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Enhanced Method of Heat Sources in Welding and Plasma Spraying (1st Report) - Overview of Simple Thermal Plasma Models -†

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Abstract

A complex model of energy flow from electrodes through ionized gas to a weld-pool (WP), stored there until solidification and subsequently transferred to a heat affected zone (HAZ), is required for the better evaluation of dimensions, microstructural composition and integrity of the sub-domain of arc welding. Such a model should operate in a chain of three sub-regions: arc column with skin layers - sheaths coating electrodes, liquid metal in the WP, and the HAZ. The description of thermal energy transfer between three sub-regions of the complex welding domain involves a large number of processes observed in gaseous electronics, thermodynamics of reacting gases, electro-dynamics of fluid, and micro-metallurgy. The first part of the paper consists a short presentation of welding plasma model based mostly on the Magneto-Hydro-Dynamics (MHD) theory. The second part consists of the review of models of welding plasma suitable for simulation of TIG and PAW welding, and plasma spraying for which the numarical codes were developed successfully.

KEY WORDS: (Thermal plasma) (Local thermodynamic equilibrium) (Micro-state) (Macro state) (Bolzmann transport equations) (Maxwell electrodynamic equations) (Magneto-hydro- dynamic equations) (Distribution functions) (Electron density)

1 Introduction

1.1 Thermal Plasma Produced by Welding Arc

All welding plasmas are classified as *hot* plasmas in the American and European literature, while Russian literature refers to *low temperature* plasmas to distinguish them from thermonuclear fusion plasmas. We consider here the plasma that occurs during TIG and PAW welding, and spraying.

Welding plasmas belong to the sub-group known as the thermal plasmas which are, by definition, in local thermodynamic equilibrium (LTE) or close to such a state.

Thermal plasmas are at near-atmospheric pressure and they are considered to be in kinetic equilibrium due to the very high number of collisions. They are not in radiative equilibrium.

Generally the state of the thermal plasma is defined by constitutive variables:

velocity	temperature	plasma composition
$\mathbf{v} = (v_r, v_z)$	T	$\mathbf{n} = (n^{(e)}, n^{(i)})$

where the plasma composition $(n^{(e)}, n^{(i)})$ is defined by

pressure	enthalpy	current density	electric potential	magnetic field
P	h	$\mathbf{J}$	V	$\mathbf{B}$

This state description is true for the so called one-fluid description of plasma flow in the frame of Magneto-Hydro-Dynamics (MHD) when the temperature of electrons is assumed to be the same as that of ions, i.e. $T^{(e)} \equiv T^{(i)} = T$

It is assumed that a plasma in LTE is in kinetic equilibrium, excitation equilibrium, and ionization equilibrium, that is summarized in tables below

Type of equilibrium	Assumptions
Kinetic equilibrium	Each of the species of the dense, collision-dominated, high-temperature plasma assumes a Maxwellian temperature distribution
Excitation equilibrium	Every process that may lead to excitation and de-excitation is taken into account: - Excitation: – electron collisions, – photo-absorptions, - De-excitation: – collisions of the second kind, – photo-emissions,
Ionization equilibrium	Only the most prominent mechanisms leading to ionization and recombination are considered - Ionization: – electron collisions, – photo-absorptions - Recombination: – three-body recombination (fast electron, heavy ion and slower electron or neutral particle combination), – photo-recombination.

Type of equilibrium	Requirements
Kinetic equilibrium	The amount of energy that electrons pick up along one mean free path has to be very small compared with thermal energy of the electrons
Excitation equilibrium	The sum of mechanisms of excitation and mechanisms of de-excitation have to be equal for dominated collisional processes (Boltzmann distribution of population density).
Ionization equilibrium	For sufficiently large electron densities the particle densities are evaluated from the Saha equation. For smaller electron densities the corona formula is used.

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The plasma composition and density are unique functions of the temperature in regions, where plasma states can be qualified as close to LTE. When a plasma interacts with a solid or liquid boundary, the boundary layers are non-equilibrium regions. Therefore, the region of the welding arc is split into various sub-regions: cathode space charge sheath, cathode pre-sheath, arc column, anode pre-sheath, anode space charge sheath. The arc column can be quite well described by two-dimensional models due to the symmetry and quasi-laminar flow in the arc root. The plasma behavior in cathode and anode regions, which are turbulent (in the sense of the magneto-hydro-dynamic fluid not the ordinary fluid) with strong deviations from LTE, should be described by three-dimensional models. In thermal plasma descriptions, two terms are necessary: magnetically induced forces $\mathbf{J} \times \mathbf{B}$ in the equation of momentum, and the Joule heating term $\mathbf{J} \cdot \mathbf{E}$ in the energy equation. The balance equations for plasma are non-linear PDEs because of the strong dependence of the thermodynamic characteristics and plasma transport properties on temperature and composition. The presence the of Joule heating term requires Ohm's law providing the relation between current density and potential, and current conservation for dc arcs or Maxwell equations for radio-frequency (rf) dischargers (important in the case of plasma spraying). The Clapeyron ideal gas law for plasmas, that are reactive (ionized) gases, is supplemented by equations giving the composition. In LTE regions, e.g. in the arc column and electrode pre-sheaths, the equation describing electron densities, the product of thermal ionization, is known as the Saha equation. It can be derived from the minimization of Gibbs's free energy. In electrode boundary layers which are non-equilibrium regions with recombination processes, a set of evolution equations for electro-chemical reactions should be used. There are several models of boundary layers given by [8], [10], [16], [17], [18]. Unfortunately, none of them is so general to be valid for all situations of thermal plasma generation.

2 Subregions of Thermal Plasma in Welding

Welding plasmas are generated by passing an electric current through a gas. The gas is not conducting a current unless a sufficient number of charge carriers is generated and then electrical breakdown establishes a non-unique conducting path between electrodes. The welding plasma exists in the area of a high intensity arc where the potential distribution drops in front of the electrodes and shows relatively small potential gradient in the arc column. This observation allows for splitting the arc into parts: solid or liquid electrodes and their surfaces, thermal plasma column considered to be in one of forms of thermal equilibrium (local or partial local), and boundary layers, where any form of equilibrium is not possible. The anode and cathode boundary layers can be split further into space charge sheaths, cathode ionization pre-sheath, and the diffusion layer of the anode produced by the vaporization of the weld pool. Regions of thermal plasma in TIG welding are listed in the following table:

Physical regions in TIG welding	Sub-regions	sub-sub-regions
tungsten cathode	tungsten rod (non-consumable) tungsten cone (disintegration)	
gas tungsten welding arc	cathode layers	space charge sheath cathode pre-sheath (ionization zone)
	arc column	
	anode layers	anode pre-sheath (diffusion layer) space charge sheath
anode (workpiece)	weld pool fusion zone heat affected zone base metal	

