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THE SUPERCRITICAL MULTI-TYPE CRUMP AND MODE AGE-DEPENDENT MODEL

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1. Introduction

In 1966, Kesten and Stigum [10] obtained necessary and sufficient conditions for the supercritical p -type Galton-Watson process (appropriately normalized) to converge to a nontrivial limit distribution. These results have been extended to other models by various authors. The age-dependent Bellman-Harris model was considered by Athreya [1] in 1969 for $p=1$; more recently, N. Kaplan [8] treated the general p -type version in 1975. The single-type ($p=1$) Crump and Mode model was considered by R. Doney [5] in 1972. In this paper, we consider the multi-type version of the Crump and Mode model. As in all of the above, the results depend upon the finiteness of $E[Y|\log Y|]$ for suitably defined random variables Y . Our proof relies heavily on the $p=1$ results and has the same flavor as a paper of Athreya's [2].

We shall first describe the model on an intuitive basis. Let $K_i(t)=(K_{i1}(t), \dots, K_{ip}(t))$, $1 \leq i \leq p$, be arbitrary vector-valued counting processes. $K_{ij}(t)$ counts the potential number of offspring of the j th type born to an individual of the i th type during the time interval $[0, t]$. We arbitrarily stop the counting process K_i at a random time L_i , the lifetime of an individual of the i th type. Set

$$N_i(t) = \begin{cases} K_i(t) & \text{if } t < L_i \\ K_i(L_i) & \text{if } t \geq L_i \end{cases}$$

and $G_i(t) = Pr\{L_i \leq t\}$. Thus N_i counts the actual number of offspring born to an individual of type i during its lifetime and G_i is its lifetime distribution. Each newborn object behaves similarly and all particles behave independently

Abstract. Let $X(t) = (X_1(t), \dots, X_p(t))$ be a p -dimensional supercritical Crump-Mode age-dependent branching process. If $Z(t) = (Z_1(t), \dots, Z_p(t))$ counts the number of objects alive at time t , we find necessary and sufficient conditions for $Z(t)e^{-\lambda t}$ to converge in distribution to a nontrivial random variable. We also investigate some of its properties. By considering the total progeny process we deduce convergence in probability to the limiting age distribution. Lastly we consider a generalized immigration model.

of all other particles. Using these ingredients, one can construct a stochastic population process $(\Omega, P, X(t))$ having the above characteristics (cf. Mode [12]). The process $X(t)=(X_1(t), \dots, X_p(t))$ not only keeps track of the number and type, but also age. Thus for each t and ω , either $X_i(t, \omega)=0$ (no particles of type i at time t), $+\infty$ (an infinite number of particles of type i at time t), or for some $n \geq 1$, $X_i(t, \omega) \in [0, \infty)^n$. In the latter case, if $X_i(t, \omega)=(x_1, \dots, x_n)$, then there are n objects of the i th type alive at time t and of ages x_1, \dots, x_n respectively.

If f and g are real-valued functions (defined on $[0, \infty)$) satisfying $|f| \leq 1$ and g nonnegative or bounded, we extend them to $\{0\} \cup \bigcup_{n=1}^{\infty} [0, \infty)^n \cup \{+\infty\}$ by

$$\hat{f}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \prod_{i=1}^n f(x_i) & \text{if } x = (x_1, \dots, x_n) \in [0, \infty)^n \\ 0 & \text{if } x = +\infty \end{cases}$$

and

$$\check{g}(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } +\infty \\ \sum_{i=1}^n g(x_i) & \text{if } x = (x_1, \dots, x_n) \in [0, \infty)^n. \end{cases}$$

If $f=(f_1, \dots, f_p)$ and $g=(g_1, \dots, g_p)$ are vectors of such functions, then we set

$$\hat{f}(X(t)) = \prod_{i=1}^p \hat{f}_i(X_i(t))$$

and

$$\check{g}(X(t)) = \sum_{i=1}^p \check{g}_i(X_i(t)).$$

Also let e_i be the p -vector $(\delta_{i1}, \dots, \delta_{ip})$ where δ_{ij} is the Kronecker delta. Furthermore, we denote the conditional expectations $E[\cdot | X(0)=e_i]$ by $E_i[\cdot]$.

