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STOCHASTIC CALCULUS RELATED TO NON-SYMMETRIC DIRICHLET FORMS

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0. Introduction

The theory of symmetric Dirichlet spaces and the probabilistic potential theory built on Hunt processes were unified by M. Fukushima [8], M.L. Silverstein [17] and others (see References in [8]). In particular, analysis based on additive functionals (AF's) and stochastic calculus related to symmetric Dirichlet spaces were developed by M. Fukushima [8], M. Fukushima and M. Takeda [9], S. Nakao [15] and M. Takeda [20]. On the other hand, the theory of non-symmetric Dirichlet spaces was studied by J. Bliedtner [3, 4], H. Kunita [10] etc.. Furthermore S. Carrillo Menendez [5] constructed the Hunt process associated with a non-symmetric Dirichlet space. Then many results in the symmetric case have been extended to the non-symmetric case by Y. Le Jan [11, 12], M.L. Silverstein [18], S. Carrillo Menendez [6] etc.. The purpose of this paper is to extend those results in [8], [9] and [20] to the non-symmetric case and thereby enlarge the range of applications of Dirichlet space theory.

1. Summary of the results

We first give a precise definition of non-symmetric Dirichlet form. Let X be a locally compact Hausdorff space with countable base and m a non-negative Radon measure on X such that $\operatorname{supp}[m]=X$. $L^2(X, m)$ denotes the real L^2-space with inner product

$$(u, v)_{L^2}=\int_X u(x)\,v(x)\,m(dx)\,,\ u, v\in L^2(X, m)\,.$$

Let $\boldsymbol{H}$ be a dense linear subspace of $L^2(X, m)$ which forms a Hilbert space with a norm $\|\ \|_{\boldsymbol{H}}$ such that for some $K>0$, $\|u\|_{\boldsymbol{H}}\geqq K\|u\|_{L^2}$ for any $u\in\boldsymbol{H}$. Moreover we assume that if $u\in\boldsymbol{H}$, then $|u|$, $u\wedge 1\in\boldsymbol{H}$. In this article we consider a bilinear form $\mathbf{a}$ on $\boldsymbol{H}\times\boldsymbol{H}$ which satisfies the following conditions;

($\mathbf{a}$.1) $\mathbf{a}_\alpha$ is coercive for any $\alpha>0$, i.e., there exists a constant $K_1=K_1(\alpha)>0$ such that $\mathbf{a}_\alpha(u, u)\geqq K_1\|u\|_{\boldsymbol{H}}^2$ for every $u\in\boldsymbol{H}$,

(**a**.2) **a** is continuous in the following sense; there exists a constant $K_2>0$ such that $|\mathbf{a}(u, v)|\leqq K_2\|u\|_{\boldsymbol{H}}\|v\|_{\boldsymbol{H}}$ for every $u, v\in\boldsymbol{H}$,
(**a**.3) $\mathbf{a}(T_1 u, u-T_1 u)\geqq 0$,
(**a**.4) $\hat{\mathbf{a}}(T_1 u, u-T_1 u)\geqq 0$.
Here $\mathbf{a}_\alpha(u, v)=\mathbf{a}(u, v)+\alpha(u, v)_{L^2}$, $\hat{\mathbf{a}}(u, v)=\mathbf{a}(v, u)$ and $T_1 u=u^+\wedge 1$ $(u^+=u\vee 0)$. A bilinear form **a** which fulfills (**a**.1)~(**a**.4) is called a *Dirichlet form* on $\boldsymbol{H}\times\boldsymbol{H}$, and (**a**, $\boldsymbol{H}$) a *Dirichlet space* on $L^2(X, m)$. Let us note that, under the conditions (**a**.1) and (**a**.2), the conditions (**a**.3) and (**a**.4) are equivalent to the next ones respectively (see [10]);
(**a**.3)′ $\mathbf{a}(u+T_1 u, u-T_1 u)\geqq 0$
and
(**a**.4)′ $\hat{\mathbf{a}}(u+T_1 u, u-T_1 u)\geqq 0$,
In particular, if **a** is symmetric, (**a**.3)′ reduces to the usual sub-Markov condition for a symmetric form that $\mathbf{a}(u, u)\geqq\mathbf{a}(T_1 u, T_1 u)$. Let $C_0(X)$ be the set of all bounded continuous functions on X with compact support. From now on we assume that $\boldsymbol{H}$ is *regular,* i.e., $C_0(X)\cap\boldsymbol{H}$ is dense in $\boldsymbol{H}$ with $\|\ \|_{\boldsymbol{H}}$ and dense in $C_0(X)$ with uniform norm.

Let us now summarize the results and methods in the present paper.

In Section 2 we give some basic notions and the Beurling-Deny formula for **a**; for $u, v\in C_0(X)\cap\boldsymbol{H}$,

$$
\begin{aligned}
(1.1)\qquad \frac{1}{2}\mathbf{a}(u, v)+\frac{1}{2}\hat{\mathbf{a}}(u, v) &= N(u, v) \\
&+\frac{1}{2}\int_{X\times X-\Delta}(u(x)-u(y))(v(x)-v(y))\,\sigma(dx, dy) \\
&+\frac{1}{2}\int_X u(x)\,v(x)\,\chi(dx)+\frac{1}{2}\int_X u(x)\,v(x)\,\hat{\chi}(dx)\,.
\end{aligned}
$$

Here N is a local symmetric form on $C_0(X)\cap\boldsymbol{H}$; $N(u, v)=0$ if v is constant on a neighbourhood of supp$[u]$ (=support of u). σ is a positive Radon measure on $X\times X-\Delta$ (Δ is diagonal) satisfying

$$\int_{X\times X-\Delta} u(x)\,v(y)\,\sigma(dx, dy) = -\mathbf{a}(u, v)$$

for $u, v\in C_0(X)\cap\boldsymbol{H}$ such that supp$[u]\cap$supp$[v]=\phi$. χ (resp. $\hat{\chi}$) is a positive Radon measure on X satisfying

$$\int_X u(x)\,\chi(dx) = \mathbf{a}(v, u)-\int_{X\times X-\Delta} u(y)\,(1-v(x))\,\sigma(dx, dy)$$

$$\left(\text{resp. } \int_X u(x)\,\hat{\chi}(dx) = \hat{\mathbf{a}}(v, u)-\int_{X\times X-\Delta} u(x)\,(1-v(y))\,\sigma(dx, dy)\right)$$

for $u, v\in C_0(X)\cap\boldsymbol{H}$ such that $v=1$ on a neighbourhood of supp$[u]$ (cf. [4]).

Moreover, in Corollary 2.16, we show that the representation (1.1) of $\mathbf{a}$ can be extended to $u, v \in \boldsymbol{H}$ (cf. [12] and [18]).

