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POLYNOMIAL BOUNDS ON THE NUMBER OF SCATTERING POLES FOR METRIC PERTURBATIONS OF THE LAPLACIAN IN R^n, $n\geq3$, ODD

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1. Introduction The purpose of this note is to obtain a polynomial bound on the number of the scattering poles associated to the operator

$$G = c(x)^{-1} \sum_{1\leq i,\,j\leq n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}) \quad \text{in} \quad R^n,$$

where $n\geq3$, odd. We consider this operator under the following assumptions on the coefficients:

(a) $c(x)\in C(R^n;R)$ and $c(x)\geq c_0>0$ for all $x\in R^n$;

(b) $g_{ij}(x)\in C^1(R^n)$ and the matrix $\{g_{ij}(x)\}$ is a strictly positive hermitian one for all $x\in R^n$, i.e.

    (b)$_1$ $g_{ij}(x) = \overline{g_{ji}(x)}$, $i,j=1,\cdots,n$, $\forall x\in R^n$;

    (b)$_2$ $\sum_{i\leq j,\,j\leq n} g_{ij}(x)\xi_i\xi_j\geq C\,|\xi|^2$, $\forall(x,\xi)\in R^n\times R^n\backslash 0$, $C>0$;

(c) there exists a $\rho_0>0$ so that $c(x)=1$ and $g_{ij}(x)=\delta_{ij}$ for $|x|\geq\rho_0$, δ_{ij} being the Kronecker's symbol.

It is well known that under the above assumptions the operator G has a self-adjoint realization, which will be again denoted by G, in the Hilbert space $H=L^2(R^n;c(x)dx)$. Note also that it follows from the assumption (b)$_2$ that the operator G is elliptic. By G_0 we shall denote the self-adjoint realization of the Laplacian Δ in the Hilbert space $H_0=L^2(R^n)$.

It is well known that, under the above assumptions, the scattering matrix corresponding to the pair $\{G,G_0\}$ admits a meromorphic continuation to the entire complex plane C. Let $\{\lambda_j\}$ be the poles of this continuation, repeating according to multiplicity, and set

$$N(r) = \#\{\lambda_j : |\lambda_j|\leq r\}.$$

It is shown in [10] that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ so that

(1) $$M(r) \leq C_\varepsilon r^{n+1+\varepsilon} + C_\varepsilon.$$

In fact, (1) is proved there for first order symmetric systems in $\boldsymbol{R}^n$, $n \geq 3$, odd, but the proof easily extends to our case. The aim of this work is to improve (1). More precisely, we have the following

Theorem. *There exists a constant $C > 0$ so that the number of the scattering poles associated to the operator G satisfies the bound*

(2) $$N(r) \leq C r^{n+1} + C.$$

Note that the desired result is to obtain the bound

(3) $$N(r) \leq C r^n + C.$$

In [7] Melrose proved (3) for the Laplacian in exterior domains with Dirichlet or Robin boundary conditions, while in [13] Zworski proved (3) for the Schrödinger operator $\Delta + V(x)$ with a potential $V \in L_0^\infty(\boldsymbol{R}^n)$. Note that a bound of the form (3) is important for obtaining Weyl asymptotics of the phase shift (see [8], where such an asymptotic is obtained for an arbitrary smooth obstacle). To our knowledge, the problem of obtaining a bound of the form (3) on the number of the scattering poles associated to the operator G is still open. Let us also mention Intissar's work [4] where a bound of the form (2) on the number of the scattering poles associated to the operator $(-i\vec{\nabla} + \vec{b}(x))^2 + a(x)$ in $\boldsymbol{R}^n$, $n \geq 3$, odd, is obtained, where $a(x) \in C_0^\infty(\boldsymbol{R}^n; \boldsymbol{R})$, $\vec{b}(x) \in C_0^\infty(\boldsymbol{R}^n; \boldsymbol{R})$. In [3] and [4] he has also obtained an analogue of (2) in the case of even n, $n \geq 4$.

