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PERFECT CATEGORIES IV (QUASI-FROBENIUS CATEGORIES)

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The author defined perfect Grothendieck categories and studied them [11]. In [12], [13] he developed [11] and determined hereditary perfect categories and hereditary perfect and QF -3 categories.

In this note, as a continuous work we define quasi-Frobenius categories (briefly QF) and generalize some properties of QF -rings.

Let \mathfrak{A} be a Grothendieck category. We always assume \mathfrak{A} contains a generating set $\{G_\alpha\}_I$ of small objects G_α , e.g. functor categories. If every projective objects in \mathfrak{A} are injective, we call \mathfrak{A} a QF -category. As we see in examples of QF -categories, some important properties of QF -rings are not inherited to QF -categories.

The object of this paper is to fill those gaps. We assume mainly that G_α 's are projective, then QF -categories are perfect. It is clear that all of results in the category \mathfrak{M}_R of modules over a ring R with identity are not valid in perfect categories \mathfrak{A} . However, modifying proofs in \mathfrak{M}_R , we sometimes succeed to extend some properties in \mathfrak{M}_R to \mathfrak{A} . All of theorems in this note are well known in \mathfrak{M}_R and so we shall give often only methods how to modify proofs in \mathfrak{M}_R .

In §1 we generalize the notion of Σ -injective [5] and obtain [5], Proposition 3 in \mathfrak{A} . We define a QF -category in §2 and generalize results in [4] and [14]. In §3 we deal with a problem whether a QF -category has the following property or not: every injectives are projective, (see [6]). In the final section, we give some supplementary results of [10].

In this paper, rings S need not to have the identity, unless otherwise stated. We refer the reader to [11], [12] and [13] for notations and definitions.

1. Σ -injective

Let \mathfrak{A} be a Grothendieck category. We always assume that \mathfrak{A} has a generating set $\{G_\alpha\}_I$ of small objects G_α .

1) See [11] and [12] for the definitions.

Let M, N be objects in \mathfrak{A} and $S=[N, N]$. Then $[M, N]$ is a left S -module. Let M_0 be a subobject of M . By $l_{[M, N]}(M_0)$ (briefly $l(M_0)$) we denote the left S -submodule of $[M, N]$ whose elements consist of all f such that $f|_{M_0}=0$. By $l(M, N)$ we denote the set of such annihilator submodules of $[M, N]$. Conversely, for any left S -submodule K of $[M, N]$ we denote the subobject $\bigcap_{k \in K} \text{Ker } k$ by $r_M(K)$ (briefly $r(K)$). Finally, by $r(M, N)$ we denote the set of such annihilator subobjects in M .

The following lemma is well known in the category \mathfrak{M}_T of T -modules over a ring T with identity and we can prove it by modifying the proof of [7], Lemma 1 in p. 136.

Lemma 1 (Baer's condition). *An object Q in \mathfrak{A} is injective if and only if any $f \in [G, Q]$ is extended to an element in $[G_\alpha, Q]$ for any subobject G of $G_\alpha, \alpha \in I$.*

Following to Faith [5], we call an object Q Σ -injective if any coproducts of Q itself are injective.

The following results are some versions of [1] and [5] in \mathfrak{A} .

Lemma 2 ([5]). *Let M, N be objects in \mathfrak{A} . We assume that $r(M, N)$ is noetherian. Then for any subobject M_1 of M there exists a small subobject M_1' of M such that $l(M_1)=l(M_1')$.*

Proof. Since $r(M, N)$ is noetherian, $l(M, N)$ is artinian. From the assumption $M_1 = \bigcup M_\alpha$, where M_α 's are small objects. Then $l(M_1) = \bigcap_\alpha l(M_\alpha) = \bigcap_{i=1}^n l(M_{\alpha_i})$, since $l(M, N)$ is artinian. Hence, $l(M_1) = l(\bigcup_1^n M_{\alpha_i})$.

Theorem 1 ([1], [5]). *Let \mathfrak{A} be a Grothendieck category with generating set $\{G_\alpha\}_I$ of small objects and let $Q, \{Q_\beta\}_J$ be a set of injective objects in \mathfrak{A} . Then*

1) *If Q is Σ -injective, $r(P, Q)$ is noetherian for any small object P . Conversely, if $r(G_\alpha, Q)$ is noetherian for all G_α , then Q is Σ -injective.*

2) *$\sum_r \bigoplus Q_\beta$ is injective if and only if for any $\alpha \in I$ and any chain $T_1 \subseteq T_2 \subseteq \dots \subseteq T_n \subseteq \dots$ of subobjects of G_α , there exist n_0 and a finite subset J_0 of J such that $[T_{n+1}/T_n, Q_r] = 0$ for all $n \geq n_0$ and $r \in J - J_0$.*