Let $\boldsymbol{M}=(P_x, X_t)$ and $\hat{\boldsymbol{M}}=(\hat{P}_x, \hat{X}_t)$ be the Hunt processes associated with $\mathbf{a}$ and $\hat{\mathbf{a}}$, respectively (see [5]). Let $\boldsymbol{S}$ be the set of all smooth measures of $\boldsymbol{M}$ and $\boldsymbol{A}_c^+$ the set of all positive continuous AF's of $\boldsymbol{M}$. S. Carrillo Menendez [6] showed that $\boldsymbol{S}$ and $\boldsymbol{A}_c^+$ are in one to one correspondence which is characterized by the following relation; for $A \in \boldsymbol{A}_c^+$ and $\mu \in \boldsymbol{S}$,

$$E_{h\cdot m}[\int_0^t f(X_s)\,dA_s] = \int_0^t \langle f\mu, \hat{p}_s h\rangle ds \tag{1.2}$$

for any non-negative Borel functions f and h on X and $t>0$ (see [8] for the symmetric case). Let the couple $(H, N(y, dx))$ be a Levy system of $\boldsymbol{M}$ (see A. Benveniste and J. Jacod [2]) and ν the smooth measure corresponding to $H \in \boldsymbol{A}_c^+$. In the same way we can consider $\hat{N}$, $\hat{\nu}$. In Section 3 we show that $\sigma(dx, dy)=N(y, dx)\,\nu(dy)$, $\chi(dx)=N(y, \delta)\,\nu(dy)$ and $\hat{\chi}(dy)=\hat{N}(y, \delta)\,\hat{\nu}(dy)$ (see Theorem 3.8 and Remark 3.9).

In Section 4, for any AF A, we define

$$e(A) = \lim_{\alpha\to\infty} \frac{\alpha^2}{2} E_m[\int_0^\infty e^{-\alpha t} A_t^2\, dt] \tag{1.3}$$

if the limit exists. $e(A)$ is called the energy of A. We define the mutual energy of AF's A and B by $e(A, B)=1/2\,(e(A+B)-e(A)-e(B))$. Then we show that $(\mathring{\mathcal{M}}, e)$ is a real Hilbert space, where $\mathring{\mathcal{M}}$ is the set of all martingale AF's of finite energy, and that the AF

$$A_t^{[u]} = \tilde{u}(X_t)-\tilde{u}(X_0) \quad (\tilde{u} \text{ is a q.c. version of } u \in \boldsymbol{H})$$

has a unique decomposition

$$A^{[u]} = M^{[u]}+N^{[u]}, \quad M^{[u]} \in \mathring{\mathcal{M}}, \quad N^{[u]} \in \mathcal{N}_c, \tag{1.4}$$

where $\mathcal{N}_c$ is the set of all continuous AF's of zero energy M. Fukushima [8] proved the above results in the symmetric case. Moreover we show that

$$e(A^{[u]}) = \mathbf{a}(u, u)-\frac{1}{2}\langle \tilde{u}^2, \hat{\chi}\rangle. \tag{1.5}$$

M. Fukushima [8] defined the energy of AF A by

$$\bar{e}(A) = \lim_{t\downarrow 0} \frac{1}{2t} E_m[A_t^2] \tag{1.6}$$

In the symmetric case it holds that

$$\lim_{t\downarrow 0} \frac{1}{t} (u-p_t u, u)_{L^2} = \mathbf{a}(u, u) \tag{1.7}$$

and hence

$$\bar{e}(A^{[u]}) = \mathbf{a}(u, u) - \frac{1}{2}\langle \tilde{u}^2, \hat{\chi}\rangle \tag{1.8}$$

(see Remark 4.6). But, in the non-symmetric case, we do not know the validity of (1.7), and hence we can not prove (1.8). It is known that $M^{[u]}$ has a decomposition;

$$M^{[u]} = \overset{c}{M}{}^{[u]} + \overset{d}{M}{}^{[u]} = \overset{c}{M}{}^{[u]} + \overset{j}{M}{}^{[u]} + \overset{k}{M}{}^{[u]}$$

where $\overset{c}{M}{}^{[u]}$ and $\overset{d}{M}{}^{[u]}$ are the continuous and purely discontinuous part of $M^{[u]}$ respectively, and $\overset{j}{M}{}^{[u]}$ and $\overset{k}{M}{}^{[u]}$ are defined by

$$\overset{k}{M}{}^{[u]} = -\widehat{\tilde{u}(X_{\zeta-})\, I_{\{\zeta\leq t\}}}, \quad \overset{j}{M}{}^{[u]} = \overset{d}{M}{}^{[u]} - \overset{k}{M}{}^{[u]}\,.$$

Here, ζ is the life time of $\boldsymbol{M}$ and for AF A, $\widehat{A}$ denotes $A - A^p$ with A^p being the dual predictable projection of A (cf. [7] and [9]). The smooth measure corresponding to $A \in \boldsymbol{A}_c^+$ is denoted by μ_A. If $A_t(\omega) = A_t^{(1)}(\omega) - A_t^{(2)}(\omega)$, $A^{(1)}$, $A^{(2)} \in \boldsymbol{A}_c^+$ and if $\mu_{A^{(i)}}$, $i=1, 2$, are bounded measures, then the measure μ_A corresponding to A is given by $\mu_{A^{(1)}} - \mu_{A^{(2)}}$. In particular, for $M^{[u]}$, $M^{[v]} \in \overset{\circ}{\mathcal{M}}$ ($u, v \in \boldsymbol{H}$), we use the abbreviations $\mu_{\langle u,v\rangle}$ and $\overset{\alpha}{\mu}_{\langle u,v\rangle}$ for $\mu_{\langle M^{[u]}, M^{[v]}\rangle}$ and $\mu_{\langle \overset{\alpha}{M}{}^{[u]}, \overset{\alpha}{M}{}^{[v]}\rangle}$, $\alpha = c$, d, j, k, respectively. The symbol $\langle M, N\rangle$ is the quadratic variation of M, $N \in \overset{\circ}{\mathcal{M}}$. In Theorem 4.9 and its collorary, we prove that for u, $v \in \boldsymbol{H}$,

$$\overset{j}{\mu}_{\langle u,v\rangle}(dy) = \int_X (\tilde{u}(x) - \tilde{u}(y))\,(\tilde{v}(x) - \tilde{v}(y))\,\sigma(dx, dy)\,, \tag{1.9}$$

$$\overset{k}{\mu}_{\langle u,v\rangle}(dy) = \tilde{u}(y)\,\tilde{v}(y)\,\chi(dy) \tag{1.10}$$

and

$$\frac{1}{2}\,\overset{c}{\mu}_{\langle u,v\rangle}(X) = N(u, v)\,. \tag{1.11}$$

In Section 5 we prove the derivation property of $\overset{c}{\mu}_{\langle u,v\rangle}$; for u, $v \in \boldsymbol{H}_b$ (=the set of the essentially bounded functions in $\boldsymbol{H}$) and $w \in \boldsymbol{H}$,

$$d\overset{c}{\mu}_{\langle uv,w\rangle}(x) = \tilde{u}(x)\, d\overset{c}{\mu}_{\langle v,w\rangle}(x) + \tilde{v}(x)\, d\overset{c}{\mu}_{\langle v,w\rangle}(x) \quad (x \in X)\,. \tag{1.12}$$

This formula was already proved by Y. Le Jan [12] (also cf. [8]), but here we give an alternative proof based on the martingale theory. As an application of the equality (1.12), we prove a stronger local property (see Corollary 5.10) of the symmetric form N.

In the beginning of Section 6 we introduce the notion of general Dirichlet form due to H. Kunita [10], for the condition (**a**.1) is too restrictive to give interesting examples. This notion is a little more general than the preceding definition of Dirichlet form in the sense that the coercivity condition (**a**.1) for $\mathbf{a}_\alpha$ is only assumed for α large enough. The basic analytical results on the Dirichlet form are known to hold for the general Dirichlet form with minor modifications ([4], [12]). It is also known that any regular general Dirichlet space admits an associated Hunt process ([5]). Further the arguments in the present paper only involve $\mathbf{a}_\alpha$ for sufficiently large α. Therefore all of the present assertions for the Dirichlet form persist to hold for the general Dirichlet form. We will give several examples of general Dirichlet forms which illustrate the usefulness of the theory of non-symmetric Dirichlet space.

