Savits, T.H.
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THE SUPERCRITICAL MULTI-TYPE CRUMP AND MODE AGE-DEPENDENT MODEL

THOMAS H. SAVITS

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1. Introduction

In 1966, Kesten and Stigum [10] obtained necessary and sufficient conditions for the supercritical p-type Galton-Watson process (appropriately normalized) to converge to a nontrivial limit distribution. These results have been extended to other models by various authors. The age-dependent Bellman-Harris model was considered by Athreya [1] in 1969 for $p=1$; more recently, N. Kaplan [8] traeted the general p-type version in 1975. The single-type ($p=1$) Crump and Mode model was considered by R. Doney [5] in 1972. In this paper, we consider the multi-type version of the Crump and Mode model. As in all of the above, the results depend upon the finiteness of $E[Y|\log Y|]$ for suitably defined random variables Y. Our proof relies heavily on the $p=1$ results and has the same flavor as a paper of Athreya's [2].

We shall first describe the model on an intuitive basis. Let $K_i(t)=(K_{i1}(t), \cdots, K_{ip}(t))$, $1\leq i\leq p$, be arbitrary vector-valued counting processes. $K_{ij}(t)$ counts the potential number of offspring of the jth type born to an individual of the ith type during the time interval $[0, t]$. We arbitrarily stop the counting process K_i at a random time L_i, the lifetime of an individual of the ith type. Set

$$N_i(t) = \begin{cases} K_i(t) & \text{if } t<L_i \\ K_i(L_i) & \text{if } t\geq L_i \end{cases}$$

and $G_i(t)=Pr\{L_i\leq t\}$. Thus N_i counts the actual number of offspring born to an individual of type i during its lifetime and G_i is its lifetime distribution. Each newborn object behaves similarly and all particles behave independently

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Abstract. Let $X(t)=(X_1(t), \cdots, X_p(t))$ be a p-dimensional supercritical Crump-Mode age-dependent branching process. If $Z(t)=(Z_1(t), \cdots, Z_p(t))$ counts the number of objects alive at time t, we find necessary and sufficient conditions for $Z(t)e^{-\lambda t}$ to converge in distribution to a nontrivial random variable. We also investigate some of its properties. By considering the total progeny process we deduce convergence in probability to the limiting age distribution. Lastly we consider a generalized immigration model.

of all other particles. Using these ingredients, one can construct a stochastic population process $(\Omega, P, X(t))$ having the above characteristics (cf. Mode [12]). The process $X(t)=(X_1(t), \cdots, X_p(t))$ not only keeps track of the number and type, but also age. Thus for each t and ω, either $X_i(t, \omega)=0$ (no particles of type i at time t), $+\infty$ (an infinite number of particles of type i at time t), or for some $n\geq 1$, $X_i(t, \omega)\in[0, \infty)^n$. In the latter case, if $X_i(t, \omega)=(x_1, \cdots, x_n)$, then there are n objects of the *ith* type alive at time t and of ages $x_1, \cdots, x_n$ respectively.

If f and g are real-valued functions (defined on $[0, \infty)$) satisfying $|f|\leq 1$ and g nonnegative or bounded, we extend them to $\{0\}\cup\bigcup_{n=1}^{\infty}[0, \infty)^n\cup\{+\infty\}$ by

$$\hat{f}(x) = \begin{cases} 1 & \text{if } x=0 \\ \prod_{i=1}^{n} f(x_i) & \text{if } x=(x_1, \cdots, x_n)\in[0, \infty)^n \\ 0 & \text{if } x=+\infty \end{cases}$$

and

$$\check{g}(x) = \begin{cases} 0 & \text{if } x=0 \text{ or } +\infty \\ \sum_{i=1}^{n} g(x_i) & \text{if } x=(x_1, \cdots, x_n)\in[0, \infty)^n \,. \end{cases}$$

If $f=(f_1, \cdots, f_p)$ and $g=(g_1, \cdots, g_p)$ are vectors of such functions, then we set

$$\hat{f}(X(t)) = \prod_{i=1}^{p} \hat{f}_i(X_i(t))$$

and

$$\check{g}(X(t)) = \sum_{i=1}^{p} \check{g}_i(X_i(t)) \,.$$

Also let e_i be the p-vector $(\delta_{i1}, \cdots, \delta_{ip})$ where δ_{ij} is the Kronecker delta. Furthermore, we denote the conditional expectations $E[\cdot\,|X(0)=e_i]$ by $E_i[\cdot]$.

