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POLYNOMIAL BOUNDS ON THE NUMBER OF SCATTERING POLES FOR METRIC PERTURBATIONS OF THE LAPLACIAN IN R^n, $n\geq 3$, ODD

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1. Introduction The purpose of this note is to obtain a polynomial bound on the number of the scattering poles associated to the operator

$$G = c(x)^{-1} \sum_{1\leq i,\, j\leq n} \partial_{x_i}(g_{ij}(x)\,\partial_{x_j}) \qquad \text{in} \quad \boldsymbol{R}^n,$$

where $n\geq 3$, odd. We consider this operator under the following assumptions on the coefficients:

(a) $c(x)\in C(\boldsymbol{R}^n; \boldsymbol{R})$ and $c(x)\geq c_0>0$ for all $x\in\boldsymbol{R}^n$;

(b) $g_{ij}(x)\in C^1(\boldsymbol{R}^n)$ and the matrix $\{g_{ij}(x)\}$ is a strictly positive hermitian one for all $x\in\boldsymbol{R}^n$, i.e.

(b)$_1$ $g_{ij}(x)=\overline{g_{ji}(x)}$, $\quad i,j=1,\cdots,n,\ \forall x\in\boldsymbol{R}^n$;

(b)$_2$ $\sum_{i\leq j,\, j\leq n} g_{ij}(x)\xi_i\xi_j\geq C\,|\xi|^2,\ \forall(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0,\ C>0$;

(c) there exists a $\rho_0>0$ so that $c(x)=1$ and $g_{ij}(x)=\delta_{ij}$ for $|x|\geq\rho_0$, δ_{ij} being the Kronecker's symbol.

It is well known that under the above assumptions the operator G has a self-adjoint realization, which will be again denoted by G, in the Hilbert space $H=L^2(\boldsymbol{R}^n; c(x)dx)$. Note also that it follows from the assumption (b)$_2$ that the operator G is elliptic. By G_0 we shall denote the self-adjoint realization of the Laplacian Δ in the Hilbert space $H_0=L^2(\boldsymbol{R}^n)$.

It is well known that, under the above assumptions, the scattering matrix corresponding to the pair $\{G, G_0\}$ admits a meromorphic continuation to the entire complex plane $\boldsymbol{C}$. Let $\{\lambda_j\}$ be the poles of this continuation, repeating according to multiplicity, and set

$$N(r)=\#\{\lambda_j: |\lambda_j|\leq r\}.$$

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It is shown in [10] that for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ so that

$$M(r)\leq C_\varepsilon r^{n+1+\varepsilon}+C_\varepsilon\,. \tag{1}$$

In fact, (1) is proved there for first order symmetric systems in $\boldsymbol{R}^n$, $n\geq 3$, odd, but the proof easily extends to our case. The aim of this work is to improve (1). More precisely, we have the following

Theorem. *There exists a constant $C>0$ so that the number of the scattering poles associated to the operator G satisfies the bound*

$$N(r)\leq C\,r^{n+1}+C\,. \tag{2}$$

Note that the desired result is to obtain the bound

$$N(r)\leq C\,r^{n}+C\,. \tag{3}$$

In [7] Melrose proved (3) for the Laplacian in exterior domains with Dirichlet or Robin boundary conditions, while in [13] Zworski proved (3) for the Schrödinger operator $\Delta+V(x)$ with a potential $V\in L_0^\infty(\boldsymbol{R}^n)$. Note that a bound of the form (3) is important for obtaining Weyl asymptotics of the phase shift (see [8], where such an asymptotic is obtained for an arbitrary smooth obstacle). To our knowledge, the problem of obtaining a bound of the form (3) on the number of the scattering poles associated to the operator G is still open. Let us also mention Intissar's work [4] where a bound of the form (2) on the number of the scattering poles associated to the operator $(-i\vec{\nabla}+\vec{b}(x))^2+a(x)$ in $\boldsymbol{R}^n$, $n\geq 3$, odd, is obtained, where $a(x)\in C_0^\infty(\boldsymbol{R}^n;\boldsymbol{R})$, $\vec{b}(x)\in C_0^\infty(\boldsymbol{R}^n;\boldsymbol{R})$. In [3] and [4] he has also obtained an analogue of (2) in the case of even n, $n\geq 4$.