As the first example we exhibit a class of bilinear forms given by

$$(1.13)\quad \mathbf{a}^0(u, v) = \sum_{i,j=1}^{d} \int_D a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^{d} \int_D b_i v \frac{\partial u}{\partial x_i} dx + \int_D cuvdx$$

for $u, v \in H^1(D)$. Here D is an open set of R^d; $H^1(D)$ is the Sobolev space of order 1; the coefficients a_{ij}, b_i, $i, j=1, 2, \cdots, d$, and c are bounded measurable functions on D satisfying certain additional conditions. Moreover we show that this class contains the form corresponding to the differential operator L, treated by H. Osada [16], given formally by

$$L = L^0 + \sum_{i \neq j}^{d} \delta(x_i - x_j) \frac{\partial}{\partial x_i},$$

where L^0 is the generator corresponding to $\mathbf{a}^0$.

Next, we consider the case with boundary conditions in an orthant. Let $D = \{x = (\xi, x_d) \in R^d : \xi \in R^{d-1}, x_d > 0\}$, $\partial D = \{x \in R^d : x_d = 0\}$ and for $u, v \in H^1(D)$,

$$(1.14)\qquad \mathbf{a}(u, v) = \mathbf{a}^0(u, v) - \sum_{i=1}^{d-1} \int_{\partial D} \beta_i(\xi)\, v(\xi, 0) \frac{u(\xi, 0)}{\partial \xi_i} d\xi,$$

where β_i, $i=1, 2, \cdots, d-1$, are functions on ∂D with bounded derivatives of first order and satisfying certain additional conditions of the same type as for b_i. Then, in Theorem 6.2, we show that $\mathbf{a}$ is a general Dirichlet form on $\bar{D} = D \cup \partial D$. If the generator corresponding to $\mathbf{a}^0$ is the Laplacian, then the process associated with $\mathbf{a}$ given by (1.14) is a Brownian motion with oblique reflection on $\bar{D}$.

Finally we give an example of non-local form which is a slight generalization of Example IV. 3.2 in [5].

2. Basic notions and the Beurling-Deny formula

Let X, m and $(\mathbf{a}, \boldsymbol{H})$ be given as in Section 1.

DEFINITION 2.1. The generator $(L, \mathcal{D}(L))$ of $\mathbf{a}$ is the operator from $\mathcal{D}(L)$

$=\{u\in \boldsymbol{H}$; there exists $w\in L^2(X, m)$ such that $\mathbf{a}(u, v)=(-w, v)_{L^2}$ for all $v\in \boldsymbol{H}\}$ to $L^2(X, m)$ and defined by

$$Lu = w \qquad \text{for} \quad u\in \mathcal{D}(L)\,.$$

We see that $(L, \mathcal{D}(L))$ is the infinitesimal generator of a strongly continuous contraction semigroup $(T_t)_{t>0}$ of operators on $L^2(X, m)$. This is contained in the following theorem.

Theorem 2.2 (H. Kunita [10]). *For each $\alpha>0$, there exists a uniquely determined bounded operator G_α from $L^2(X, m)$ to $\boldsymbol{H}$ such that*
(i) $\mathbf{a}_\alpha(G_\alpha u, v)=(u, v)_{L^2}$ $(u\in L^2(X, m),\ v\in \boldsymbol{H})$,
(ii) $G_\alpha - G_\beta + (\alpha-\beta)\, G_\alpha G_\beta = 0$ *for any* $\alpha, \beta>0$,
(iii) $||\alpha G_\alpha u||_{L^2}\leqq ||u||_{L^2}$ $(u\in L^2(X, m))$,
(iv) $\lim_{\alpha\to\infty} \alpha G_\alpha u = u$ *in $\boldsymbol{H}$ as in* $L^2(X, m)$,
(v) $(\alpha I - L)^{-1} = G_\alpha$
(vi) *αG_α is sub-Markovian, i.e.,* $0\leqq \alpha G_\alpha u\leqq 1$ *m-a.e. whenever* $u\in L^2(X, m)$, $0\leqq u\leqq 1$ *m-a.e.*.

The family $(G_\alpha)_{\alpha>0}$ is called the *resolvent* of $\mathbf{a}$. By the same method we can define the *co-resolvent* $(\hat{G}_\alpha)_{\alpha>0}$ of $\hat{\mathbf{a}}$ such that

$$\hat{\mathbf{a}}_\alpha(\hat{G}_\alpha v, u) = \mathbf{a}_\alpha(u, \hat{G}_\alpha v) = (u, v)_{L^2}\,.$$

REMARK 2.3. Let

$$T_t\, u = \lim_{\alpha\to\infty} e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} (\alpha G_\alpha)^n\, u,\ u\in L^2(X, m)\,.$$

Then $(T_t)_{t>0}$ determines a strongly continuous contraction sub-Markovian semigroup of $\mathbf{a}$.

DEFINITION 2.4. Let Γ be a non-empty convex closed subset of $\boldsymbol{H}$. u is called the *$\mathbf{a}_\alpha$-projection* of $w\in \boldsymbol{H}$, written by $u=\prod_\Gamma^{\mathbf{a}_\alpha}(w)$, if $u\in\Gamma$ and $\mathbf{a}_\alpha(u-w, v-u)\geqq 0$ for any $v\in\Gamma$ $(\alpha>0)$.

DEFINITION 2.5. A positive Radon measure μ on X is said to be of *finite energy integral* if

$$\int_X |v(x)|\,\mu(dx)\leqq c||v||_H \quad (v\in C_0(X)\cap \boldsymbol{H})$$

for some constant $c>0$.

We denote by $\boldsymbol{S}_0$ the family of all positive Radon measures of finite energy integral. $\mu\in \boldsymbol{S}_0$ if and only if there exists for each $\alpha>0$ a unique function

$U_\alpha\mu\in\boldsymbol{H}$ such that

$$\mathbf{a}_\alpha(U_\alpha\mu, v)=\int_X v(x)\,\mu(dx) \quad (v\in C_0(X)\cap\boldsymbol{H})\,.$$

We call $U_\alpha\mu$ an *$\mathbf{a}_\alpha$-potential.* Similarly we can define an *$\hat{\mathbf{a}}_\alpha$-potential* $\hat{U}_\alpha\mu$.

REMARK 2.6. By Stampacchia's representation theorem (see [3]), there exist the $\mathbf{a}_\alpha$-projection and the $\mathbf{a}_\alpha$-potential.

Let A be an open set of X and define a closed convex subset $\boldsymbol{H}_A$ of $\boldsymbol{H}$ by

$$\boldsymbol{H}_A=\{v\in\boldsymbol{H}\colon v\geqq 1\ m\text{-a.e. on } A\}\,.$$

If $\boldsymbol{H}_A\neq\phi$, then there exists $u_A^\alpha=\prod_{\boldsymbol{H}_A}^{\mathbf{a}_\alpha}(0)\in\boldsymbol{H}_A$ and u_A^α is an a_α-potential ($\alpha>0$).