It is intuitively clear and can be rigorously shown that the following representations are valid. Let ${}_iX(t)$ denote the process $X(t)$ given that we start with an object of type i . Then if f and g are vector valued functions we have

$$\hat{f}({}_iX(t)) = \exp\{\delta(t-L_i)\log f_i(t)\} \prod_{k=1}^p \prod_{l=1}^{N_{ik}^{(t)}} \hat{f}_k(X(t-t_{il}^k))$$

and

$$\check{g}({}_iX(t)) = \delta(t-L_i)g_i(t) + \sum_{k=1}^p \sum_{l=1}^{N_{ik}^{(t)}} \check{g}_k(X(t-t_{il}^k)),$$

where $0 \leq t_{i1}^k \leq t_{i2}^k \leq \dots$ are the successive times at which the process $N_{ik}(t)$ increases by one, $\delta(t)=0$ or 1 accordingly as $t \geq 0$ or < 0 , and all the processes $\{{}_kX(t-t_{il}^k), t \geq t_{il}^k\}_{k,l}$ are conditionally independent given the process $\{N_i(t); t \geq 0\}$. Consequently, if $u_i(t)=E_i[\hat{f}({}_iX_t)]$ and $v_i(t)=E_i[\check{g}({}_iX_t)]$, then

$$(1.1) \quad u_i(t) = f_i(t) \int_t^\infty H_i^y[u^t] dG_i(y) + \int_0^t H_i^y[u^t] dG_i(y)$$

and

$$(1.2) \quad v_i(t) = g_i(t)[1 - G_i(t)] + \sum_{j=1}^p \int_0^t v_j(t-y) dF_{ij}(y),$$

$1 \leq i \leq p$, where H_i^y is the conditional probability generating functional given by

$$H_i^y(\phi) = H_i^y((\phi_1, \dots, \phi_p)) = E_i[\exp\{\sum_{j=1}^p \int_0^\infty \log \phi_j(x) dN_{ij}(x)\} \mid L_i = y]$$

(cf. Doney [5]), $u^t(y) = (u_1^t(y), \dots, u_p^t(y))$ with $u_j^t(y) = u_j(t-y)$ if $t \geq y$ and $= 1$ if $t < y$, and $F_{ij}(x) = E[N_{ij}(x)]$. By \int_a^b we mean $\int_{(a, b]}$.

2. Assumptions and statement of results

Again, let $F_{ij}(x) = E[N_{ij}(x)]$ and set $m_{ij} = F_{ij}(+\infty)$.

(2.1) Assumptions:

- (i) $F(x) = [F_{ij}(x)]$ is a non-lattice matrix of Borel measures (see Crump [3]) and $F(0+) = 0$.
- (ii) $H(s) = (H_1(s), \dots, H_p(s))$ is nonsingular (see Harris [6]), where $H_i(s) = E[s^{N_i(+\infty)}]$.
- (iii) $m_{ij} < \infty$ all i, j and $M = [m_{ij}]$ is positively regular.
- (iv) Since M is positively regular, it has a positive eigenvalue ρ of maximum modulus. We suppose that $\rho > 1$.

Assumption (iii) guarantees that the process $X(t)$ is regular; i.e., no explosion (see Mode [12]). Assumption (iv) just says that we are in the supercritical case. In the supercritical case, it is known that the extinction probability $q = (q_1, \dots, q_p)$ is strictly less than $1 = (1, \dots, 1)$ and is the smallest nonnegative root of $q = H(q) = (H_1(q), \dots, H_p(q))$. Furthermore, if q^* is any other nonnegative root, then either $q^* = q$ or $q^* = 0$. Note also that

$$H_i(s) = \int_0^\infty H_i^y(s) dG_i(y)$$

where $s = (s_1, \dots, s_p)$ and $|s_i| \leq 1$ all i .

Let us define a new matrix $M(\alpha)$ by

$$m_{ij}(\alpha) = \int_0^\infty e^{-\alpha x} dN_{ij}(x).$$

Since $M(\alpha)$ is also positively regular, it has a positive eigenvalue $\rho(\alpha)$ of maximum modulus. We choose $\alpha > 0$ such that $\rho(\alpha) = 1$ and set $\lambda = \alpha$. It is known that such an α exists since $\rho = \rho(0) > 1$. λ is called the Malthusian parameter.

From the Frobenius theory, it follows that corresponding to λ there exists strictly positive left and right eigenvectors of $M(\lambda)$, μ and ν respectively,

satisfying $\langle \mu, 1 \rangle = 1$ and $\langle \mu, \nu \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product. Lastly, we set $m_{i,j}^* = m_{i,j}(\lambda)$ and $M^* = M(\lambda)$.

According to Crump [3], it then follows from our assumptions that given $g = (g_1, \dots, g_p)$ such that each g_i is bounded on finite t -intervals, (1.2) has a unique solution $v = (v_1, \dots, v_p)$ which is bounded on finite t -intervals; moreover, if each $\bar{g}_i(t) = [1 - G_i(t)]e^{-\lambda t} g_i(t)$ is directly Riemann integrable, then

$$(2.2) \quad v_i(t)e^{-\lambda t} \rightarrow d\nu_i \sum_{j=1}^p \mu_j \int_0^\infty \bar{g}_j(t) dt$$

as $t \rightarrow \infty$, where d is a positive constant independent of g .