It is intuitively clear and can be rigorously shown that the following representations are valid. Let ${}_iX(t)$ denote the process $X(t)$ given that we start with an object of type i. Then if f and g are vector valued functions we have

$$\hat{f}({}_iX(t)) = \exp\{\delta(t-L_i)\log f_i(t)\} \prod_{k=1}^{p} \prod_{l=1}^{N_{ik}(t)} \hat{f}({}_kX(t-t_{il}^k))$$

and

$$\check{g}({}_iX(t)) = \delta(t-L_i)g_i(t)+\sum_{k=1}^{p} \sum_{l=1}^{N_{ik}(t)} \check{g}({}_kX(t-t_{il}^k)) \,,$$

where $0\leq t_{i1}^k\leq t_{i2}^k\leq\cdots$ are the successive times at which the process $N_{ik}(t)$ increases by one, $\delta(t)=0$ or 1 accordingly as $t\geq 0$ or <0, and all the processes $\{{}_kX(t-t_{il}^k),\ t\geq t_{il}^k\}_{k,l}$ are conditionally independent given the process $\{N_i(t);\ t\geq 0\}$. Consequently, if $u_i(t)=E_i[\hat{f}(X_t)]$ and $v_i(t)=E_i[\check{g}(X_t)]$, then

$$u_i(t) = f_i(t)\int_t^{\infty} H_i^y[u^t]dG_i(y)+\int_0^t H_i^y[u^t]dG_i(y) \tag{1.1}$$

and

(1.2) $$v_i(t) = g_i(t)[1-G_i(t)]+\sum_{j=1}^{p}\int_0^t v_j(t-y)dF_{ij}(y),$$

$1\leq i\leq p$, where H_i^y is the conditional probability generating functional given by

$$H_i^y(\phi) = H_i^y((\phi_1, \cdots, \phi_p)) = E_i[\exp\{\sum_{j=1}^{p}\int_0^\infty \log \phi_j(x)dN_{ij}(x)\} \mid L_i = y]$$

(cf. Doney [5]), $u^t(y)=(u_1^t(y), \cdots, u_p^t(y))$ with $u_j^t(y)=u_j(t-y)$ if $t\geq y$ and $=1$ if $t<y$, and $F_{ij}(x)=E[N_{ij}(x)]$. By $\int_a^b$ we mean $\int_{(a,b]}$.

2. Assumptions and statement of results

Again, let $F_{ij}(x)=E[N_{ij}(x)]$ and set $m_{ij}=F_{ij}(+\infty)$.

(2.1) **Assumptions:**

(i) $F(x)=[F_{ij}(x)]$ is a non-lattice matrix of Borel measures (see Crump [3]) and $F(0+)=0$.

(ii) $H(s)=(H_1(s), \cdots, H_p(s))$ is nonsingular (see Harris [6]), where $H_i(s) =E[s^{N_i(+\infty)}]$.

(iii) $m_{ij}<\infty$ all i,j and $M=[m_{ij}]$ is positively regular.

(iv) Since M is positively regular, it has a positive eigenvalue ρ of maximum modulus. We suppose that $\rho>1$.

Assumption (iii) guarantees that the process $X(t)$ is regular; i.e., no explosion (see Mode [12]). Assumption (iv) just says that we are in the supercritical case. In the supercritical case, it is known that the extinction probability $q=(q_1, \cdots, q_p)$ is strictly less than $1=(1, \cdots, 1)$ and is the smallest nonnegative root of $q=H(q)=(H_1(q), \cdots, H_p(q))$. Furthermore, if q^* is any other nonnegative root, then either $q^*=q$ or $q^*=0$. Note also that

$$H_i(s) = \int_0^\infty H_i^y(s)dG_i(y)$$

where $s=(s_1, \cdots, s_p)$ and $|s_i|\leq 1$ all i.

Let us define a new matrix $M(\alpha)$ by

$$m_{ij}(\alpha) = \int_0^\infty e^{-\alpha x}dN_{ij}(x).$$

Since $M(\alpha)$ is also positively regular, it has a positive eigenvalue $\rho(\alpha)$ of maximum modulus. We choose $\alpha>0$ such that $\rho(\alpha)=1$ and set $\lambda=\alpha$. It is known that such an α exists since $\rho=\rho(0)>1$. λ is called the Malthusian parameter.

From the Frobenius theory, it follows that corresponding to λ there exists strictly positive left and right eigenvectors of $M(\lambda)$, μ and ν respectively, satisfying $\langle\mu, 1\rangle=1$ and $\langle\mu, \nu\rangle=1$. Here $\langle\cdot, \cdot\rangle$ denotes the inner product. Lastly, we set $m_{ij}^*=m_{ij}(\lambda)$ and $M^*=M(\lambda)$.

According to Crump [3], it then follows from our assumptions that given $g=(g_1, \cdots, g_p)$ such that each g_i is bounded on finite t-intervals, (1.2) has a unique solution $v=(v_1, \cdots, v_p)$ which is bounded on finite t-intervals; moreover, if each $\bar{g}_i(t)=[1-G_i(t)]e^{-\lambda t}g_i(t)$ is directly Riemann integrable, then

$$v_i(t)e^{-\lambda t} \to d\nu_i \sum_{j=1}^p \mu_j \int_0^\infty \bar{g}_j(t)dt \tag{2.2}$$

as $t\to\infty$, where d is a positive constant independent of g.