To prove (2) we exploit the fact that the scattering poles, with multiplicity, coincide with the poles of the meromorphic continuation of the cutoff resolvent $R_\chi(z) = \chi R(z) \chi$ from $\{z \in \boldsymbol{C} : \operatorname{Im} z > 0\}$ to the entire complex plane $\boldsymbol{C}$, where $R(z) = (G + z^2)^{-1}$ for $\operatorname{Im} z > 0$, and $\chi \in C_0^\infty(\boldsymbol{R}^n)$ is such that $\chi = 1$ for $|x| \leq \rho_0 + 1$, $\chi = 0$ for $|x| \geq \rho_0 + 2$ (see [5]). This enables us to characterize the scattering poles as the poles of a meromorphic function of the form $(1 - K(z))^{-1}$ where $K(z)$ is an entire family of compact operators on H such that $K(z)^p$, $p = (n+1)/2$, is trace class. Then, following [4], [6], [7] and [13], we deduce that the scattering poles, with multiplicity, are among the zeros of the entire function $h(z) = \det(1 - K(z)^p)$, and hence, to prove (2) it suffices to show that the order of $h(z)$ is less than or equal to $n+1$. However, in our case the operator $K(z)$ is much more complicated and therefore it is not so easy to obtain the desired order of $h(z)$. To overcome the difficulties we use precise estimates of the cutoff free resolvent for $\operatorname{Im} z \geq 0$ combined with an application of the Phragmen-Lindelof principle.

discussions during the preparation of this work.

2. Representation of the cutoff resolvent

First, we shall introduce some notations. Given two Hilbert spaces X and Y, $\mathcal{L}(X,Y)$ will denote the space of all linear bounded operators acting from X into Y. Given any $s>0$, H^s will denote the usual Sobolev space $H^s(\mathbf{R}^n)$. Finally, given a compact operator $\mathcal{A}$, $\mu_j(\mathcal{A})$ will denote the characteristic values of $\mathcal{A}$, i.e. the eigenvalues of $(\mathcal{A}^*\mathcal{A})^{1/2}$, ordered, with multiplicity, to form a nonincreasing sequence.

Denote by $R_0(z)$ the outgoing resolvent of Δ, i.e. that one with kernel $E(x-y,z)$ where $E(x,z)$ is the outgoing fundamental solution of the operator $\Delta+z^2$. Then we have $R_0(z)=(G_0+z^2)^{-1}\in\mathcal{L}(H_0, H_0)$ for Im $z>0$. Moreover, it is well known that the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$(4) \qquad R_0(z)(x,y) = -(i/4)(2\pi)^{(n-2)/2}(z/|x-y|)^{(n-2)/2}H^{(1)}_{(n-2)/2}(z|x-y|).$$

It follows easily from this representation that $\mathcal{X}R_0(z)\mathcal{X}$ forms an entire family of compact pseudodifferential operators of order -2 in $\mathcal{L}(H_0, H_0)$, $\mathcal{X}$ being the function introduced in the previous section. Using this we shall build the meromorphic continuation of $R_x(z)$. Set $Q=G_0-G$ and fix a $z_0\in\mathbf{C}$, Im $z_0>0$. Clearly, for Im $z>0$, we have

$$(5) \qquad\qquad R(z) = R_0(z)+R(z)QR_0(z)$$

and

$$(6) \qquad\qquad R(z) = R(z_0)+(z_0^2-z^2)R(z)R(z_0).$$

Combining these identities yields

$$R(z)(1-(z_0^2-z^2)QR_0(z)R(z_0)) = R(z_0)+(z_0^2-z^2)R_0(z)R(z_0)$$

for Im $z>0$. Multiplying the both sides of this identity by $\mathcal{X}$, since $Q=\mathcal{X}Q$, we get

$$(7) \qquad R_x(z)(1-K(z)) = R_x(z_0)+K_1(z) \qquad \text{for} \quad \text{Im } z>0,$$

where $K(z)=(z_0^2-z^2)QR_0(z)R(z_0)\mathcal{X}$ and $K_1(z)=(z_0^2-z^2)\mathcal{X}R_0(z)R(z_0)\mathcal{X}$. We need now the following

Lemma 1. *The operator-valued functions $K(z)$ and $K_1(z)$ have analytic continuations from $\{z\in\mathbf{C}: \text{Im } z>0\}$ to the entire $\mathbf{C}$ with values in the compact operators in $\mathcal{L}(H,H)$. Moreover, there exists a constant $C>0$ so that*

$$(8) \qquad\qquad \mu_j(K(z))\leq C\exp(C|z|), \quad \forall z\in\mathbf{C}, \forall j;$$

$$(9) \quad \mu_j(K(z))\leq C(1+|z|)^3(1+|\text{Im } z|)^{-1}j^{-2/n}, \quad \forall z\in\mathbf{C}, \text{ if } j\geq C(1+|z|)^n.$$

