((s + t)^2 + (s - t)^2 + \mu^2))^2 - b^2((s + t)^4 + (s - t)^4\\ &+ \mu^4 - (s + t)^2(s - t)^2 - (s + t)^2\mu^2 - (s - t)^2\mu^2) = 0,\end{aligned} \tag{4.8}$$

which, when written in a more explicit way, is

$$\begin{aligned}&(3b^2 - 8bc + 4c^2)t^4\\ &+ (4b - 4c - 6b^2s^2 - 16bcs^2 + 8c^2s^2 + 6b^2\mu^2 - 8bc\mu^2 + 4c^2\mu^2)t^2\\ &+ (1 + (b - c)(2s^2 + \mu^2))^2 - b^2(s^4 - 2s^2\mu^2 + \mu^4) = 0.\end{aligned} \tag{4.9}$$

To study curvature, we now write Γ_μ locally as the graph of some function $s \to \chi(s)$, which means that we have to replace t by $\chi(s)$ in (4.9). The implicit equation for χ is therefore

$$\begin{aligned}&(3b^2 - 8bc + 4c^2)\chi(s)^4\\ &+ (4b - 4c - 6b^2s^2 - 16bcs^2 + 8c^2s^2 - 8bc\mu^2 + 6b^2\mu^2 + 4c^2\mu^2)\chi(s)^2\\ &+ (1 + (b - c)(2s^2 + \mu^2))^2 - b^2(s^4 - 2s^2\mu^2 + \mu^4) \equiv 0.\end{aligned} \tag{4.10}$$

Recall that here μ corresponds to ξ_3 and that $|\mu| < 1/\sqrt{c-2b}$.

The value of $\chi(0)$ is easy to calculate in terms of μ in that the equation (4.10) factors for $s = 0$, as a consequence of (3.2), into

$$(3b\chi^2(0) + 1 - c\mu^2 - 2c\chi^2(0))(b\chi^2(0) + 2b\mu^2 + 1 - c\mu^2 - 2c\chi^2(0)). \tag{4.11}$$

This gives

$$\chi^2(0) = \frac{1 - c\mu^2}{2c - 3b} \quad \text{or} \quad \chi^2(0) = \frac{1 + (2b - c)\mu^2}{2c - b}. \tag{4.12}$$

Since we are only interested in positive values for $\chi(0)$, this makes sense only when the expressions in (4.12) are positive for the μ under consideration. This is trivial for the second expression, but will only happen when $|\mu| \le 1/\sqrt{c}$ in the case of the first expression. (The geometric interpretation of these calculations is clear: since the inner sheet is symmetric and convex, only the hyperplanes $\xi_3 = \mu$ with $|\mu| \le 1/\sqrt{c}$ can have a nontrivial intersection with the interior sheet. Alternatively we may look at the ellipses defined in (4.11).) We are also interested to understand which of the two values (4.12) is bigger. Again we may look at the ellipses in (4.11) but we may also argue analytically. Since both $2c - b$ and $2c - 3b$ are positive by the assumptions on b, c, we have to evaluate the sign of $(1 - c\mu^2)(2c - b) - (1 + (2b - c)\mu^2)(2c - 3b)$. The sign of this is (since we assume $b \ge 0$) equal to the sign of $1 - (3c - 3b)\mu^2$, which shows that

$$\begin{aligned} &\frac{1 - c\mu^2}{2c - 3b} \ge \frac{1 + (2b - c)\mu^2}{2c - b} \quad \text{when} \quad |\mu| \le \frac{1}{\sqrt{3c - 3b}}, \\ &\frac{1 - c\mu^2}{2c - 3b} \le \frac{1 + (2b - c)\mu^2}{2c - b} \quad \text{when} \quad \frac{1}{\sqrt{3c - 3b}} \le |\mu| \le \frac{1}{\sqrt{c - 2b}}. \end{aligned} \tag{4.13}$$

We shall denote by $\chi_1(0)$ the smaller of the two values and by $\chi_2(0)$ the bigger one. Note that the point $(0, \chi_1(0))$ lies then on the inner curve and $(0, \chi_2(0))$ on the outer curve. Since the inner curve is strictly convex, we are again foremost interested in $\chi_2(0)$. According to the above,

$$\begin{aligned} &\chi_2^2(0) = \frac{1 - c\mu^2}{2c - 3b} \quad \text{when} \quad |\mu| \le \frac{1}{\sqrt{3c - 3b}}, \\ &\text{respectively} \quad \chi_2^2(0) = \frac{1 + (2b - c)\mu^2}{2c - b} \quad \text{when} \quad |\mu| \ge \frac{1}{\sqrt{3c - 3b}}. \end{aligned} \tag{4.14}$$

We can now calculate $\chi''(0)$ by derivating (4.10) twice in s and setting $s = 0$ afterwards. After some calculations, we obtain the equation

$$C\chi(0)\chi''(0) + D = 0, \tag{4.15}$$

where

$$C = 4(3b^2 - 8bc + 4c^2)\chi^2(0) + 2(4b - 4c + 6b^2\mu^2 - 8bc\mu^2 + 4c^2\mu^2),$$
$$D = 8(1 + (b - c)\mu^2)(b - c) + 4b^2\mu^2 + (8(2b - 2c)(b - c) - 28b^2)\chi^2(0).$$

Lemma 4.8. *When $\mu^2 = (3c - 3b)^{-1}$, then C vanishes. This corresponds to the fact that Γ_μ has singular points in that case. When $\mu^2 \neq (3c-3b)^{-1}$, then C is always positive. The sign of $\chi_2''(0)$ will therefore be (-1)-times the sign of D when $\chi(0) = \chi_2(0)$. The sign of $\chi_2''(0)$ is:*

I. *In the case $\mu^2 < 1/(3c - 3b)$:*
   - i) *negative for $b < 2c/9$, when in addition $\mu^2 < (2c - 9b)/(6c^2 - 21bc + 9b^2)$,*
   - ii) *positive for $b < 2c/9$, when actually $(2c - 9)/(6c^2 - 21bc + 9b^2) < \mu^2 < 1/(3c - 3b)$,*
   - iii) *positive for $2c/9 \le b \le c/3$,*
   - iv) *and positive also for $c/3 < b < c/2$.*

II. *In the case $\mu^2 > 1/(3c - 3b)$:*
   - v) *positive for $1/(3c - 3b) \le \mu^2 < (2c + 5b)/(6c^2 - 21bc + 9b^2)$,*
   - vi) *negative for $(2c + 5b)/(6c^2 - 21bc + 9b^2) \le \mu^2 < 1/(c - 2b)$.*

   We also mention as case

III. *When $\mu^2 = (2c - 9b)/(6c^2 - 21bc + 9b^2)$, then $\chi_2''(0) = 0$.*

Proof. The statement on C follows again from Remark 4.5. As for D, the problem is to study for a given b the sign of

$$8(1 + (b - c)\mu^2)(b - c) + 4b^2\mu^2 + (8(2b - 2c)(b - c) - 28b^2)\chi_2^2(0) \tag{4.16}$$

as a function of μ. The sign of this is the same with the sign of

$$\begin{aligned} &2(1 + (b - c)\mu^2)(b - c) + b^2\mu^2 + (4(b - c)^2 - 7b^2)\chi_2^2(0) \\ &= (-3b^2 - 8bc + 4c^2)\chi_2^2(0) + 2b - 2c + 3b^2\mu^2 - 4bc\mu^2 + 2c^2\mu^2. \end{aligned} \tag{4.17}$$