Processes occurring in subregions in TIG welding plasma, weld pool and heat affected zone (HAZ) have a thermo-mechanical or an electrical nature and can be split into two groups:

Physical Subregions	Thermodynamics and Fluid Dynamics	Electricity and Gaseous Electronics
tungsten rod	• conduction and Ohmic heating, • radiation and convection	
cathode surface (solid body), tungsten cone	• conduction and Ohmic heating, • black body radiation • convection • energy flux towards the cathode surface balanced by heat conduction into the solid	• cooling due to thermionic emission of electrons from the surface, • heating due to ion emission from the plasma impacting on the cathode, • ion recombination

Physical Subregions	Thermodynamics and Fluid Dynamics	Electricity and Gaseous Electronics
cathode sheath (charge imbalance area, Debye length, collision free)	• convection, • radiation	• space charge zone screening off the wall potential, • electrical boundary layer, • sheath potential drop, • acceleration of ions towards cathode, • electron emission at the sheath edge, • electrons repelled by a sheath potential
cathode pre-sheath (collision dominated)	• Local Thermodynamic Equilibrium (LTE) of plasma at the plasma side of pre-sheath, • constant thermal pressure	• Ionization in electron collision with neutrals, • electron-ion collisions, • energy transfer between the beam of electrons (emitted by cathode) and heavy particles (ions), • potential drop, • three-body recombination, • electron self-diffusion, • thermal conductivity of electrons
arc column	• laminar fluid flow, • turbulent flow at the arc fringes, • LTE • governing equations deduced from magneto-hydro-dynamics (MHD), • convection (loss), • radiation (loss),	• free of space charges, • current driven by the electric field and the Hall-current, • transport of ionization energy, • thermal split of electrons and heavy ions,
arc column cont.	• two separate fluids flow (electrons and ions), • transport of metal vapor (from TIG anode), • thermo-diffusion of metal vapor into the shielding gas, • heat flux given by conduction, enthalpy transport (by the current carrying electrons), and thermo-diffusion,	
anode pre-sheath	• LTE would prevail throughout this zone, • transport of metal vapor (from anode), • convection, • radiation	• diffusion of charge carriers, • potential drop, • ion-electron collisions
anode sheath (charge imbalance area, Debye length)	• convection, • radiation • metal evaporation from the weld pool surface	• space charge zone screening off the wall potential, • electrical boundary layer, • strong electric field due to deviations from the quasi-neutrality, • almost zero electrical conductivity in the front of anode,
weld pool	• buoyancy driven by spatial variation of liquid metal density and variations of local composition (temperature dependent), • Marangoni convective flow, • heat transfer,	• generation of magnetic field by divergent current path, • Lorentz force, • viscous drag from plasma,

Physical Subregions	Thermodynamics and Fluid Dynamics	Electricity and Gaseous Electronics
weld pool cont.	• Marangoni force driven by spatial (gradient) variation of surface tension, • surface tension,	
fusion zone	• solidification, • solid-phase transformations, • conduction, • convection, • radiation,	

3 Comprehensive theory of thermal plasma

3.1 General notions

Several processes occur in the plasma and they are listed below respectively for the principal species:

particle	process
electrons	Ionization excitation penning ionization elastic scattering dissociation dissociative ionization dissociative attachment

particle	process
ions	charge exchange elastic scattering ionization excitation recombination dissociation chemical reaction

particle	process
photons	photo excitation photo dissociation photo ionization elastic scattering ionization dissociation photo emission

Plasma [2], [12] consists of a very large number of interacting particles and a statistical approach is appropriate to reduce the amount of information required for the development of a phenomenological model of thermal plasma and to provide a macroscopic description of plasma phenomena. The distribution function for specific particle species is defined as the density of particles in phase space $f(\mathbf{x}, \mathbf{v}, t) = dn(\mathbf{x}, \mathbf{v}, t)/d\mathbf{x}d\mathbf{v}$, where $f(\mathbf{x}, \mathbf{v}, t) \in \mathcal{C}$, f is finite for any t, $f \to 0$ as $\mathbf{v} \to \infty$. All macroscopic variables, in the thermal plasma model, are deduced from the distribution function because the moments of this statistical function are related to: number density $n(\mathbf{x}, \mathbf{v}, t)$, average velocity, momentum of flow , and energy of flow.

The Boltzmann equation [3], [4] gives the dependence of the distribution function on the independent variables $\{\mathbf{x}, \mathbf{v}, t\}$. The Boltzmann equation for particles of species i is

$$\frac{\partial}{\partial t} f_j + \nabla_{\mathbf{x}}(\mathbf{v}_j f_j) + \nabla_{\mathbf{v}}(\frac{\mathbf{F}_j}{m_j} f_j) = \sum_j C_{jk} \quad (1)$$

$\nabla_{\mathbf{x}}(\mathbf{v}_j f_j)$	net flow
$\nabla_{\mathbf{v}}(\frac{\mathbf{F}_j}{m_j} f_j)$	external forces
C_{jk}	net rate of increase of particles in the control volume as a result of collisions between particles of species j with particles of species k
F_j	either electric or magnetic forces acting perpendicular to $\mathbf{v}$
$\mathbf{v}_j$	velocity of species j
m_j	mass of j species
$\nabla_{\mathbf{v}}$, $\nabla_{\mathbf{x}}$	divergence operator defined in Cartesian coordinates $\mathbf{x}$ or $\mathbf{v}$

Assuming f to be a function of energy $\mathcal{E}$, the solution of Eq.(1) is called the Boltzmann distribution of energy and describes the energy distribution $f(\mathcal{E})$ among classical, eg. distinguishable particles

$$f(\mathcal{E}) = Ae^{-\mathcal{E}/k_B T} \quad (2)$$

where A is a normalization constant. It can be used to evaluate the average energy $\langle \mathcal{E} \rangle = k_B T$ per particle when there is no energy-dependent density of states to skew the statistics of the distribution.

The dynamics of plasma [2], [4] can be approximately described considering that the motion of plasma particles is controlled by the applied external fields in addition to the macroscopic average fields (smooth in space and time) generated to the presence and motion of all plasma particles.