In particular, if we take $g=e_j$ (considered as a vector of functions, then

$$m_{ij}(t) = E_i[\check{e}_j(X_t)] = E_i[Z_j(t)] \sim \nu_i\mu_j c_j e^{\lambda t}$$

as $t\to\infty$, where $c_j=d\int_0^\infty e^{-\lambda t}[1-G_j(t)]dt$. Here $Z_j(t)$ just counts the number of particles of type j alive at time t.

Let us now define $W_i(t)=Z_i(t)/c_ie^{\lambda t}$ and $W(t)=(W_1(t), \cdots, W_p(t))$. Set $W^*(t)=\langle\nu, W(t)\rangle$. Then we shall prove the following.

Theorem 2.1. *Define the random variables* $Y_{ij}=\int_0^\infty e^{-\lambda x}dN_{ij}(x)$. *Consider*

$$\sup_{ij} E[Y_{ij}|\log Y_{ij}|]\,. \tag{*}$$

Then $W^*(t)$ *converges in distribution to a nontrivial random variable* W^* *iff* (*) *is finite; moreover, in this case,* $P_i(W^*=0)=q_i$ *and* $E_i[W^*]=\nu_i$ *all* i.

Corollary 1°. *If* (*) *is finite, then* $W(t)\to\mu W^*$ *in distribution.*

Let $Z_i(x; t)$ be the number of particles of type i alive at time t and of age $\leq x$. Set $W_i(x; t)=Z_i(x; t)/c_i(x)e^{\lambda t}$ where $c_i(x)=d\int_0^x e^{-\lambda t}[1-G_i(t)]dt$. If $x=(x_1, \cdots, x_p)$, we set $W(x; t)=(W_1(x_1; t), \cdots, W_p(x_p; t))$.

Corollary 2°. *If* (*) *is finite,* $W(x; t)\to\mu W^*$ *in distribution.*

Theorem 2.2. *Assume that* (*) *is finite. If in addition we assume that for at least one index* i, *the random variable*

$$\sum_{j=1}^p Y_{ij}\nu_j \tag{**}$$

can take on at least two values with positive probability, then W^* *has a continuous density on* $(0, \infty)$.

Let $Y_i(t)$ denote the total number of objects of type i born in $[0, t]$ including the ancestor if it is also of type i. Set $V_i(t)=\lambda Y_i(t)/de^{\lambda t}$ and $V(t)=(V_1(t), \cdots, V_p(t))$.

Theorem 2.3. *If* (*) *is finite,* $(W(x;t), V(t)) \to (\mu W^*, \mu W^*)$ *in distribution.*

Corollary. *Under the hypothesis of Theorem 2.2, we have that* $Z_i(x;t)/|Y(t)| \to \mu_i c_i(x)\lambda d^{-1}$ *in probability off*

$$Q = \{Z(t) = (Z_1(t), \cdots, Z_p(t)) \to 0\}. \quad \text{Here } |Y(t)| = \textstyle\sum_{k=1}^{p} Y_k(t)\,.$$

Furthermore, if we start from a particle of type j then $Z_i(x;t)/Y_j(t) \to \mu_i \lambda c_i(x)/\mu_j d$ in probability off Q.

REMARK. Since convergence in probability is preserved under addition and mulitplication, it follows for example that $Z_i(x;t)/Z_k(y;t) \to \mu_i A_i(x)/\mu_k A_k(y)$, $|Z(t)|/|Y(t)| \to \lambda d^{-1}\sum_{k=1}^{p}\mu_k c_k$ and $Z_i(x;t)/|Z(t)| \to \mu_i c_i(x)/\sum_{k=1}^{p}\mu_k c_k$ in probability off Q. If we start from a particle of type j, then we also have $Y_j(t)/|Y(t)| \to \mu_j$ in probability off. Q. Here $A_i(x) = c_i(x)/c_i$ is the limiting age distribution.

In section 4 we consider a generalized immigration model. Basically it is a $(p+1)$—type Crump and Mode process corresponding to $(N_0(t), N_1(t), \cdots, N_p(t))$ in which $(N_1(t), \cdots, N_p(t))$ produces no particles of type 0. $N_0(t)$ can thus be considered as the immigration component. Under the assumptions of section 4 we have the following

Theorem 2.4. *In the supercritical case, all of the preceeding results remain valid for this immigration model (provided we don't divide by* μ_0 *in Remark of Theorem* 2.3. *since* $\mu_0 = 0$*). In particular, starting with a particle of type* 0,

$$(W_1(t), \cdots, W_p(t)) \to \bar{\mu} W^*$$

in distribution to a nontrivial random variable iff (*) *is finite; in this case,* $P_0(W^*) = q_0$ *and* $E_0(W^*) = \nu_0$. *Fruthermore, if* (**) *is also true,* W^* *has a continuous density on* $(0, \infty)$.