Proof. We assume that Q is Σ -injective and $r(P, Q)$ is not noetherian for a small object P . Let $P_1 \subseteq P_2 \subseteq \dots \subseteq P_n \subseteq \dots$ be a chain in $r(P, Q)$. Put $P_0 = \bigcup P_i$ and let $f_i \in l(P_i) - l(P_{i+1})$. Then $f_i(P_j) = 0$ for $j \leq i$ and $f_i(P_k) \neq 0$ for $k \geq i+1$. Put $f = \prod f_i \in [P_0, \prod Q]$. Since $f(P_i) \subset \sum \bigoplus Q$ and $P_0 = \lim P_i, f(P_0) \subset \sum \bigoplus Q$. However, P is small and so $\text{Im } f \subset \sum_1^m \bigoplus Q$, which contradicts to a fact $f|_{P_{m+1}} = \sum_{i=1}^{m+1} f_i|_{P_{m+1}} \not\subset \sum_1^m \bigoplus Q$. Hence, $r(P, Q)$ is noetherian. Conversely, we

assume that $r(G_\alpha, Q)$ is noetherian for all $\alpha \in I$. We consider a diagram for a subobject P of G_α

$$\begin{array}{ccccc} 0 & \rightarrow & P & \rightarrow & G_\alpha \\ & & \downarrow f & & \\ & & \sum_\gamma \oplus Q & & \end{array}$$

Let π_γ be the projection of $\sum_\gamma \oplus Q$ to the γ -th component Q . From Lemma 2 we obtain a small subobject P' of G_α such that $l(P) = l(P')$. Since P' is small, $\pi_\gamma f|P' = 0$ for almost all $\gamma \in J$. Hence, $\pi_\gamma f|P = 0$ for almost all γ , which means that $\text{Im } f \subset \sum_1^m \oplus Q$. Therefore, f is extended to an element in $[G_\alpha, \sum_\gamma \oplus Q]$, since Q is injective. Hence, $\sum_\gamma \oplus Q$ is injective by Lemma 1. We can prove 2) similarly to the case of modules.

Corollary 1. *Let Q be a Σ -injective and small object in \mathfrak{A} . Then $[Q, Q]$ is a semi-primary ring.*

Proof. It is clear from Theorem 1 and [10], Theorem 1.

Corollary 2 ([2]). *Let $\{Q_\alpha\}_I$ be a set of Σ -injectives. If $\sum_\gamma \oplus Q_\alpha$ is injective, $\sum_\gamma \oplus Q_\alpha$ is Σ -injective.*

Proof. It is clear from Theorem 1.

From Chase's method [3] and Theorem 1 we obtain

Corollary 3 ([3]). *Let \mathfrak{A} be as above. Then every injectives are Σ -injective if and only if \mathfrak{A} is locally noetherian.*

2. QF-categories

We have many characterizations of quasi-Frobenius rings R with identities. The categorical ones among them are

- I *Every projective modules is injective* [5] and
- II *Every injective module is projective* [6].

We shall define a quasi-Frobenius category by taking the property I. Let \mathfrak{A} be a Grothendieck category with generating set $\{G_\alpha\}_I$ of small objects. \mathfrak{A} is called *QF* if every projectives are injective.

First, we have

Proposition 1. *Let \mathfrak{A} be as above and G_α projective for all $\alpha \in I$. Then \mathfrak{A} is QF if and only if \mathfrak{A} is perfect and $\sum_I \oplus G_\alpha$ is Σ -injective.*

Proof. We assume \mathfrak{A} is QF . Then G_α is Σ -injective. Hence, G_α is a coproduct of completely indecomposable objects $\{P_\alpha^{(i)}\}$ by Corollary 1 to Theorem 1. Furthermore, since $\sum_K \oplus G_\alpha$ is injective, for any K , $\{P_\alpha^{(i)}\}_{\alpha,i}$ is a right T -nilpotent system by [9], Corollary to Proposition 10. Hence, \mathfrak{A} is perfect from [11], Corollary 1 to Theorem 4. Conversely, if \mathfrak{A} is perfect, \mathfrak{A} contains a generating set $\{P'_\alpha\}_{I'}$ of small projectives and every projectives are coproduct of some family of P'_α [11], §3. On the other hand $\sum_{I'} \oplus P'_\alpha$ is Σ -injective if so is $\sum_I \oplus G_\alpha$. Therefore, \mathfrak{A} is QF .

We know many interesting properties of a QF -ring and in this note we shall generalize some of them in \mathfrak{A} .

First, we shall give examples of QF -Grothendieck categories.

EXAMPLE 1. Let $\{K_i\}$ be a family of QF -rings. Then $\prod_i \mathfrak{M}_{R_i}$ is QF .

The following example is a slight modification of [18], p. 379.

EXAMPLE 2. Let K be a field and R be a vector space over K with basis $\{e_i, f_i\} : R = \sum_1^\infty \oplus (e_i K \oplus f_i K)$. We define a multiplication in R as follows:

$$e_i e_j = \delta_{i,j} e_i, e_i f_j = \delta_{i,j} f_i, f_i e_j = \delta_{i,j-1} f_i \quad \text{and} \quad f_i f_j = 0,$$

where $\delta_{i,j}$ is the Kronecker δ .