DEFINITION 2.7. We define the *capacity* of an open set $A\subset X$ as the number

$$\operatorname{cap}(A)=\begin{cases}\infty & \text{if } \boldsymbol{H}_A=\phi\\ \mathbf{a}_1(u_A^1, u_A^1) & \text{if } \boldsymbol{H}_A\neq\phi\,.\end{cases}$$

u_A^1 is called the capacitary potential of A. And for any subset $E\subset X$ we define

$$\operatorname{cap}(E)=\begin{cases}\infty & \text{if } \boldsymbol{H}_E=\phi\\ \mathbf{a}_1(u_E^1, u_E^1) & \text{if } \boldsymbol{H}_E\neq\phi\,,\end{cases}$$

where

$$\boldsymbol{H}_E=\overline{\bigcup_{\substack{E\subset A\\ A \text{ is open}}}\boldsymbol{H}_A} \quad\text{and}\quad u_E^1=\prod\nolimits_{\boldsymbol{H}_E}^{\mathbf{a}_1}(0)\,.$$

Let $\mathcal{O}(E)$ be the set of open sets containing E. S. Carrillo Menendez [5] has proved that if there exists $A\in\mathcal{O}(E)$ such that $\boldsymbol{H}_A\neq\phi$, then $\operatorname{cap}(E)=\lim_{\mathcal{O}(E)}\operatorname{cap}(A)$. It is also known that if A and B are open sets and $A\subset B$, then $\operatorname{cap}(A)\leqq\operatorname{cap}(B)$ (see [3]). Then it follows that if $E\subset F$, then $\operatorname{cap}(E)\leqq\operatorname{cap}(F)$ and hence $\operatorname{cap}(E)=\inf_{\substack{E\subset A\\ A \text{ is open}}}\operatorname{cap}(A)$.

The *co-capacity* $\widehat{\operatorname{cap}}(E)$ of $E\subset X$ is defined by using $\hat{\mathbf{a}}$ instead of $\mathbf{a}$.

Set

$$\boldsymbol{S}_{00}=\{\mu\in\boldsymbol{S}_0\colon \mu(X)=1\,,\ \|U_1\mu\|_\infty<\infty\}\,,$$

and

$$\hat{\boldsymbol{S}}_{00}=\{\mu\in\boldsymbol{S}_0\colon \mu(X)=1\,,\ \|\hat{U}_1\mu\|_\infty<\infty\}\,.$$

The following lemma can be proved by the same method as Theorem 3.3.2 in [8].

Lemma 2.8. *The following conditions are equivalent for a Borel set* $A\subset X$;
(i) $\mathrm{cap}(A)=0$,
(ii) $\widehat{\mathrm{cap}}(A)=0$,
(iii) $\mu(A)=0$ *for any* $\mu\in \boldsymbol{S}_0$,
(iv) $\mu(A)=0$ *for any* $\mu\in \boldsymbol{S}_{00}$,
(v) $\mu(A)=0$ *for any* $\mu\in \hat{\boldsymbol{S}}_{00}$.

Let A be a subset of X. A statement depending on $x\in X$ is said to hold q.e. on A if there exists a set $N\subset A$ of zero capacity such that the statement is true for every $x\in X-N$. "q.e." is an abbreviation of "quasi-everywhere".

DEFINITION 2.9. An extended real valued function f on X is called *quasi-continuous* (q.c.) if for any $\varepsilon>0$ there exists an open set A such that
(i) $\mathrm{cap}(A)<\varepsilon$
(ii) $f|_{X-A}$ is continuous on $X-A$.

REMARK 2.10. In the same way as in the symmetric case [8] it is shown that any element of $\boldsymbol{H}$ has a q.c. version.

Let

$$\mathbf{a}^{(\alpha)}(u, v) = \alpha(u-\alpha G_\alpha u, v)_{L^2}$$

for any $\alpha>0$ and $u, v\in L^2(X, m)$.

Lemma 2.11 (H. Kunita [10]). (i) *Let* $u\in L^2(X, m)$. *Then* $u\in \boldsymbol{H}$ *if and only if* $\sup_{\alpha>0} a^{(\alpha)}(u, u)<\infty$.
(ii) *For any* $u, v\in \boldsymbol{H}$, $\lim_{\alpha\to\infty} \mathbf{a}^{(\alpha)}(u, v)=\mathbf{a}(u, v)$.

Let

$$\bar{\mathbf{a}}(u, v) = \frac{1}{2}(\mathbf{a}(u, v)+\hat{\mathbf{a}}(u, v)) \quad (u, v\in \boldsymbol{H}).$$

Clearly $(\bar{\mathbf{a}}, \boldsymbol{H})$ is a symmetric Dirichlet space and it holds that $\bar{\mathbf{a}}(u, u)=\mathbf{a}(u, u)=\hat{\mathbf{a}}(u, u)$. Hence we have the following lemma (see [8]).

Lemma 2.12. (i) *If* $u\in \boldsymbol{H}$, *then* $Tu\in \boldsymbol{H}$ *and* $\mathbf{a}(u, u)\geqq \mathbf{a}(Tu, Tu)$ *for any normal contraction* T.
(ii) *If* u, v *are bounded functions in* $\boldsymbol{H}$, *then* $uv\in \boldsymbol{H}$ *and* $\mathbf{a}(uv, uv)^{1/2}\leqq ||u||_\infty \mathbf{a}(v, v)^{1/2}+||v||_\infty \mathbf{a}(u, u)^{1/2}$.
(iii) *If* $u\in \boldsymbol{H}$ *and* $u_n=(-n\vee u)\wedge n$, *then* $u_n\in \boldsymbol{H}$ *and* $u_n\to u$ *as* $n\to\infty$ *with respect to* $||\ ||_{\boldsymbol{H}}$.
(iv) *If* $u\in \boldsymbol{H}$ *and* $u^{(\varepsilon)}=u-((-\varepsilon)\vee u)\wedge\varepsilon$ $(\varepsilon>0)$, *then* $u^{(\varepsilon)}\in \boldsymbol{H}$ *and* $u^{(\varepsilon)}\to u$ *as* $\varepsilon\to 0$ *with respect to* $||\ ||_{\boldsymbol{H}}$.
(v) *Let* D *be an open set. Then* $C_0(D)\cap \boldsymbol{H}$ *is dense in* $\boldsymbol{H}^D$ *with* $||\ ||_{\boldsymbol{H}}$, *where*

$\boldsymbol{H}^D=\{v\in\boldsymbol{H}\colon v=0$ q.e. *on* $D^C\}$.

Lemma 2.13 ([8]). *If the resolvent* $(G_\alpha)_{\alpha>0}$ *is positive, then there exists a unique measure* σ_α *on* $X\times X$ *such that*

$$\alpha^2\int_X G_\alpha u(x)\, v(x)\, m(dx) = \int_{X\times X} u(x)\, v(y)\, \sigma_\alpha(dx, dy) \tag{2.1}$$

for every Borel functions u, $v\in L^2(X, m)$.

Using Lemma 2.11, we can define a unique Radon measure σ on $X\times X-\Delta$ such that $\sigma_\alpha\to\sigma$ vaguely on $X\times X-\Delta$ as $\alpha\to\infty$ and

$$\int_{X\times X-\Delta} u(x)\, v(y)\, \sigma(dx, dy) = -\mathbf{a}(u, v) \tag{2.2}$$

for any u, $v\in C_0(X)\cap\boldsymbol{H}$ such that $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\phi$. By the same method, we can define the measures $\hat{\sigma}_\alpha$ and $\hat{\sigma}$ if we replace G_α and $\mathbf{a}$ by $\hat{G}_\alpha$ and $\hat{\mathbf{a}}$ in (2.1) and (2.2). It is easy to see that $\sigma(dx, dy)=\hat{\sigma}(dy, dx)$.