In particular, if we take $g = e_j$ (considered as a vector of functions, then

$$m_{i,j}(t) = E_i[\dot{e}_j(X_t)] = E_i[Z_j(t)] \sim \nu_i \mu_j c_j e^{\lambda t}$$

as $t \rightarrow \infty$, where $c_j = d \int_0^\infty e^{-\lambda t} [1 - G_j(t)] dt$. Here $Z_j(t)$ just counts the number of particles of type j alive at time t .

Let us now define $W_i(t) = Z_i(t)/c_i e^{\lambda t}$ and $W(t) = (W_1(t), \dots, W_p(t))$. Set $W^*(t) = \langle \nu, W(t) \rangle$. Then we shall prove the following.

Theorem 2.1. Define the random variables $Y_{i,j} = \int_0^\infty e^{-\lambda x} dN_{i,j}(x)$. Consider

$$(*) \quad \sup_{i,j} E[Y_{i,j} | \log Y_{i,j} |] .$$

Then $W^*(t)$ converges in distribution to a nontrivial random variable W^* iff $(*)$ is finite; moreover, in this case, $P_i(W^* = 0) = q_i$ and $E_i[W^*] = \nu_i$ all i .

Corollary 1°. If $(*)$ is finite, then $W(t) \rightarrow_\mu W^*$ in distribution.

Let $Z_i(x; t)$ be the number of particles of type i alive at time t and of age $\leq x$. Set $W_i(x; t) = Z_i(x; t)/c_i(x) e^{\lambda t}$ where $c_i(x) = d \int_0^x e^{-\lambda t} [1 - G_i(t)] dt$. If $x = (x_1, \dots, x_p)$, we set $W(x; t) = (W_1(x_1; t), \dots, W_p(x_p; t))$.

Corollary 2°. If $(*)$ is finite, $W(x; t) \rightarrow_\mu W^*$ in distribution.

Theorem 2.2. Assume that $(*)$ is finite. If in addition we assume that for at least one index i , the random variable

$$(**) \quad \sum_{j=1}^p Y_{i,j} \nu_j$$

can take on at least two values with positive probability, then W^* has a continuous density on $(0, \infty)$.

Let $Y_i(t)$ denote the total number of objects of type i born in $[0, t]$ including the ancestor if it is also of type i . Set $V_i(t) = \lambda Y_i(t) / de^{\lambda t}$ and $V(t) = (V_1(t), \dots, V_p(t))$.

Theorem 2.3. *If (*) is finite, $(W(x; t), V(t)) \rightarrow (\mu W^*, \mu W^*)$ in distribution.*

Corollary. *Under the hypothesis of Theorem 2.2, we have that $Z_i(x; t) / |Y(t)| \rightarrow \mu_i c_i(x) \lambda d^{-1}$ in probability off*

$$Q = \{Z(t) = (Z_1(t), \dots, Z_p(t)) \rightarrow 0\}. \text{ Here } |Y(t)| = \sum_{k=1}^p Y_k(t).$$

Furthermore, if we start from a particle of type j then $Z_i(x; t) / Y_j(t) \rightarrow \mu_i \lambda c_i(x) / \mu_j d$ in probability off Q .

REMARK. Since convergence in probability is preserved under addition and multiplication, it follows for example that $Z_i(x; t) / Z_k(y; t) \rightarrow \mu_i A_i(x) / \mu_k A_k(y)$, $|Z(t)| / |Y(t)| \rightarrow \lambda d^{-1} \sum_{k=1}^p \mu_k c_k$ and $Z_i(x; t) / |Z(t)| \rightarrow \mu_i c_i(x) / \sum_{k=1}^p \mu_k c_k$ in probability off Q . If we start from a particle of type j , then we also have $Y_j(t) / |Y(t)| \rightarrow \mu_j$ in probability off Q . Here $A_i(x) = c_i(x) / c_i$ is the limiting age distribution.

In section 4 we consider a generalized immigration model. Basically it is a $(p+1)$ -type Crump and Mode process corresponding to $(N_0(t), N_1(t), \dots, N_p(t))$ in which $(N_1(t), \dots, N_p(t))$ produces no particles of type 0. $N_0(t)$ can thus be considered as the immigration component. Under the assumptions of section 4 we have the following

Theorem 2.4. *In the supercritical case, all of the preceding results remain valid for this immigration model (provided we don't divide by μ_0 in Remark of Theorem 2.3. since $\mu_0 = 0$). In particular, starting with a particle of type 0,*

$$(W_1(t), \dots, W_p(t)) \rightarrow \bar{\mu} W^*$$

in distribution to a nontrivial random variable iff () is finite; in this case, $P_0(W^*) = q_0$ and $E_0(W^*) = v_0$. Furthermore, if (**) is also true, W^* has a continuous density on $(0, \infty)$.*