To prove (2) we exploit the fact that the scattering poles, with multiplicity, coincide with the poles of the meromorphic continuation of the cutoff resolvent $R_\chi(z)=\chi R(z)\chi$ from $\{z\in\boldsymbol{C}\colon \operatorname{Im} z>0\}$ to the entire complex plane $\boldsymbol{C}$, where $R(z)=(G+z^2)^{-1}$ for $\operatorname{Im} z>0$, and $\chi\in C_0^\infty(\boldsymbol{R}^n)$ is such that $\chi=1$ for $|x|\leq\rho_0+1$, $\chi=0$ for $|x|\geq\rho_0+2$ (see [5]). This enables us to characterize the scattering poles as the poles of a meromorphic function of the form $(1-K(z))^{-1}$ where $K(z)$ is an entire family of compact operators on H such that $K(z)^p$, $p=(n+1)/2$, is trace class. Then, following [4], [6], [7] and [13], we deduce that the scattering poles, with multiplicity, are among the zeros of the entire function $h(z)=\det(1-K(z)^p)$, and hence, to prove (2) it suffices to show that the order of $h(z)$ is less than or equal to $n+1$. However, in our case the operator $K(z)$ is much more complicated and therefore it is not so easy to obtain the desired order of $h(z)$. To overcome the difficulties we use precise estimates of the cutoff free resolvent for $\operatorname{Im} z\geq 0$ combained with an application of the Phragmen-Lindelof principle.

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2. Representation of the cutoff resolvent

First, we shall introduce some notations. Given two Hilbert spaces X and Y, $\mathcal{L}(X,Y)$ will denote the space of all linear bounded operators acting from X into Y. Given any $s>0$, H^s will denote the usual Sobolev space $H^s(\boldsymbol{R}^n)$. Finally, given a compact operator $\mathcal{A}$, $\mu_j(\mathcal{A})$ will denote the characteristic values of $\mathcal{A}$, i.e. the eigenvalues of $(\mathcal{A}^*\mathcal{A})^{1/2}$, ordered, with multiplicity, to form a nonincreasing sequence.

Denote by $R_0(z)$ the outgoing resolvent of Δ, i.e. that one with kernel $E(x-y,z)$ where $E(x,z)$ is the outgoing fundamental solution of the operator $\Delta+z^2$. Then we have $R_0(z)=(G_0+z^2)^{-1}\in\mathcal{L}(H_0, H_0)$ for $\operatorname{Im} z>0$. Moreover, it is well known that the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$(4)\qquad R_0(z)(x,y)=-(i/4)(2\pi)^{(n-2)/2}(z/|x-y|)^{(n-2)/2}H^{(1)}_{(n-2)/2}(z|x-y|)\,.$$

It follows easily from this representation that $\chi R_0(z)\chi$ forms an entire family of compact pseudodifferential operators of order -2 in $\mathcal{L}(H_0, H_0)$, χ being the function introduced in the previous section. Using this we shall build the meromorphic continuation of $R_\chi(z)$. Set $Q=G_0-G$ and fix a $z_0\in\boldsymbol{C}$, $\operatorname{Im} z_0>0$. Clearly, for $\operatorname{Im} z>0$, we have

$$(5)\qquad R(z)=R_0(z)+R(z)QR_0(z)$$

and

$$(6)\qquad R(z)=R(z_0)+(z_0^2-z^2)R(z)R(z_0)\,.$$

Combining these identities yields

$$R(z)(1-(z_0^2-z^2)QR_0(z)R(z_0))=R(z_0)+(z_0^2-z^2)R_0(z)R(z_0)$$

for $\operatorname{Im} z>0$. Multiplying the both sides of this identity by χ, since $Q=\chi Q$, we get

$$(7)\qquad R_\chi(z)(1-K(z))=R_\chi(z_0)+K_1(z)\qquad\text{for}\quad \operatorname{Im} z>0\,,$$

where $K(z)=(z_0^2-z^2)QR_0(z)R(z_0)\chi$ and $K_1(z)=(z_0^2-z^2)\chi R_0(z)R(z_0)\chi$. We need now the following