REMARK. For the immigration process, we take the sup over all $1 \le i, j \le p$ in (*) and we only consider the random variables $\sum_{j=1}^{p} Y_{ij}\nu_j$, $1 \le i \le p$, for (**)

3. Proofs

Let $\Phi_i(u,t) = E_i[\exp(-uW^*(t))]$ be the Laplace transform of $W^*(t)$. It follows from (1.1) that Φ_i satisfies

$$\Phi_i(u,t) = \exp\{-ue^{-\lambda t}\nu_i/c_i\}\int_t^\infty H_i^y[\Phi^t(ue^{-\lambda\cdot}, \cdot)]dG_i(y)$$
$$+\int_0^t H_i^y[\Phi^t(ue^{-\lambda\cdot}, \cdot)]dG_i(y)\,.$$

By $\Phi^t(ue^{-\lambda\cdot}, \cdot)$ we mean the vector function $\Phi(ue^{-\lambda x}, t-x)$ if $t \ge x$ and the vector 1 if $t < x$.

Since $E_i[W^*(t)]\to\nu_i$ as $t\to\infty$, $\{W^*(t)\}$ is tight (with respect to each P_i). Suppose now that $W^*(t)\to W^*$ in distribution. Then $\Phi_i(u)=E_i[\exp\{-uW^*\}]$ satisfies

$$(3.1)\qquad \Phi_i(u)=\int_0^\infty H_i^y[\Phi(ue^{-\lambda\cdot})]dG_i(y)=H_i[\Phi(ue^{-\lambda\cdot})]\,.$$

If we now let $u\uparrow\infty$, then we see that $q_i^*=P_i(W^*=0)$ satisfies

$$q^*=H(q^*)\,.$$

Hence either $q^*=1$ or $q^*=q<1$. In the former case, it follows that $W^*=0$ a.s. (P_i), all i.

For θ a strictly positive p-vector, let

$C(\theta)=\{\phi=(\phi_1,\cdots,\phi_p)$: ϕ_i is the Laplace transform of a probability measure on $[0,\infty)$ and $\lim_{u\downarrow 0}u^{-1}[1-\phi_i(u)]=\theta_i\}$ and set $C=\bigcup_{\theta>0}C(\theta)$. According to the above, we see that either $\Phi\equiv 1$ or $\Phi\in C$ (Actually, we can say in this case that $\Phi\in C(\theta)$ for some $\theta\le\nu$).

Before we can proceed further we shall need a few preliminaries.

Lemma 3.1. *If $0\le\phi\le\psi\le 1$ as vector functions, then $H_i(\phi)\le H_i(\psi)$ and*

$$|H_i(\phi)-H_i(\psi)|\le\sum_{j=1}^p\int_0^\infty|\phi_j(x)-\psi_j(x)|\,dF_{ij}(x),\ 1\le i\le p\,.$$

The proof is similar to the one-dimensional version given in Doney [5]. Let $\phi=(\phi_1,\cdots,\phi_p)$ with $0\le\phi_i\le 1$ and set

$$A_i(\phi)=H_i(\phi)-1+\sum_{j=1}^p\int_0^\infty[1-\phi_j(x)]dF_{ij}(x)\,.$$

Again as in Doney [5], we have that if $0\le\phi\le\psi\le 1$, then $0\le A_i(\psi)\le A_i(\phi)$. We define $\bar{A}_i(\phi)=A_i((\phi,\cdots,\phi))$ and set $A^*(\phi)=\sum_{i=1}^p\mu_i\bar{A}_i(\phi)$. It is not hard to see that A^* corresponds to N^* exactly as A corresponds to N in Doney, where $N^*(t)$ is the counting process which with probability μ_i looks like $\sum_{j=1}^p N_{ij}(t)$. Since

$$E\Big[\int_0^\infty e^{-\lambda x}dN^*(x)\Big]=\sum_{i=1}^p\mu_i\sum_{j=1}^p\int_0^\infty e^{-\lambda x}dF_{ij}(x)=\sum_{i,j=1}^p\mu_i m_{ij}{}^*=1\,,$$

we are in a position to use the one-dimensional results of Doney. Let $\psi^*(u)=u^{-1}A^*(\exp\{-ue^{-\lambda\cdot}\})$ for $u>0$. Then

Lemma 3.2. *For every $\delta<0$ and $0<r<1$,*

$$\sum_{n=0}^\infty\psi^*(\delta r^n)<\infty\quad\textit{and}\quad\lim_{\delta\downarrow 0}\sum_{n=0}^\infty\psi^*(\delta r^n)=0$$

iff $E[Y^*|\log Y^*|]<\infty$, where $Y^*=\int_0^\infty e^{-\lambda x}dN^*(x)$; moreover, $E[Y^*|\log Y^*|]<\infty$ iff $\sup_{i,j}E[Y_{ij}|\log Y_{ij}|]<\infty$.