Assuming that the conclusions of Lemma 1 are fulfilled, we shall complete the proof of (2). Obviously, $1-K(z)$ is invertable in $\mathcal{L}(H,H)$ at $z=z_0$, and since $K(z)$ is an entire family of compact operators, $(1-K(z))^{-1}$ is a meromorphic $\mathcal{L}(H,H)$-valued function on C. By (7) we deduce that so is true for $R_\chi(z)$ and the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1-(K(z))^{-1}$, and hence among the poles of $(1-K(z)^p)^{-1}$ where $p=(n+1)/2$. On the other hand, it follows from (9) and the well known inequality

$$(10) \qquad \mu_{pj}(K(z)^p) \leq \mu_j(K(z))^p \qquad \forall j,$$

that $K(z)^p$ is trace class for all $z \in C$. Hence we can introduce the entire function

$$h(z) = \det(1-K(z)^p)$$

and conclude that the poles of $R_\chi(z)$, with multiplicity, are among the zeros of $h(z)$. Hence, (2) will be proved if we show that

$$(11) \qquad |h(z)| \leq C \exp(C|z|^{n+1}) \qquad \forall z \in C.$$

By (8), (9) and (10) we obtain with some constant $C'>0$:

$$\mu_j(K(z)^p) \leq C' \exp(C'|z|) \qquad \forall z \in C, \forall j,$$

and

$$\mu_j(K(z)^p) \leq C'(1+|z|)^{3p}(1+|\operatorname{Im} z|)^{-p} j^{-(n+1)/n} \qquad \forall z \in C,$$

if $j \geq C'(1+|z|)^n$. Now by Weyl's convexity estimate we get

$$|h(z)| \leq \prod_{j=1}^{\infty} (1+\mu_j(K(z)^p))$$

$$\leq \left(\prod_{j \leq C'(1+|z|)^n} C' \exp(C'|z|)\right) \exp\left(\sum_{j \geq C'(1+|z|)^n} \mu_j(K(z)^p)\right)$$

$$\leq \exp(C''(1+|z|)^{n+1}) \exp\left(C'(1+|z|)^{3p}(1+|\operatorname{Im} z|)^{-p} \sum_{j=1}^{\infty} j^{-(n+1)/n}\right).$$

Thus we have obtained the estimate

$$(12) \qquad |h(z)| \leq C \exp(C(1+|z|)^{3p}(1+|\operatorname{Im} z|)^{-p}) \qquad \forall z \in C,$$

with some constant $C>0$. We shall show that this estimate implies (11). Clearly, by (12) we have

$$(13) \qquad |h(z)| \leq C \exp(C|z|^{3p}) \qquad \forall z \in C,$$

with possibly a greater constant $C>0$. Introduce the sets $S^\pm = \{z \in C : |\operatorname{Im} z| < \pm\gamma \operatorname{Re} z\}$, where $\gamma = \operatorname{tg}(\pi/8p)$, and set $S = S^+ \cup S^-$. It is easy to see that

$$(14) \qquad (1+|z|)(1+|\operatorname{Im} z|)^{-1} \leq 1+\gamma^{-1} \qquad \forall z \in C \setminus S.$$

Thus, by (12) and (14), we obtain

$$(15) \qquad |h(z)| \leq C \exp(C|z|^{n+1}) \qquad \forall z \in C \backslash S.$$

Now we are going to show that such an estimate holds on S. To this end we need the following fundamental lemma (for the proof, see [9]).

Lemma 2. *Let* $\Lambda = \{z \in C : \theta_1 < \arg z < \theta_2\}$ *and let* $\alpha < 1/2$ *be such that* $\theta_2 - \theta_1 = \pi/\alpha$. *Let the function* $f(z)$ *be holomorphic in a neighbourhood of* Λ *and satisfy the conditions:*

$$(i) \qquad |f(z)| \leq M \qquad for \quad \arg z = \theta_k, \ k = 1, 2;$$

$$(ii) \qquad |f(z)| \leq M'(\exp(M'|z|^\beta) \qquad z,$$

with a β *such that* $0 < \beta < \alpha$.
Then

$$|f(z)| \leq M \qquad \forall z \in \Lambda.$$

Clearly, $S^+ = \{z \in C : -\pi/8p < \arg z < \pi/8p\}$. Introduce the function $f(z) = h(z)\exp(q\,z^{2p})$ where $q \in R$ is a parameter to be chosen later on. In view of (13) we have