Lemma 1. *The operator-valued functions $K(z)$ and $K_1(z)$ have analytic continuations from $\{z\in\boldsymbol{C}\colon \operatorname{Im} z>0\}$ to the entire $\boldsymbol{C}$ with values in the compact operators in $\mathcal{L}(H,H)$. Moreover, there exists a constant $C>0$ so that*

$$(8)\qquad \mu_j(K(z))\le C\exp(C|z|),\ \forall z\in\boldsymbol{C},\ \forall j;$$

$$(9)\qquad \mu_j(K(z))\le C(1+|z|)^3(1+|\operatorname{Im} z|)^{-1}j^{-2/n},\ \forall z\in\boldsymbol{C},\ \mathit{if}\ j\ge C(1+|z|)^n\,.$$

Assuming that the conclusions of Lemma 1 are fulfilled, we shall complete the proof of (2). Obviously, $1-K(z)$ is invertable in $\mathcal{L}(H,H)$ at $z=z_0$, and since $K(z)$ is an entire family of compact operators, $(1-K(z))^{-1}$ is a meromorphic $\mathcal{L}(H,H)$-valued function on $\boldsymbol{C}$. By (7) we deduce that so is true for $R_\chi(z)$ and the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1-(K(z))^{-1}$, and hence among the poles of $(1-K(z)^p)^{-1}$ where $p=(n+1)/2$. On the other hand, it follows from (9) and the well known inequality

$$\mu_{pj}(K(z)^p)\leq\mu_j(K(z))^p \qquad \forall j, \tag{10}$$

that $K(z)^p$ is trace class for all $z\in\boldsymbol{C}$. Hence we can introduce the entire function

$$h(z)=\det(1-K(z)^p)$$

and conclude that the poles of $R_\chi(z)$, with multiplicity, are among the zeros of $h(z)$. Hence, (2) will be proved if we show that

$$|h(z)|\leq C\exp(C|z|^{n+1}) \qquad \forall z\in\boldsymbol{C}. \tag{11}$$

By (8), (9) and (10) we obtain with some constant $C'>0$:

$$\mu_j(K(z)^p)\leq C'\exp(C'|z|) \qquad \forall z\in\boldsymbol{C},\ \forall j,$$

and

$$\mu_j(K(z)^p)\leq C'(1+|z|)^{3p}(1+|\mathrm{Im}\, z|)^{-p}j^{-(n+1)/n} \qquad \forall z\in\boldsymbol{C},$$

if $j\geq C'(1+|z|)^n$. Now by Weyl's convexity estimate we get

$$|h(z)|\leq\prod_{j=1}^{\infty}(1+\mu_j(K(z)^p))$$

$$\leq\Big(\prod_{j\leq C'(1+|z|)^n}C'\exp(C'|z|)\Big)\exp\Big(\sum_{j\geq C'(1+|z|)^n}\mu_j(K(z)^p)\Big)$$

$$\leq\exp(C''(1+|z|)^{n+1})\exp\Big(C'(1+|z|)^{3p}(1+|\mathrm{Im}\, z|)^{-p}\sum_{j=1}^{\infty}j^{-(n+1)/n}\Big).$$

Thus we have obtained the estimate

$$|h(z)|\leq C\exp(C(1+|z|)^{3p}(1+|\mathrm{Im}\, z|)^{-p}) \qquad \forall z\in\boldsymbol{C}, \tag{12}$$

with some constant $C>0$. We shall show that this estimate implies (11). Clearly, by (12) we have

$$|h(z)|\leq C\exp(C|z|^{3p}) \qquad \forall z\in\boldsymbol{C}, \tag{13}$$

with possibly a greater constant $C>0$. Introduce the sets $S^{\pm}=\{z\in\boldsymbol{C}\colon |\mathrm{Im}\, z|<\pm\gamma\,\mathrm{Re}\, z\}$, where $\gamma=\mathrm{tg}(\pi/8p)$, and set $S=S^+\cup S^-$. It is easy to see that

$$(1+|z|)(1+|\mathrm{Im}\, z|)^{-1}\leq 1+\gamma^{-1} \qquad \forall z\in\boldsymbol{C}\backslash S. \tag{14}$$

Thus, by (12) and (14), we obtain

$$(15) \qquad |h(z)| \le C \exp(C|z|^{n+1}) \qquad \forall z \in \boldsymbol{C} \setminus S .$$

Now we are going to show that such an estimate holds on S. To this end we need the following fundamental lemma (for the proof, see [9]).