The second "iff" can be verified as in Athreya [2].

Lemma 3.3. *Let Φ be a solution of* (3.1). *Then $\Phi\in C$ only if* (*) *is finite.*

Proof. Suppose $\Phi\in C$. Then there are constants $c>0$, $\delta>0$ such that for all $u\le\delta$, $1-\Phi_j(u)\ge cu$. Let $g_i(u)=u^{-1}[1-\Phi_i(u)]$ for $u>0$. From (3.1) we have that

$$\begin{aligned} g_i(u) &= u^{-1}(1-H_i[\Phi(ue^{-\lambda\cdot})]) \\ &= \textstyle\sum_{j=1}^{p}\int_0^\infty g_j(ue^{-\lambda x})e^{-\lambda x}dF_{ij}(x)-u^{-1}A_i[\Phi(ue^{-\lambda\cdot})] \\ &\le \textstyle\sum_{j=1}^{p} m_{ij}^*\int_0^\infty g_j(ue^{-\lambda x})dF_{ij}^*(x)-u^{-1}\bar{A}_i[\exp\{-cue^{-\lambda\cdot}\}] \end{aligned}$$

for all $0<u\le\delta$, where F_{ij}^* is a probability measure (if $m_{ij}^*=0$, let F_{ij}^* be any nontrivial probability measure on $[0,\infty)$ having finite mean). Since each $g_j(ue^{-\lambda x})\uparrow$ as $x\uparrow$ we can find a nondegenerate probability measure $\tilde{G}$ such that $\int_0^\infty g_j(ue^{-\lambda x})dF_{ij}^*(x)\le\int_0^\infty g_j(ue^{-\lambda x})d\tilde{G}(x)$ all i, j. Now set $g(u)=\sum_{i=1}^{p}\mu_i g_i(u)$. Then

$$g(u)\le\int_0^\infty g(ue^{-\lambda x})d\tilde{G}(x)-c\psi^*(cu)\,.$$

Proceeding as in Doney, we have the desired result.

Lemma 3.4. *Let $I_i(u,t)=u^{-1}E_i[\exp\{-uW^*(t)\}+uW^*(t)-1]$ for $u>0$. Then if* (*) *is finite,* $\lim_{u\downarrow 0}\sup_{t\ge 0}|I_i(u,t)|=0$, $1\le i\le p$.

Proof. Define $m_i^*(t)=E_i[W^*(t)]$. Using (1.1) and (1.2) we can rewrite I_i as

$$\begin{aligned} I_i(u,t) &= u^{-1}[1-G_i(t)]\,[\exp\{-uv_ie^{-\lambda t}/c_i\}+uv_ie^{-\lambda t}/c_i-1] \\ &\quad +u^{-1}[1-\exp\{-uv_ie^{-\lambda t}/c_i\}]\int_t^\infty\{1-H_i^y[\Phi^t(ue^{-\lambda\cdot},\,\cdot)]\}dG_i(y) \\ &\quad +u^{-1}A_i[\Phi^t(ue^{-\lambda\cdot},\,\cdot)]+\textstyle\sum_{j=1}^{p}m_{ij}^*\int_0^t I_j(ue^{-\lambda y},\,t-y)dF_{ij}^*(y)\,. \end{aligned}$$

Since $I_i(u,t)\ge 0$, $\Phi_i(u,t)\ge 1-u\,m_i^*(t)$. Recalling that $m_i^*(t)\to v_i$ as $t\to\infty$ and is bounded on finite t-intervals, it follows that there exist positive constants c, η, and δ such that $1-\Phi_i(u,t)\le cu$ all u, t, i and $\Phi_i(u,t)\ge e^{-\eta u}$ all $0<u<\delta$, i and t. Consequently, for $0<u<\delta$

$$\begin{aligned} 1-H_i[\Phi^t(ue^{-\lambda\cdot},\,\cdot)] &\le \textstyle\sum_{j=1}^{p}\int_0^t[1-\Phi_j(ue^{-\lambda y},\,t-y)]dF_{ij}(y) \\ &\le uc\textstyle\sum_{j=1}^{p}\int_0^\infty e^{-\lambda y}dF_{ij}(y) = uc\sum_{j=1}^{p}m_{ij}^* \end{aligned}$$

and

$$A_i[\Phi^t(ue^{-\lambda\cdot}, \cdot)] \leq \bar{A}_i[\exp\{-\eta ue^{-\lambda\cdot}\}] .$$