Similar approximate methods for derivation of macroscopic variables are needed because of difficulties in solution of the time dependent Boltzmann equation. Directly from this equation and without solving it, one can derive differential equations governing the temporal and spatial variation of the macroscopic variables. These differential equations are called the macroscopic transport equations and can be obtained by taking moments of the Boltzmann equation, Eq.(1). The first three moments are

balance equation	equation type	obtained by multiplying LHS & RHS of Eq.(1) by
conservation of mass	continuity equation	m
conservation of momentum	equation of motion	$m\mathbf{v}$
conservation of energy	energy equation	$\frac{m\mathbf{v}^2}{2}$

Unfortunately, the resulting set of transport equations is not complete for each stage of moments hierarchy. Attempting to obtain a complete set of transport equations for a higher moment of the Boltzmann equation leads to the introduction of a new macroscopic variable:

equation	added new macroscopic variable
equation of motion	dyad of kinetic pressure
energy equation	heat flow vector

Therefore, it is necessary to introduce a simplifying assumption concerning the highest moment of the distribution function that appears in the system. Such an assumption can truncate the system of equations at some stages of moments hierarchy and create the closed system of transport equations.

The basic plasma theories can be classified according to the level of complication in the basic equations and number of macroscopic variables

level of difficulty	plasma theory	equations for each particle species
low	cold plasma	conservation of mass conservation of momentum
medium	warm plasma	conservation of mass conservation of momentum balance of adiabatic energy
high	hot plasma	conservation of mass, conservation of momentum conservation of energy

level of difficulty	macroscopic variables	approximations
low	number density mean velocity	kinetic pressure dyad $i, j = 1...K, p_{ij} = 0$ temperature = 0
medium	number density scalar pressure	heat flux vector = 0 non-diagonal terms of pressure dyad $i \neq j, p_{ij} = 0$ diagonal terms of pressure dyad
high	number density mean velocity scalar pressure temperature	$i \neq j, p_{ij} = 0$ $i = j, p_{ij} = p$

Using simplified forms of Boltzmann's transport equations and Maxwell's electrodynamic equations: Faraday's law, Ampere's law, Poisson's equation, and the continuity of magnetic field equation, the magneto-hydro-dynamic (MHD) theory can be developed. Further approximations could be done either on the cold or warm plasma levels:

type of plasma or particles	approximations and assumptions
isotropic plasma	no external magnetic field
anisotropic plasma	external magnetic field present
collisional plasma	wave dumping
plasma particles	only electron gas considered or electron and one or more ion species gas mixture considered or whole plasma considered as a conducting fluid

3.2 Transmission of plasma characteristics from the molecular- via micro- to the macro-theory of ionized fluid

Analysis of kinetic equations defined on the molecular level in plasma and based on the concept of probability density in the six-dimensional phase space (6-D PS) combining three-dimensional real geometrical space and three-dimensional velocity space is rather difficult and not very practical for the simulation of manufacturing by welding. Therefore, thermal plasma theories for the analysis on the macroscopic level are more attractive for this purpose. The above table is illustrating the transmission from the molecular- to macro-level

molecular state state	state of a particle defined by quantum numbers,
quantum numbers $\hat{n}_j = \{n_{jx}, n_{jy}, n_{jz}\}$	defined for the permitted values of particle momentum, i.e. $p_j = \hat{n}_j \frac{h}{2L}$ where L - side length of a control volume, h - Planck's constant
energy levels	can be imagined as the set of shelves at different elevations, defined for different possible values of $\hat{n}_j^2$, issued from the kinetic energy $E_{kj} = \frac{p_j^2}{2m} = \hat{n}_j^2 \frac{h^2}{8mL^2}$,
compartment	can be considered as the box of particles with different states, but all with the same energy,
energy states	can be shown as the set of compartments on each shelve,
degeneracy g_j of energy level j	number of compartments of the corresponding shelf,
micro-state	specification of the total number of particles in each **energy state**
macro-state	specification of the total number of particles N_j in each **energy level**

and supports the table where items appropriate for various levels of plasma modelling are listed subsequently from the molecular level up to the macro-state

Notion	Definition
state of every particle	Position and momentum of every particle in the system of N particles can be represented by a point in six-dimensional phase space $\{\mathbf{X} \times \mathbf{p}\} = \{x_1, x_2, x_3, p_1, p_2, p_3\}$ or $\{\mathbf{X} \times \mathbf{v}\} = \{x_1, x_2, x_3, v_1, v_2, v_3\}$ $\mathbf{x}$ -position vector, $\mathbf{p} = m\mathbf{v}$ -momentum, m -mass of a particle
compartment in phase space	Specified by six coordinates within a cell by $g = \frac{(d\mathcal{V}_{\updownarrow})_{min}}{h^3} \gg 1$, where $(d\mathcal{V}_{\updownarrow})_{min}$ is the minimum size of volume element in phase space. Within the compartments only the number of phase points can be specified. There is no specification of coordinates of an individual phase point within a compartment.
cell in 6-D space $\{\mathbf{X} \times \mathbf{p}\}$	Consists of many compartments g. Small volume element $dV_l = dx_1 dx_2 dx_3 dp_1 dp_2 dp_3, l = 1...k$ but large enough to contain a large number of phase points necessary for application of statistical laws
micro-state of the system in quantum statistics	Defined by a complete specification of coordinates of compartments in cells. Coordinates of phase points can not be specified. All microstates are equally possible.
macro-state of the system in Newtonian mechanics	Given macro-state corresponds to large number of various micro-states. Distribution of phase points in phase space for $\sum N_k = N$ is such that N_1 phase points fall into cell number 1, N_j phase points fall into cell number j, N_k phase points fall into cell number k. Only the number of phase points per cell is specified. Individual coordinates of phase points within a cell are not specified.
thermodynamic probability $\mathcal{W}$	The number of micro-states $\mathcal{W} = \prod_{\|} \frac{(\}_{\|} + \mathcal{N}_{\|} - \infty)!}{(\}_{\|} - \infty)! \mathcal{N}_{\|}!}$ is associated with any given macro-state, where g_k represents the number of compartments in cell k or the multiplicity (degeneracy) of energy state E_k, or statistical weight of excited atoms in quantum state k. Max. thermodynamic probability defines the max. number of micro-states for the particular macro-state that corresponds to the state of max. entropy $S = k_B \log \mathcal{W}$ that defines the equilibrium state.

cont. Notion	cont. Definition
Maxwell-Boltzmann distribution of energy of identical but distinguishable particles multiplier	Describes the fractional population of excited states in terms of phase points $N_k \Rightarrow \frac{N_k}{N} = \frac{g_k}{Q}\exp(-E_k/k_BT)$ or number densities $n \Rightarrow \frac{n_k}{n} = \frac{g_k}{Q}\exp(-E_k/k_BT)$ $n_k = \frac{N_k}{\mathcal{V}}$ - number density $\mathcal{V}$ - volume of the system $Q = \sum_k g_k \exp(-\beta E_k)$ -partition function, $\beta = 1/k_BT$ $[J^{-1}]$-Lagrangian

The partition functions $Q_j, j \in (1, ..., \mathcal{J})$ for the plasma composed with $\mathcal{J}$ species establish the link between the six-dimensional, 6-D, microscopic system and macroscopic thermodynamic plasma properties in three-dimensional, 3-D, space.