Lemma 2.14. (i) *For any relatively compact open set* $D\subset X$, *there exists a unique Radon measure* χ_D *on* X, *supported in* D, *such that for every* $u\in C_0(D)$

$$\int_X u(x)\, \chi_D(dx) = \lim_{\alpha\to\infty}\alpha\int_X u(x)\, (1-\alpha G_\alpha I_D(x))\, m(dx)\,, \tag{2.3}$$

and it holds that if D' *is a relatively compact open set such that* $D\subset D'$, *then* $\chi_D\geqq\chi_{D'}$ *on* D.
(ii) *There exists a unique Radon measure* χ *on* X *such that for any* $u\in C_0(X)$,

$$\int_X u(x)\, \chi(dx) = \lim_{D\uparrow X}\int_X u(x)\, \chi_D(dx)\,. \tag{2.4}$$

Moreover, if u, $v\in C_0(X)\cap\boldsymbol{H}$ *and* $v=1$ *on a neighbourhood of* $\mathrm{supp}[u]$, *then the measure* χ *satisfies the equality*

$$\int_X u(x)\, \chi(dx) = \mathbf{a}(v, u)-\int_{X\times X-\Delta} u(y)\, (1-v(x))\, \sigma(dx, dy)\,. \tag{2.5}$$

Proof. Let $u\in C_0(X)\cap\boldsymbol{H}$ and $v=1$ on a neighbourhood of $\mathrm{supp}[u]$. Then, by Lemma 2.13, we have

$$\begin{aligned}\mathbf{a}^{(\alpha)}(u, v) &= \int_{D\times D-\Delta} u(y)\, (1-v(x))\, \sigma_\alpha(dx, dy)\\ &\quad+\alpha\int_X u(x)\, (1-\alpha G_\alpha I_D(x))\, m(dx)\end{aligned} \tag{2.6}$$

which implies that the measure $\alpha(1-\alpha G_\alpha I_D(x))\, m(dx)$ is uniformly bounded on D. Hence there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$ and a positive Radon measure χ_D on D such that

$$\alpha_n(1-\alpha_n G_{\alpha_n} I_D)\, m \to \chi_D \quad \text{vaguely on } D \text{ as } \alpha_n \to \infty\,.$$

Thus we have

$$\int_X u(x)\, \chi_D(dx) = \lim_{\alpha_n\to\infty} [\mathbf{a}^{(\alpha_n)}(v, u) - \int_{D\times D-\Delta} u(y)\,(1-v(x))\, \sigma_{\alpha_n}(dx, dy)]$$
$$= \mathbf{a}(v, u) - \int_{D\times D-\Delta} u(y)\,(1-v(x))\, \sigma(dx, dy)\,.$$

This show that χ_D is uniquely determined independently of the choice of $\{\alpha_n\}$ and satisfies the equality (2.3). Let D' be a relatively compact open set such that $D\subset D'$. Then it is clear that $\chi_D \geqq \chi_{D'}$ on D. It is easy to see that (ii) follows from (i). The proof is complete.

By the same method, we can define $\hat{\chi}_D$ and $\hat{\chi}$ if we replace G_α by $\hat{G}_\alpha$ in the above lemma. Then $\hat{\chi}$ satisfies for the above u and v,

$$\int_X u(x)\, \hat{\chi}(dx) = \mathbf{a}(u, v) - \int_{X\times X-\Delta} u(x)\,(1-v(y))\, \sigma(dx, dy)\,. \tag{2.7}$$

Theorem 2.15 (Beurling-Deny formula). *The regular Dirichlet form* **a** *on $L^2(X, m)$ can be represented for $u, v\in C_0(X)\cap \boldsymbol{H}$ as follows;*

$$\frac{1}{2}\mathbf{a}(u, v) + \frac{1}{2}\hat{\mathbf{a}}(u, v) = N(u, v) \tag{2.8}$$
$$+\frac{1}{2}\int_{X\times X-\Delta} (u(x)-u(y))\,(v(x)-v(y))\,\sigma(dx, dy)$$
$$+\frac{1}{2}\int_X u(x)\,v(x)\,\chi(dx) + \frac{1}{2}\int_X u(x)\,v(x)\,\hat{\chi}(dx)\,.$$

Here N is a symmetric form with domain $C_0(X)\cap \boldsymbol{H}$ and satisfies the following condition;

(2.9) *$N(u, v)=0$ for $u, v\in C_0(X)\cap \boldsymbol{H}$ such that v is constant on a neighbourhood of* supp$[u]$. *σ is a positive Radon measure on $X\times X-\Delta$ given by* (2.2). χ (resp. $\hat{\chi}$) *is a positive Radon measure on X given by* (2.5) (resp. (2.7)).

Proof. Since $(\bar{\mathbf{a}}, \boldsymbol{H})$ is a regular symmetric Dirichlet space on $L^2(X, m)$, we get for $u, v\in C_0(X)\cap \boldsymbol{H}$,

$$\bar{\mathbf{a}}(u, v) = \bar{N}(u, v) \tag{2.10}$$
$$+\frac{1}{2}\int_{X\times X-\Delta} (u(x)-u(y))\,(v(x)-v(y))\,\bar{\sigma}(dx, dy)$$

$$+\int_X u(x)\, v(x)\, \bar{\chi}(dx)\,,$$

where $\bar{N}$ is a symmetric form satisfying (2.9), $\bar{\sigma}$ is a symmetric positive Radon measure on $X\times X-\Delta$ such that

$$\int_{X\times X-\Delta} u(x)\, v(y)\, \bar{\sigma}(dx, dy) = -\bar{\mathbf{a}}(u, v) \tag{2.11}$$

for any u, $v\in C_0(X)\cap \boldsymbol{H}$ with disjoint supports, and $\bar{\chi}$ is a positive Radon measure on X such that

$$\int_X u(x)\, \bar{\chi}(dx) = \bar{\mathbf{a}}(u, v)-\frac{1}{2}\int_{X\times X-\Delta} (u(x)-u(y))\,(v(x)-v(y))\, \bar{\sigma}(dx, dy) \tag{2.12}$$

for any u, $v\in C_0(X)\cap \boldsymbol{H}$ such that $v=1$ on a neighbourhood of supp$[u]$. But by the definition of σ and $\hat{\sigma}$, we can see that $\bar{\sigma}=\frac{1}{2}(\sigma+\hat{\sigma})$. And (2.12) implies that

$$\begin{aligned}\int_X u(x)\, \bar{\chi}(dx) &= \bar{\mathbf{a}}(u, v)-\int_{X\times X-\Delta} u(y)\,(v(y)-v(x))\, \bar{\sigma}(dx, dy)\\ &= \frac{1}{2}\,\mathbf{a}(u, v)-\frac{1}{2}\int_{X\times X-\Delta} u(y)\,(1-v(y))\, \sigma(dx, dy)\\ &\quad +\frac{1}{2}\,\hat{\mathbf{a}}(u, v)-\frac{1}{2}\int_{X\times X-\Delta} u(y)\,(1-v(y))\, \hat{\sigma}(dx, dy)\\ &= \frac{1}{2}\int_X u(x)\hat{\chi}(dx)+\frac{1}{2}\int_X u(x)\, \chi(dx)\,.\end{aligned}$$

Hence $\bar{\chi}=\frac{1}{2}(\chi+\hat{\chi})$. Now let $N=\bar{N}$, then N satisfies the condition (2.9) and

$$\begin{aligned}\frac{1}{2}\,\mathbf{a}(u, v)+\frac{1}{2}\,\hat{\mathbf{a}}(u, v) &= \bar{\mathbf{a}}(u, v)\\ &= N(u, v)\\ &\quad +\frac{1}{4}\int_{X\times X-\Delta} (u(x)-u(y))\,(v(x)-v(y))\, \sigma(dx, dy)\\ &\quad +\frac{1}{4}\int_{X\times X-\Delta} (u(x)-u(y))\,(v(x)-v(y))\, \hat{\sigma}(dx, dy)\\ &\quad +\frac{1}{2}\int_X u(x)\, v(x)\, \chi(dx)+\frac{1}{2}\int_X u(x)\, v(x)\, \hat{\chi}(dx)\,.\end{aligned} \tag{2.13}$$

Since the second and third terms of the right hand side of (2.13) are the same, (2.13) implies (2.8). The proof is complete.