Now set $I_i^T(u)=\sup_{0\leq t\leq T} I_i(u, t)$. Then there is a constant $M>0$ such that for all $0<u\leq\delta$,

$$I_i^T(u) \leq uM+u^{-1}\bar{A}_i[\exp\{-\eta ue^{-\lambda\cdot}\}]+\sum_{j=1}^p m_{ij}^* \int_0^T I_j^T(ue^{-\lambda y})dF_{ij}^*(y) .$$

Since each $I_i^T(u)$ is nondecreasing in u, we can find a non-degenerate probability measure G such that $\int_0^T I_j^T(ue^{-\lambda y})dF_{ij}^*(y) \leq \int_0^T I_j^T(ue^{-\lambda y})dG(y)$ for all $i, j, u>0$, $T>0$. Thus if we set $I^T(u)=\sum_{i=1}^p \mu_i I_i^T(u)$, then

$$I^T(u) \leq uM+\eta\psi^*(\eta u)+\int_0^T I^T(ue^{-\lambda y})dG(y) .$$

Now proceed as in Doney [5].

Lemma 3.5. *If* Φ^1, $\Phi^2 \in \mathcal{C}(\theta)$ *and both satisfy* (3.1), *then* $\Phi^1=\Phi^2$.

Proof. Let $g_i(u)=u^{-1}|\Phi_i^1(u)-\Phi_i^2(u)|$ for $u>0$. Then

$$\begin{aligned} g_i(u) &\leq \sum_{j=1}^p m_{ij}^* \int_0^\infty g_j(ue^{-\lambda y})dF_{ij}^*(y) \\ &= \sum_{j=1}^p m_{ij}^* E[g_j(ue^{-\lambda X_{ij}})] \end{aligned}$$

where X_{ij} is a random variable with distribution function F_{ij}^*; moreover, we may assume that they are independent. Iterating yields,

$$g_i(u) \leq \sum_{1\leq j_1,\cdots,j_k\leq p} m_{ij_1}^* m_{j_1j_2}^* \cdots m_{j_{k-1}j_k}^* E[g_{j_k}(ue^{-\lambda S_k(j_0,j_1,\cdots,j_k)})]$$

where $j_0=i$ and

$$S_k(j_0, j_1, \cdots, j_k) = \sum_{l=1}^k X_{j_{l-1},j_l}^l \geq \sum_{l=1}^k \min_{i,j}(X_{ij}^l) .$$

The supercript l refers to independent copies of the same random variable. Since $E[\min_{i,j}(X_{i,j}^1)]>0$ we can now proceed as in Kaplan [8] to deduce that $g_i=0$ all $u>0$ and hence $\Phi^1=\Phi^2$.

Proof of Theorem 2.1. Suppose (*) is infinite and $W^*(t)\to W^*$ in distribution. Then it follows from Lemma 3.3. that $W^*=0$ w.p. 1. On the other hand suppose (*) is finite. Set

$$K_i(u) = \limsup_{t\to\infty} \sup_{s\geq 0} u^{-1}|\Phi_i(u, t+s)-\Phi_i(u, t)| \text{ for } u>0 .$$

It is an easy consequence of Lemma 3.4 that $\lim_{u\downarrow 0} K_i(u)=K_i(0+)=0$ all i. Now making use of the equation that Φ_i satisfies it is not hard to show (cf. Athreya [1]) that

$$K_i(u) \leq \sum_{j=1}^{p} m_{ij}^* E[K_j(ue^{-\lambda X_{ij}})]$$

where X_{ij} is as in the proof of Lemma 3.5. It follows then that $K_i(u)=0$ for $u>0$ and all i. Consequently $\lim_{t\to\infty}\Phi_i(u, t)=\Phi_i(u)$ exists and satisfies (3.1). Because of tightness we conclude that $W^*(t)$ converges in distribution to a nonnegative random variable W^*; furthermore, it follows from Lemma 3.4 that $E_i[W^*]=\nu_i$ and hence is nontrivial.

Proof of Corollary 1° of Theorem 2.1. All we need show is that $\langle\eta, W(t)\rangle \to\langle\eta, \mu\rangle W^*$ in distribution for any nonnegative p-vector η. First observe that $E_i[\langle\eta, W(t)\rangle]\to\langle\eta, \mu\rangle\nu_i$ as $t\to\infty$. Secondly, it follows from (1.1) that if we do have convergence in distribution, then the transform of the limit random variable is a solution of (3.1). Lastly, we see that there exists a positive constant K such that $0\leq\langle\eta, W(t)\rangle\leq K\langle\nu, W(t)\rangle=KW^*(t)$. Since $B(x)=e^{-x}+x-1$ increases in x for $x\geq 0$,

$$u^{-1}E_i[B(u\langle\eta, W(t)\rangle)]\leq K(uK)^{-1}E_i[B(uKW^*(t))] = KI_i(uK, t)\,.$$

Hence $\lim_{u\downarrow 0}\sup_{t\geq 0}u^{-1}E_i[B(u\langle\eta, W(t)\rangle)]=0$ all i if (*) is finite. Now proceed as in the proof of Theorem 2.1.