We now insert the value of $\chi_2(0)$ into (4.17) and obtain

$$\begin{aligned} &\frac{(-3b^2 - 8bc + 4c^2)(1 - c\mu^2)}{2c - 3b} + (3b^2 - 4bc + 2c^2)\mu^2 + 2b - 2c \\ &= \frac{-b(9b - 21\mu^2 bc - 2c + 6\mu^2c^2 + 9b^2\mu^2)}{2c - 3b}, \end{aligned} \tag{4.18}$$

in the first case in (4.14), respectively

$$\begin{aligned} &\frac{(-3b^2 - 8bc + 4c^2)(1 + 2b\mu^2 - c\mu^2)}{2c - b} + (3b^2 - 4bc + 2c^2)\mu^2 + 2b - 2c \\ &= \frac{-b(9b^2\mu^2 + 5b + 3bc\mu^2 + 2c - 6c^2\mu^2)}{2c - b}, \end{aligned} \tag{4.19}$$

in the second. The values of μ^2 for which these expressions vanish are when

$$E = 9b - 2c - 21bc\mu^2 + 6c^2\mu^2 + 9b^2\mu^2 = 9b - 2c + (6c^2 - 21bc + 9b^2)\mu^2 = 0$$

respectively when

$$F = 9b^2\mu^2 + 5b + 3bc\mu^2 + 2c - 6c^2\mu^2 = 2c + 5b + (9b^2 + 3bc - 6c^2)\mu^2 = 0,$$

which gives formally

$$\mu^2 = \frac{2c - 9b}{6c^2 - 21bc + 9b^2}, \quad \text{respectively} \quad \mu^2 = \frac{2c + 5b}{6c^2 - 3bc - 9b^2}.$$

Of course, and this is why we say "formally", for this to make sense, the quantities $(2c - 9b)/(6c^2 - 21bc + 9b^2)$ and $(2c + 5b)(6c^2 - 3bc - 9b^2)$ must be positive. We now introduce the following quantities:

$$E' = 9b - 2c + (6c^2 - 21bc + 9b^2)\nu, \quad F' = 2c + 5b + (9b^2 + 3bc - 6c^2)\nu, \quad \nu \in \mathbb{R},$$

which correspond to E and F when we replace μ^2 by ν. We then have:

REMARK 4.9. The quantities E', F' are linear in ν. They vanish when $\nu = (2c - 9b)/(6c^2 - 21bc + 9b^2)$, respectively $\nu = (2c + 5b)/(6c^2 - 3bc - 9b^2)$ and there is a change of sign at those values. (We assume here that the quantities $6c^2 - 21bc + 9b^2$, $6c^2 - 3bc - 9b^2$ do not vanish.) The sign of $\chi_2''(0)$ is that of $E'(\nu)$ for $\nu = \mu^2$), respectively $F'(\nu)$, again for $\nu = \mu^2$).

Note that $6c^2 - 21bc + 9b^2$ vanishes for $b = c/3$, respectively $b = 2c$, whereas $6c^2 - 3bc - 9b^2$ vanishes for $b = -c$ respectively $b = 2c/3$.

This means that in the region $0 \leq b < c/2$ (which is the region of interest for us)

$$6c^2 - 21bc + 9b^2 \quad \text{is positive precisely when} \quad b < \frac{c}{3} \tag{4.20}$$

whereas

$$9b^2 + 3bc - 6c^2 \quad \text{is always negative.} \tag{4.21}$$

Moreover (again when $0 \leq b < c/2$), the following statements hold:

$(2c - 9b)/(6c^2 - 21bc + 9b^2)$ is a positive number precisely for $b \in (-\infty, 2c/9) \cup (c/3, 2c)$,

$$\begin{aligned} &\frac{2c - 9b}{6c^2 - 21bc + 9b^2} \leq \frac{1}{3c - 3b} \quad \text{for} \quad b < c/3, \\ &\frac{2c - 9b}{6c^2 - 21bc + 9b^2} \geq \frac{1}{3c - 3b} \quad \text{for} \quad c/3 < b < c/2, \\ &\frac{1}{3c - 3b} \leq \frac{2c + 5b}{6c^2 - 3bc - 9b^2} \leq \frac{1}{c - 2b}. \end{aligned} \tag{4.22}$$

(Thus for example $(2c+5b)/(6c^2-3bc-9b^2) \leq 1/(c-2b)$ is always true since $(2c+5b)(c-2b)-(6c^2-3bc-9b^2) = -4c^2+4bc-b^2$ is always negative.) $\square$

We now discuss the sign of E, F, in a number of situations which correspond exactly to the regions mentioned in the statement of Lemma 4.8. (Note that the sign of E, F is -1 times the sign of D, so it is already the sign of $\chi_2''(0)$. We should keep in mind that ν corresponds to μ^2.) The "case" III in Lemma 4.8 is of course clear.

I) The case $\mu^2 < 1/(3c-3b)$. The relevant quantity is E.

a) $b < 2c/9$. In this case $6c^2-21bc+9b^2$, the coefficient of ν in E', is positive and E' is negative for $\nu = \mu^2 < \theta = (2c-9b)/(6c^2-21bc+9b^2)$, respectively positive when $\theta < \nu < 1/(3c-3b)$.

b) $2c/9 < b < c/3$. In this case $6c^2-21bc+9b^2$ is still positive, and E', changes sign from $-$ to $+$ at $\nu = (2c-9b)/(6c^2-21bc+9b^2)$, which is negative. E is therefore positive for all $0 \leq \mu^2 = \nu < 1/(3c-3b)$.

c) $c/3 < b < c/2$. In this case $6c^2-21bc+9b^2$ is negative, whereas $(2c-9b)/(6c^2-21bc+9b^2)$ is larger than $1/(3c-3b)$. Therefore E is positive for all $0 \leq \mu^2 = \nu < 1/(3c-3b)$.

II) The case $\mu^2 \geq 1/(3c-3b)$. We have to discuss the sign of F. Since $\theta' = (2c+5b)/(6c^2-3bc-9b^2) \geq 1/(3c-3b)$, and since $-6c^2+3bc+9b^2$ is always negative, F always has a change of sign from "$+$" to "$-$" at $\mu^2 = \theta'$.

We can now return to the proof of Theorem 4.1. In fact, we shall have inflection points precisely when $\chi_2''(0) \geq 0$. (Recall that $\chi_2''(0)$ refers to the coordinates (s, t), in which $(0, \chi_2(0))$ is a point on the diagonal in the initial variables ξ'.) This is a consequence of Zeuthen's theorem and can be proved with an argument used in a similar situation in section 6 in [7]. We repeat the argument for the convenience of the reader. In fact, by symmetry (under reflection with respect to $\xi_1 = 0$, $\xi_2 = 0$, $\xi_1 = \xi_2$, $\xi_1 = -\xi_2$) we see that if Γ_μ has some inflection point, then it also must have an inflection point in the region $\{\xi'; 0 < \xi_2 \leq \xi_1\}$ and that, more generally, the number of inflection points in each of the regions $\{\xi'; 0 < \pm\xi_2 \leq \pm\xi_1\}$, $\{\xi'; 0 < \pm\xi_1 \leq \pm\xi_2\}$, is the same. Since all in all there are 8 such regions in the plane, the number of inflection points in $\{\xi'; 0 < \xi_2 < \xi_1\}$ must be zero or one: we use here the fact that Zeuthen's theorem limits the total number of inflection points to 8. Now, there is certainly one inflection point in this region when $\chi_2''(0) > 0$, since the sign of $h''(0)$ is negative. (We recall that $(0, h(0))$ corresponded to the "north pole". On the other hand, there can be no inflection points in this region when $\chi_2''(0) < 0$, since in the opposite case, we would have at least two inflection points in the region, which is excluded by the remark just made. Also the case when $\chi_2''(0) = 0$ is easy: there is then an inflection point on $\xi_1 = \xi_2$ and by Zeuthen's theorem there can be no further inflection points in $0 < \xi_2 \leq \xi_1$.

REMARK 4.10. We shall say that Γ_μ is of "type I" if $\Gamma_{\mu,o}$ has no inflection points and that it is of "type II" in the opposite case. Furthermore, we say that μ^0

is a critical value if Γ_μ changes type at μ^0. It is a consequence of Lemma 4.8 that for $2c - 9b > 0$ the critical values are $\mu^2 = (2c - 9b)/(6c^2 - 21bc + 9b^2)$, respectively $\mu^2 = (2c + 5b)/(6c^2 - 21bc + 9b^2)$. At these values, $\chi_2''(0) = 0$ and $\chi_2''(0) \neq 0$nearby, with a change of sign at the critical values. For $2c - 9b < 0$ all μ with $\mu^2 < (2c - 9b)/(6c^2 - 21bc + 9b^2)$ are of type II.