Let D be any relatively compact open subset of X. For non-negative

$u\in C_0(X)\cap \boldsymbol{H}$ with $\operatorname{supp}[u]\subset X-\bar{D}$, we define the positive measures σ_u, $\hat{\sigma}_u$ and $\bar{\sigma}_u$ on X by

$$\sigma_u(dy) = I_D(y)\int_X u(x)\,\sigma(dx, dy)\,,$$

$$\hat{\sigma}_u(dy) = I_D(y)\int_X u(x)\,\hat{\sigma}(dx, dy)$$

and

$$\bar{\sigma}_u(dy) = I_D(y)\int_X u(x)\,\bar{\sigma}(dx, dy)\,.$$

We use the notation $\boldsymbol{H}^D=\{v\in\boldsymbol{H}\colon v=0 \text{ q.e. on } D^c\}$. Since $\bar{\sigma}_u\in\boldsymbol{S}_0$ with respect to the restriction $\mathbf{a}^D=\mathbf{a}\,|_{\boldsymbol{H}^D\times\boldsymbol{H}^D}$ and $\bar{\sigma}=\frac{1}{2}(\sigma+\hat{\sigma})$, so do σ_u and $\hat{\sigma}_u$. The equality (2.10) implies that for any $v\in C_0(X)\cap\boldsymbol{H}$,

$$\int_X v(x)^2\,\bar{\chi}(dx)\leqq\bar{\mathbf{a}}(v, v)\,.$$

This implies that $I_K\cdot\bar{\chi}\in\boldsymbol{S}_0$ and hence $I_K\cdot\chi$, $I_K\cdot\hat{\chi}\in\boldsymbol{S}_0$ for any compact set $K\subset X$. Thus, using the same method as Lemma 4.5.4 in [8], we have the following corollary.

Corollary 2.16. *The representation* (2.8) *of Dirichlet form* **a** *extends to* $u, v\in\boldsymbol{H}$;

$$\begin{aligned}(2.14)\qquad \frac{1}{2}\,\mathbf{a}(u, v)+\frac{1}{2}\,\hat{\mathbf{a}}(u, v) &= N(u, v)\\ &+\frac{1}{2}\int_{X\times X-\Delta}(\tilde{u}(x)-\tilde{u}(y))\,(\tilde{v}(x)-\tilde{v}(y))\,\sigma(dx, dy)\\ &+\frac{1}{2}\int_X\tilde{u}(x)\,\tilde{v}(x)\,\chi(dx)+\frac{1}{2}\int_X\tilde{u}(x)\,\tilde{v}(x)\,\hat{\chi}(dx)\,,\end{aligned}$$

where $\tilde{u}$ and $\tilde{v}$ denote q.e. versions and N is a symmetric form on $\boldsymbol{H}\times\boldsymbol{H}$ satisfying, for u, $v\in\boldsymbol{H}$ with compact support, $N(u, v)=0$ if v is constant on a neighbourhood of supp$[u]$.

3. Jumping and killing measures

Let $\boldsymbol{M}=(\Omega, P_x, \zeta, X_t)$ and $\hat{\boldsymbol{M}}=(\hat{\Omega}, \hat{P}_x, \hat{\zeta}, \hat{X}_t)$ be the Hunt processes associated with **a** and $\hat{\mathbf{a}}$, respectively, which were constructed by S. Carrillo Menendez [5]. We denote by $\boldsymbol{S}$ the family of all smooth measures (see [8]). It is known that $\mu\in\boldsymbol{S}$ if and only if there exists a nest $\{F_n\}$ of μ such that $I_{F_n}\cdot\mu\in\boldsymbol{S}_0$ for each n (see [8]; Theorem 3.2.3) and that a set $N\subset X$ is exceptional if and only if cap$(N)=0$ ([11]).

The transition function and resolvent of $\boldsymbol{M}$ is denoted by $(p_t)_{t>0}$ and $(R_\alpha)_{\alpha>0}$ respectively. Then for any $u\in L^2(X, m)$, $p_t u$ and $R_\alpha u$ are q.c. versions of $T_t u$ and $G_\alpha u$ $(t, \alpha>0)$ respectively (see [5]). The restriction $\boldsymbol{M}^D$ of $\boldsymbol{M}$ by an open set D is the Hunt process associated with $(\mathbf{a}^D, \boldsymbol{H}^D)$, where $\boldsymbol{H}^D=\{u\in\boldsymbol{H}: u=0$ q.e. on $D^c\}$ and $\mathbf{a}^D=\mathbf{a}\,|_{\boldsymbol{H}^D\times\boldsymbol{H}^D}$. Let $(R^D_\alpha)_{\alpha>0}$ be the resolvent of $\boldsymbol{M}^D$. We define

$$H^M_\alpha(x, E) = E_x[\exp(-\alpha\sigma_M\, I_E(X_{\sigma_M})] \quad (\alpha>0,\ x\in X,\ E\in\mathcal{B}(X)),$$

where $M=D^c$ and $\sigma_M=\inf\{t>0: X_t\in M\}$. Then we have the following relations;

$$H^M_\alpha - H^M_\beta = (\beta-\alpha)\,R^D_\alpha\, H^M_\beta \tag{3.1}$$

and

$$R_\alpha = R^D_\alpha + H^M_\alpha\, R_\alpha \tag{3.2}$$

Similarly we can define the kernels $\hat{R}_\alpha$, $\hat{R}^D_\alpha$, $\hat{p}_t$ and $\hat{H}^M_\alpha$ related to $\hat{\boldsymbol{M}}$.

Lemma 3.1. *There exists a unique positive Radon measure k on X charging no exceptional set such that for any $u\in C_0(X)\cup\boldsymbol{H}_0$,*

$$\begin{aligned}\langle\tilde{u}, k\rangle &= \lim_{\alpha\to\infty}\alpha\int_X u(x)\,(1-\alpha R_\alpha\,1(x))\,m(dx)\\ &= \lim_{t\downarrow 0}\frac{1}{t}\int_X u(x)\,(1-p_t\,1(x))\,m(dx),\end{aligned}\tag{3.3}$$

where $\boldsymbol{H}_0$ is a set of all elements of $\boldsymbol{H}$ with compact support in X and $\tilde{u}$ is any q.c. version of u.

Similarly $\hat{k}$ is given by replacing $\hat{R}_\alpha$ and $\hat{p}_t$ for R_α and p_t in (3.3).