REMARK. For the immigration process, we take the sup over all $1 \leq i, j \leq p$ in (*) and we only consider the random variables $\sum_{j=1}^p Y_{ij} \nu_j$, $1 \leq i \leq p$, for (**)

3. Proofs

Let $\Phi_i(u, t) = E_i[\exp(-uW^*(t))]$ be the Laplace transform of $W^*(t)$. It follows from (1.1) that Φ_i satisfies

$$\begin{aligned} \Phi_i(u, t) = \exp\{-ue^{-\lambda t} \nu_i / c_i\} & \int_t^\infty H_i^y[\Phi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y) \\ & + \int_0^t H_i^y[\Phi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y). \end{aligned}$$

By $\Phi^t(ue^{-\lambda \cdot}, \cdot)$ we mean the vector function $\Phi(ue^{-\lambda x}, t-x)$ if $t \geq x$ and the vector 1 if $t < x$.

Since $E_i[W^*(t)] \rightarrow \nu_i$ as $t \rightarrow \infty$, $\{W^*(t)\}$ is tight (with respect to each P_i). Suppose now that $W^*(t) \rightarrow W^*$ in distribution. Then $\Phi_i(u) = E_i[\exp\{-uW^*\}]$ satisfies

$$(3.1) \quad \Phi_i(u) = \int_0^\infty H_i^y[\Phi(ue^{-\lambda \cdot})] dG_i(y) = H_i[\Phi(ue^{-\lambda \cdot})].$$

If we now let $u \uparrow \infty$, then we see that $q_i^* = P_i(W^* = 0)$ satisfies

$$q^* = H(q^*).$$

Hence either $q^* = 1$ or $q^* = q < 1$. In the former case, it follows that $W^* = 0$ a.s. (P_i), all i .

For θ a strictly positive p -vector, let

$\mathcal{C}(\theta) = \{\phi = (\phi_1, \dots, \phi_p) : \phi_i \text{ is the Laplace transform of a probability measure on } [0, \infty) \text{ and } \lim_{u \rightarrow 0} u^{-1}[1 - \phi_i(u)] = \theta_i\}$ and set $\mathcal{C} = \cup_{\theta > 0} \mathcal{C}(\theta)$. According to the above, we see that either $\Phi \equiv 1$ or $\Phi \in \mathcal{C}$ (Actually, we can say in this case that $\Phi \in \mathcal{C}(\theta)$ for some $\theta \leq \nu$).

Before we can proceed further we shall need a few preliminaries.

Lemma 3.1. *If $0 \leq \phi \leq \psi \leq 1$ as vector functions, then $H_i(\phi) \leq H_i(\psi)$ and*

$$|H_i(\phi) - H_i(\psi)| \leq \sum_{j=1}^p \int_0^\infty |\phi_j(x) - \psi_j(x)| dF_{i,j}(x), \quad 1 \leq i \leq p.$$

The proof is similar to the one-dimensional version given in Doney [5]. Let $\phi = (\phi_1, \dots, \phi_p)$ with $0 \leq \phi_i \leq 1$ and set

$$A_i(\phi) = H_i(\phi) - 1 + \sum_{j=1}^p \int_0^\infty [1 - \phi_j(x)] dF_{i,j}(x).$$

Again as in Doney [5], we have that if $0 \leq \phi \leq \psi \leq 1$, then $0 \leq A_i(\psi) \leq A_i(\phi)$. We define $\bar{A}_i(\phi) = A_i((\phi, \dots, \phi))$ and set $A^*(\phi) = \sum_{i=1}^p \mu_i \bar{A}_i(\phi)$. It is not hard to see that A^* corresponds to N^* exactly as A corresponds to N in Doney, where $N^*(t)$ is the counting process which with probability μ_i looks like $\sum_{j=1}^p N_{i,j}(t)$. Since

$$E\left[\int_0^\infty e^{-\lambda x} dN^*(x)\right] = \sum_{i=1}^p \mu_i \sum_{j=1}^p \int_0^\infty e^{-\lambda x} dF_{i,j}(x) = \sum_{i,j=1}^p \mu_i m_{i,j}^* = 1,$$

we are in a position to use the one-dimensional results of Doney. Let $\psi^*(u) = u^{-1} A^*(\exp\{-ue^{-\lambda \cdot}\})$ for $u > 0$. Then

Lemma 3.2. *For every $\delta < 0$ and $0 < r < 1$,*

$$\sum_{n=0}^\infty \psi^*(\delta r^n) < \infty \quad \text{and} \quad \lim_{\delta \downarrow 0} \sum_{n=0}^\infty \psi^*(\delta r^n) = 0$$

iff $E[Y^ | \log Y^*] < \infty$, where $Y^* = \int_0^\infty e^{-\lambda x} dN^*(x)$; moreover, $E[Y^* | \log Y^*] < \infty$ iff $\sup_{i,j} E[Y_{i,j} | \log Y_{i,j}] < \infty$.*

The second “iff” can be verified as in Athreya [2].