Finally, if Γ_μ is of type II and if μ is not a critical value, then it has exactly one inflection point in each of the sectors $\{\xi' \in \mathbb{R}^2;\ 0 < \xi_2 < \xi_1\}$, $\{\xi' \in \mathbb{R}^2;\ 0 < \xi_1 < \xi_2\}$.

Proof of Theorem 4.1: end. It is a consequence of Lemma 4.8 that Γ_μ can have inflection points only when μ^2 lies in the interval with endpoints at the critical values. The conclusion of the theorem will follow therefore if we can check that the distance of these critical values to $1/(3c - 3b)$ is of order $\sim b$. We have indeed

$$\frac{2c - 9b}{6c^2 - 21bc + 9b^2} - \frac{1}{3c - 3b} = \frac{2}{3}\frac{b(-2c + 3b)}{(2c^2 - 7bc + 3b^2)(c - b)}, \tag{4.23}$$

respectively

$$\frac{2c + 5b}{6c^2 - 3bc - 9b^2} - \frac{1}{3c - 3b} = \frac{2}{3}\frac{b(2c - b)}{(2c^2 - bc - 3b^2)(c - b)}. \tag{4.24}$$

Both are of order "b", so the proof is complete.

5. The Hessian of $\rho_1 + \rho_2$ at the conically singular points

In this section we shall use the notation P for the conically singular point on S in the region $\mathbb{R}^3_+ = \{\xi;\ \xi_j \geq 0,\ j = 1, 2, 3\}$, i.e., $P = (P_1, P_2, P_3)$ with $P_i = 1/\sqrt{3c - 3b})$. We shall also write P' for (P_1, P_2) and denote, for the appropriate ξ', by $\xi' \to \rho_1(\xi')$, $\xi' \to \rho_2(\xi')$ the positive roots of $\rho \to p(\xi', \rho)$ ordered in such a way that $\rho_1(\xi') \leq \rho_2(\xi')$. Our aim is to calculate curvature properties of the surface $S_{mean} = \{\xi;\ \xi_3 = [\rho_1(\xi') + \rho_2(\xi')]/2,\ |\xi' - P'| < \delta\}$, δ small. S_{mean} is thus a surface which locally near P lies between S_o and S_i. There are two reasons why we are interested in this study, one geometrical and one technical. The geometric reason is that if we write ρ_j, $j = 1,2$, as $\rho_j(\xi') = [\rho_1(\xi') + \rho_2(\xi') + (-1)^j \sqrt{(\rho_2(\xi') - \rho_1(\xi'))^2}]/2$, then $[\sqrt{(\rho_2(\xi') - \rho_1(\xi'))^2}]/2$ describes (intuitively speaking) the conically singular behavior of S near P, whereas $\rho_1(\xi') + \rho_2(\xi')$ describes the underlying smooth structure at P. The curvature of S_{mean} thus shows us how much S is bent if we disregard the contribution of the conical singularity and we shall use the information we obtain in the proof of Theorem 1.2. While this is for the moment only an intuitive assessment, it is closely related to our technical interest in the curvature of S_{mean}. In fact, the main motivation for a geometric study of S is to understand the asymptotic behavior of Fourier transforms of densities which live on $\tilde{S}$ as described in (1.1). Locally near P we shall then have the two contributions

$$I_j(x) = \int_{|\xi'-P'|\le\delta} e^{i(x_1\xi_1+x_2\xi_2+x_3\rho_j(\xi'))} h'_j(\xi')\, d\xi', \quad j = 1, 2,$$

for some functions h'_j. After a linear change of variables and a translation, and denoting the new variables by $\eta = (\eta', \eta_3)$, we may assume that $P = 0$ and that $\nabla_{\eta'}(\rho_1 + \rho_2)(0) = 0$. The next step in the argument, at least if one is to follow the general line of argument outlined in [1], is then to choose smooth coordinates (denoted again by η') in a neighborhood of $0 \in \mathbb{R}^2$ in which $[(\rho_1 + \rho_2)(\eta')]/2 = -|\eta'|^2$. This is possible by the Morse lemma if we can show that the Hessian of $\rho_1 + \rho_2$ in the initial variables ξ' at P' is negative definite: it is this information we want to obtain in this section. We shall also insist on explicit calculations, since we want to be able, at least in principle, to determine thresholds for certain statements to hold. (For sufficiently small b these properties are obvious from the fact that the surface is then a small perturbation of a double sphere, the perturbation being made of course within the class of surfaces of type “S”.)

The calculation of the Hessian H of $\rho_1 + \rho_2$ at P' leads to expressions which are notationally complicated. We shall therefore prepare them by remaining as long as possible in a more abstract setting. In the beginning of the argument we shall consider three analytic functions $\xi' \to A(\xi')$, $\xi' \to B(\xi')$, $\xi \to q(\xi)$, defined in complex neighborhoods of some point $P' \in \mathbb{R}^2$, respectively $P = (P', P_3) \in \mathbb{R}^3$. We shall assume that P_3 is a double root of $\xi_3 \to \xi_3^2 + A(\xi')\xi_3 + B(\xi')$ at $\xi' = P'$ and denote $q(\xi)[\xi_3^2 + A(\xi')\xi_3 + B(\xi')]$ by f. When we return to our original situation, f shall be the defining function p of S and P the conically singular point considered above. We want to calculate the derivatives of order less or equal than two of A and B at P' in terms of derivatives of f. In fact, our main interest is for the Hessian of A, but it turns out that in order to calculate it, we also need the second derivatives of B. We can obtain all this information by derivating the equality $f(\xi) = q(\xi)(\xi_3^2 + A(\xi')\xi_3 + B(\xi'))$ and putting in the end $\xi' = P'$, $\xi_3 = P_3$. We obtain:

- $q(P) = 2^{-1}\partial_{\xi_3}^2 f(P)$ (by derivating twice in ξ_3),
- $\partial_{\xi_1} A(P') = \partial_{\xi_1}\partial_{\xi_3} f(P)/q(P)$ (by derivating once in ξ_1 and once in ξ_3),
- $\partial_{\xi_3} q(P) = 6^{-1}\partial_{\xi_3}^3 f(P)$ (by derivating three times in ξ_3),
- $\partial_{\xi_1} q(P) = 2^{-1}\partial_{\xi_3}^2\partial_{\xi_1} f(P) - \partial_{\xi_3} q(P)\partial_{\xi_1} A(P')$ for (by derivating twice in ξ_3 and once in ξ_1),
- $\partial_{\xi_i}\partial_{\xi_j}(P_3 A(P') + B(P')) = \partial_{\xi_i}\partial_{\xi_j} f(P)/q(P)$ for $i, j \neq 3$ (by derivating once in ξ_i and once in ξ_j),
- $(P_3\partial_{\xi_3} q(P) + q(P))\partial_{\xi_i}^2 A(P') + \partial_{\xi_3} q(P)\partial_{\xi_i}^2 B(P') = \partial_{\xi_3}\partial_{\xi_i}^2 f(P) - 2\partial_{\xi_i} q(P)\partial_{\xi_i} A(P')$ for $i \neq 3$ (by derivating twice in ξ_i and once in ξ_3),
- $(P_3\partial_{\xi_3} q(P) + q(P))\partial_{\xi_1}\partial_{\xi_2} A(P') + \partial_{\xi_3} q(P)\partial_{\xi_1}\partial_{\xi_2} B(P') = \partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3} f(P) - \partial_{\xi_1} q(P)\partial_{\xi_2} A(P') - \partial_{\xi_2} q(P)\partial_{\xi_1} A(P')$ (by derivating once in each of the variables ξ_1, ξ_2, ξ_3).