Thermodynamic and transport properties of plasma depend on the plasma composition. For singly ionized atoms the composition of plasma is defined by the Saha-Eggert equation, Dalton's law, and the condition for plasma quasi-neutrality. Formulaes for plasma composition are given in the table

	Name of the relationship	Equation
1	Saha-Eggert equilibrium for thermal ionization derived by minimizing of Gibbs free energy	$\frac{n^{(e)}n^{(i)}}{n} = \frac{2Q^{(i)}}{Q}(\frac{2\pi m^{(e)}k_BT}{h^2})\exp(-\frac{E^{(i)}}{k_BT})$ $n^{(e)}$ -electron number density, $[m^3]$, $n^{(i)}$ -ion number density, $[m^3]$ n -neutral number density, $h = 6.6261 \times 10^{-34}$, $[Js]$ -Planck's const., $E^{(i)}$ -ionization energy, Q, $Q^{(i)}$ -partition functions of ions and neutrals. $Q^{(i)} = \sum_s g_s^{(i)} \exp(-E_s^{(i)}/k_BT)$ $Q = \sum_s g_s \exp(-E_s/k_BT)$ $g_s^{(i)}$, g_s -statistical weights of energy levels of ions and neutrals $E_s^{(i)}$, E_s -energy levels of ions and neutrals $m^{(e)}$ -mass of electrons k_B -Boltzmann constant
2	Dalton's law	$P = (n^{(e)} + n^{(i)} + n)k_BT$
3	quasi-neutrality of plasma	$n^{(e)} = n^{(i)}$

The Saha-Eggert equation can be used for evaluation of $n^{(e)}$ by using the following algorithm

- define function $f(T)$ =RHS of the Saha-Eggert equation, ie. $f(T) = \frac{2Q^{(i)}}{Q}(\frac{2\pi m^{(e)}k_BT}{h^2})\exp(-\frac{E^{(i)}}{k_BT})$
- assume an initial value of $n^{(e)} = {}^{ini}n^{(e)}$
- find $E^{(i)}$
- solve equation $(n^{(e)})^2 + 2f(T)n^{(e)} - \frac{Pf(T)}{k_BT} = 0$
- find the new ${}^kn^{(e)}$
- iterate unless ${}^{(k+1)}n^{(e)} - {}^kn^{(e)} < \delta_{iter}$

Thermodynamic properties of plasma are defined by: mass density, internal energy, enthalpy, specific heat, and entropy. The following table shows expressions for these properties both for the classical and statistical (plasma) thermodynamics .

	property	classical thermodynamic variables and parameters
1	mass density	ρ - density independent variable
2a	internal energy for system described by $T\&\mathcal{V}$	$U = F + TS$ U -energy needed to create the system F-Helmholtz free energy T -absolute temperature S -final entropy TS-energy imported from system's environment by heating
2b	internal energy for system described by $T\&p$	$U = G + TS - P\mathcal{V}$ U -energy needed to create the system G -Gibbs free energy P -absolute pressure $\mathcal{V}$ -final volume $P\mathcal{V}$ - work to give the system final volume $\mathcal{V}$ at constant pressure P
3	specific heat	$c = \frac{q}{m\Delta T}$ q -heat added m -mass ΔT -change in temperature
4a	enthalpy	$H = U + P\mathcal{V}$ energy for creation of the system plus the work needed to make room for it
5	entropy	$\Delta S = \frac{q}{T}$ measure of energy amount which is unavailable to do work $\frac{1}{T} = (\frac{\partial S}{\partial \mathcal{V}})_{V,N}$ -alternative definition of temperature

	cont. property	forms for statistical thermodynamics variables and parameters
1	mass density	$\rho = \Sigma_l n_l m_l$ n_l - number density of various species in plasma m_l - species mass
2a	internal energy for system described by $T\&\mathcal{V}$	Helmholtz free energy [9] $F - F_0 = -\sum_j N_j k_B T_j (1 + \ln(Q_j/N_j)) - \frac{k_BT\mathcal{V}}{12\pi\lambda_D^3}$ F_0 -reference energy N_j -total number of particles of species j k_B-Boltzmann constant, 1.3807×10^{-23} $[JK^{-1}]$ T -kinetic temperature $\frac{3}{2}k_BT_j = \frac{1}{2}m_j \bar{\mathbf{v}}_j^{2\,1/2}$ $\bar{\mathbf{v}}_j^{2\,1/2}$ - rms or effective velocity of particle j Q_j -partition function of species $\mathcal{V}$ -volume of the plasma $[m^3]$ λ_D^3 - Debye length $[m]$ $\frac{k_BT\mathcal{V}}{12\pi\lambda_D^3}$ -Debye correction for interaction energy due to long-range Coulomb interactions between species
2b	internal energy for system described by $T\&p$	Gibbs free energy [9] $G - G_0 = p\mathcal{V} + \mathcal{F} - \mathcal{F}_l$ $= -\sum_j N_j k_B T \ln(Q_j/N_j) - \frac{k_BT\mathcal{V}}{8\pi\lambda_D^3}$ G_0 -reference energy $p = \frac{N_j k_B T}{\mathcal{V}} - \frac{k_BT}{24\pi\lambda_D^3}$ -pressure