Lemma 3.3. *Let Φ be a solution of (3.1). Then $\Phi \in \mathcal{C}$ only if (*) is finite.*

Proof. Suppose $\Phi \in \mathcal{C}$. Then there are constants $c > 0, \delta > 0$ such that for all $u \leq \delta, 1 - \Phi_j(u) \geq cu$. Let $g_i(u) = u^{-1}[1 - \Phi_i(u)]$ for $u > 0$. From (3.1) we have that

$$\begin{aligned} g_i(u) &= u^{-1}(1 - H_i[\Phi(ue^{-\lambda \cdot})]) \\ &= \sum_{j=1}^p \int_0^\infty g_j(ue^{-\lambda x}) e^{-\lambda x} dF_{ij}(x) - u^{-1} A_i[\Phi(ue^{-\lambda \cdot})] \\ &\leq \sum_{j=1}^p m_{ij}^* \int_0^\infty g_j(ue^{-\lambda x}) dF_{ij}^*(x) - u^{-1} \bar{A}_i[\exp\{-cue^{-\lambda \cdot}\}] \end{aligned}$$

for all $0 < u \leq \delta$, where F_{ij}^* is a probability measure (if $m_{ij}^* = 0$, let F_{ij}^* be any nontrivial probability measure on $[0, \infty)$ having finite mean). Since each $g_j(ue^{-\lambda x}) \uparrow$ as $x \uparrow$ we can find a nondegenerate probability measure \tilde{G} such that $\int_0^\infty g_j(ue^{-\lambda x}) dF_{ij}^*(x) \leq \int_0^\infty g_j(ue^{-\lambda x}) d\tilde{G}(x)$ all i, j . Now set $g(u) = \sum_{i=1}^p \mu_i g_i(u)$. Then

$$g(u) \leq \int_0^\infty g(ue^{-\lambda x}) d\tilde{G}(x) - c\psi^*(cu).$$

Proceeding as in Doney, we have the desired result.

Lemma 3.4. *Let $I_i(u, t) = u^{-1} E_i[\exp\{-uW^*(t)\} + uW^*(t) - 1]$ for $u > 0$. Then if (*) is finite, $\lim_{u \downarrow 0} \sup_{t \geq 0} |I_i(u, t)| = 0, 1 \leq i \leq p$.*

Proof. Define $m_i^*(t) = E_i[W^*(t)]$. Using (1.1) and (1.2) we can rewrite I_i as

$$\begin{aligned} I_i(u, t) &= u^{-1}[1 - G_i(t)] [\exp\{-uv_i e^{-\lambda t}/c_i\} + uv_i e^{-\lambda t}/c_i - 1] \\ &\quad + u^{-1}[1 - \exp\{-uv_i e^{-\lambda t}/c_i\}] \int_t^\infty \{1 - H_i^v[\Phi^t(ue^{-\lambda \cdot}, \cdot)]\} dG_i(y) \\ &\quad + u^{-1} A_i[\Phi^t(ue^{-\lambda \cdot}, \cdot)] + \sum_{j=1}^p m_{ij}^* \int_0^t I_j(ue^{-\lambda y}, t-y) dF_{ij}^*(y). \end{aligned}$$

Since $I_i(u, t) \geq 0, \Phi_i(u, t) \geq 1 - u m_i^*(t)$. Recalling that $m_i^*(t) \rightarrow v_i$ as $t \rightarrow \infty$ and is bounded on finite t -intervals, it follows that there exist positive constants c, η , and δ such that $1 - \Phi_i(u, t) \leq cu$ all u, t, i and $\Phi_i(u, t) \geq e^{-\eta u}$ all $0 < u < \delta, i$ and t . Consequently, for $0 < u < \delta$

$$\begin{aligned} 1 - H_i[\Phi^t(ue^{-\lambda \cdot}, \cdot)] &\leq \sum_{j=1}^p \int_0^t [1 - \Phi_j(ue^{-\lambda y}, t-y)] dF_{ij}(y) \\ &\leq uc \sum_{j=1}^p \int_0^\infty e^{-\lambda y} dF_{ij}(y) = uc \sum_{j=1}^p m_{ij}^* \end{aligned}$$

and

$$A_i[\Phi^i(ue^{-\lambda}, \cdot)] \leq \bar{A}_i[\exp\{-\eta ue^{-\lambda}\}].$$