It is easily seen that R is an associative ring and $R = \sum \oplus e_i R = \sum \oplus R e_i$. Since $e_i R = e_i K \oplus f_i K$ is an artinian and noetherian R -module, \mathfrak{M}_R^{+1} is a locally artinian and noetherian perfect Grothendieck category from [11], §3. We shall show that $e_i R$ is injective in \mathfrak{M}_R^+ . Every $e_i R$ has only one proper submodule $f_i K$. Let g be in $[f_j K, e_i R]$. It is clear that $g=0$ if $i \neq j$. If $i=j$, $g(f_i) = f_i k$ for some k in K . Hence, $e_i k \in [e_i R, e_i R]$ and $e_i k | f_i K = g$. Therefore, $e_i R$ is injective by Lemma 1. Hence, \mathfrak{M}_R^+ is a perfect QF -category by Corollary 3 and Proposition 1.

Next, we consider ${}_R \mathfrak{M}^+$. Let g be an element in $[K f_1, R e_1]$ such that $g(f_1) = e_1$. Then it is clear that ag is not extended to an element in $[R e_2, R e_1]$. Hence, $R e_1$ is not injective in ${}_R \mathfrak{M}^+$. On the other hand, all of other $R e_i$ are injective as above. Thus, ${}_R \mathfrak{M}^+$ is a QF -3 perfect category from [13], but not QF . Furthermore, R is a cogenerator in ${}_R \mathfrak{M}^+$, but not in \mathfrak{M}_R^+ .

This example shows that a perfect QF -category does not inherit some properties of QF -rings, Furthermore, the example given in [9], p. 331 is a QF -Grothendieck category with generator and cogenerator object, however it is neither locally noetherian nor artinian (this category does not contain a generating set of small objects).

We do not know whether QF -categories with generating set of small objects are locally noetherian (or artinian).

Let \mathfrak{A} be the category as above. Put $G = \sum_I \oplus G_\alpha$ and $S = [G, G] = \prod_\alpha [G_\alpha, G]$. Then S contains the ring $R = \sum_{\alpha, \beta} \oplus [G_\alpha, G_\beta]$. Let $\prod_\alpha f_\alpha, \prod_\alpha g_\alpha$ be elements in S . Since G_α is small, $\sum_\gamma f_\gamma g_\alpha$ is in $[G_\alpha, G]$. Hence, $(\prod f_\alpha)(\prod g_\alpha) = \prod_\alpha (\sum_\gamma f_\gamma g_\alpha)$. If $\prod g_\alpha$ is in R , $g_\alpha = 0$ for almost all α . Hence, $(\prod f_\alpha)(\prod g_\alpha) \in R$ and $SR \subset R$. For any subobject Q of G we put $l_R(Q) = l_S(Q) \cap R$.

Lemma 3. *Let G and G_α be as above. For any subobject Q of G_α we have $r(l_R(Q)) = r(l_S(Q))$.*

Proof. Put $G_\alpha = G_1, G_2 = \sum_{\alpha \neq \beta} \oplus G_\beta$ and $S_{ij} = [G_j, G_i]$. Then $S = \sum_{i,j=1}^2 \oplus S_{ij}$ and S_{i1} are in R . $l_S(Q) = \sum_{i=1}^2 S_{i2} \oplus \sum_{i=1}^2 (l_S(Q) \cap S_{i1})$. Then $r(l_S(Q)) = r(\sum_i \oplus S_{i2}) \cap r(T) = G_1 \cap r(T)$, where $T = \sum_{i=1}^2 (l_S(Q) \cap S_{i1})$. On the other hand, $l_R(Q) = (R \cap \sum_{i=1}^2 S_{i2}) \oplus T$ and $r(l_R(Q)) = G_1 \cap r(T) = r(l_S(Q))$.

Proposition 2. *Let \mathfrak{A} be the Grothendieck category with G_α . If G_α is Σ -injective for all α and G is an injective cogenerator, then \mathfrak{A} is locally noetherian.*

Proof. Let S and R be as above. Then $r(l_S(Q)) = Q$ for any subobject Q of G by the assumption, (cf. [10], §2). Put $R = [G_\alpha, G] \oplus \sum_{\alpha \neq \beta} \oplus [G_\beta, G]$. Then for any subobject Q' of G_α , $l_R(Q') = l_S(Q') \cap [G_\alpha, G] \oplus \sum_{\alpha \neq \beta} \oplus [G_\beta, G]$. Hence, $Q' = r(l_S(Q')) = r(l(Q')) = r(l_R(Q') \cap [G_\alpha, G]) \cap r(\sum_{\alpha \neq \beta} \oplus [G_\beta, G]) = G_\alpha \cap r(l_{[G_\alpha, G]}(Q')) = r_{G_\alpha}(l_{[G_\alpha, G]}(Q'))$. Since G is Σ -injective, G_α is noetherian from Theorem 1.

Corollary. *Let $R = \sum_I \oplus e_\alpha R = \sum_I \oplus Re_\alpha$ be the induced ring from a category.¹⁾ We assume that \mathfrak{M}_R^+ is QF and R is a cogenerator in \mathfrak{M}_R^+ . If for a given α , $e_\alpha Re_\beta = 0$ for almost all $\beta \in I$, $e_\alpha R$ is artinian and noetherian.*

Proof. There exists an idempotent $E = e_\alpha + e_{\alpha_2} + \dots + e_{\alpha_n}$ such that $e_\alpha R = e_\alpha RE \subset ERE$. ER is noetherian by Proposition 2. Hence, ERE is right noetherian and semi-primary by [10], Theorem 1. Therefore, ERE is right artinian and so $e_\alpha R$ is artinian as an R -module.