	cont. property	forms for statistical thermodynamics variables and parameters
2b cont.		$\frac{k_B T \mathcal{V}}{8\pi\lambda_D^3}$ -Debye correction $\lambda_D = \frac{\varepsilon_0 k_B T \mathcal{V}}{e^2 \sum_{j=1}^{2} Z_j^2 N_j}$ ε_0 -permittivity of free space (vacuum) e -elementary charge, 1.6022×10^{-19}[C] Z_j -number of ionic charges of species j
3	specific heat	$c_p = \frac{\partial H_g}{\partial T}\|_p$ $H_g = \frac{\sum_{j=1}^{k} x_j H_j}{\sum_{j=1}^{k} x_j M_j}$ -specific enthalpy, $[kJ/kg]$ $x_j = N_j/N_{tot}$ -molar fraction of chemical species j N_{tot} -total number of all species H_j -enthalpy of one mole of species j M_j -mass of one mole of species j
4a	enthalpy	1. when c_p available from tables $H - H^0 = \int_0^T c_p(T)dT$ H -total enthalpy of mixture at T and P H^0 -total enthalpy at reference state $T = 0$, $P = P_a$ P_a -ambient pressure $c_p(T)$ -specific heat (capacity), $[kJ/kgK]$
4b	enthalpy	2. when c_p calculated through partition functions Example for nitrogen N_2: composition of 1 mole of N_2 at (T,p) gives $N_2 \rightarrow \mathcal{N}_{N_2}^{mol} N_2 + \mathcal{N}_N^{mol} N + \mathcal{N}_{N+}^{mol} N^+ + $ $+\mathcal{N}_e^{mol} e^-$ $\mathcal{N}_j^{mol} = N_j/N_A$ -number of moles of species j N_A - Avogadro number Only two reactions occur: $N_2 \rightarrow 2N$ -dissociation $N \rightarrow N^+ + e^-$ -ionization Total enthalpy at (T,p) $H = \mathcal{N}_{N_2}^{mol} H_{N_2} + \mathcal{N}_N^{mol} H_N + \mathcal{N}_{N+}^{mol} H_{N+} +$ $+\mathcal{N}_{(e)}^{mol} H^{(e)}$ Enthalpy change to produce plasma $\triangle H = H - H_{N_2}^0$ $H_{N_2}^0$ -enthalpy at (T_0, p_0) $\triangle H = \triangle H_{N_2} + \frac{1}{2}(\mathcal{N}_N^{mol} + \mathcal{N}_{N+}^{mol})H_N^D +$ $+\mathcal{N}_{N+}^{mol} H_{N+}^I$ $\triangle H_{N_2}$ - frozen enthalpy with no reaction while N_2 heated from T_0 to T H_N^D - reaction enthalpy due to dissociation H_{N+}^I - reaction enthalpy due to ionization
5	entropy	$S = k_B \log \mathcal{N}W$ and also $S = \int_0^T \frac{c_p(T)}{T}dT$ when c_p is available from tables

3.3 Fluid and MHD theory of thermal plasma for the arc beam

The plasma state in the arc column area is close to LTE and such a state is called the partial local thermodynamic equilibrium (PLTE) and following that observation it can be approximated by one of MHD theories. Unfortunately the plasma theory for electrode sheaths is much more complicated and can be based on the analysis presented in [2], [13] [14] and [15]. The theory of plasma in sheaths is not general and refers deeply to the molecular physics and the nature of plasma transport coefficients which are related to the Coulomb collisions of species.

3.3.1 Distribution functions

In this theory the following distribution functions are fundamental issues:

distribution	expression-definition
particle distribution function	total number of particles $f(\mathbf{x},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ in differential six-dimensional phase space element $d^3\mathbf{x}d^3\mathbf{v}$
particle number density	number of particles per unit volume $n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}$

The other quantities

quantity	definition
fluid velocity	$\mathbf{u}_s = \langle \mathbf{v}_s \rangle$
mean thermal velocity	$V_s = \langle (\mathbf{v}_s - \mathbf{u}_s)^2 \rangle^{\frac{1}{2}}$
mass density	$\rho = \Sigma_s m_s n_s$
vector of mean mass velocity	$\mathcal{U}_s = \frac{1}{\rho}\Sigma_s m_s n_s \mathbf{u}_s$
velocity of particle relative to mean mass velocity	$\mathbf{w}_s = \mathbf{v}_s - \mathcal{U}_s$; $\langle \mathbf{w}_s \rangle = \mathbf{u}_s - \mathcal{U}_s$
pressure tensor	$P_{s,jk} = m_s n_s \langle w_{s,j} w_{s,k} \rangle$

are defined for particle species s, relatively to these two basic notions as moments, following the general definition of a moment of quantity $\mathcal{Q}(\mathbf{v})$:

$$\langle \mathcal{Q}(\mathbf{v}) \rangle = \frac{1}{n(\mathbf{x},t)} \int f(\mathbf{x},\mathbf{v},t)\mathcal{Q}(\mathbf{v})d^3\mathbf{v} \quad (3)$$

The distribution function $f_s(\mathbf{x},\mathbf{v},t)$ for species s satisfies the Boltzmann equation [4] that is written here both in the vector and tensor form

$$\frac{\partial f_s}{\partial t}|_{coll} = \frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}_s} + \mathbf{g} \cdot \frac{\partial f_s}{\partial \mathbf{v}_s} \quad (4)$$

$$\frac{\partial f_s}{\partial t}|_{coll} = \frac{\partial f_s}{\partial t} + v_j \frac{\partial f_s}{\partial x_j} + \frac{q_s}{m_s}(E_j + \epsilon_{jkl} v_k B_l)\frac{\partial f_s}{\partial v_j} + g_j \frac{\partial f_s}{\partial v_j} \quad (5)$$

where

- total time derivative of the distribution function, $\frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B} + \frac{m_s}{q_s}\mathbf{g}) \cdot \frac{\partial f_s}{\partial \mathbf{v}_s}$
- five-fold integral term accounting for the change in f_s due to molecular collisions, $\frac{\partial f_s}{\partial t}|_{coll}$
- Lorentz force on charge q_s, $q_s(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
- electric field $[volt/m]$, $\mathbf{E}$
- magnetic induction $[tesla]$, $\mathbf{B}$
- charge of species s, $[coulomb]$ q_s
- mass of species s, $[kg]$, m_s
- velocity of species s and velocity component, $\mathbf{v}_s$, v_j
- mass force per unit mass (eg. gravitational), $\mathbf{g}$, g_j
- unit permutation tensor, ϵ_{jkl}
- subscript for collision-related quantities, $_{coll}$

3.3.2 Basic integro-differential system of equations

When external fields **E** and **B** are known, Eq.(4) can be solved as a linear differential equation. However in plasma case, fields **E** and **B** are self-consistent and then the Boltzmann equation is associated with Maxwell's equations which describe how charge and current densities affect the magnetic and electric fields. The velocity of a particle injected into a plasma varies under the influence of **E** and **B** fields due to interacting forces and that induce currents which in turn alter the external fields. Maxwell's equations read in the form appropriate for SI-unit system

$\nabla \times \mathbf{E} =$	$-\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law	(6)
$\nabla \times \mathbf{H} =$	$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$	Ampere's law	(7)
$\nabla \cdot \mathbf{D} =$	ρ_{ch}	Poisson's equation	(8)
$\nabla \cdot \mathbf{B} =$	0	absence of magnetic monopoles	(9)

with constitutive relations

$\epsilon \mathbf{E} = \mathbf{D}$	transformation of electric field $\mathbf{E}$, $[V/m]$ to displacement $\mathbf{D}$, $[C/m^2]$
$\mu \mathbf{H} = \mathbf{B}$	transformation of magnetic induction $\mathbf{B}$, $[T]$ to magnetic intensity $\mathbf{H}$, $[H]$

and symbols

symbol	physical quantity	Value	Units
μ_0, μ	magnetic permeability	$\mu \approx \mu_0 = 4\pi \times 10^{-7}$	Hm^{-1}, H - henry, magnetic inductance
ϵ_0, ϵ	electric permittivity	$\epsilon \approx \epsilon_0 = 8.8542 \times 10^{-12}$	Fm^{-1}, F -farad, electric capacitance

and charge and current densities defined by

$\rho_{ch} =$	$\Sigma_s q_s n_s$	charge density	(10)
$\mathbf{J}(\mathbf{x}, t) =$	$\Sigma_s q_s n_s \langle \mathbf{v}_s \rangle$	current density	(11)

Integro-differential Eqs. (4) (or (5)), (7), (8) and Eqs.(6), (9) with relations Eq.(10) and (11) are basic expressions both for the kinetic and the fluid theories of plasma.