The quantities $\partial_{\xi_i}\partial_{\xi_j}A(P')$, $\partial_{\xi_i}\partial_{\xi_j}B(P')$, $i=j=1,2$, respectively $i,j=1,2$, therefore satisfy the systems:

$$\begin{aligned}
&(P_3\partial_{\xi_3}q(P)+q(P))\partial^2_{\xi_i}A(P')+\partial_{\xi_3}q(P)\partial^2_{\xi_i}B(P')\\
&=\partial_{\xi_3}\partial^2_{\xi_i}f(P)-2\partial_{\xi_i}q(P)\partial_{\xi_i}A(P'),
\end{aligned}$$

$$P_3\partial^2_{\xi_i}A(P')+\partial^2_{\xi_i}B(P')=\frac{\partial^2_{\xi_i}f(P)}{q(P)}, \tag{5.1}$$

$$\begin{aligned}
&(P_3\partial_{\xi_3}q(P)+q(P))\partial_{\xi_1}\partial_{\xi_2}A(P')+\partial_{\xi_3}q(P)\partial_{\xi_1}\partial_{\xi_2}B(P')\\
&=\partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3}f(P)-\partial_{\xi_1}q(P)\partial_{\xi_2}A(P')-\partial_{\xi_2}q(P)\partial_{\xi_1}A(P'),
\end{aligned}$$

$$P_3\partial_{\xi_1}\partial_{\xi_2}A(P')+\partial_{\xi_1}\partial_{\xi_2}B(P')=\frac{\partial_{\xi_1}\partial_{\xi_2}f(P)}{q(P)}. \tag{5.2}$$

We can extract from the last two sets of relations the following formulas for the second derivatives of A and B:

$$\begin{aligned}
\partial^2_{\xi_i}A(P')&=\frac{1}{q(P)}\left[\partial_{\xi_3}\partial^2_{\xi_i}f(P)-2\partial_{\xi_i}q(P)\partial_{\xi_i}A(P)-\frac{\partial_{\xi_3}q(P)\partial^2_{\xi_i}f(P)}{q(P)}\right],\\
\partial^2_{\xi_i}B(P')&=-\frac{1}{q(P)}\left[P_3(\partial_{\xi_3}\partial^2_{\xi_i}f(P)-2\partial_{\xi_i}q(P)\partial_{\xi_i}A(P))\right.\\
&\qquad\left.-\frac{(P_3\partial_{\xi_3}q(P)+q(P))\partial^2_{\xi_i}f(P)}{q(P)}\right],\\
\partial_{\xi_1}\partial_{\xi_2}A(P')&=\frac{1}{q(P)}\left[\partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3}f(P)-\partial_{\xi_1}q(P)\partial_{\xi_2}A(P')-\partial_{\xi_2}q(P)\partial_{\xi_1}A(P')\right.\\
&\qquad\left.-\frac{\partial_{\xi_3}q(P)\partial_{\xi_1}\partial_{\xi_2}f(P)}{q(P)}\right],\\
\partial_{\xi_1}\partial_{\xi_2}B(P')&=-\frac{1}{q(P)}\left[P_3(\partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3}f(P)-\partial_{\xi_1}q(P)\partial_{\xi_2}A(P')-\partial_{\xi_2}q(P)\partial_{\xi_1}A(P'))\right.\\
&\qquad\left.-\frac{(P_3\partial_{\xi_3}q(P)+q(P))\partial_{\xi_1}\partial_{\xi_2}f(P)}{q(P)}\right].
\end{aligned}$$

(The determinant of the two systems which determine the values of $\partial_{\xi_i}\partial_{\xi_j}(A,B)$ is in both cases $q(P)$.)

We now apply this for the case when f is the defining function of S and when P is the conically singular point of S recalled at the beginning of the section. If S is given near P by the graph of the two root functions $\xi'\to\rho_1(\xi')$, $\xi'\to\rho_2(\xi')$, then we have

$$p(\xi)=q(\xi)(\xi_3-\rho_1(\xi'))(\xi_3-\rho_2(\xi'))=q(\xi)(\xi_3^2+A(\xi')\xi_3+B(\xi')), \tag{5.3}$$

where q, $A=-\rho_1-\rho_2$, $B=\rho_1\rho_2$, are analytic functions in their respective variables, defined near P, respectively near $P'=(1/\sqrt{3c-3b},1/\sqrt{3c-3b})$. Since $\rho_1(P')=$

$\rho_2(P') = P_3 = 1/\sqrt{3c-3b}$, we have that

$$A(P') = -\frac{2}{\sqrt{3c-3b}}, \quad B(P') = \frac{1}{3c-3b}. \tag{5.4}$$

We next give the expressions of the derivatives of p needed in the above calculations for the case at hand:

$$\begin{aligned}
\frac{\partial^2 p(\xi)}{\partial \xi_3^2} &= 8(b-c)^2\xi_3^2 + 4(1+(b-c)(\xi_1^2+\xi_2^2+\xi_3^2))(b-c) \\
&\quad - b^2(12\xi_3^2 - 2\xi_1^2 - 2\xi_2^2), \\
\frac{\partial^2 p(\xi)}{\partial \xi_1 \partial \xi_2} &= 8(b-c)^2\xi_1\xi_2 + 4b^2\xi_1\xi_2, \\
\frac{\partial^3 p(\xi)}{\partial \xi_3^3} &= 24(b-c)^2\xi_3 - 24b^2\xi_3, \\
\frac{\partial^3 p(\xi)}{\partial \xi_1 \partial \xi_3^2} &= 8(b-c)^2\xi_1 + 4b^2\xi_1, \\
\frac{\partial^3 p(P)}{\partial \xi_1 \partial \xi_2 \partial \xi_3} &= 0.
\end{aligned}$$

This leads to a complete knowledge of second and third order derivatives of p at P if we also use some obvious symmetry relations:

$$\begin{aligned}
\frac{\partial^2 p(P)}{\partial \xi_i^2} &= \frac{8(b-c)^2}{3c-3b} + 4\left(1 + \frac{3(b-c)}{3c-3b}\right)(b-c) - \frac{8b^2}{3c-3b} \\
&= \frac{8c(c-2b)}{3(c-b)}, \quad i = 1, 2, 3, \\
\frac{\partial^2 p(P)}{\partial \xi_i \partial \xi_j} &= \frac{4(2c^2-4bc+3b^2)}{3(c-b)}, \quad \text{if} \quad i \neq j, \\
\frac{\partial^3 p(P)}{\partial \xi_i \partial \xi_j^2} &= \frac{4(2c^2-4bc+3b^2)}{\sqrt{3c-3b}}, \quad \text{if} \quad i \neq j, \\
\frac{\partial^3 p(P)}{\partial \xi_3^3} &= \frac{24c(c-2b)}{\sqrt{3c-3b}}.
\end{aligned}$$

We now obtain the following expressions for the values of the derivatives of q and A, respectively B:

$$\begin{aligned}
&q(P) = \frac{4c(c-2b)}{3(c-b)}, \quad \partial_{\xi_3} q(P) = \frac{4c(c-2b)}{\sqrt{3c-3b}}, \\
&\partial_{\xi_1} A(P') = \partial_{\xi_2} A(P') = \frac{4(2c^2-4bc+3b^2)}{3(c-b)} \bigg/ \frac{4c(c-2b)}{3(c-b)} = \frac{2c^2-4bc+3b^2}{c(c-2b)},
\end{aligned}$$

$$\begin{aligned}
\partial_{\xi_1} q(P) &= \frac{2(2c^2-4bc+3b^2)}{\sqrt{3c-3b}} - \frac{4c(c-2b)}{\sqrt{3c-3b}}\frac{2c^2-4bc+3b^2}{c(c-2b)} \\
&= -\frac{2(2c^2-4bc+3b^2)}{\sqrt{3c-3b}}, \\
\partial^2_{\xi_1} A(P') = \partial^2_{\xi_2} A(P') &= \frac{-3(-30b^3c+31b^2c^2+9b^4-16bc^3+4c^4)(b-c)}{c^2\sqrt{3c-3b}(2b-c)^2} \\
&= \frac{-3(-2c+b)(-2c+3b)(3b^2-2bc+c^2)(b-c)}{c^2\sqrt{3c-3b}(2b-c)^2}, \\
\partial_{\xi_1}\partial_{\xi_2} A(P') &= \frac{-3(3b^2-2bc+c^2)(3b^2-4bc+2c^2)(b-c)}{c^2\sqrt{3c-3b}(2b-c)^2}, \\
\partial^2_{\xi_1} B(P') = \partial^2_{\xi_2} B(P') &= -\frac{9b^4-30b^3c+23b^2c^2-8bc^3+2c^4}{c^2(2b-c)^2}, \\
\partial_{\xi_1}\partial_{\xi_2} B(P') &= -\frac{3b^2(3b^2-4bc+2c^2)}{c^2(2b-c)^2}. \qquad (5.5)
\end{aligned}$$