Proof. Let D, D_1 and D_2 be relatively compact open subsets of X such that $\bar{D}\subset D_1$ and $\bar{D}_1\subset D_2$. If $u\in C_0(X)\cap\boldsymbol{H}^{D_1}$ and $v\in C_0(X)\cap\boldsymbol{H}^{D_2}$, then

$$\begin{aligned}\mathbf{a}^{(\alpha)}(v, u) = &\int_{D_2\times D_2} u(y)\,(v(y)-v(x))\,\sigma_\alpha(dx, dy)\\ &+\alpha\int_{D_1} u(x)\,v(x)\,(1-\alpha R_\alpha\, I_{D_2}(x))\,m(dx)\,.\end{aligned}$$

Fix v satisfying $v=1$ on D_1. Then

$$\begin{aligned}\mathbf{a}^{(\alpha)}(v, u) = &\int_{D_2\times D_2} u(y)\,(1-v(x))\,\sigma_\alpha(dx, dy)\\ &+\alpha\int_{D_1} u(x)\,(1-\alpha R_\alpha\, I_{D_2}(x))\,m(dx)\,.\end{aligned}$$

This implies that the family of maps

$$(3.4)\qquad F_\alpha^D: u \mapsto \alpha\int_X u(x)\,(1-\alpha R_\alpha I_{D_2}(x))\, m(dx)$$

is equicontinuous on $\boldsymbol{H}^{D_1}$. In fact, for any $u\in C_0(X)\cap \boldsymbol{H}^{D_1}$,

$$\begin{aligned}(3.5)\quad |F_\alpha^D(u)| &\leqq F_\alpha^D(|u|)\leqq \mathbf{a}^{(\alpha)}(v,\,|u|)\\ &\leqq C_1\{\mathbf{a}^{(\alpha)}(v,v)+(v,v)_{L^2}\}^{1/2}\,\mathbf{a}_1(|u|,\,|u|)^{1/2}\quad\text{by Lemma 3.1 of [10]}\\ &\leqq C\,\mathbf{a}_1(u,u)^{1/2}\quad\text{by Lemma 2.11,}\end{aligned}$$

with some constants C_1 and C. It is trivial that $\{F_\alpha^D\}$ is equicontinuous on $C_0(D)$. Let

$$F_\alpha(u) = \alpha\int_X u(x)\,(1-\alpha R_\alpha 1(x))\, m(dx)$$

Then, since $F_\alpha(u)\leqq F_\alpha^D(u)$ for any positive Borel function u, $\{F_\alpha\}$ is equicontinuous both on $\boldsymbol{H}^D$ and on $C_0(D)$. Consider the case that $u=\hat{R}_\beta^D f$, $\beta>0$, $f\in C_0(D)$. Then, by (3.2), we have

$$\begin{aligned}(3.6)\qquad F_\alpha(u) &= \alpha\int_X \hat{R}_\beta^D f(x)\,(1-\alpha R_\alpha^D 1(x)-\alpha H_\alpha^M R_\alpha 1(x))\, m(dx)\\ &= \alpha\int_X f(x)\,(R_\beta^D 1(x)-\alpha R_\beta^D R_\alpha^D 1(x))\, m(dx)\\ &\quad -\alpha^2\int_X f(x)\, R_\beta^D H_\alpha^M R_\alpha 1(x)\, m(dx)\,.\end{aligned}$$

Using the resolvent equation and (3.1), the right hand side of (3.6) is equal to

$$\alpha\int_X f(x)\,(R_\alpha^D 1(x)-\beta R_\beta^D R_\alpha^D 1(x))\, m(dx)-\alpha^2\int_X f(x)\, R_\alpha^D H_\beta^D R_\alpha 1(x)\, m(dx)$$

and this converges to

$$\int_X f(x)\,(1-\beta R_\beta^D 1(x))\, m(dx)-\int_X f(x)\, H_\beta^M 1(x)\, m(dx)$$

as $\alpha\to\infty$. Since the range of $\hat{R}_\beta^D$, $\beta>0$, is dense in $\boldsymbol{H}^D$ and $\{F_\alpha\}$ is equicontinuous on $\boldsymbol{H}^D$, $F_\alpha(u)$ converges to some limit as $\alpha\to\infty$ for any $u\in\boldsymbol{H}^D$. By the regularity of $(\mathbf{a}^D, \boldsymbol{H}^D)$, $\boldsymbol{H}^D\cap C_0(D)$ is dense in $C_0(D)$. Again by the equicontinuity of $\{F_\alpha\}$ on $C_0(D)$, $F_\alpha(u)$ converges as $\alpha\to\infty$ for any $u\in C_0(D)$. Since D is arbitrary, there exists a unique positive Radon measure k such that for any $u\in C_0(X)$, $\langle u, k\rangle=\lim_{\alpha\to\infty} F_\alpha(u)$. And for each D,

$$|\langle u, k\rangle| = |\lim_{\alpha\to\infty} F_\alpha(u)| \leqq C\,\mathbf{a}_1(u,u)^{1/2}\quad (u\in C_0(D)\cap\boldsymbol{H}^D)\,.$$

This shows that k charges no set of zero capaicty in D and $\langle\tilde{u}, k\rangle=\lim_{\alpha\to\infty} F_\alpha(u)$ for any $u\in\boldsymbol{H}^D$. Since D is arbitrary, k charges no exceptional set and $\langle\tilde{u}, k\rangle=$

$\lim_{\alpha\to\infty} F_\alpha(u)$ for any $u\in \boldsymbol{H}_0$. Hence the measure k satisfies the first equality of (3.3). The second equality of (3.3) holds by Karamata's theorem (see D.V. Widder [22]; Theorem 5.4.3). The proof is complete.

REMARK 3.2. In the proof of the above lemma, we have seen that the family of maps F_α^D in (3.4) is equicontinuous both on $\boldsymbol{H}^D$ and $C_0(D)$. Hence, by Lemma 2.14 and the argument similar to that for the family $\{F_\alpha\}$, it follows that

$$(3.7)\qquad \langle \tilde{u}, \chi_{D_2}\rangle = \lim_{\alpha\to\infty} \alpha\int_X u(x)\,(1-\alpha R_\alpha I_{D_2}(x))\, m(dx) \quad \text{for any } u\in \boldsymbol{H}^D\,.$$

Lemma 3.3 (cf. [8]; Lemma 4.5.2). *The measure k in Lemma* 3.1 *satisfies the following conditions*;
(i) *For any* $u\in \boldsymbol{H}$,

$$(3.8)\qquad \langle \tilde{u}^2, k\rangle = \lim_{\alpha\to\infty} \alpha\int_X u(x)^2\,(1-\alpha R_\alpha 1(x))\, m(dx)$$
$$= \lim_{t\downarrow 0} \frac{1}{t}\int_X u(x)^2\,(1-p_t 1(x))\, m(dx)\,,$$

(ii) *For any* $f,\ h\in \boldsymbol{B}^+(X)$ (*=the set of all positive Borel functions*) *and* $t>0$,

$$(3.9)\qquad E_{h\cdot m}[f(X_{\zeta-}) \colon \zeta\leqq t] = \int_0^t \langle fk, \hat{p}_s h\rangle\, ds\,,$$

(iii) $E_x[e^{-\alpha\zeta} f(X_{\zeta-})]$ *is a q.c. version of* $U_\alpha(fk)$ *for* $\alpha>0,\ f\in C_0^+(X)$.