The following theorem is a version of [14] in \mathfrak{A} .

Theorem 2 ([14]). *Let \mathfrak{A} be a locally noetherian category with generating set $\{P_\alpha\}_I$ of small projectives. We put $P = \sum_I \oplus P_\alpha$ and $S = [P, P]$. Then \mathfrak{A} is QF if and only if*

- 1) *For any $\alpha \in I$ and any finitely generated S -module \mathfrak{I} of $[P_\alpha, P]$ $l(r(\mathfrak{I})) = \mathfrak{I}$.*
- 2) *For any $\alpha \in I$ and any subobjects P_1, P_2 in P_α $l(P_1 \cap P_2) = l(P_1) + l(P_2)$ in $[P_\alpha, P]$.*

Proof. Let Q be an injective object. Then 1) and 2) are valid if we replace P and S by Q and $[Q, Q]$ (cf. [10]). We assume 1) and 2) and show that P_ω is injective. Let P_1 be a subobject of P_β such that $P_1 = \text{Im } x; x \in [P_\gamma, P_\beta]$. We may assume $x \in [P_\gamma, P]$. Let f be in $[P_1, P_\omega]$. $x: P_\gamma \xrightarrow{x'} P_1 \xrightarrow{i} P_\beta$ and put $K = \text{Ker } x'$. Then $r_{P_\gamma}(x) = K$. Since $l(K) = l(r(x)) = Sx$ and $fx_1 \in l(K), fx' = sx$ for some $s \in S$. We may assume $s \in [P_\beta, P_\omega]$. Then, $f = si$ and $s|P = f$. Since P_β is noetherian, every subobject of P_β is of form $\bigcup_1^n \text{Im } x_i; x_i \in [P_\gamma, P]$. We can prove, analogously to the case of modules, from 2) that every element in $[P', P_\omega]$ is extended to one in $[P_\beta, P_\omega]$, (cf. [10]). Hence, P_ω is injective by Lemma 1. Since \mathfrak{A} is locally noetherian, \mathfrak{A} is perfect from Corollary 3 to Theorem 1 and [9], Corollary to Proposition 10. Hence, \mathfrak{A} is QF by Proposition 1.

Let T be a ring with identity. If T is right artinian and self injective as a right T -module, then T is QF and T is left artinian and self injective as a left T -module. However, as shown in Example 2, this fact is not true for \mathfrak{A} .

Theorem 3. *Let $R = \sum_I \oplus e_\alpha R = \sum_I \oplus Re_\alpha$ be the induced ring from the category \mathfrak{A} . Then the following are equivalent.*

- 1) \mathfrak{M}_R^+ and ${}_R\mathfrak{M}^+$ are QF .
- 2) \mathfrak{M}_R^+ is locally noetherian and R is injective in \mathfrak{M}_R^+ and ${}_R\mathfrak{M}^+$.
- 3) \mathfrak{M}_R^+ is QF and R is injective in ${}_R\mathfrak{M}^+$.
- 4) \mathfrak{M}_R^+ is QF and locally artinian and R is a cogenerator in \mathfrak{M}_R^+ , (cf. [4]).

Proof. We first show the following fact. If \mathfrak{M}_R^+ is QF and R is injective in ${}_R\mathfrak{M}^+$, then R is a cogenerator in \mathfrak{M}_R^+ . We may assume e_α 's are primitive. From the remark before Lemma 3 and the first part of the proof of Theorem 2, we have $rl(r') = r'$ for any finitely generated right R -module r' in $[Re_\alpha, R] = e_\alpha R$. Let r be any right R -module in $e_\alpha R$. Then $l(r) = \bigcap l(r')$, where r' runs through all finitely generated R -modules. Since \mathfrak{M}_R^+ is perfect from the assumption and Proposition 1, ${}_R\mathfrak{M}^+$ is semi-artinian by [11], Theorem 5. Hence, Re_α contains a unique minimal submodule S_α . Since $l(r') \neq 0, l(r) = \bigcap l(r') \supseteq S_\alpha \neq 0$. Therefore, $e_\alpha R/r$ is contained in R and hence, R is a cogenerator in \mathfrak{M}_R^+ .

1) \rightarrow 2), 3) and 4). Since ${}_R\mathfrak{M}^+$ is perfect, \mathfrak{M}_R^+ is semi-artinian. On the other hand, R is a cogenerator in \mathfrak{M}_R^+ from the above. Hence, \mathfrak{M}_R^+ is locally noetherian and artinian by Proposition 2. Therefore, 1) implies 2), 3) and 4).