$$(16) \qquad |f(z)| \leq C' \exp(C'|z|^{2p}) \qquad \forall z \in C.$$

Writing $z = r\,e^{i\varphi}$, $r = |z|$, $\varphi = \arg z$, for $\varphi = \pm \pi/8p$, in view of (15), we have

$$|f(z)| \leq C \exp(C\,r^{2p})\,|\exp(q\,r^{2p}\,e^{i2p}\varphi)|$$
$$= C \exp(C\,r^{2p}) \exp(q\,r^{2p} \cos(\pi/4)),$$

and taking $q = -C/\cos(\pi/4)$, we deduce

$$(17) \qquad |f(z)| \leq C \qquad for \quad \arg z = \pm \pi/8p.$$

Now, in view of (16) and (17), we can apply Lemma 2 with $\Lambda = S^+$, $\alpha = 4p$, $\beta = 3p$, to conclude that

$$|f(z)| \leq C \qquad \forall z \in S^+,$$

which in turn yields

$$|h(z)| \leq C\,|\exp(-q\,z^{2p})| \leq C \exp(C'|z|^{2p}) \qquad \forall z \in S^+.$$

Similarly, so is true for all $z \in S^-$, and hence for all $z \in S$. This together with (15) imply (11) since $2p = n+1$.

3. Proof of Lemma 1

Since $R(z_0) = R_0(z_0) + R_0(z_0)QR(z_0)$, we have

(18) $K(z) = (z_0^2 - z^2)QR_0(z)R_0(z_0)\mathcal{X}K_2$ for   Im $z > 0$,

where $K_2 = 1 + QR(z_0)\mathcal{X}$. Since G is an elliptic second order differential operator, we have $R(z_0) \in \mathcal{L}(H, H^2)$, and hence $K_2 \in \mathcal{L}(H, H)$. Choose functions $\mathcal{X}_1$, $\mathcal{X}_2 \in C_0^\infty(\mathbf{R}^n)$ such that $\mathcal{X}_1 = 1$ on supp Q, $\mathcal{X}_2 = 1$ on supp $\mathcal{X}_1$ and $\mathcal{X} = 1$ on supp $\mathcal{X}_2$. Now, using (5) with $R_0(z)$, after an easy computation, we obtain from (18):

(19) $K_3(z) = (K + (z_0^2 - z^2)K_4)\mathcal{X}R_0(z)\mathcal{X}K_2 + K_5$ for   Im $z > 0$,

where

$$K_3 = QR_0(z_0)[G_0, \mathcal{X}_1]R_0(z_0)[G_0, \mathcal{X}_2] ,$$

$$K_4 = QR_0(z_0)\mathcal{X}_1 + QR_0(z_0)[G_0, \mathcal{X}_1]R_0(z_0)\mathcal{X}_2 ,$$

$$K_5 = -K_3R_0(z_0)\mathcal{X}K_2 .$$

Here [,] stands for the commutator. Clearly, $K_3, K_4 \in \mathcal{L}(H, H)$ and $K_5 \in \mathcal{L}(H, H^2)$. Hence K_5 is a compact operator in $\mathcal{L}(H, H)$. Furthermore, as mentioned above, $\mathcal{X}R_0(z)\mathcal{X}$ is an entire family of compact operators in $\mathcal{L}(H_0, H_0)$, and hence, by (19), $K(z)$ can be continued analytically to the entire $\mathbf{C}$ with values in the compact operators in $\mathcal{L}(H, H)$. Clearly, so is true for $K_1(z)$. To prove (8) and (9) we need the following

Lemma 3. *There exists a constant $C > 0$ so that*

(20) $\|\mathcal{X}R_0(z)\mathcal{X}\|_{\mathcal{L}(H_0, H_0)} \leq C \exp(C|z|)$ $\forall z \in \mathbf{C}$;

(21) $\|\mathcal{X}R_0(z)\mathcal{X}\|_{\mathcal{L}(H_0, H_0)} \leq C(1 + |z|)^{-1}(1 + \text{Im } z)^{-1}$ *for*   Im $z \geq 0$;

(22) $\|\mathcal{X}R_0(z)\mathcal{X}\|_{\mathcal{L}(H_0, H^2)} \leq C(1 + |z|)(1 + \text{Im } z)^{-1}$ *for*   Im $z \geq 0$.