3.3.3 Fluid theory

Assuming that only flow of electrons and ions is involved in the transportation of energy in a welding plasma beam, equations of motion for the two-fluid flow theory are derived here. These equations are fundamental for the formulation of MHD theory.

By taking moments of Boltzmann's equation Eq.(5), the particle velocity distribution is replaced by values averaged over velocity space in the description of plasma as a fluid. Moment equations are obtained by multiplying Eq.(5) by an arbitrary function of velocity $\mathcal{Q}(\mathbf{v})$ and integrating each term of the equation following the Eq.(3). Moments of three LHS terms of Eq.(5) are following

$$\int \frac{\partial f_s}{\partial t} \mathcal{Q}(\mathbf{v}) d^3\mathbf{v} = \frac{\partial}{\partial t}(\mathbf{n_s}\langle \mathcal{Q} \rangle) \quad (12)$$

$$\int v_j \frac{\partial f_s}{\partial x_j} \mathcal{Q}(\mathbf{v}) d^3\mathbf{v} = \frac{\partial}{\partial \mathbf{x_j}}(\mathbf{n_s}\langle \mathbf{v_j} \mathcal{Q} \rangle) \quad (13)$$

$$\int \frac{q_s}{m_s}(E_j + \frac{\epsilon_{jkl} v_k B_l}{c} + \frac{m_s}{q_s} g_j) \frac{\partial f_s}{\partial v_j} \mathcal{Q}(\mathbf{v}) d^3\mathbf{v} =$$
$$-\frac{q_s}{m_s} E_j n_s \langle \frac{\partial \mathcal{Q}}{\partial v_j} \rangle - \frac{q_s}{m_s c} \varepsilon_{jkl} B_l \langle \frac{\partial \mathcal{Q}}{\partial v_j} v_k \rangle - g_j n_s \langle \frac{\partial \mathcal{Q}}{\partial v_j} \rangle \quad (14)$$

The general moment equation for the Boltzmann Eq.(5) called also the general equation of change [1] reads

$$\frac{\partial}{\partial t}(n_s \langle \mathcal{Q} \rangle) + \frac{\partial}{\partial x_j}(n_s \langle v_j \mathcal{Q} \rangle) - \frac{q_s}{m_s} E_j n_s \langle \frac{\partial \mathcal{Q}}{\partial v_j} \rangle$$
$$-\frac{q_s}{m_s c} \varepsilon_{jkl} B_l \langle \frac{\partial \mathcal{Q}}{\partial v_j} v_k \rangle - g_j n_s \langle \frac{\partial \mathcal{Q}}{\partial v_j} \rangle =$$
$$\int (\frac{\partial f_s}{\partial t})|_{coll} \mathcal{Q} d^3\mathbf{v} \quad (15)$$

The zeroth moment, useful in derivation of two conservation equations, is obtained by assuming $\mathcal{Q}(\mathbf{v}) = \mathbf{1}$ in Eq.(15) and is expressed in the form

$$\frac{\partial n_s}{\partial t} + \frac{\partial (n_s u_j)}{\partial t} = 0 \quad (16)$$

where u_j is a component of fluid velocity and RHS is zero assuming ideal plasma, ie. ignoring ionization, three-body recombination and charge exchange effects that can be expressed by relations: $(\frac{\partial n^i}{\partial t})|_{coll} = 0$, $(\frac{\partial n^e}{\partial t})|_{coll} = 0$, where superscript i stands for ions and e for electrons. Two conservation equations: a mass conservation equation, and a charge conservation equation, are obtained by multiplying Eq.(16) by mass m_s or charge q_s, respectively, and summation over s:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathcal{U}) = 0 \quad (17)$$

with mass density $\rho = \sum_j n_j m_j$

$$\frac{\partial \rho_{ch}}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (18)$$

The first moment of the Boltzmann equation with $\mathcal{Q} = \mathbf{v}$ is obtained by multiplying Eq.(5) by v_m and integrating over velocity space,

$$\frac{\partial (m_s n_s u_m)}{\partial t} + \frac{\partial (m_s n_s \langle v_j v_m \rangle)}{\partial x} -$$
$$n_s q_s (E_m - \epsilon_{mjk} u_j B_k) - n_s g_m = \pm \mathcal{P}_m \quad (19)$$

where the sign of the RHS momentum density is opposite for electrons and ions and reads

$$\mathcal{P}_m = m^{(i)} \int (\frac{\partial f^{(i)}}{\partial t})|_{coll} v_m^{(i)} d^3\mathbf{v}^{(i)} = m^{(e)} \int (\frac{\partial f^{(e)}}{\partial t})|_{coll} v_m^{(e)} d^3\mathbf{v}^{(e)} \quad (20)$$

with quantities appropriate for electrons and ions marked with the upper-script e or i respectively. Eq.(20) can be written when assuming that the total momentum density of the system do not vary due to collisions between electrons and ions.

The fluid equation of motion can be derived from Eq.(19) by using fluid quantities:

notation	components of
u_j	fluid velocity
$\mathcal{U}_j$	mean mass velocity
w_j	particle velocity relative to mean mass velocity
P_{jk}	pressure tensor

together with the relation for velocity dyad

$$\langle v_j v_k \rangle = \langle (\mathcal{U}_j + w_j)(\mathcal{U}_k + w_k) \rangle = \frac{1}{nm} P_{jk} + \mathcal{U}_j u_k + \mathcal{U}_k u_j - \mathcal{U}_j \mathcal{U}_k. \quad (21)$$

and finally it can be written in forms appropriate for ions and electrons

$$\pm \mathcal{P}_m^{(i)} = \frac{\partial}{\partial t}(m^{(i)} n^{(i)} u_m) + \frac{\partial}{\partial x_m}(P_{jm}^{(i)} + m^{(i)} n^{(i)} (\mathcal{U}_j u_m + \mathcal{U}_m u_j - \mathcal{U}_j \mathcal{U}_m)) - n^{(i)} q^{(i)} E_m - n^{(i)} q^{(i)} \epsilon_{mjk} u_j B_k - n^{(i)} g_m \quad (22)$$

$$\pm \mathcal{P}_m^{(e)} = \frac{\partial}{\partial t}(m^{(e)} n^{(e)} u_m) + \frac{\partial}{\partial x_m}(P_{jm}^{(e)} + m^{(e)} n^{(e)} (\mathcal{U}_j u_m + \mathcal{U}_m u_j - \mathcal{U}_j \mathcal{U}_m)) - n^{(e)} q^{(e)} E_m - n^{(e)} q^{(e)} \epsilon_{mjk} u_j B_k - n^{(e)} g_m \quad (23)$$