2) \rightarrow 3) and 4). Since $e_\alpha R$ is injective and noetherian, $e_\alpha Re_\alpha$ is semi-primary by [10], Theorem 1. Furthermore, we may assume that $e_\alpha R$'s are indecomposable. Then so are the Re_α 's. Since $R = \sum_I \oplus Re_\alpha$ is injective, $\{Re_\alpha\}_I$ is a semi- T -nilpotent system by [9], Corollary to Proposition 10. However, $e_\alpha Re_\alpha$ is semi-

primary and hence, $\{Re_\alpha\}_I$ is a T -nilpotent system. Therefore, ${}_R\mathfrak{M}^+$ is perfect. Similarly, we obtain from Corollary 3 to Theorem 1 that \mathfrak{M}_R^+ is QF . Hence 2) implies 3) and 4) from the first statement. 3) \rightarrow 2). R is a cogenerator in \mathfrak{M}_R^+ from the first remark. Hence, \mathfrak{M}_R^+ is locally noetherian. 4) \rightarrow 1). We may assume that $e_\alpha R$ is perfect for all α . We note $[e_\alpha R, R]=Re_\alpha$ and $[Re_\alpha, R]=e_\alpha R$. Since R is an injective cogenerator in \mathfrak{M}_R^+ , $r_{e_\alpha R}(l_{Re_\alpha}(\mathfrak{r}))=\mathfrak{r}$ for any R -submodule \mathfrak{r} in $e_\alpha R$ and $l_{Re_\alpha}(r_{e_\alpha R}(\mathfrak{l}))=\mathfrak{l}$ for a finitely generated left R -submodule \mathfrak{l} of Re_α . Hence, Re_α is noetherian by the assumption and artinian from Proposition 2 and the above. Moreover, the above facts imply, from Theorem 2 and the remark, that ${}_R\mathfrak{M}^+$ is QF .

Corollary. *Let R be as above. We assume that R is a cogenerator in \mathfrak{M}_R^+ and ${}_R\mathfrak{M}^+$, and \mathfrak{M}_R^+ is locally noetherian. Then the following are equivalent.*

- 1) R is injective in ${}_R\mathfrak{M}^+$.
- 2) ${}_R\mathfrak{M}^+$ is locally noetherian.
- 3) \mathfrak{M}_R^+ is locally artinian.

In those cases \mathfrak{M}_R^+ and ${}_R\mathfrak{M}^+$ are QF , (cf. [17], p. 406).

Proof. We first show that \mathfrak{M}_R^+ is QF from the assumption. We quote here the idea of Kasch [17]. Let E be an injective hull of R in \mathfrak{M}_R^+ . We put $\mathfrak{r}=\cup \text{Im } f, f\in[E, R]$, then \mathfrak{r} is a two-sided ideal of R . If $\mathfrak{r}\neq R$, there exists $s\neq 0$ in $[R, R]$ as left R -modules such that $r^s=0$, since R is a cogenerator in ${}_R\mathfrak{M}^+$. We take an idempotent e_α in R such that $e_\alpha^s\neq 0$. Then for any $f\in[E, R]$, $0=(f(e_\alpha))^s=(f(e_\alpha)e_\alpha)^s=f(e_\alpha)e_\alpha^s=f(e_\alpha^s)$. On the other hand, R is a cogenerator in \mathfrak{M}_R^+ . Hence, we have shown $\mathfrak{r}=R$, which implies that R is a retract of $\sum_{[E, R]} \oplus E$. Since R is locally noetherian, R is injective in \mathfrak{M}_R^+ . Therefore, \mathfrak{M}_R^+ is QF . Similarly, we can prove that ${}_R\mathfrak{M}^+$ is QF if ${}_R\mathfrak{M}^+$ is locally noetherian. Hence, 2) implies 1) and 3). The remaining parts are clear from Theorem 3.

3. Property II

In this section, we shall study a relation between the property II and a QF -category. Faith and Walker [6] showed that a ring T with identity is QF if and only if II is satisfied. However, the following examples show that the above fact is not true for Grothendieck categories.

EXAMPLE 3. In Example 2 we replace relations $f_i e_j = \delta_{i, j+1} f_i$ and $e_i f_i = f_i$ by $e_j f_i = \delta_{j-1, i} f_i$ and $f_i e_i = f_i$, respectively. Then $R = \sum \oplus e_i R = \sum \oplus R e_i$ is perfect and locally artinian and noetherian. We can show that $e_i R$ for $i \geq 2$ are injective. Let E be an injective object in \mathfrak{M}_R^+ . Then E contains non-zero homomorphic image of some $e_i R$. Hence, E contains $e_i R$ or $e_{i+1} R$ as an isomorphic image. Therefore, $E \approx \sum_{i \geq 2} \oplus e_i R^{(\alpha)}$, since R is locally noetherian. Thus,

R satisfies II and $\sum_{i \geq 2} \oplus e_i R$ is an injective cogenerator in \mathfrak{M}_R^+ . On the other hand, $e_1 R$ is not injective and hence, \mathfrak{M}_R^+ is not QF .

In Example 2 an injective hull $E(e_1 R/e_1 N)$ of $e_1 R/e_1 N$ is not projective and hence, \mathfrak{M}_R^+ does not satisfy II. On the other hand,

EXAMPLE 4. Let R be a vector space over a field K with basis $\{e_i, f_i\}_{i \geq 1}$ and define the multiplication in R in Example 2. Then \mathfrak{M}_R^+ and ${}_R \mathfrak{M}^+$ are QF and R is a cogenerator in \mathfrak{M}_R^+ and ${}_R \mathfrak{M}^+$.