Now set $I_i^T(u) = \sup_{0 \leq t \leq T} I_i(u, t)$. Then there is a constant $M > 0$ such that for all $0 < u \leq \delta$,

$$I_i^T(u) \leq uM + u^{-1} \bar{A}_i[\exp\{-\eta ue^{-\lambda}\}] + \sum_{j=1}^p m_{i,j}^* \int_0^T I_j^T(ue^{-\lambda y}) dF_{i,j}^*(y).$$

Since each $I_i^T(u)$ is nondecreasing in u , we can find a non-degenerate probability measure G such that $\int_0^T I_j^T(ue^{-\lambda y}) dF_{i,j}^*(y) \leq \int_0^T I_j^T(ue^{-\lambda y}) dG(y)$ for all $i, j, u > 0, T > 0$. Thus if we set $I^T(u) = \sum_{i=1}^p \mu_i I_i^T(u)$, then

$$I^T(u) \leq uM + \eta \psi^*(\eta u) + \int_0^T I^T(ue^{-\lambda y}) dG(y).$$

Now proceed as in Doney [5].

Lemma 3.5. *If $\Phi^1, \Phi^2 \in \mathcal{C}(\theta)$ and both satisfy (3.1), then $\Phi^1 = \Phi^2$.*

Proof. Let $g_i(u) = u^{-1} |\Phi_i^1(u) - \Phi_i^2(u)|$ for $u > 0$. Then

$$\begin{aligned} g_i(u) &\leq \sum_{j=1}^p m_{i,j}^* \int_0^\infty g_j(ue^{-\lambda y}) dF_{i,j}^*(y) \\ &= \sum_{j=1}^p m_{i,j}^* E[g_j(ue^{-\lambda X_{ij}})] \end{aligned}$$

where X_{ij} is a random variable with distribution function $F_{i,j}^*$; moreover, we may assume that they are independent. Iterating yields,

$$g_i(u) \leq \sum_{1 \leq j_1, \dots, j_k \leq p} m_{i,j_1}^* m_{j_1,j_2}^* \dots m_{j_{k-1},j_k}^* E[g_{j_k}(ue^{-\lambda S_k^{j_0, j_1, \dots, j_k}})]$$

where $j_0 = i$ and

$$S_k(j_0, j_1, \dots, j_k) = \sum_{l=1}^k X_{j_{l-1}, j_l}^l \geq \sum_{l=1}^k \min_{i,j} (X_{ij}^l).$$

The superscript l refers to independent copies of the same random variable. Since $E[\min_{i,j} (X_{ij}^1)] > 0$ we can now proceed as in Kaplan [8] to deduce that $g_i = 0$ all $u > 0$ and hence $\Phi^1 = \Phi^2$.

Proof of Theorem 2.1. Suppose (*) is infinite and $W^*(t) \rightarrow W^*$ in distribution. Then it follows from Lemma 3.3. that $W^* = 0$ w.p. 1. On the other hand suppose (*) is finite. Set

$$K_i(u) = \limsup_{t \rightarrow \infty} \sup_{s \geq 0} u^{-1} |\Phi_i(u, t+s) - \Phi_i(u, t)| \text{ for } u > 0.$$

It is an easy consequence of Lemma 3.4 that $\lim_{u \downarrow 0} K_i(u) = K_i(0+) = 0$ all i . Now making use of the equation that Φ_i satisfies it is not hard to show (cf. Athreya [1]) that

$$K_i(u) \leq \sum_{j=1}^p m_{ij}^* E[K_j(ue^{-\lambda X_{ij}})]$$

where X_{ij} is as in the proof of Lemma 3.5. It follows then that $K_i(u) = 0$ for $u > 0$ and all i . Consequently $\lim_{t \rightarrow \infty} \Phi_i(u, t) = \Phi_i(u)$ exists and satisfies (3.1). Because of tightness we conclude that $W^*(t)$ converges in distribution to a nonnegative random variable W^* ; furthermore, it follows from Lemma 3.4 that $E_i[W^*] = \nu_i$ and hence is nontrivial.

Proof of Corollary 1° of Theorem 2.1. All we need show is that $\langle \eta, W(t) \rangle \rightarrow \langle \eta, \mu \rangle W^*$ in distribution for any nonnegative p -vector η . First observe that $E_i[\langle \eta, W(t) \rangle] \rightarrow \langle \eta, \mu \rangle \nu_i$ as $t \rightarrow \infty$. Secondly, it follows from (1.1) that if we do have convergence in distribution, then the transform of the limit random variable is a solution of (3.1). Lastly, we see that there exists a positive constant K such that $0 \leq \langle \eta, W(t) \rangle \leq K \langle \nu, W(t) \rangle = KW^*(t)$. Since $B(x) = e^{-x} + x - 1$ increases in x for $x \geq 0$,

$$u^{-1} E_i[B(u \langle \eta, W(t) \rangle)] \leq K(uK)^{-1} E_i[B(uKW^*(t))] = KI_i(uK, t).$$

Hence $\lim_{u \downarrow 0} \sup_{t \geq 0} u^{-1} E_i[B(u \langle \eta, W(t) \rangle)] = 0$ all i if (*) is finite. Now proceed as in the proof of Theorem 2.1.