Proposition 5.1. *The Hessian H of $\rho_1+\rho_2$ at P is negative definite for $b<c/2$. Moreover, if we fix $0<c'<1/2$, then we can find $c''>0$, which in principle can be effectively calculated in terms of c', so that $H \le -c''I$ when $b \le c'c$.*

In fact, the Hessian of $\rho_1+\rho_2$ is proportional to the Hessian of $-A$. Here $\partial^2_{\xi_1}A(P')$ is clearly positive and the determinant of the Hessian of A is $d[(2c-b)^2(2c-3b)^2 - (2c^2-4bc+3b^2)^2] = 12d(c-2b)(c-b)^2c$, where d is a positive constant. This is trivially positive when $b<c/2$. The quantitative estimate follows looking at the expressions for the second derivatives of A.

REMARK 5.2. It is an important feature of the Hessian that for small b it is uniformly in b negative definite. This corresponds to the fact that the graph of $(\rho_1+\rho_2)/2$ is a smooth perturbation of a piece of a sphere.

6. The Hessian of the discriminant of the second order polynomial $\tau \to d_0\tau^2 + d_2\tau + d_4$ at P

With notations introduced in (2.1), the expression of the discriminant Δ of the polynomial in the title of the section at the conically singular point P is

$$\begin{aligned}
\Delta = (d_2)^2 - 4d_0d_4 = {}&[(3b^2+2c^2-4bc)|\xi'|^2+2b-2c]^2 \\
&- 4c(c-2b)[(1+(b-c)|\xi'|^2)^2 - b^2(\xi_1^4+\xi_2^4-\xi_1^2\xi_2^2)].
\end{aligned} \qquad (6.1)$$

If ρ_1,ρ_2, are the positive roots of $p(\xi',\rho)$, then $\tau_1=\rho_1^2$, $\tau_2=\rho_2^2$, are the roots of $\tau \to d_0\tau^2+d_2\tau+d_4$. In particular, $\tau_1-\tau_2=(\rho_1-\rho_2)(\rho_1+\rho_2)$. We are mostly interested in the Taylor expansion to second degree terms of Δ at $P' = (1/\sqrt{3c-3b}, 1/\sqrt{3c-3b})$ with respect to the variables (ξ_1, ξ_2). To simplify calculations we shift the origin to $(1/\sqrt{3c-3b}, 1/\sqrt{3c-3b})$, which can be done by performing the change of variables

$$s = \xi_1 - 1/\sqrt{3c-3b}, \quad t = \xi_2 - 1/\sqrt{3c-3b}. \tag{6.2}$$

Since $|\xi_j| \leq 1/\sqrt{c-2b}$ we shall have $|s| + |t| \leq 1/\sqrt{c-2b} + 1/\sqrt{3c-3b}$.

After some calculations we now have

(6.3)
$$\begin{aligned}\Delta = \big[&-42b^2cs^2t^2 + 36bc^2s^2t^2 - 12b^2s^2t + 32bc\sqrt{3c-3b}s^3 - 12b^2\sqrt{3c-3b}st^2\\ &+ 32bc\sqrt{3c-3b}t^3 - 8c^2\sqrt{3c-3b}s^2t - 8c^2\sqrt{3c-3b}st^2 + 18b^3s^2t^2 - 33b^2cs^4\\ &- 33b^2ct^4 + 36bc^2t^4 + 32bcs^2 + 32bct^2 - 24b^2st - 16c^2st - 12c^3s^4 - 12c^3t^4\\ &+ 16b\sqrt{3c-3b}cst^2 - 16c^2t^3\sqrt{3c-3b} - 16c^2s^3\sqrt{3c-3b} + 9b^3t^4 + 9b^3s^4\\ &- 12b^2s^3\sqrt{3c-3b} - 12b^2t^3\sqrt{3c-3b} + 16bc\sqrt{3c-3b}s^2t + 36bc^2s^4\\ &- 12b^2s^2 - 12b^2t^2 - 16c^2s^2 - 16c^2t^2 + 32bsct - 12c^3s^2t^2\big]\frac{b^2}{b-c}.\end{aligned}$$

We see in particular that there are no terms of degree 0 or 1 in (s, t). In fact, we obtain that Δ is equal to

$$\frac{b^2(16c^2 - 32cb + 12b^2)}{c-b}(s^2 + t^2) + \frac{b^2(16c^2 - 32cb + 24b^2)}{c-b}st + \frac{b^2 O(|(s,t)|^3)}{c-b},$$

for $|(s,t)| \to 0$, i.e., we calculate modulo terms of order 3. (The terms, summarized in "$O(|(s,t)|^3)$", but explicitly known from (6.3), also depend on b.) The Hessian $\tilde{H}$ of the discriminant is thus

$$\tilde{H} = \frac{b^2}{c-b}\begin{pmatrix} 2(16c^2 - 32cb + 12b^2) & 16c^2 - 32cb + 24b^2 \\ 16c^2 - 32cb + 24b^2 & 2(16c^2 - 32cb + 12b^2) \end{pmatrix}. \tag{6.4}$$

This is positively definite in the admissible region $b > 0$, $b < c/2$ since both $16c^2 - 32cb + 12b^2$ and the determinant of $\tilde{H}$ are positive there. (The determinant of $\tilde{H}$ is $2^6(b^4/(c-b)^2)(12c^4 - 48bc^3 + 60c^2b^2 - 24cb^3)$. It vanishes for $c/b \in \{0, 1, 2\}$.)

With the notation $(s,t)^\perp = \begin{pmatrix} s \\ t \end{pmatrix}$ we have proved:

Proposition 6.1. *For every $c' < 1/2$ there is an explicitly computable constant c'' such that if $b < c'c$, then we have that $|\Delta - (1/2)\langle \tilde{H} \cdot (s,t)^\perp, (s,t)^\perp \rangle| \leq c''b^2|(s,t)|^3$ for (s,t) in some previously fixed bounded region. ($\tilde{H} \cdot (s,t)^\perp$ is the matrix $\tilde{H}$ multiplied with the vector $(s,t)^\perp$.)*

The exact structure of $\Delta - (1/2)\langle \tilde{H} \cdot (s, t)^{\perp}, (s, t)^{\perp}\rangle$ is interesting if we want to obtain quantitatively sharp estimates. The following result can be read off from the explicit expression of Δ:

REMARK 6.2. $T(s,t) = ((c-b)/b^2)(\Delta - (1/2)\langle \tilde{H} \cdot (s,t)^{\perp}, (s,t)^{\perp}\rangle)$ has the following form:

$$\sqrt{3c-3b}[(-12b^2 + 32bc - 16c^2)(s^3 + t^3) + (16bc - 12b^2 - 8c^2)(s^2 t + st^2)]$$
$$+ (9b^3 + 36bc^2 - 33cb^2 - 12c^3)(s^4 + t^4) + (18b^3 - 42b^2 c + 36bc^2 - 12c^3)s^2 t^2.$$

This gives for $c = 1$, $0 \le b < 1/2$, the following (somewhat rough) estimate:

$$|T| \le 28\sqrt{3}(|s|^3 + s^2|t| + |s|t^2 + |t^3|) + 60(s^4 + t^4 + s^2 t^2). \tag{6.5}$$

We shall from now on not any more carry terms of order 4 and explicit numerical constants with us. They could be interesting if one wants to obtain sharp thresholds, but we do not try to find these thresholds explicitly.

REMARK 6.3. It is an important feature here that $\tilde{H}$ and the remainder term in the right hand side of the preceding relation are uniformly small of order b^2.