Let \mathfrak{A} be a Grothendieck category with a generating set $\{P_\alpha\}_I$ of small projectives. We assume \mathfrak{A} satisfies II. Then considering the induced ring from \mathfrak{A} , we can show from [6], Theorem 1.1 that \mathfrak{A} is locally noetherian. Thus, we have from the argument in Example 3

Proposition 3. *Let \mathfrak{A} be as above. Then \mathfrak{A} satisfies II if and only if \mathfrak{A} is locally noetherian and every indecomposable injective object is projective.*

Corollary 1. *Let \mathfrak{A} be as above. If \mathfrak{A} satisfies II, $P = \sum_I \oplus P_\alpha$ is a cogenerator in \mathfrak{M}_R^+ . Conversely, if \mathfrak{A} is locally noetherian and artinian and P is a cogenerator, then \mathfrak{A} satisfies II.*

Proof. We assume \mathfrak{A} satisfies II. Then for any minimal object S_α in \mathfrak{M}_R^+ , $E_\alpha = E(S_\alpha)$ is projective indecomposable. Hence, E_α is isomorphic to a retract of some P_β by [21], Lemma 2. Since P_α 's are finitely generated, $\sum \oplus E_\alpha$ is a cogenerator. Therefore, P is a cogenerator. Conversely, we assume \mathfrak{A} is locally noetherian and artinian. Every indecomposable injective E is the injective hull of its socle. If P is a cogenerator, E is a retract of P . Hence, E is projective. Therefore, \mathfrak{A} satisfies II by Proposition 3.

Corollary 2. *Let \mathfrak{A} be as above. We assume \mathfrak{A} is QF and semi-artinian. Then $P = \sum_I \oplus P_\alpha$ is a cogenerator if and only if \mathfrak{A} satisfies II.*

Proof. If P is a cogenerator, \mathfrak{A} is locally noetherian, and hence, locally artinian by the assumption.

The following lemma is essentially due to Faith and Walker [6]. However, we shall give the proof as an application of [9], Theorem 1.

Lemma 4 ([6]). *Let R be the induced ring from a category and let $\{E_\alpha\}_L$ be a set of projective, injective and indecomposable objects in \mathfrak{M}_R^+ . Then every coproducts P of any family of E_α 's are injective if and only if $E(P)$ is projective for all P .*

Proof. "Only if" part is clear. We denote the cardinal number of a set K' by $|K'|$. Let $\zeta = |R|$ and K a countably infinite set. We put $M = \sum_{i \in K} \oplus E_i$;

$E_i \in \{E_i\}_L$, (E_i may be equal to E_j). Since E_i is projective, $E_\alpha \approx f_\alpha R$ for some primitive idempotent f_α in R by [11], Corollary to Lemma 2. Let J be a set of $|J| = \max(\zeta, \aleph_0) = \xi$ and put $M^\xi = \sum_{\alpha \in J} \oplus M^{(\alpha)}$; $M^{(\alpha)} \approx M$. Then $M^\xi = \sum_{\beta \in K} \sum_{\delta \in J_\beta} \oplus (f_\beta R)^{(\delta)}$; $(f_\beta R)^{(\delta)} \approx f_\beta R$ and $|J_\beta| = \xi$. Let $E = E(M^\xi)$. Since E is projective by the assumption, E is a retract of a form $\sum_{\varepsilon \in T} \oplus e_\varepsilon R$. Hence, $E \approx \sum \oplus g_\varepsilon R$ and $e_\varepsilon R = g_\varepsilon R \oplus g_{\varepsilon'} R$ by [21], Lemma 2. Now, we consider those injective modules in the category \mathfrak{C} of injective modules modulo the radical of \mathfrak{C} defined in [9], §1. Then $\sum_{\kappa} \sum_{J_\beta} \oplus (f_\beta R)^{(\delta)} = \sum_{\mathcal{T}} \oplus \overline{g_\varepsilon R}$, where $\overline{f_\beta R}$ and $\overline{g_\varepsilon R}$ mean the residue classes of $f_\beta R$ and $g_\varepsilon R$, respectively. Since $\overline{f_\beta R}$ is minimal in \mathfrak{C} , $\overline{g_\varepsilon R} \approx \sum_{\kappa \in \beta} \sum_{J_\beta^{(\varepsilon)}} \oplus \overline{f_\beta R}^{(\delta)}$, $J_\beta'(\varepsilon) \subseteq J_\beta$, which means that $g_\varepsilon R = E(\sum_{\kappa} \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)})$. Hence, every element φ in $[\sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}]$ is extended to φ' in $[g_\varepsilon R, g_\varepsilon R] = g_\varepsilon R g_\varepsilon$. On the other hand, $[\sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}] = \prod_{J_\beta'} [(f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}] \supseteq \prod_{J_\beta'} f_\beta R f_\beta$ and $\zeta \geq |g_\varepsilon R g_\varepsilon| \geq 2^{|J_\beta'(\varepsilon)|}$. Hence, $|J_\beta'(\varepsilon)| < \zeta \leq \xi$. Next, we take an index β in K and consider the subset $T_\beta = \{\varepsilon \in T, J_\beta'(\varepsilon) \neq \emptyset\}$. Since $J_\beta = \bigcup_{\varepsilon \in T_\beta} J_\beta'(\varepsilon)$, $|T_\beta| = |J_\beta| = \xi \geq \aleph_0$. Hence, for each β we can find an index $\varepsilon(\beta)$ in T_β such that $\varepsilon(\beta) \neq \varepsilon(\beta')$ if $\beta \neq \beta'$. Therefore, $\sum_{\alpha \in K} \oplus f_\alpha R$ is a retract of E and hence, M is injective. Thus, we have proved the lemma by virtue of the proof of Theorem 2, (see [5]).