Everything that we have done above can be extended to the following situation. Let $g = (g_1, \dots, g_p)$ be a vector of nonnegative bounded functions which are directly Riemann integrable and set $c_i(g_i) = d \int_0^\infty e^{-\lambda t} [1 - G_i(t)] g_i(t) dt$. Assume for the moment that each $c_i(g_i) > 0$. Set

$$W_i(g_i; t) = \check{g}_i(X_i(t)) / c_i(g_i) e^{-\lambda t} \text{ and } W(g; t) = (W_1(g_1; t), \dots, W_p(g_p; t)).$$

Since for each nonnegative p -vector η , there is a constant $K > 0$ such that $\langle \eta, W(g; t) \rangle \leq KW^*(t)$ and $E_i[\langle \eta, W(g; t) \rangle] \rightarrow \langle \eta, \mu \rangle \nu_i$ as $t \rightarrow +\infty$, we deduce as in the proof of Corollary 1 that $W(g; t) \rightarrow \mu W^*$ in distribution. Equivalently, we can say that

$$(\check{g}_1(X_1(t)), \dots, \check{g}_p(X_p(t))) e^{-\lambda t} \rightarrow \mu(g) W^*$$

in distribution, where $\mu(g)$ is the p -vector with components $\mu_i(g) = \mu_i c_i(g_i)$. This latter statement remains valid even if some of the terms $c_i(g_i)$ are zero.

Proof of Corollary 2° of Theorem 2.1. Take $g_i(y) = 1_{[0, x_i]}(y)$, $1 \leq i \leq p$, in the above.

Proof of Theorem 2.2. One can modify the proof given in Doney [5] for the one-dimensional case along the same lines that Kaplan [8] used for the Bellman-Harris model. The details will be omitted.

Proof of Theorem 2.3. Although this result is not a corollary of The-

orem 2.1, it is a corollary of its proof as we shall now show. Recall that $Y_j(t)$ is the total number of objects of type j born in $[0, t]$ including its ancestor if it is also of type j . As in section 1, we can show that the following representation is valid.

$${}_iY_j(t) = \delta_{ij} + \sum_{k=1}^p \sum_{i=1}^{N_{i1}(t)} n_{kj}(t-t_{ik}^i).$$

Consequently, $n_{ij}(t) = E_i[Y_j(t)]$ satisfies

$$n_{ij}(t) = \delta_{ij} + \sum_{k=1}^p \int_0^t n_{kj}(t-y) dF_{ik}(y).$$

Hence, $n_{ij}(t) \sim \lambda^{-1} d\nu_i \mu_j e^{\lambda t}$ as $t \rightarrow +\infty$. We set $V_i(t) = \lambda Y_i(t) / de^{\lambda t}$ and $V(t) = (V_1(t), \dots, V_p(t))$. To prove our theorem, it suffices to consider sums of the form $U(t) = \langle \xi, V(t) \rangle + \langle \eta, W(x; t) \rangle$ for nonnegative p -vectors ξ and η . Note that $E_i[U(t)] \rightarrow \nu_i \langle \xi + \eta, \mu \rangle$ as $t \rightarrow \infty$. If $\psi_i(u, t) = E_i[\exp\{-uU(t)\}]$, then from our representations, it follows that

$$\begin{aligned} \psi_i(u, t) &= \exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i / c_i(x_i))\} \int_t^\infty H_i^\nu[\psi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y) \\ &\quad + \exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\} \int_0^t H_i^\nu[\psi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y). \end{aligned}$$

Hence if $U(t) \rightarrow U$ in distribution, its Laplace transform is a solution of (3.1). Everything now follows as before once we rewrite $J_i(u, t) = E_i[B(uU(t))]$ as

$$\begin{aligned} J_i(u, t) &= u^{-1} [1 - G_i(t)] [\exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i / c_i(x_i))\} \\ &\quad + ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i / c_i(x_i)) - 1] \\ &\quad + u^{-1} G_i(t) [\exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\} + ue^{-\lambda t} \xi_i \lambda d^{-1} - 1] \\ &\quad + u^{-1} [1 - \exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i / c_i(x_i))\}] \\ &\quad \times \int_t^\infty \{1 - H_i^\nu[\psi^t(ue^{-\lambda \cdot}, \cdot)]\} dG_i(y) \\ &\quad + u^{-1} [1 - \exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\}] \int_0^t \{1 - H_i^\nu[\psi^t(ue^{-\lambda \cdot}, \cdot)]\} dG_i(y) \\ &\quad + u^{-1} A_i[\psi^t(ue^{-\lambda \cdot}, \cdot)] + \sum_{k=1}^p m_{ik}^* \int_0^t J_k(ue^{-\lambda x}, t-x) dF_{ik}^*(x). \end{aligned}$$

Proof of Corollary of Theorem 2.3. Apply the same technique as in Doney [4].

