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## A NOTE ON STABLE CLIFFORD EXTENSIONS OF MODULES

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### Abstract

Let  $H$  be a normal subgroup of  $G$ . Let  $W$  be a  $G$ -invariant indecomposable  $RH$ -module with vertex  $Q$ . Let  $V$  be an indecomposable direct summand of the induced module  $W^G$ . Let  $W'$  and  $V'$  be the Green correspondents of  $W$  and  $V$  in  $N_H(Q)$  and  $N_G(Q)$  respectively. Then we prove that  $\text{rank}_R V / \text{rank}_R W = \text{rank}_R V' / \text{rank}_R W'$ .

Let  $\mathcal{O}$  be a complete discrete valuation ring, and let  $F$  be the residual field of  $\mathcal{O}$  of characteristic  $p > 0$ . We assume that  $\mathcal{O}$  and  $F$  are big enough. Let  $R$  be  $\mathcal{O}$  or  $F$ .

Let  $G$  be a finite group. Let  $H$  be a normal subgroup of  $G$ . Let  $W$  be a  $G$ -invariant indecomposable  $RH$ -module with vertex  $Q$ . Let  $W'$  be the Green correspondent of  $W$  with respect to  $(H, Q, N_H(Q))$ . Then  $G = HN_G(Q)$ , and  $W'$  is  $N_G(Q)$ -invariant. We write  $W^G$  for the induction of  $W$  to  $G$ . Set  $E = \text{End}_{RG}(W^G)$  and  $\Lambda = \text{End}_{RH}(W)$ . We can write  $E$  in the form  $E = \sum_{\bar{x} \in X} \bigoplus E_{\bar{x}}$  where  $X = G/H$  and  $E_{\bar{x}}$  is the  $R$ -submodule of  $E$  mapping  $W = W \otimes 1$  to  $W \otimes x$  inside  $W^G$ , and  $\text{Hom}_{RH}(W, Wx) \cong E_{\bar{x}}$  (as  $R$ -module) by [4, Chap. 4, Lemma 6.4]. Clearly  $E_{\bar{x}}E_{\bar{y}} \subset E_{\bar{x}\bar{y}}$ , for  $\bar{x}, \bar{y} \in X$ . Also we can use the stability hypothesis to choose an element  $\varphi_{\bar{x}} \in E_{\bar{x}}$  mapping  $W \otimes 1$  isomorphically onto  $W \otimes x$ ; it follows that  $\varphi_{\bar{x}}$  is a unit in  $E$ . Since  $E_{\bar{1}}$  can be identified with  $\Lambda$ , we have  $E_{\bar{x}} = \Lambda\varphi_{\bar{x}} = \varphi_{\bar{x}}\Lambda$ . Thus  $E_{\bar{x}}$  can also be identified with  $\text{Hom}_{RH}(W, Wx)$ . We do so in this paper. Set  $E' = \text{End}_{RN_G(Q)}((W')^{N_G(Q)})$ , and  $\Lambda' = \text{End}_{RN_H(Q)}(W')$ . It is well-known that  $J(\Lambda)E \subset J(E)$  (resp.  $J(\Lambda')E' \subset J(E')$ ), and  $E/J(\Lambda)E$  (resp.  $E'/J(\Lambda')E'$ ) is a twisted group algebra.

In [3], the author proves that  $E'/J(\Lambda')E'$  is isomorphic to  $E/J(\Lambda)E$ , which already appears without proof in Cline [2]. Here we will give an application of the above isomorphism. The following is our main result.

**Theorem.** *Keep notation and assumptions as above. Then the Green correspondence gives a one-to-one correspondence between the set of non-isomorphic indecomposable direct summands of  $W^G$  and that of  $(W')^{N_G(Q)}$ , and keeps multiplicities. Let*

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$V$  be an indecomposable direct summand of  $W^G$ . Let  $V'$  be the Green correspondent of  $V$  in  $N_G(Q)$ . Then  $\text{rank}_R V/\text{rank}_R W = \text{rank}_R V'/\text{rank}_R W'$ .

Proof. From the proof of the main theorem in [3], we do not know whether the isomorphism given there is compatible with the Green correspondence. Here we will first construct an isomorphism from  $E'/J(\Lambda')E'$  to  $E/J(\Lambda)E$ , which is compatible with the Green correspondence.

Let  $L$  be a subgroup of  $G$ . For  $RL$ -modules  $W_1$  and  $W_2$ , there is a homomorphism

$$\text{Tr}_L^G: \text{Hom}_{RL}(W_1, W_2) \rightarrow \text{Hom}_{RG}((W_1)^G, (W_2)^G),$$

such that the following holds: for  $f \in \text{Hom}_{RL}(W_1, W_2)$  and  $\sum_{x \in L \setminus G} u_x \otimes x \in (W_1)^G$ ,

$$\text{Tr}_L^G(f) \left( \sum_{x \in L \setminus G} u_x \otimes x \right) = \sum_{x \in L \setminus G} f(u_x) \otimes x.$$