7. Estimates near P'

The results in the preceding two sections were mostly about the Hessians of $\rho_1 + \rho_2$ and of Δ, defined in section 6 at P', with (P', P_3) the conically singular point of S in $\mathbb{R}^3_+$. We shall now look for similar information for points in a full, perhaps small, neighborhood of P'. Again we are interested in quantitative expressions, but we shall not look for explicit sharp estimates. (In principle, such estimates can be obtained, but lead to complicated formulas.) It also seems justified to normalize acoustical constants in such a way that $c = 1$.

The estimates themselves will depend on the size of ξ. We know already that on S $|\xi_j| \le 1/\sqrt{c-2b}$, which for $c = 1$ comes to $|\xi_j| \le 1/\sqrt{1-2b}$. To keep $1/\sqrt{1-2b}$ bounded by some constant we shall often assume that $b \le c'$ for some constant $c' < 1/2$. (A reasonable choice could be to work from the very beginning for $b \le 1/3$.)

Our first remark is that we can estimate derivatives of any order of $\rho_1 + \rho_2$ starting from $\rho_1 + \rho_2 = (\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2)^{1/2}$ and noticing that with notations introduced in (2.1), $\rho_1^2 + \rho_2^2 = -d_2/d_0$ and $\rho_1^2\rho_2^2 = d_4/d_0$. In particular, $\rho_1^2 + \rho_2^2$ and $\rho_1^2\rho_2^2$ are polynomials in ξ' with explicitly calculable coefficients. It follows that there is a (calculable) constant χ such that these coefficients can be estimated by $1/(1-2b)$. From this we can now estimate derivatives of $\rho_1 + \rho_2$ of any previously fixed order, provided we remain in a region where $\rho_1 > \delta$, $\rho_2 > \delta$ for some suitably fixed $\delta > 0$. Since we know from (1.7) that $|(\xi', \rho_i(\xi'))| > 1/\sqrt{c} = 1$, this will be the case if we remain in a fixed open convex cone Γ containing the singular point in $\mathbb{R}^3_+$ which intersects the plane $\xi_3 = 0$ only at 0. Clearly, δ can be calculated in terms of Γ.

REMARK 7.1. We assume that a closed convex cone Γ containing the conically singular point in $\mathbb{R}^3_+$ with $\Gamma \cap \{\xi\,;\,\xi_3 = 0\} = \{0\}$ has been fixed. Then there is a constant κ such that for $c = 1$

$$\sum_{|\alpha|\le 5} |\partial^\alpha_{\xi'}(\rho_1(\xi') + \rho_2(\xi'))| \le \frac{\kappa}{\sqrt{1-2b}} \quad \text{whenever} \quad (\xi', \rho_j(\xi')) \in \Gamma, \quad j = 1, 2. \tag{7.1}$$

In view of this remark, we can now estimate second derivatives of $(\rho_1 + \rho_2)(\xi')$ using Taylor's formula at P' and information about remainder terms. This is particularly easy on the lines $L(\alpha, \beta)$ which pass through P' defined for some given direction (α, β), $\alpha^2 + \beta^2 = 1$, by

$$\tau \to \left(\alpha\tau + \frac{1}{\sqrt{3-3b}}, \beta\tau + \frac{1}{\sqrt{3-3b}}\right). \tag{7.2}$$

In fact we denote by $M(\alpha, \beta, \tau)$ the function

$$M(\alpha, \beta, \tau) = (\rho_1 + \rho_2)\left(\alpha\tau + \frac{1}{\sqrt{3-3b}}, \beta\tau + \frac{1}{\sqrt{3-3b}}\right), \tag{7.3}$$

and notice that (with $\tau \in \mathbb{R}$ and H again the Hessian of $\rho_1 + \rho_2$ at P')

$$\frac{d^2}{d\tau^2}\langle H \cdot (\alpha\tau, \beta\tau)^\perp, (\alpha\tau, \beta\tau)^\perp\rangle = 2\langle H \cdot (\alpha, \beta)^\perp, (\alpha, \beta)^\perp\rangle.$$

Then we have, derivating Taylor's formula up to terms of degree two for M at $\tau = 0$ twice, and using (7.1),

$$\forall\alpha, \forall\beta \quad \text{with} \quad \alpha^2 + \beta^2 = 1, \quad \left|\frac{d^2}{d\tau^2}M(\alpha, \beta, \tau) - \langle H \cdot (\alpha, \beta)^\perp, (\alpha, \beta)^\perp\rangle\right| \le c_1|\tau|, \tag{7.4}$$

for some $c_1 > 0$.

We have now studied $\rho_1 + \rho_2$ and it remains to say a few words about the function $\rho_2 - \rho_1$. We have of course $\rho_2 - \rho_1 = \sqrt{\Delta}/(\rho_1 + \rho_2)d_0$, with "$\sqrt{\Delta}$" the positive square root of Δ. (Recall that $\Delta = d_0^2(\rho_1^2 - \rho_2^2)^2$.) In this notation we have mixed the coordinates ξ' with the coordinates (s, t): the ρ_j are functions of ξ', whereas Δ is in Section 6 a function of (s, t). However, ξ' is just a translation of (s, t), so we can argue for a moment using both coordinate systems simultaneously. Since Δ vanishes to order two at P' (in the coordinates (s, t), P' corresponds to the point $(0, 0)$), whereas $\rho_1 + \rho_2$ does not vanish there, $\Delta/(\rho_1 + \rho_2)^2$ is a $\mathcal{C}^\infty$ function which vanishes of order two at P'. Moreover, its Hessian at P' is the one of Δ divided by the number

$(\rho_1(P') + \rho_2(P'))^2 = 4/(3-3b)$. (When we calculate the Hessian, we work of course in one fixed sets of coordinates.)

We observe next that we have in view of Section 6

Proposition 7.2. *Fix $\tilde{c} < 1/2$. There is $d > 0$ and for every $b \le \tilde{c}$ a positive definite 2×2 matrix $\tilde{H}$ such that with (s,t) denoting $(\xi_1 - 1/\sqrt{3-3b}, \xi_2 - 1/\sqrt{3-3b})$, $(\rho_2 - \rho_1)(\xi') = b\sqrt{\langle \tilde{H}\cdot (s,t)^{\perp}, (s,t)^{\perp}\rangle + O(|(s,t)|^3)}$ if $|(s,t)| \le d$.*

The main idea in the following calculations is now that when restricted to the lines $L(\alpha, \beta)$ (defined in (7.2)), the positive square root $\sqrt{\Delta}$ of Δ is $\mathcal{C}^\infty$ smooth up to the singular point. (Here we use that when we consider a positive $\mathcal{C}^\infty$ function h defined on $(-\delta, \delta)$ which satisfies $h(0) = 0$, $h(t) > 0$ for $t \ne 0$, then $\sqrt{h}$ is $\mathcal{C}^\infty$ on $[0, \delta)$.)

Corollary 7.3. *Fix $\tilde{c} < 1/2$. There are calculable constants $c_i > 0$, $i = 1, 2, 3$, such that if $0 \le b \le \tilde{c}$ and if α, β are real numbers chosen with $\alpha^2 + \beta^2 = 1$, then the function $F(\alpha, \beta, \tau) = (\rho_2 - \rho_1)^2(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b})$ is of form $F(\alpha,\beta,\tau) = b^2 d(\alpha,\beta)\tau^2 + b^2 T(\alpha,\beta,\tau)$ for some function T which is $\mathcal{C}^\infty$ in τ for $\tau \in [0, c_1)$ and vanishes of order 3 at $\tau = 0$. Here $d(\alpha,\beta)$ is for fixed α,β a constant $\ge c_2$ and $|T(\alpha,\beta,\tau)| \le c_3|\tau|^3$. It follows that $(\rho_2 - \rho_1)(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b}) = b|\tau|[\sqrt{d(\alpha,\beta)} + b\nu(\alpha,\beta,\tau)]$, where ν is a $\mathcal{C}^\infty$ function for small $\tau \ge 0$ up to $\tau = 0$ which is bounded uniformly in (α,β) and vanishes at $\tau = 0$.*

8. Proof of Theorem 1.2

For the convenience of the reader, we prove explicitly the following simple

REMARK 8.1. Let Σ be a $\mathcal{C}^2$-surface given in a neighborhood U of some point $Q = (Q', Q_3)$ as a graph of a function $\xi' \to z(\xi')$ and assume that in the ξ'-plane we are given a line L such that the plane curve $\mathcal{K} = \{(\xi', z(\xi'));\ \xi' \in L,\ (\xi', z(\xi')) \in U\}$ has no inflection point at Q. Then the Gaussian and mean curvatures of Σ cannot vanish simultaneously at Q.