Proposition 4. *Let \mathfrak{A} be the Grothendieck category with generating set $\{P_\alpha\}_I$ of small projectives. We assume \mathfrak{A} satisfies II. Then the representative class $\{S_\gamma\}_K$ of the minimal objects is a set. Furthermore, $E(\sum_K \oplus S_\gamma) / (J(E(\sum_K \oplus S_\gamma))) \approx \sum_K \oplus S_\gamma$ if and only if \mathfrak{A} is QF and every projective contains the non-zero socle, where $J(\)$ means the Jacobson radical.*

Proof. “Only if”. We take the induced ring R from \mathfrak{A} . Let S_α be an minimal object (cf. [11], Proposition 2) and $E_\alpha = E(S_\alpha)$. Then $E_\alpha \approx f_\alpha R$ by the assumption. Hence, $\{S_\alpha\}_K$ is a set. It is clear that $\bigcup_K E_\alpha = \sum_K \oplus E_\alpha$ and $\sum \oplus E_\alpha$ is injective by Lemma 4. Therefore, $E(\sum \oplus S_\alpha) = \sum \oplus E_\alpha$. We assume any $S_\alpha \approx E_\alpha' / J(E_\alpha') = f_\alpha' R / f_\alpha' J(R)$. Let P be projective. Then P contains a maximal subobject P_0 by [11], Proposition 2. $P/P_0 \approx E_\alpha / J(E_\alpha)$ for some α by the assumption. Since E_α is perfect, E_α is a retract of P . We consider the set of submodules in R which are coproducts of some E_α 's. Using the Zorn's lemma and the above fact, we know $R = \sum \oplus E_\alpha$. Hence, \mathfrak{M}_R^+ is QF and every projective contains a minimal module. “If”. We assume the above properties, then $E = E(\sum_K \oplus S_\alpha) \approx \sum_K \oplus e_\alpha R$ and every indecomposable projective P is isomorphic to $e_\alpha R$ for some $\alpha \in K$. Hence, $E/J(E) \approx \sum \oplus S_\alpha$.

We shall apply the above to a ring with identity.

Theorem 4 ([6]). *Let S be a ring with identity. Then S is a QF-ring if and only if II is satisfied in \mathfrak{M}_R .*

Proof. “Only if” part is clear from Corollary 1 to Proposition 3. We assume II. Then $\sum_K \oplus E_\alpha$ is a direct summand of S as the proof of Proposition

4. Hence, K is finite. Therefore, $E/J(E) \approx \sum_K \oplus S_\alpha$.

Finally, we shall consider the category of covariant additive functors (\mathfrak{C}, Ab) , where \mathfrak{C} is a small abelian category.

Proposition 5. *Let \mathfrak{C} be a small abelian category. Then the following are equivalent.*

- 1) (\mathfrak{C}, Ab) is semi-simple (completely reducible).
- 2) (\mathfrak{C}, Ab) is QF.
- 3) (\mathfrak{C}, Ab) satisfies II.

In such a case, every object in \mathfrak{C} is a finite coproduct of minimal objects.

Proof. Put $\mathfrak{A} = (\mathfrak{C}, Ab)$ and $H^C = [C, -]$ for $C \in \mathfrak{C}$, then $\{H^C\}_{C \in \mathfrak{C}}$ is a generating set of small projectives of \mathfrak{A} . We assume \mathfrak{A} is QF. Then \mathfrak{A} is perfect By Theorem 1. Hence, every object C in \mathfrak{C} is a finite coproduct of completely indecomposable objects $\{C_\alpha\}$ by [11], Proposition 5. Furthermore, since H^C is injective in \mathfrak{A} , C is projective in \mathfrak{C} by [15], p. 100, Proposition 2.3. Hence, every C_α is minimal and \mathfrak{A} is semi-simple by [20], Proposition 5. Next, we assume \mathfrak{A} satisfies II. Then \mathfrak{A} is locally noetherian by Proposition 3. Hence, \mathfrak{C} is artinian. Let C be minimal in \mathfrak{C} . Then we can easily see that H^C is minimal in \mathfrak{A} , (cf. [20]). Let $C \supset C_1$ be objects in \mathfrak{C} and C_1 minimal. Then $0 \rightarrow H^{C/C_1} \rightarrow H^C \rightarrow H^{C_1} \rightarrow 0$ is exact, since H^{C_1} is minimal. Hence, C_1 is a retract of C , since H^{C_1} is projective. Therefore, \mathfrak{C} is semi-simple and artinian, which implies \mathfrak{A} is semi-simple from [20], Proposition 5.