Assume that  $(W')^H = W \oplus M$  for some  $RH$ -module  $M$ . Then  $(W')^G = W^G \oplus M^G$ . Let  $\Omega = \{P: P < Q \cap Q^x, \text{ for some } x \in H - N_H(Q)\}$ . Then  $M$  and  $M^G$  are  $\Omega$ -projective. Let  $\iota_{W^G}: W^G \rightarrow (W')^G$  and  $\pi_{W^G}: (W')^G \rightarrow W^G$  be the inclusion map and the projection map, respectively. We have the following algebra homomorphism:

$$\begin{aligned} \beta: \text{End}_{RN_G(Q)}((W')^{N_G(Q)}) &\rightarrow \text{End}_{RG}(W^G) \\ f &\mapsto \pi_{W^G} \cdot \text{Tr}_{N_G(Q)}^G(f) \cdot \iota_{W^G}. \end{aligned}$$

Since  $M^G$  is  $\Omega$ -projective,

$$\text{End}_{RG}((W')^G) = \text{End}_{RG}(W^G) + \text{Tr}_\Omega^G(\text{End}_R((W')^G)).$$

It is easy to see that

$$\text{End}_{RG}(W^G) \cap \text{Tr}_\Omega^G(\text{End}_R((W')^G)) = \text{Tr}_\Omega^G(\text{End}_R(W^G)).$$

By [4, Chapter 4, Lemma 5.3], the map  $\text{Tr}_{N_G(Q)}^G$  induces the following algebra isomorphism:

$$\text{End}_{RN_G(Q)}((W')^{N_G(Q)})/\text{Tr}_\Omega^{N_G(Q)}(\text{End}_R((W')^{N_G(Q)})) \rightarrow \text{End}_{RG}((W')^G)/\text{Tr}_\Omega^G(\text{End}_R((W')^G)),$$

so  $\beta$  induces the following algebra isomorphism:

$$\beta: \text{End}_{RN_G(Q)}((W')^{N_G(Q)})/\text{Tr}_\Omega^{N_G(Q)}(\text{End}_R((W')^{N_G(Q)})) \rightarrow \text{End}_{RG}(W^G)/\text{Tr}_\Omega^G(\text{End}_R(W^G)).$$

By [4, Chapter 4, Lemma 5.1 (Dade)], the following isomorphism holds:

$$\sum_{x \in H \setminus G} \bigoplus \text{Tr}_\Omega^H(\text{Hom}_R(W, Wx)) = \text{Tr}_\Omega^H(\text{Hom}_R(W, W^G)) \cong \text{Tr}_\Omega^G(\text{End}_R(W^G)).$$

Let  $I = \text{Tr}_\Omega^H(\text{End}_R(W))$  and  $I' = \text{Tr}_\Omega^{N_H(Q)}(\text{End}_R(W'))$ . It is easy to see that  $\text{Tr}_\Omega^H(\text{Hom}_R(W, Wx_i)) = \text{Hom}_{RH}(W, Wx_i) \cdot I$ . Thus by identification,  $\text{Tr}_\Omega^G(\text{End}_R(W^G))$  is just the graded ideal  $EI$  of  $E$  generated by  $I$ . Let  $E'I'$  be the graded ideal of  $E'$  generated by  $I'$ . Then  $\text{Tr}_\Omega^G(\text{End}_R(W^G)) = EI$  and  $\text{Tr}_\Omega^{N_G(Q)}(\text{End}_R(W'^{N_G(Q)})) = E'I'$ . Thus  $\beta$  gives an isomorphism from  $E'/E'I'$  to  $E/EI$ . Since both  $W$  and  $W'$  are not  $\Omega$ -projective,  $I \subseteq J(\Lambda)$  and  $I' \subseteq J(\Lambda')$ . Note that by restriction  $\beta$  induces an algebra isomorphism from  $\Lambda'/I'$  to  $\Lambda/I$ . Thus  $\beta$  induces an isomorphism from  $\Lambda'/J(\Lambda')$  to  $\Lambda/J(\Lambda)$ . So  $\beta$  sends  $J(\Lambda')$  to  $J(\Lambda)$ . As  $\beta$  is an isomorphism from  $E'/E'I'$  to  $E/EI$ , we have that  $\beta$  is an isomorphism from  $E'/J(\Lambda')E'$  to  $E/J(\Lambda)E$ .

The first statement of the Theorem is obvious. Thus  $V'$  is an indecomposable direct summand of  $(W')^{N_G(Q)}$ . Let  $e'$  be a (primitive) idempotent of  $E'$  correspondent to  $V'$ . Let  $e$  be a primitive idempotent of  $E$  such that  $\beta(\bar{e}') = \bar{e}$ . Set  $\tilde{V} = eW^G$ . Then  $\tilde{V}$  is of vertex  $Q$ , and is a direct summand of  $\text{Tr}_{N_G(Q)}^G(e')(W')^G = (e'(W')^{N_G(Q)})^G = (V')^G$ . We must have that  $\tilde{V}$  is the Green correspondent of  $V'$ . Thus  $V \cong \tilde{V}$ . By Cline [1, Corollary 3.15],  $V \cong W \otimes_\Lambda eE$  and  $V' \cong W' \otimes_{\Lambda'} e'E'$ . Since  $eE/J(\Lambda)eE = \beta(e'E'/J(\Lambda')e'E')$ , we have  $\dim_F e'E'/J(\Lambda')e'E' = \dim_F eE/J(\Lambda)eE$ . So

$$\text{rank}_R V/\text{rank}_R W = \dim_F eE/J(\Lambda)eE = \dim_F e'E'/J(\Lambda')e'E' = \text{rank}_R V'/\text{rank}_R W',$$

as desired. □

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