3.3.4 Magneto-hydro-dynamic (MHD) equations

Magneto-hydro-dynamic theory of plasma is the further simplification of fluid theory. Two simplifications of MHD are the most popular: two-fluid hydrodynamics, and one-fluid hydrodynamics. In both of them the following assumptions are involved:

ions and electron fluids	are combined and possess a common flow velocity $\mathcal{U}$
relevant time scales	long in comparison to microscopic particle motion time scales
spatial scale lengths	long in comparison to the Debye length λ_D long in comparison to the thermal ion gyro-radius

The equation of motion for the MHD fluid can be derived by adding Eq.(22) and (23). The MHD equation of motion can be expressed both in terms of components or vectors and reads in forms

$$\frac{\partial}{\partial t}(\rho \mathcal{U}_m) + \frac{\partial}{\partial x_m}(P_{jm} + \rho \mathcal{U}_j \mathcal{U}_m) - \rho_{ch} E_m - \epsilon_{mjk} J_j B_k - \rho g_m = 0 \quad (24)$$

$$\rho \left[\frac{\partial \mathcal{U}}{\partial t} + (\mathcal{U} \cdot \nabla)\mathcal{U} \right] = -\nabla P + \rho_{ch}\mathbf{E} + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g} \quad (25)$$

where pressure is a scalar P when velocity distribution is sufficiently random and $P = P_{jm}^{(e)} + P_{jm}^{(i)} = P_{jm}$. The Eulerian velocity of fluid $\mathcal{U}(\mathbf{x}, \mathbf{t})$ refers to the velocity of fluid element and it is contrasted with the Langrangian velocity of fluid which is related to the velocity of individual particles constituting that fluid element at any time. The Lagrangian velocity is the time derivative of the position vector of a particle and is only a function of time.

The relation linking the current density $\mathbf{J}$ with $\mathbf{B}, \mathbf{E}$ and $\mathcal{U}$, ie. the generalized Ohm's law, can be derived from Eq. 22 and 23 in the form

$$\mathbf{J} + \frac{\sigma m^{(i)} m^{(e)}}{\rho e^2} \frac{\partial \mathbf{J}}{\partial t} + \frac{\sigma m^{(i)}}{\rho e} \mathbf{J} \times \mathbf{B} = \sigma(\mathbf{E} + \mathcal{U} \times \mathbf{B} + \frac{m^{(i)}}{\rho e} \nabla P) \quad (26)$$

following the procedure:

multiply Eq.(22) by $(-e/m^{(e)})$ multiply Eq.(23) by $(e/m^{(i)})$
add two equations and ignore terms with $\mathcal{U}_j u_k, \mathcal{U}_k u_j, \mathcal{U}_j \mathcal{U}_k$
write the equation $\frac{\partial J_m}{\partial t} = \frac{e}{m^{(e)}} \frac{\partial P^{(e)}}{\partial x_m} - \frac{e}{m^{(i)}} \frac{\partial P^{(i)}}{\partial x_m} + e^2(\frac{n^{(i)}}{m^{(i)}} + \frac{n^{(e)}}{m^{(e)}})E_m$ $\frac{e^2}{c} \varepsilon_{mjk} (\frac{n^{(i)} u_j^{(i)}}{m^{(i)}} + \frac{n^{(e)} u_j^{(e)}}{m^{(e)}}) B_k + e(\frac{1}{m^{(i)}} + \frac{1}{m^{(e)}}) \mathcal{P}_m$
approximate $n^{(e)} \approx n^{(i)} \approx \frac{\rho}{m^{(i)}}, u_m^{(i)} \approx \mathcal{U}_m$ $u_m^{(e)} \approx \mathcal{U}_m - \frac{m^{(i)} c}{\rho e} J_m$
assume   momentum exchange between electrons and ions is proportional to the relative velocity
approximate $\mathcal{P}_m = \frac{e \rho J_m}{m^{(i)} \sigma}$

where

electrical conductivity $\sigma = \epsilon_0 \frac{\omega_{pi}^2}{\omega_{pe}}, \quad S/m$
permittivity $\epsilon_0 = 8.8542 \times 10^{-12}, \quad F/m$
ion plasma frequency $\omega_{pi} = 4.20\pi \times 10^2 Z \mu_{i/p}^{-1/2} (n^{(i)})^{1/2}, \quad Hz$
electron plasma frequency $\omega_{pe} = 18.96\pi \times 10^3 (n^{(e)})^{1/2}, \quad Hz$
ion charge state Z
ion-proton mass ratio $\mu_{i/p} = \frac{m^{(i)}}{m^{(p)}}$
proton mass $m^{(p)} = 1.6726 \times 10^{-27}, kg$
electron mass $m^{(e)} = 9.1094 \times 10^{-31}, kg$
ion mass $m^{(i)}$
elementary charge (charge of an electron) $e = 1.6022 \times 10^{-19}, \quad C$
speed of light in vacuum $c = 2.9979 \times 10^8, \quad ms^{-1}$

Ohm's law Eq.(26) can be further simplified assuming

assumption	approximation
$\omega_{pi} \ll \omega_{pe}$	$\frac{\sigma m^{(i)} m^{(e)}}{\rho e^2} \frac{\partial \mathbf{J}}{\partial t} = 0$
cyclotron frequency $\Omega^{(e)} \ll \omega_{pe}$	$\frac{\sigma m^{(i)}}{\rho e} \mathbf{J} \times \mathbf{B} = 0$
insignificant pressure gradient $\nabla P \approx 0$	$\frac{m^{(i)}}{\rho e} \nabla P = 0$

and then written in the form

$$\mathbf{J} = \sigma(\mathbf{E} + \mathcal{U} \times \mathbf{B}) \tag{27}$$

Assuming that the internal energy of a fluid element does not change when it propagates with pressure P proportional to $\rho^{\mathcal{I}}$ ($\mathcal{I}$ -adiabatic index), ie. assuming the adiabatic process and taking the second order moment of Boltzmann equation Eq.(4), the equation of energy conservation can be expressed in the form

$$\frac{\partial}{\partial t}\left(\rho\|\mathcal{U}\|^2 + \frac{2P}{\mathcal{I}-1} + \frac{\|B\|^2}{\mu_0} + \epsilon_0\|E\|^2\right) + \nabla\cdot\left(\rho\|\mathcal{U}\|^2\mathcal{U} + \frac{2\mathcal{I}}{\mathcal{I}-1}P\mathcal{U} + \frac{2}{\mu_0}\mathbf{E}\times\mathbf{B}\right) = 0 \tag{28}$$

where the adiabatic index for a mono-atomic gas is $\mathcal{I} = \frac{5}{3}$ and $\|\cdot\|$ is the vector quadratic norm.
The formulation of complete system of equations in MHD requires further assumptions:

- neglecting the displacement term $\frac{\partial \mathbf{D}}{\partial t} \approx 0$ in Ampere's law gives

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{29}$$

- taking the divergence of Eq.(29) gives LHS equal to zero and $\nabla \cdot \mathbf{J} = 0$
- using $\nabla\cdot\mathbf{J} = 0$ in Eq.(18) leads to $\frac{\partial \rho_{ch}}{\partial t} \equiv \dot{\rho_{ch}} = 0$
- eliminate $\mathbf{E}$ from the approximate Ohm's law Eq.(27) and Faraday's law from Eqs.(6
  - apply the operator curl $\nabla\times$ to Eq.(27), ie. $\nabla \times \mathbf{J} = \sigma(\nabla \times \mathbf{E} + \nabla \times (\mathcal{U} \times \mathbf{B}))$
  - $\nabla \times \mathbf{J}$ in terms of $\mathbf{B}$ can be obtained by taking $\nabla^2\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla\times\nabla\times\mathbf{B}$
  - the first RHS term $\nabla(\nabla \cdot \mathbf{B}) = 0$ because of $\nabla \cdot \mathbf{B} = 0$ from Eq.(9)
  - the second RHS term $\nabla\times\nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{J}$ because of Eq.(29) and $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$
  - then $\nabla \times \mathbf{J} = \frac{1}{\mu_0}\nabla^2\mathbf{B}$
  - finally the induction equation can be expressed in terms of $\mathbf{B}$ or $\mathbf{H}$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathcal{U} \times \mathbf{B}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{B} \tag{30}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathcal{U} \times \mathbf{H}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{H} \tag{31}$$

- assuming the charge neutrality $\rho_{ch} = 0$ (with $\frac{\partial \rho_{ch}}{\partial t} = 0$)the Poisson's equation Eq.(8) is not contributing to the final system of equations
- Eq.(9) can be treated as the initial condition because
  - taking divergence of Eq.(6) $\nabla \cdot \nabla \times \mathbf{E} = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t}$, note that LHS equals zero
  - RHS can be re-formulated $\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B})$ to show that $\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$ during the process

The final set of MHD equations [5] which determine the time evolution reads:

conservation of mass

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathcal{U} \times \mathbf{B}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}$$

induction equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathcal{U}) = 0$$

equation of motion

$$\rho\left[\frac{\partial \mathcal{U}}{\partial t} + (\mathcal{U}\cdot\nabla)\mathcal{U}\right] = -\nabla P + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g}$$

energy conservation

$$\frac{\partial}{\partial t}\left(\rho\|\mathcal{U}\|^2 + \frac{2P}{\mathcal{I}-\infty} + \frac{\|B\|^2}{\mu_0} + \epsilon_0\|E\|^2\right) + \nabla\cdot\left(\rho\|\mathcal{U}\|^2\mathcal{U} + \frac{2\mathcal{I}}{\mathcal{I}-\infty}P\mathcal{U} + \frac{2}{\mu_0}\mathbf{E}\times\mathbf{B}\right) = 0 \tag{32}$$

with two scalar quantities: P and ρ, and two vector quantities: $\mathbf{B}$ (or $\mathbf{H}$) and $\mathcal{U}$, as unknown "evolution" quantities. The electric field $\mathbf{E}$ and the current density $\mathbf{J}$ are determined by

modification of Eq.(29)

$$\mu_0\mathbf{J} = \nabla \times \mathbf{B} \tag{33}$$

modification of Eq.(27)

$$\mathbf{E} = -\mathcal{U} \times \mathbf{B} \tag{34}$$

if the magnetic Reynolds number fulfils the requirement $R_m = \frac{\|\mathcal{U}\|\|\mathbf{B}\|}{\sigma\|\mathbf{J}\|} \gg 1$.
The auxiliary relations, that are subject of the plasma-fluid approximation, in the case of the single-fluid MHD theory, are expressed by

mass density

$$n^{(i)} = n^{(e)} = n \Rightarrow \rho = m^{(i)}n \tag{35}$$

fluid velocity

$$\mathcal{U} = \mathcal{U}^{(i)} \tag{36}$$

temperature

$$T = T^{(e)} + T^{(i)} \tag{37}$$

pressure

$$P = P^{(e)} + P^{(i)} = k_B(n^{(e)}T^{(e)} + n^{(i)}T^{(i)}) = nk_BT \tag{38}$$

In MHD plasma literature there are several variations of MHD theory with various sets of variables:

variables	references
single-fluid "evolution" variables ρ, P, $\mathcal{U}$, $\mathbf{J}$, T	[6]
two-fluid "evolution" variables n, $\mathcal{U}^{(i)}$, $\mathcal{U}^{(e)}$, $P^{(i)}$, $P^{(i)}$, $\mathbf{J}$, $T^{(i)}$, $T^{(e)}$	[6]
one-fluid all variables ρ, P, $\mathcal{U}$,$\mathbf{B}$, $\mathbf{E}$, $\mathbf{J}$	[11]
single-fluid "evolution" variables ρ, P, $\mathcal{U}$, $\mathbf{B}$	[5]

4 Conclusions

The MHD theory is not a powerful concept for modelling of the microscopic phenomena observed in laboratory plasma but it seems to be suitable for modelling of welding and plasma spraying, where particular velocity distribution and wave-particle interactions are not the prime objectives. It has the advantage [11] that the macroscopic dynamics of the self-magnetized plasma can be analyzed for problems formulated in three-dimensional space. In spite of that, the three important ideas of modern physics [6]: relativity and quantum mechanics, and the Maxwell equation with wave propagation, are not contributing in MHD. Following this, processes in plasma such as: radiation, rf heating, micro-instabilities of particular motions, classical transport, and resistive instabilities, are not adequately described. But in welding plasma such flaws of MHD can be seen as the secondary issues. Unfortunately, the situation can be different in some cases of plasma spraying. The most sophisticated models should be used in electrode "skin" layers, where the energy lost is significant. In this case the most engineering oriented model of turbulent plasma can follow the concept refined in [15]. The second part of this paper is devoted to presentation of several "working" models of welding and spraying as well as to numerical methods effective in the solution of conducting fluid problems.

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