Lemma 2. *Let* $\Lambda = \{z \in \boldsymbol{C} : \theta_1 < \arg z < \theta_2\}$ *and let* $\alpha < 1/2$ *be such that* $\theta_2 - \theta_1 = \pi/\alpha$. *Let the function* $f(z)$ *be holomorphic in a neighbourhood of* Λ *and satisfy the conditions:*

$$\text{(i)} \qquad |f(z)| \le M \quad \textit{for} \quad \arg z = \theta_k,\ k = 1,2 ;$$

$$\text{(ii)} \qquad |f(z)| \le M'(\exp(M'|z|^{\beta}) \qquad z ,$$

with a β *such that* $0 < \beta < \alpha$.
Then

$$|f(z)| \le M \qquad \forall z \in \Lambda .$$

Clearly, $S^+ = \{z \in \boldsymbol{C} : -\pi/8p < \arg z < \pi/8p\}$. Introduce the function $f(z) = h(z)\exp(q\, z^{2p})$ where $q \in \boldsymbol{R}$ is a parameter to be chosen later on. In view of (13) we have

$$(16) \qquad |f(z)| \le C' \exp(C'|z|^{2p}) \qquad \forall z \in \boldsymbol{C} .$$

Writing $z = re^{i\varphi}$, $r = |z|$, $\varphi = \arg z$, for $\varphi = \pm\pi/8p$, in view of (15), we have

$$\begin{aligned} |f(z)| &\le C \exp(C r^{2p}) |\exp(q\, r^{2p} e^{i2p}\varphi)| \\ &= C \exp(C r^{2p}) \exp(q\, r^{2p} \cos(\pi/4)) , \end{aligned}$$

and taking $q = -C/\cos(\pi/4)$, we deduce

$$(17) \qquad |f(z)| \le C \qquad \text{for} \quad \arg z = \pm\pi/8p .$$

Now, in view of (16) and (17), we can apply Lemma 2 with $\Lambda = S^+$, $\alpha = 4p$, $\beta = 3p$, to conclude that

$$|f(z)| \le C \qquad \forall z \in S^+ ,$$

which in turn yields

$$|h(z)| \le C\ |\exp(-q\, z^{2p})| \le C \exp(C'|z|^{2p}) \qquad \forall z \in S^+ .$$

Similarly, so is true for all $z \in S^-$, and hence for all $z \in S$. This together with (15) imply (11) since $2p = n+1$.

3. Proof of Lemma 1

Since $R(z_0) = R_0(z_0) + R_0(z_0) Q R(z_0)$, we have

$$K(z) = (z_0^2 - z^2) Q R_0(z) R_0(z_0) \chi K_2 \quad \text{for} \quad \mathrm{Im}\, z > 0\,, \tag{18}$$

where $K_2 = 1 + QR(z_0)\chi$. Since G is an elliptic second order differential operator, we have $R(z_0) \in \mathcal{L}(H, H^2)$, and hence $K_2 \in \mathcal{L}(H, H)$. Choose functions χ_1, $\chi_2 \in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_1 = 1$ on supp Q, $\chi_2 = 1$ on supp χ_1 and $\chi = 1$ on supp χ_2. Now, using (5) with $R_0(z)$, after an easy computation, we obtain from (18):

$$K_3(z) = (K + (z_0^2 - z^2) K_4) \chi R_0(z) \chi K_2 + K_5 \quad \text{for} \quad \mathrm{Im}\, z > 0\,, \tag{19}$$