Assume for a moment that the conclusions of Lemma 3 are fulfilled. Now (8) immediately follows from (19), (20) and the well known inequality $\mu_j(\mathcal{A}) \leq \|\mathcal{A}\|$, $\forall j$. Turn to the proof of (9). Set $B = \{x \in \mathbf{R}^n : |x| \leq \rho_0 + 3\}$ and denote by Δ_B the self-adjoint realization of the Laplacian Δ with domain $D(\Delta) = C_0^\infty(B)$ in the Hilbert space $L^2(B)$. It is well known that

(23) $\mu_j((1 - \Delta_B)^{-m}) \leq C_B^m \cdot j^{-2m/n}$, $\forall j$, $\forall$ integer $m \geq 1$.

First, we shall prove (9) for Im $z \geq 0$. Using the well known inequalities

(24) $\mu_{2j+1}(\mathcal{A} + \mathcal{B}) \leq \mu_j(\mathcal{A}) + \mu_j(\mathcal{B})$ $\forall j$,

and

(25) $\mu_j(\mathcal{A}\mathcal{B}) \leq \begin{cases} \|\mathcal{A}\| \; \mu_j(\mathcal{B}), \; \forall j , \\ \|\mathcal{B}\| \; \mu_j(\mathcal{A}), \; \forall j , \end{cases}$

by (19), we obtain

$$\mu_{2j+1}((K(z)) \le \mu_j(K_5) + \|K_3 + (z_0^2 - z^2)K_4\| \mu_j(\mathcal{X}R_0(z)\mathcal{X}) \|K_2\|$$
$$\le C \mu_j(\mathcal{X}R_0(z_0)\mathcal{X}) + C(1 + |z|^2)(\mu_j(\mathcal{X}R_0(z)\mathcal{X}),$$

where $\|\cdot\|$ is the norm in $\mathcal{L}(H_0, H_0)$. On the other hand, by (22), (23) and (25) we have

$$\mu_j(\mathcal{X}R_0(z)\mathcal{X}) \le \mu_j((1 - \Delta_B)^{-1}) \|(1 - \Delta)\mathcal{X}R_0(z)\mathcal{X}\|_{\mathcal{L}(H_0, H_0)}$$
$$\le C j^{-2/n}(1 + |z|)(1 + \mathrm{Im}\ z)^{-1} \quad \text{for}\ \ \mathrm{Im}\ z \ge 0, \forall j.$$

Now, in this case, (9) follows from the above estimates at once.

Turn to the proof of (9) for Im $z \le 0$. By (24) we have

(26) $$\mu_{2j+1}(K(z)) \le \mu_j(K(-z)) + \mu_j(\tilde{K}(z)),$$

where $\tilde{K}(z) = K(z) - K(-z)$. We have already seen above that, when Im $z \le 0$, $\mu_j(K(-z))$ has the desired bound. To estimate the other term we shall proceed as in [13]. Set $\tilde{R}_0(z) = R_0(z) - R_0(-z)$. It follows from (4) that the kernel of $\tilde{R}_0(z)$ is given by

$$\tilde{R}_0(z)(x, y) = (i/2)(2\pi)^{-n+1} z^{n-2} \int_{S^{n-1}} e^{iz\langle x-y, w \rangle} dw,$$

where S^{n-1} denotes the unit sphere in $\mathbf{R}^n$. Now it is easy to see that for any multiindex α we have

(27) $$\sup_{\substack{|x| \le \rho_0 + 3 \\ |y| \le \rho_0 + 3}} |\partial_x^\alpha \tilde{R}_0(z)(x, y)| \le C^{|\alpha|+1} |z|^{|\alpha|} e^{C|z|}$$

with some constant $C > 0$. By Theorem 1.4.2 of [2], for any integer $m \ge 1$ there exists a function $\mathcal{X}_m \in C_0^\infty(\mathbf{R}^n)$ such that $\mathcal{X}_m = 1$ on supp $\mathcal{X}$, $\mathcal{X}_m = 0$ for $|x| \ge \rho_0 + 3$ and $|\partial_x^\alpha \mathcal{X}_m| \le C^{|\alpha|+1} |\alpha|!$ for $|\alpha| \le 2m$ with some constant $C > 0$. Using this together with (19), (23), (25) and (27), we get