To see why this is so, we may assume without loss of generality that $Q = 0$ and that L is the ξ_1-axis. Next, we consider, with notations which are standard in differential geometry (and are somewhat in conflict with the notations in the other parts of the paper), the quantities:

$$\tilde{p} = \frac{\partial z(Q)}{\partial \xi_1}, \quad \tilde{q} = \frac{\partial z(Q)}{\partial \xi_2}, \quad \tilde{r} = \frac{\partial^2 z(Q)}{\partial \xi_1^2}, \quad \tilde{s} = \frac{\partial^2 z(Q)}{\partial \xi_1 \partial \xi_2}, \quad \tilde{t} = \frac{\partial^2 z(Q)}{\partial \xi_2^2}.$$

The Gaussian curvature K and the mean curvature K_{mean} at Q are then given by the quantities

$$K = \frac{\tilde{r}\tilde{t} - \tilde{s}^2}{(1 + \tilde{p}^2 + \tilde{q}^2)^2}, \quad K_{mean} = \frac{\tilde{r}(1 + \tilde{q}^2) - 2\tilde{p}\tilde{q}\tilde{s} + \tilde{t}(1 + \tilde{p}^2)}{2(\sqrt{1 + \tilde{p}^2 + \tilde{q}^2})^3},$$

and we know that $\tilde{r} \neq 0$. If $K = 0$ we must have $\tilde{r}\tilde{t} = \tilde{s}^2$. We claim that then $\tilde{r}K_{mean} \neq 0$. In fact, the nominator in $\tilde{r}K_{mean}$ can be written as $T = \tilde{r}^2(1 + \tilde{q}^2) - 2\tilde{p}\tilde{q}\tilde{s}\tilde{r} + \tilde{s}^2(1 + \tilde{p}^2)$ and there are no real $(\tilde{r}, \tilde{s}) \neq 0$ for which $T = 0$.

We now turn to the proof of Theorem 1.2. We shall at first apply the results from Section 4. Since the statements there distinguish between the cases $9b < 2c$ and $9b \geq 2c$, and since we intend to work with some relatively small b, we may assume that $9b < 2c$.

We shall argue by contradiction and assume that for some fixed (b, c) there are points on S_o at which the Gaussian and the mean curvature both vanish. For symmetry reasons, it is no loss of generality to assume that some of these points lie in the region $\mathbb{R}^3_+ = \{\xi \in S;\, \xi_1 \geq 0,\ \xi_2 \geq 0,\ \xi_3 \geq 0\}$. Actually, no such point can lie in one of the planes $\xi_i = 0$. (This is a consequence of the results in Section 4: we shall argue for $\xi_1 = 0$, the other cases being symmetrical. Consider then a point $(0, \xi_2, \mu) \in S_o$ and denote $\Gamma_\mu = \{\xi \in S_o;\, \xi_3 = \mu\}$ the curve associated with μ in Section 4. Then we know from that section that Γ_μ has no inflection point when $\xi_1 = 0$ and therefore (e.g., by Remark 8.1) the Gaussian and the mean curvature cannot vanish at $(0, \xi_2, \mu)$ simultaneously.)

Actually, the same argument gives that points where the Gaussian and mean curvature vanish simultaneously can only occur, if at all, when we have $|\xi_j - 1/\sqrt{3c - 3b}| \leq \tilde{c}b$, $j = 1, 2, 3$, where $\tilde{c}$ is the constant in Theorem 4.1. We recall that this constant is calculable. We conclude that if we fix some closed convex cone Γ which contains the conically singular point in $\mathbb{R}^3_+$, then there can be no points on $\mathbb{R}^3_+ \setminus \Gamma$ where the Gaussian and the mean curvature vanish simultaneously provided b is sufficiently small. (Once we have fixed Γ, we can calculate b in terms of Γ and the constant $\tilde{c}$.)

We must now exclude the possibility that there are points on S_o on which both curvatures vanish in Γ, where Γ is some closed convex cone which contains the singular direction in $\mathbb{R}^3_+$ in its interior. Moreover, we are allowed to fix Γ as we please in the remaining part of the argument. We shall argue by checking that the plane curves

$$G(\alpha, \beta, c') = \left\{ \left(\alpha\tau + \frac{1}{\sqrt{3-3b}}, \beta\tau + \frac{1}{\sqrt{3-3b}}, \rho_2\left(\alpha\tau + \frac{1}{\sqrt{3-3b}}, \beta\tau + \frac{1}{\sqrt{3-3b}} \right) \right);\ |\tau| \leq c' \right\}$$

(which lie in S_o) have no inflection points if c' is fixed small enough. Once this is done, we can then apply Remark 8.1 and can then conclude the argument with

REMARK 8.2. Fix some small $c' > 0$. Then we can find a closed convex cone Γ which contains the singular direction in $\mathbb{R}^3_+$ in its interior such that $\Gamma \cap S_o \subset \bigcup_{\alpha,\beta} G(\alpha, \beta, c')$, the union being for $\alpha^2 + \beta^2 = 1$.

We are thus left with the study of inflection points of the curves $G(\alpha, \beta, c')$. We shall now rely on the results in Section 7, assuming, as we may after a normalization, that $c = 1$, so the first condition on b shall be $b < 2/9$. Since $G(\alpha, \beta, c')$ is the graph of the function ρ_2 over (part of) the line $L(\alpha, \beta)$ introduced in (7.2), it suffices then to show that $(d/d\tau)^2\rho_2(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b}) \neq 0$ if the point $(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b}, \rho_2(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b}))$ stays in Γ.

Here we write ρ_2 as $[\rho_2 + \rho_1 + (\rho_2 - \rho_1)]/2$. Also recall the notation $M(\alpha, \beta, \tau)$ for $(\rho_1 + \rho_2)(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b}))$ (see (7.3)). In view of (7.4) we shall have that $(d/d\tau)^2 M(\alpha, \beta, \tau) = \langle H\cdot(\alpha, \beta)^\perp, (\alpha, \beta)^\perp\rangle + O(|\tau|)$ for $\tau \to 0$ (with calculable constants). It follows if we also use Proposition 5.1 that $(d/d\tau)^2 M(\alpha, \beta, \tau) \leq -c_1$ for some constant $c_1 > 0$ if $|\tau| \leq c_2$ and b is sufficiently small. Here c_1, c_2 are independent of b once b is small.

On the other hand, we can estimate $(d/d\tau)^2(\rho_2 - \rho_1)$ using Corollary 7.3. It is clear in particular that $(d/d\tau)^2(\rho_2 - \rho_1)(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b})| \leq \tilde{c}b$ if $|\tau| \leq c_3$. This shows that if c' is sufficiently small, then $(d/d\tau)^2\rho_2(\alpha\tau + 1/\sqrt{3-3b}, \beta\tau + 1/\sqrt{3-3b})$ is strictly negative for small $|\tau| \leq c'$. This concludes the argument.