Everything that we have done above can be extended to the following situation. Let $g=(g_1, \cdots, g_p)$ be a vector of nonnegative bounded functions which are directly Riemann integrable and set $c_i(g_i)=d\int_0^\infty e^{-\lambda t}[1-G_i(t)]g_i(t)dt$. Assume for the moment that each $c_i(g_i)>0$. Set

$$W_i(g_i; t) = \check{g}_i(X_i(t))/c_i(g_i)e^{\lambda t} \text{ and } W(g; t) = (W_1(g_1; t), \cdots, W_p(g_p; t))\,.$$

Since for each nonnegative p-vector η, there is a constant $K>0$ such that $\langle\eta, W(g; t)\rangle\leq KW^*(t)$ and $E_i[\langle\eta, W(g; t)\rangle]\to\langle\eta, \mu\rangle\nu_i$ as $t\to+\infty$, we deduce as in the proof of Corollary 1 that $W(g; t)\to\mu W^*$ in distribution. Equivalently, we can say that

$$(\check{g}_1(X_1(t)), \cdots, \check{g}_p(X_p(t)))e^{-\lambda t} \to \mu(g)W^*$$

in distribution, where $\mu(g)$ is the p-vector with components $\mu_i(g)=\mu_i\, c_i(g_i)$. This latter statement remains valid even if some of the terms $c_i(g_i)$ are zero.

Proof of Corollary 2° of Theorem 2.1. Take $g_i(y)=1_{[0,x_i]}(y)$, $1\leq i\leq p$, in the above.

Proof of Theorem 2.2. One can modify the proof given in Doney [5] for the one-dimensional case along the same lines that Kaplan [8] used for the Bellman-Harris model. The details will be omitted.

Proof of Theorem 2.3. Although this result is not a corollary of Theorem 2.1, it is a corollary of its proof as we shall now show. Recall that $Y_j(t)$ is the total number of objects of type j born in $[0, t]$ including its acnestor if it is also of type j. As in section 1, we can show that the following representation is valid.

$$ {}_iY_j(t) = \delta_{ij} + \sum_{k=1}^{p} \sum_{l=1}^{N_{ik}(t)} {}_kY_j(t-t_{ik}^l) . $$

Consequently, $n_{ij}(t)=E_i[Y_j(t)]$ satisfies

$$ n_{ij}(t) = \delta_{ij} + \sum_{k=1}^{p} \int_0^t n_{kj}(t-y) dF_{ik}(y) . $$

Hence, $n_{ij}(t) \sim \lambda^{-1} d v_i \mu_j e^{\lambda t}$ as $t \to +\infty$. We set $V_i(t)=\lambda Y_i(t)/de^{\lambda t}$ and $V(t)=(V_1(t), \cdots, V_p(t))$. To prove our theorem, it suffices to consider sums of the form $U(t)=\langle \xi, V(t)\rangle + \langle \eta, W(x; t)\rangle$ for nonnegative p-vectors ξ and η. Note that $E_i[U(t)] \to v_i \langle \xi+\eta, \mu\rangle$ as $t\to\infty$. If $\psi_i(u, t)=E_i[\exp\{-uU(t)\}]$, then from our representations, it follows that

$$ \begin{aligned} \psi_i(u, t) &= \exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i/c_i(x_i))\} \int_t^\infty H_i^y[\psi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y) \\ &\quad + \exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\} \int_0^t H_i^y[\psi^t(ue^{-\lambda \cdot}, \cdot)] dG_i(y) . \end{aligned} $$

Hence if $U(t)\to U$ in distribution, its Laplace transform is a solution of (3.1). Everything now follows as before once we rewrite $J_i(u, t)=E_i[B(uU(t))]$ as

$$ \begin{aligned} J_i(u, t) = {} & u^{-1}[1-G_i(t)] [\exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i/c_i(x_i))\} \\ & + ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i/c_i(x_i)) - 1] \\ & + u^{-1} G_i(t) [\exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\} + ue^{-\lambda t} \xi_i \lambda d^{-1} - 1] \\ & + u^{-1}[1-\exp\{-ue^{-\lambda t}(\xi_i \lambda d^{-1} + \eta_i/c_i(x_i))\}] \\ & \times \int_t^\infty \{1 - H_i^y[\psi^t(ue^{-\lambda \cdot}, \cdot)]\} dG_i(y) \\ & + u^{-1}[1-\exp\{-ue^{-\lambda t} \xi_i \lambda d^{-1}\}] \int_0^t \{1 - H_i^y[\psi^t(ue^{-\lambda \cdot}, \cdot)]\} dG_i(y) \\ & + u^{-1} A_i[\psi^t(ue^{-\lambda \cdot}, \cdot)] + \sum_{k=1}^{p} m_{ik}^* \int_0^t J_k(ue^{-\lambda x}, t-x) dF_{ik}^*(x) . \end{aligned} $$

Proof of Corollary of Theorem 2.3. Apply the same technique as in Doney [4].