4. Immigration processes

Let $(N_0(t), N_1(t), \dots, N_p(t))$ generate a $(p+1)$ dimensional Crump and Mode process. We assume that each $N_i(t)$ process ($1 \leq i \leq p$) cannot give birth to objects of type O , but $N_0(t)$ gives birth to at least one object of type $1, 2, \dots$ or p ; i.e., we assume that $m_{i0} = 0$ for $i = 1, \dots, p$, and that there exists at least one

$j \neq 0$ such that $m_{0j} \neq 0$. $N_0(t)$ can thus be considered as an immigration component. We will call such a process a p -type age-dependent branching process with immigration. This model seems to include all immigration models that have appeared in the literature. For example, let $N_0(t) = (1, 0, \dots, 0) = e_0$ for $t < L$ and $= (1, \xi)$ for $t \geq L$ where ξ is a p -dimensional random variable independent of L having probability generating function $h(s_1, \dots, s_p)$ and let $(N_1(t), \dots, N_p(t))$ generate a p -dimensional Bellman-Harris model. The case $p=1$ was originally studied by Jagers [7] while the general p -dimensional version was recently considered by Kaplan and Pakes [9]. If we want the times of immigration to obey a Poisson distribution, let $N_0(t) = (0, N_{01}(t), \dots, N_{0p}(t))$ where $(N_{01}(t), \dots, N_{0p}(t))$ is a nonhomogeneous compound Poisson process.

The study of immigration processes thus reduces to the study of such $(p+1)$ -type models where we start with a particle of type 0. The only thing different about these processes is that now the corresponding mean matrix M is reducible; specifically, M has the form

$$M = \left(\begin{array}{c|ccc} m_{00} & m_{01} & \dots & m_{0p} \\ \hline 0 & & & \\ \vdots & & \bar{M} & \\ 0 & & & \end{array} \right)$$

where \bar{M} is the $p \times p$ matrix corresponding to the p -dimensional process generated by $(N_1(t), \dots, N_p(t))$. The eigenvalue ρ of maximum modulus is given by $\rho = \max(m_{00}, \bar{\rho})$ where $\bar{\rho}$ is the eigenvalue of maximum modulus corresponding to \bar{M} . From now on we shall assume that $(N_1(t), \dots, N_p(t))$ satisfies assumptions (2.1). We also assume that $m_{0i} < \infty$ all $i=0, 1, \dots, p$ and that $F_{00}(x)$ is a non-lattice Borel measure satisfying $F_{00}(0+) = 0$. In addition we shall make the following assumption.

(4.1) Assumption. $1 \geq m_{00}$

Consequently it follows that $\rho = \bar{\rho}$ and if we choose $\alpha > 0$ such that $\rho(\alpha) = 1$, then $\rho(\alpha) = \bar{\rho}(\alpha)$. Hence the Malthusian parameter $\lambda = \alpha$ corresponds to that of the process generated by $(N_1(t), \dots, N_p(t))$. This assumption (4.1) is satisfied for the supercritical immigration processes that have been considered in the literature. Without (4.1) it is conceivable that $\rho(\alpha) = m_{00}^* = 1 > \bar{\rho}(\alpha)$ even if we assume that $\rho = \bar{\rho}$. This possibility will be investigated in the future as well as the critical and subcritical cases for this model. For more information on the reducible case, see Kesten and Stigum [11] and Mode [12].

Let $\bar{\mu}$ and $\bar{\nu}$ be the strictly positive left and right eigenvectors respectively of $\bar{M}(\lambda)$ satisfying $\langle \bar{\mu}, \bar{\nu} \rangle = 1$ and $\langle \bar{\mu}, 1 \rangle = 1$. Setting $\mu = (0, \bar{\mu})$ and $\nu = (\nu_0, \bar{\nu})$ where $\nu_0 = (1 - m_{00}^*)^{-1} \sum_{k=1}^p m_{0k}^* \bar{\nu}_k$ we see that μ and ν are left and right eigenvectors respectively of $M(\lambda) = M^*$ also satisfying $\langle \mu, \nu \rangle = 1$ and $\langle \mu, 1 \rangle = 1$.

It is not difficult to show in this case that all of the results in section 2 remain valid. The proofs make substantial use of the known results for the p -type process $\bar{X}(t)$ and of the fact that $m_{00}^* < 1$. The details of Theorem 2.4. will be omitted, however.

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