9. Final comments

We write, using the notation d for $b - a$, and dropping the assumption "$a = -2b$", (1.3) in the form

$$\prod_{j=1}^{3}(\tau^2 - c|\xi|^2 + d\xi_j^2) - \sum_{j=1}^{3} b\xi_j^2(\tau^2 - c|\xi|^2 + d\xi_{j+1}^2)(\tau^2 - c|\xi|^2 + d\xi_{j+2}^2) = 0 \tag{9.1}$$

(indices are counted modulo 3) and denote by q the polynomial in three variables obtained when in (9.1) we put $\tau = 1$:

$$q(\xi) = \prod_{j=1}^{3}(1 - c|\xi|^2 + d\xi_j^2) - \sum_{j=1}^{3} b\xi_j^2(1 - c|\xi|^2 + d\xi_{j+1}^2)(1 - c|\xi|^2 + d\xi_{j+2}^2). \tag{9.2}$$

In the arguments of this paper we have used in an essential way that for $a = -2b$, q splits into the product of two factors of lower degree. In this way our problems reduced to studying algebraic surfaces defined by polynomials of degree 4, rather than of degree 6. There are two other cases when q given by (9.2) is known to split into a product of two simpler factors: when $b = 0$ and when $b = a$. Indeed, when $b = 0$, then q is of form $q(\xi) = \prod_{j=1}^{3}(1 - c|\xi|^2 - a\xi_j^2)$ and when $b = a$, (which corresponds to the isotropic case) then $q(\xi) = (1-c|\xi|^2)^2(1-(c+b)|\xi|^2)$. In these cases the geometry of the slowness surface is of course trivial. We claim that apart from the above situations, there are no other cases in which q decomposes into factors of strictly lower degree.

To prove this, let us then start with a decomposition of q in the form $q = q_1q_2$, where q_1 and q_2 have degrees strictly smaller than 6 and which are with real coefficients. They cannot be of odd degree however, since real valued polynomials of odd degree have an unbounded set of real zeros. We may thus assume that the degree of q_1 is two and that that of q_2 is 4. We assume that $a \neq b$, $b \neq 0$, $a \neq -2b$, and restrict q to the coordinate plane $\xi_3 = 0$. It follows that we must have $q(\xi', 0) = q_1(\xi', 0)q_2(\xi', 0)$.

As a consequence of (9.2) there is, on the other hand, a natural decomposition of $q(\xi', 0)$ into a product of two factors, namely

$$q(\xi', 0) = (1 - c|\xi'|^2) \times q_3(\xi'), \tag{9.3}$$

where

$$\begin{aligned} q_3(\xi') = {} & b\xi_1^2(1 - c|\xi'|^2 + d\xi_2^2) + b\xi_2^2(1 - c|\xi'|^2 + d\xi_1^2) \\ & - (1 - c|\xi'|^2 + d\xi_1^2)(1 - c|\xi'|^2 + d\xi_2^2). \end{aligned}$$

We next want to see how these two decompositions of $q(\xi', 0)$ are related. We denote $\{\xi' \in \mathbb{R}^2;\, q_1(\xi', 0) = 0\}$ by S_1, $\{\xi' \in \mathbb{R}^2;\, q_2(\xi') = 0\}$ by S_2 and $\{\xi' \in \mathbb{R}^2;\, q_3(\xi') = 0\}$ by S_3. Here S_1 is then an ellipse and S_2, S_3, are quartics. We shall also assume for the moment that, in addition to $a \neq b$, $b \neq 0$, we have $a \neq -b$ and claim that $1 - c|\xi'|^2$ and $q_1(\xi', 0)$ must be proportional.

In fact, otherwise the ellipse S_1 and the circle $1 - c|\xi'|^2 = 0$ could have only finitely many points in common and therefore, by density, $1 - c|\xi'|^2 = 0$, would imply $q_3(\xi') = 0$. A particular point on the circle $1-c|\xi'|^2 = 0$ is $\xi'^0 = (1/\sqrt{2c}, 1/\sqrt{2c})$ and we have $q_3(\xi'^0) = -(1/4)(b^2-a^2)/c$. Since by our assumptions (which imply $a^2 \neq b^2$), this does not vanish we have then proved that $1-c|\xi'|^2 = 0$ and S_1 have infinitely many points in common and $1 - c|\xi'|^2$ and $q_1(\xi_1', 0)$ must therefore be proportional. In particular, $q_1(\xi)$ contains no terms of type $\xi_1\xi_2$. The same is then true for mixed terms of form $\xi_2\xi_3$, $\xi_3\xi_1$, and it follows easily that $q_1(\xi)$ and $1-c|\xi|^2$ are proportional, i.e., we may assume that in (9.3), $1 - c|\xi|^2$ is $q_1(\xi)$. We can sum up what we have obtained so far by saying that, when $a \neq -b$, then the assumption "$q = q_1q_2$" implies that the sphere $\{\xi;\, c|\xi|^2-1 = 0\}$ is contained in the slowness surface. This is in fact what happens for the case $a = -2b$, and there are no other cases (with $a^2 \neq b^2$) when it is true. Indeed, for example the point $\tilde{\xi} = (1/\sqrt{3c}), 1/\sqrt{3c}, 1/\sqrt{3c})$ satisfies $c|\tilde{\xi}|^2-1 = 0$, but is not a point of the slowness surface: it is in fact a point on the space diagonal $\xi_1 = \xi_2 = \xi_3$, and we already have calculated all points of the slowness surface on the space diagonals. They are $(\pm 1/\sqrt{3c-d}), \pm 1/\sqrt{3c-d}, \pm 1/\sqrt{3c-d})$ (which are double points) and the points $(\pm 1/\sqrt{3c-d+3b}), \pm 1/\sqrt{3c-d+3b}, \pm 1/\sqrt{3c-d+3b})$, and none of them is equal to $\tilde{\xi}$, since we have ruled out the cases $d = 0$, $d - 3b = 0$. This concludes the argument if we assume that $a \neq -b$ and it remains to see what happens when $a = -b$. This case is still more elementary since then $q(\xi', 0) = (-1 + c|\xi'|^2)^2((b-c)|\xi'|^2 - 1)$. q_1 is then immediately seen to be one of the three factors in this decomposition, and again it has no mixed terms of form $\xi_i \xi_j$, $i \neq j$. We can then continue the argument as above.

References

[1] A. Bannini and O. Liess: *Estimates for Fourier transforms of surface carried densities on surfaces with singular points*, II, Ann. Univ. Ferrara, Sez. VII, Sci. Mat. **52** (2006), 211–232.

[2] R. Courant and D. Hilbert: *Methods of mathematical physics*, II, Partial differential equations, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1989.

[3] G.F.D. Duff: *The Cauchy problem for elastic waves in an anisotropic medium*, Phil. Transactions Royal Soc. London, Ser. A **252** (1960), 249–273.

[4] B. Gross and J. Harris: *Real algebraic curves*, Ann. E.N.S., 4^e série t.**14**, (1981), 157–182.

[5] L.D. Landau and E.M. Lifshits: Theoretical Physics; Elasticity Theory, first Russian edition, Mir, Moscow, 1953. (Available also in many other editions and translations in many languages.)

[6] O. Liess: Conical Refraction and Higher Microlocalization, Springer Lecture Notes in Mathematics **1555**, 1993.

[7] O. Liess: *Estimates for Fourier transforms of surface-carried densities on surfaces with singular points*, Asymptotic analysis **37** (2004), 329–363.

[8] O. Liess: *Decay estimates for the solutions of the system of crystal acoustics for cubic crystals*, Kokyoroku series of the RIMS in Kyoto **1412** (2005), 1–13.

[9] O. Liess: *Curvature properties of the slowness surface of the system of crystal acoustics for cubic crystals*, Osaka J. Math. **45** (2008), 173–210.

[10] O. Liess: *Decay estimates for the solutions of the system of crystal acoustics for cubic crystals*, Asymptotic analysis **64** (2009), 1–27.

[11] G.F. Miller and M.J.P. Musgrave: *On the propagation of elastic waves in aleotropic media*, III, Media of cubic symmetry, Proc. Royal Soc. London **236** (1956), 352–383.

[12] M.J.P. Musgrave: Crystal Acoustics, Holden and Day, San Francisco, 1979.

[13] N. Ortner and P. Wagner: *Fundamental matrices of homogeneous hyperbolic systems. Applications to crystal optics, elastodynamics, and piezoelectromagnetism*, ZAMM **84** (2004), 314–346.

[14] H.G. Zeuthen: *Sur les courbes du quatrième ordre*, Mathematische Annalen **7**, (1874), 410–432.

Otto Liess
Dipartimento di Matematica
Università di Bologna
Italy
e-mail: ottoedwin.liess@unibo.it

Tetsuya Sonobe
Passed away on July 2004