where

$$K_3 = QR_0(z_0)[G_0, \chi_1] R_0(z_0)[G_0, \chi_2]\,,$$

$$K_4 = QR_0(z_0)\chi_1 + QR_0(z_0)[G_0, \chi_1] R_0(z_0) \chi_2\,,$$

$$K_5 = -K_3 R_0(z_0) \chi K_2\,.$$

Here [,] stands for the commutator. Clearly, $K_3, K_4 \in \mathcal{L}(H, H)$ and $K_5 \in \mathcal{L}(H, H^2)$. Hence K_5 is a compact operator in $\mathcal{L}(H, H)$. Furthermore, as mentioned above, $\chi R_0(z) \chi$ is an entire family of compact operators in $\mathcal{L}(H_0, H_0)$, and hence, by (19), $K(z)$ can be continued analytically to the entire $\boldsymbol{C}$ with values in the compact operators in $\mathcal{L}(H, H)$. Clearly, so is true for $K_1(z)$. To prove (8) and (9) we need the following

Lemma 3. *There exists a constant $C > 0$ so that*

$$\|\chi R_0(z) \chi\|_{\mathcal{L}(H_0, H_0)} \leq C \exp(C|z|) \qquad \forall z \in \boldsymbol{C}\,; \tag{20}$$

$$\|\chi R_0(z) \chi\|_{\mathcal{L}(H_0, H_0)} \leq C(1 + |z|)^{-1}(1 + \mathrm{Im}\, z)^{-1} \quad \textit{for} \quad \mathrm{Im}\, z \geq 0\,; \tag{21}$$

$$\|\chi R_0(z) \chi\|_{\mathcal{L}(H_0, H^2)} \leq C(1 + |z|)(1 + \mathrm{Im}\, z)^{-1} \quad \textit{for} \quad \mathrm{Im}\, z \geq 0\,. \tag{22}$$

Assume for a moment that the conclusions of Lemma 3 are fulfilled. Now (8) immediately follows from (19), (20) and the well known inequality $\mu_j(\mathcal{A}) \leq \|\mathcal{A}\|$, $\forall j$. Turn to the proof of (9). Set $B = \{x \in \boldsymbol{R}^n \colon |x| \leq \rho_0 + 3\}$ and denote by Δ_B the self-adjoint realization of the Laplacian Δ with domain $D(\Delta) = C_0^\infty(B)$ in the Hilbert space $L^2(B)$. It is well known that

$$\mu_j((1 - \Delta_B)^{-m}) \leq C_B^m \cdot j^{-2m/n}, \ \forall j, \ \forall \text{integer } m \geq 1\,. \tag{23}$$

First, we shall prove (9) for $\mathrm{Im}\, z \geq 0$. Using the well known inequalities

$$\mu_{2j+1}(\mathcal{A} + \mathcal{B}) \leq \mu_j(\mathcal{A}) + \mu_j(\mathcal{B}) \qquad \forall j\,, \tag{24}$$

and

$$\mu_j(\mathcal{A}\mathcal{B}) \leq \begin{cases} \|\mathcal{A}\|\ \mu_j(\mathcal{B}), \ \forall j\,, \\ \|\mathcal{B}\|\ \mu_j(\mathcal{A}), \ \forall j\,, \end{cases} \tag{25}$$

by (19), we obtain

$$\begin{aligned}\mu_{2j+1}((K(z))&\le \mu_j(K_5)+\|K_3+(z_0^2-z^2)K_4\|\mu_j(\chi R_0(z)\chi)\|K_2\|\\&\le C\,\mu_j(\chi R_0(z_0)\chi)+C(1+|z|^2)(\mu_j(\chi R_0(z)\chi)\,,\end{aligned}$$

where $\|\cdot\|$ is the norm in $\mathcal{L}(H_0,H_0)$. On the other hand, by (22), (23) and (25) we have

$$\begin{aligned}\mu_j(\chi R_0(z)\chi)&\le \mu_j((1-\Delta_B)^{-1})\|(1-\Delta)\chi R_0(z)\chi\|_{\mathcal{L}(H_0,H_0)}\\&\le C j^{-2/n}(1+|z|)(1+\operatorname{Im} z)^{-1}\qquad \text{for}\quad \operatorname{Im} z\ge 0,\ \forall j\,.\end{aligned}$$

Now, in this case, (9) follows from the above estimates at once.