4. Immigration processes

Let $(N_0(t), N_1(t), \cdots, N_p(t))$ generate a $(p+1)$ dimensional Crump and Mode process. We assume that each $N_i(t)$ process $(1 \le i \le p)$ cannot give birth to objects of type O, but $N_0(t)$ gives birth to at least one object of type $1, 2, \cdots$ or p; *i.e.*, we assume that $m_{i0}=0$ for $i=1, \cdots, p$, and that there exists at least one

$j \neq 0$ such that $m_{0j} \neq 0$. $N_0(t)$ can thus be considered as an immigration component. We will call such a process a p-type age-dependent branching process with immigration. This model seems to include all immigration models that have appeared in the literature. For example, let $N_0(t)=(1, 0, \cdots, 0)=e_0$ for $t<L$ and $=(1, \xi)$ for $t \geq L$ where ξ is a p-dimensional random variable independent of L having probability generating function $h(s_1, \cdots, s_p)$ and let $(N_1(t), \cdots, N_p(t))$ generate a p-demensional Bellman-Harris model. The case $p=1$ was originally studied by Jagers [7] while the general p-dimensional version was recently considered by Kaplan and Pakes [9]. If we want the times of immigration to obey a Poisson distribution, let $N_0(t)=(0, N_{01}(t), \cdots, N_{0p}(t))$ where $(N_{01}(t), \cdots, N_{0p}(t))$ is a nonhomogeneous compound Poisson process.

The study of immigration processes thus reduces to the study of such $(p+1)$-type models where we start with a particle of type 0. The only thing different about these processes is that now the corresponding mean matrix M is reducible; specifically, M has the form

$$M=\left(\begin{array}{c|c} m_{00} & m_{01} \cdots m_{0p} \\ \hline 0 & \\ \vdots & \bar{M} \\ 0 & \end{array}\right)$$

where $\bar{M}$ is the $p \times p$ matrix corresponding to the p-dimensional process generated by $(N_1(t), \cdots, N_p(t))$. The eigenvalue ρ of maximum modulus is given by $\rho=\max(m_{00}, \bar{\rho})$ where $\bar{\rho}$ is the eigenvalue of maximum modulus corresponding to $\bar{M}$. From now on we shall assume that $(N_1(t), \cdots, N_p(t))$ satisfies assumptions (2.1). We also assume that $m_{0i}<\infty$ all $i=0, i, \cdots, p$ and that $F_{00}(x)$ is a non-lattice Borel measure satisfying $F_{00}(0+)=0$. In addition we shall make the following assumption.

(4.1) **Assumption.** $1 \geq m_{00}$

Consequently it follows that $\rho=\bar{\rho}$ and if we choose $\alpha>0$ such that $\rho(\alpha)=1$, then $\rho(\alpha)=\bar{\rho}(\alpha)$. Hence the Malthusian parameter $\lambda=\alpha$ corresponds to that of the process generated by $(N_1(t), \cdots, N_p(t))$. This assumption (4.1) is satisfied for the supercritical immigration processes that have been considered in the literature. Without (4.1) it is conceivable that $\rho(\alpha)=m_{00}^*=1>\bar{\rho}(\alpha)$ even if we assume that $\rho=\bar{\rho}$. This possibility will be investigated in the future as well as the critical and subcritical cases for this model. For more information on the reducible case, see Kesten and Stigum [11] and Mode [12].

Let $\bar{\mu}$ and $\bar{\nu}$ be the strictly positive left and right eigenvectors respectively of $\bar{M}(\lambda)$ satisfying $\langle\bar{\mu}, \bar{\nu}\rangle=1$ and $\langle\bar{\mu}, 1\rangle=1$. Setting $\mu=(0, \bar{\mu})$ and $\nu=(\nu_0, \bar{\nu})$ where $\nu_0=(1-m_{00}^*)^{-1}\sum_{k=1}^{p} m_{0k}^*\bar{\nu}_k$ we see that μ and ν are left and right eigenvectors respectively of $M(\lambda)=M^*$ also satisfying $\langle\mu, \nu\rangle=1$ and $\langle\mu, 1\rangle=1$.

It is not difficult to show in this case that all of the results in section 2 remain valid. The proofs make substantial use of the known results for the p-type process $\bar{X}(t)$ and of the fact that $m_{00}^{*}<1$. The details of Theorem 2.4. will be omitted, however.

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