$$\mu_j(\tilde{K}(z)) \le \|(K_3 + (z_0^2 - z^2)K_4)\mathcal{X}\| \mu_j(\mathcal{X}_m \tilde{R}_0(z)\mathcal{X}) \|K_2\|$$
$$\le C'(1 + |z|^2)\mu_j((1 - \Delta_B)^{-m}) \|(1 - \Delta)^m \mathcal{X}_m \tilde{R}_0(z)\mathcal{X}\|$$
$$\le C''(1 + |z|^2)\mu_j((1 - \Delta_B)^{-m}) \sup_{x, y} |(1 - \Delta_x)^m (\mathcal{X}_m(x)\tilde{R}_0(z)(x, y)\mathcal{X}(y))|$$
$$\le C^{2m+1}(|z|^{2m} + (2m)^{2m})e^{C|z|} j^{-2m/n}, \forall z \in C, \forall j, \forall m \ge 1,$$

with some constant $C > 0$. Here $\|\cdot\|$ again denotes the norm in $\mathcal{L}(H_0, H_0)$. Now, taking $2m = |z|$ we can easily arrange for $j \ge q|z|^n$, $|z| \gg 1$, with large q depending only on C, that

$$\mu_j(\tilde{K}(z)) \le C j^{-2/n}$$

with possibly another constant $C > 0$. Now, in this case, (9) follows from this

estimate, (26) and the estimate of $\mu_j(K(-z))$ obtained above.

4. Proof of Lemma 3

First, we shall derive (22) from (21). Choose functions $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_1 = 1$ on supp χ, $\chi_2 = 1$ on supp χ_1. As above, we have

$$(28) \qquad \chi R_0(z)\chi = R_0(z_0)A(z)$$

where

$$A(z) = ((z_0^2 - z^2)(\chi + [G_0, \chi]R_0(z_0)\chi_1) + \\ + [G_0, \chi]R_0(z_0)[G_0, \chi_1]\chi_2 R_0(z)\chi + [G_0, \chi]R_0(z_0)\chi_1 + \chi^2 \, .$$

Since $R_0(z_0) \in \mathcal{L}(H_0, H^2)$, clearly $A(z) \in \mathcal{L}(H_0, H_0)$ and by (28) we get

$$\|\chi R_0(z)\chi\|_{\mathcal{L}(H_0, H^2)} \leq C\|A(z)\|_{\mathcal{L}(H_0, H_0)}$$
$$\leq C'(1 + |z|^2)\|\chi_2 R_0(z)\chi_2\|_{\mathcal{L}(H_0, H_0)} + C'$$
$$\leq C''(1 + |z|)(1 + \operatorname{Im} z)^{-1} \qquad \text{for} \quad \operatorname{Im} z \geq 0 \, ,$$

provided (21) is fulfilled with χ replaced by χ_2.

To prove (20) and (21) we shall exploit the following resolvent formula

$$(29) \qquad R_0(z)f = \int_0^\infty e^{itz}U_0(t)f \, dt \qquad \text{for} \quad \operatorname{Im} z > 0, f \in H_0 \, ,$$

where $U_0(t)$ is the propagator of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta)U_0(t)f(x) = 0 & \text{in} \quad \mathbf{R}_t \times \mathbf{R}_x^n \, , \\ U_0(0)f(x) = 0, \quad \partial_t U_0(0)f(x) = f(x) \, . \end{cases}$$

It is easy to see that

$$(30) \qquad \|\partial_t U_0(t)f\|_{H_0} \leq \|f\|_{H_0} \qquad \forall f \in H_0, \quad \forall t \, .$$

Now, integrating by parts in (29) and using that by Huygens' principle there exists a $T > 0$ so that $\chi U_0(t)\chi = 0$ for $t \geq T$, we get

$$(31) \qquad z\chi R_0(z)\chi f = i \int_0^T e^{itz}\chi \partial_t U_0(t)\chi f \, dt \qquad \text{for} \quad \operatorname{Im} z > 0, \quad f \in H_0 \, ,$$

which clearly extends analytically to the entire complex plane $\mathbf{C}$. By (30) and (31) we have

$$\|z\chi R_0(z)\chi\|_{\mathcal{L}(H_0, H_0)} \leq C\int_0^T e^{-t\operatorname{Im} z} \, dt \leq \begin{cases} C\,T\,e^{T|z|}, & \forall z \in \mathbf{C} \, , \\ C\,T & \text{for} \quad \operatorname{Im} z \geq 0 \, , \\ C(\operatorname{Im} z)^{-1} & \text{for} \quad \operatorname{Im} z \geq 1 \, . \end{cases}$$

Now (20) and (21) follow from these estimates at once.

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