Turn to the proof of (9) for $\operatorname{Im} z\le 0$. By (24) we have

$$\mu_{2j+1}(K(z))\le \mu_j(K(-z))+\mu_j(\tilde{K}(z))\,, \tag{26}$$

where $\tilde{K}(z)=K(z)-K(-z)$. We have already seen above that, when $\operatorname{Im} z\le 0$, $\mu_j(K(-z))$ has the desired bound. To estimate the other term we shall proceed as in [13]. Set $\tilde{R}_0(z)=R_0(z)-R_0(-z)$. It follows from (4) that the kernel of $\tilde{R}_0(z)$ is given by

$$\tilde{R}_0(z)(x,y)=(i/2)(2\pi)^{-n+1}z^{n-2}\int_{S^{n-1}}e^{iz\langle x-y,w\rangle}dw\,,$$

where S^{n-1} denotes the unit sphere in $\boldsymbol{R}^n$. Now it is easy to see that for any multiindex α we have

$$\sup_{\substack{|x|\le\rho_0+3\\|y|\le\rho_0+3}}|\partial_x^\alpha\tilde{R}_0(z)(x,y)|\le C^{|\alpha|+1}|z|^{|\alpha|}e^{C|z|} \tag{27}$$

with some constant $C>0$. By Theorem 1.4.2 of [2], for any integer $m\ge 1$ there exists a function $\chi_m\in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_m=1$ on supp χ, $\chi_m=0$ for $|x|\ge\rho_0+3$ and $|\partial_x^\alpha\chi_m|\le C^{|\alpha|+1}|\alpha|!$ for $|\alpha|\le 2m$ with some constant $C>0$. Using this together with (19), (23), (25) and (27), we get

$$\begin{aligned}\mu_j(\tilde{K}(z))&\le\|(K_3+(z_0^2-z^2)K_4)\chi\|\mu_j(\chi_m\tilde{R}_0(z)\chi)\|K_2\|\\&\le C'(1+|z|^2)\mu_j((1-\Delta_B)^{-m})\|(1-\Delta)^m\chi_m\tilde{R}_0(z)\chi\|\\&\le C''(1+|z|^2)\mu_j((1-\Delta_B)^{-m})\sup_{x,y}|(1-\Delta_x)^m(\chi_m(x)\tilde{R}_0(z)(x,y)\chi(y))|\\&\le C^{2m+1}(|z|^{2m}+(2m)^{2m})e^{C|z|}j^{-2m/n},\ \forall z\in\boldsymbol{C},\ \forall j,\ \forall m\ge 1\,,\end{aligned}$$

with some constant $C>0$. Here $\|\cdot\|$ again denotes the norm in $\mathcal{L}(H_0,H_0)$. Now, taking $2m=|z|$ we can easily arrange for $j\ge q|z|^n$, $|z|\gg 1$, with large q depending only on C, that

$$\mu_j(\tilde{K}(z))\le C j^{-2/n}$$

with possibly another constant $C>0$. Now, in this case, (9) follows from this estimate, (26) and the estimate of $\mu_j(K(-z))$ obtained above.

4. Proof of Lemma 3

First, we shall derive (22) from (21). Choose functions $\chi_1, \chi_2 \in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_1=1$ on supp χ, $\chi_2=1$ on supp χ_1. As above, we have

$$\chi R_0(z)\chi = R_0(z_0)A(z) \tag{28}$$

where

$$A(z) = ((z_0^2-z^2)(\chi+[G_0,\chi]R_0(z_0)\chi_1)+ \\ +[G_0,\chi]R_0(z_0)[G_0,\chi_1]\chi_2R_0(z)\chi+[G_0,\chi]R_0(z_0)\chi_1+\chi^2\,.$$

Since $R_0(z_0)\in\mathcal{L}(H_0,H^2)$, clearly $A(z)\in\mathcal{L}(H_0,H_0)$ and by (28) we get

$$\begin{aligned}||\chi R_0(z)\chi||_{\mathcal{L}(H_0,H^2)}&\leq C||A(z)||_{\mathcal{L}(H_0,H_0)}\\ &\leq C'(1+|z|^2)||\chi_2R_0(z)\chi_2||_{\mathcal{L}(H_0,H_0)}+C'\\ &\leq C''(1+|z|)(1+\operatorname{Im} z)^{-1}\qquad \text{for}\quad \operatorname{Im} z\geq 0\,,\end{aligned}$$

provided (21) is fulfilled with χ replaced by χ_2.