We note that every perfect Grothendieck category \mathfrak{A} is equivalent to (\mathfrak{C}°, Ab) by [11], Theorem 4, where \mathfrak{C} is a small amenable preadditive category. Hence, if \mathfrak{A} is non semi-simple QF, \mathfrak{C} is not abelian.

4. Projective and injective objects

From the definition of a QF-category, every projectives are injective and so we shall study, in this section, projective, injective objects in the Grothendieck category \mathfrak{A} with generating set $\{G_\alpha\}_I$ of small objects. Which is a supplement of [10].

As a dual of weakly distinguished objects [9], we define a weakly co-distinguished object. If an object P in \mathfrak{A} has a property $[P, P_1/P_2] \neq 0$ for

any subobjects $P_1 \supset P_2$ of P such that P_1/P_2 is minimal, then P is called *weakly co-distinguished*. Since \mathfrak{A} has $\{G_\alpha\}$, if P is projective, then P is weakly co-distinguished if and only if $[P, P_1] \cong [P, P_2]$ for any subobjects $P_1 \cong P_2$ of P .

Put $S = [P, P]$. For any subset T of S $r_S(T) = \{s \in S, Ts = 0\}$, $l_S(T) = \{s \in S, sT = 0\}$ and $TP = \bigcup_{f \in T} \text{Im } f$.

Lemma 5. *Let P be projective and $S = [P, P]$. For any left ideal I and right ideal \mathfrak{r} of S , $r_S(I) = [P, r_P(I)]$ and $l_S(\mathfrak{r}P) = l_S(\mathfrak{r}P)$.*

Proof. It is clear that $r_S(I)P \subseteq r_P(I)$ and $r_S(I) \subseteq [P, r_P(I)]$. Let f be in $[P, r_P(I)]$. Then $I f(P) \subseteq I r_P(I) = 0$. Hence $f \in r_S(I)$. The last statement is clear.

Proposition 6. *Let P be projective and weakly co-distinguished in \mathfrak{A} and $S = [P, P]$. Then*

- 1) $r_P(I) = r_S(I)P$ for any left ideal I in S .
- 2) $P_0 = [P, P_0]P$ for any subobject P_0 of P .

Furthermore, we assume P is injective and weakly distinguished, then

- 3) $l_S(r_S(I)) = l_S(r_P(I)) = I$ for any finitely generated left ideal I of S .
- 4) $r_S(l_S(\mathfrak{r})) = \mathfrak{r}$ for any finitely generated right ideal \mathfrak{r} of S .
- 5) $r_P(l_S(\mathfrak{r})) = \mathfrak{r}P$ for any right ideal \mathfrak{r} of S .

Proof. We assume that P is projective and co-distinguished. We have from Lemma 5 that $r_S(I) \subseteq [P, r_S(I)P] \subseteq [P, r_P(I)] = r_S(I)$. Hence, $r_P(I) = r_S(I)P$. Similarly, we have 2). We further assume P is injective. Then $l_S(r_S(I)) = l_S(r_S(I)P) = l_S(r_P(I))$ by Lemma 5 and 1). If I is finitely generated, $l_S(r_P(I)) = I$, (Theorem 2). Finally, we further assume P is injective and distinguished. $r_P(l_S(\mathfrak{r})) = r_P(l_S(\mathfrak{r}P)) = \mathfrak{r}P$ for any right ideal \mathfrak{r} by Lemma 5 and [10]. Hence, if \mathfrak{r} is finitely generated, $\mathfrak{r} = [P, \mathfrak{r}P] = [P, r_P(l_S(\mathfrak{r}))] = r_S(l_S(\mathfrak{r}))$ by Lemma 5 and [8], Lemma 2.6.

Corollary. *Let P and S be as above. Then P is artinian if and only if S is right artinian. Furthermore, if P is injective, the following are equivalent.*

- 1) P is artinian.
- 2) P is noetherian.
- 3) S is right noetherian, (artinian).

If P is projective, injective, weakly distinguished and co-distinguished, then the following are equivalent.

- 1)~3).
- 4) S is left noetherian, (artinian).
- 5) S is a QF-ring. (cf. [10], Theorem 2, [16], Satz and [19], §3).

Proof. The first statement is clear from Proposition 6, 2) and [11], Corollary 2 to Lemma 2. We assume P is injective. 1)→3). Since S is

right artinian from the above, S is noetherian. 3) \leftrightarrow 2). It is evident from Proposition 6, 2) and [8], Proposition 2.7. 2) \rightarrow 1). S is semi-primary by [10], Theorem 1 and hence, S is right artinian. Therefore, P is artinian from the first statement. Finally, we assume further that P is weakly distinguished. 1) \rightarrow 4). It is clear from Proposition 6, 3). Furthermore, S is left artinian, since S is semi-primary. 4) \rightarrow 1). P is artinian, since P is injective and weakly distinguished. 1) \leftrightarrow 5). It is clear from the proof of Theorem 2 and Proposition 6, 3) and 4).

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