To prove (20) and (21) we shall exploit the following resolvent formula

$$R_0(z)f=\int_0^\infty e^{itz}U_0(t)f\,dt \qquad \text{for}\quad \operatorname{Im} z>0, f\in H_0\,, \tag{29}$$

where $U_0(t)$ is the propagator of the Cauchy problem

$$\begin{cases}(\partial_t^2-\Delta)U_0(t)f(x)=0 & \text{in}\quad \boldsymbol{R}_t\times\boldsymbol{R}_x^n\,,\\ U_0(0)f(x)=0,\quad \partial_tU_0(0)f(x)=f(x)\,.\end{cases}$$

It is easy to see that

$$||\partial_tU_0(t)f||_{H_0}\leq||f||_{H_0}\qquad \forall f\in H_0,\quad \forall t\,. \tag{30}$$

Now, integrating by parts in (29) and using that by Huygens' principle there exists a $T>0$ so that $\chi U_0(t)\chi=0$ for $t\geq T$, we get

$$z\chi R_0(z)\chi f=i\int_0^T e^{itz}\chi\partial_tU_0(t)\chi f\,dt \qquad \text{for}\quad \operatorname{Im} z>0,\quad f\in H_0\,, \tag{31}$$

which clearly extends analytically to the entire complex plane $\boldsymbol{C}$. By (30) and (31) we have

$$||z\chi R_0(z)\chi||_{\mathcal{L}(H_0,H_0)}\leq C\int_0^T e^{-t\operatorname{Im} z}\,dt\leq\begin{cases}C\,T\,e^{T|z|}, & \forall z\in\boldsymbol{C}\,,\\ C\,T & \text{for}\quad \operatorname{Im} z\geq 0\,,\\ C(\operatorname{Im} z)^{-1} & \text{for}\quad \operatorname{Im} z\geq 1.\end{cases}$$

Now (20) and (21) follow from these estimates at once.

References

[1] C. Bardos, J. Guillot and J. Ralston: *La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la theorie de la diffusion,* Comm. Partial Differential Equations **7** (1982), 905–958.

[2] L. Hörmander: The analysis of linear partial differential operators, I, Distribution theory and Fourier analysis, Springer Verlag, Berlin, 1983.

[3] A. Intissar: *A polynomial bound on the number of scattering poles for a potential in even dimensional space* $\boldsymbol{R}^n$, Comm. Partial Differential Equations **11** (1986), 367–396.

[4] A. Intissar: *On the value distribution of the scattering poles associated to the Schrödinger operator* $H=(-i\vec{\nabla}+\vec{b}(x))^2+a(x)$ *in* $\boldsymbol{R}^n$, $n\geq 3$, preprint.

[5] P.D. Lax and R.S. Phillips: Scattering theory, Academic Press, 1967.

[6] R.B. Melrose: *Polynomial bounds on the number of scattering poles,* J. Func. Analy. **53** (1983), 287–303.

[7] R.B. Melrose: *Polynomial bounds on the distribution of poles in scattering by an obstacle,* Journees "Equations aux Derivees Partielle", Saint Jean de Montes, 1984.

[8] R.B. Melrose: *Weyl asymptotics for the phase in obstacle scattering,* Comm. Partial Differential Equations **13** (1988), 1431–1439.

[9] E.C. Titchmarsh: The theory of functions, Oxford University Press, 1968.

[10] G. Vodev: *Polynomial bounds on the number of scattering poles for symmetric systems, preprint.*

[13] M. Zworski: *Distribution of poles for scattering on the real line,* J. Func. Anal. **73** (1987), 277–296.

[12] M. Zworski: *Sharp polynomial bounds on the number of scattering poles of radial potentials,* J. Func. Anal. **82** (1989), 370–403.

[13] M. Zworski: *Sharp polynomial bounds on the number of scattering poles,* Duke Math. J. **59** (1989), 311–323.

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