Proof. Since $\alpha\hat{R}_\alpha$ is sub-Markovian, it holds that

$$(3.10)\qquad \alpha\int_X u(x)^2\,(1-\alpha R_\alpha 1(x))\, m(dx) \leqq 2\alpha\int_X u(x)\,(u(x)-\alpha R_\alpha u(x))\, m(dx)$$

for all $u\in \boldsymbol{H}$ (see the proof of Proposition 1.3.3 in [12]). By Lemma 2.11 and Lemma 3.1, we have

$$\langle u^2, k\rangle \leqq 2\mathbf{a}(u, u) \quad \text{for any } u\in C_0(X)\cap \boldsymbol{H}\,.$$

By Fatou's lemma, we have

$$(3.11)\qquad \langle \tilde{u}^2, k\rangle \leqq 2\mathbf{a}(u, u) \quad \text{for any } u\in \boldsymbol{H}\,.$$

From Lemma 3.1 and (3.11), the statements (i), (ii) and (iii) follow by the same way as Lemma 4.5.2 in [8]. The proof is complete.

REMARK 3.4. The statement (3.9) is equivalent to the following;

$$(3.12)\qquad E_{h\cdot m}[e^{-\alpha\zeta} f(X_{\zeta-})] = \langle fk, \hat{R}_\alpha h\rangle,\ f,\ h\in \boldsymbol{B}^+(X),\ \alpha>0\,.$$

We can give a direct proof of (3.12). In fact, let D be any relatively compact open set of X and $f\in C_0^+(X)$ and supp $[f]\subset D$. Then $f\hat{R}_\alpha h\in \boldsymbol{H}^D$ $(h\in C_0^+(X))$ and it holds that

$$\begin{aligned}\langle fk, \hat{R}_\alpha h\rangle &= \langle k, f\hat{R}_\alpha h\rangle \\ &= \lim_{\beta\to\infty}\beta\int_X f\hat{R}_\alpha h(1-\beta R_\beta 1)\, dm \\ &= \lim_{\beta\to\infty}\beta\int_X hR_\alpha[f(1-\beta R_\beta 1)]\, dm\,.\end{aligned}$$

But we have

$$\begin{aligned}\beta R_\alpha[f(1-\beta R_\alpha 1)]\,(x) &= \beta E_x[\int_0^\infty e^{-\alpha t} f(X_t)\, E_{X_t}[e^{-\beta\zeta}]\, dt] \\ &= \beta E_x[\int_0^\infty e^{-\alpha t} f(X_t)\, e^{-\beta(\zeta-t)}\, dt]\end{aligned}$$

and this converges to $E_x[e^{-\alpha\zeta} f(X_{\zeta-})]$ as $\beta\to\infty$.

Theorem 3.5. *$k=\chi$ and $\hat{k}=\hat{\chi}$.*

Proof. By the definition of χ and k, it is clear that $k\leqq\chi$. Hence we prove that $k\geqq\chi$. Take D and D_2 as in the proof of Lemma 3.1. For any $f\in C_0^+(X)\cap\boldsymbol{H}$ such that supp $[f]\subset D$ and any $h\in C_0^+(X)$,

$$\begin{aligned}\langle f\chi, \hat{R}_\alpha^D h\rangle &= \langle \chi, f\hat{R}_\alpha^D h\rangle \leqq \langle \chi_{D_2}, f\hat{R}_\alpha^D h\rangle \\ &= \lim_{\alpha\to\infty}\alpha\int_X f\hat{R}_\alpha^D h(1-\alpha R_\alpha I_{D_2})\, dm \quad \text{by (3.7)} \\ &= \lim_{\alpha\to\infty}\alpha\int_X hR_\alpha^D[f(1-\alpha R_\alpha I_{D_2})]\, dm \\ &\leqq \lim_{\alpha\to\infty}\alpha\int_X hR_\alpha^D[f(1-\alpha R_\alpha^D I_D)]\, dm \\ &= E_{h\cdot m}[e^{-(\zeta\wedge\tau_D)} f(X_{(\zeta\wedge\tau_D)-})]\end{aligned}$$

where $\tau_D=\sigma_{X-D}$. The last equality follows from the same calculus as Remark 3.4. Letting $D\uparrow X$, we have

$$\langle f\chi, \hat{R}_\alpha h\rangle \leqq E_{h\cdot m}[e^{-\zeta} f(X_{\zeta-})] = \langle fk, \hat{R}_\alpha h\rangle\,.$$

This shows that $\chi\leqq k$. Similarly we can see $\hat{\chi}=\hat{k}$. The proof is complete.

In the present paper, the definitions of *additive functionals* and related concepts (*continuous, square integrable* etc.) are taken in the sense of Fukushima [8]; Chap. 5. Let $\boldsymbol{A}_c^+$ be the set of all positive continuous AF's. Henceforth we use the notations;

$$U_A^\alpha f(x) = E_x[\int_0^\infty e^{-\alpha t} f(X_t)\, dA_t] \quad \text{and} \quad (fA)_t = \int_0^t f(X_s)\, dA_s$$

for $A \in \boldsymbol{A}_c^+$ and $f \in \boldsymbol{B}^+(X)$. The set of all *smooth measures* on X is denoted by $\boldsymbol{S}$. The following theorem is proved in [6].

Theorem 3.6. *For $\mu \in \boldsymbol{S}$ and $A \in \boldsymbol{A}_c^+$, the following conditions are equivalent to each other;*

(i) *$U_A^\alpha f$ is a q.c. version of $U_\alpha(f\mu)$ for any $f \in \boldsymbol{B}^+(X)$ such that $f\mu \in \boldsymbol{S}_0$.*
(ii) $(h, U_A^\alpha f)_{L_2} = \langle f\mu, \hat{R}_\alpha h \rangle$, $\alpha > 0, f, h \in \boldsymbol{B}^+(X)$.
(iii) $E_{h\cdot m}[(fA)_t] = \int_0^t \langle f\mu, \hat{p}_s h \rangle\, ds$, $t > 0, f, h \in \boldsymbol{B}^+(X)$.
(iv) $\lim_{\alpha \to \infty} \uparrow \alpha(h, U_A^{\alpha+\gamma} f) = \langle f\mu, h \rangle$ *for any γ-co-excessive function h and $f \in \boldsymbol{B}^+(X)$.*
(v) $\lim_{t \downarrow 0} \frac{1}{t} E_{h\cdot m}[(fA)_t] = \langle f\mu, h \rangle$ *for any γ-co-excessive function h and $f \in \boldsymbol{B}^+(X)$.*

Moreover, $\boldsymbol{S}$ and $\boldsymbol{A}_c^+$ are in one to one correspondence which is characterized by one of above statements.

Let $D \subset X$ be an open set, $\boldsymbol{A}_c^+(D)$ the set of all positive continuous AF's of $\boldsymbol{M}^D$ and $\boldsymbol{S}^D$ the set of all smooth measures related to $\mathbf{a}^D$. Theorem 3.6 holds on D, i.e., $\boldsymbol{S}^D$ and $\boldsymbol{A}_c^+(D)$ correspond to each other in a unique manner. For $A \in \boldsymbol{A}_c^+$, let $A_t^D = A_{\sigma_{\boldsymbol{M}} \wedge t} \in \boldsymbol{A}_c^+(D)$.

Lemma 3.7. *If μ is the smooth measure corresponding to A, then $\mu|_D$ is the measure corresponding to A^D.*

Proof. In the same way as in the proof of Lemma 5.1.5 in [8], we can show that Theorem 3.6 (iv) is valid on D, and this completes the proof.

By A. Benveniste and J. Jacod [2], there exists a couple $(H, N(y, dx))$, so called Levy system of $\boldsymbol{M}$, of $H \in \boldsymbol{A}_c^+$ and a kernel N such that

$$E_x[\sum_{0<s\leq t} f(X_s, X_{s-})\, I\{X_s \neq X_{s-}\}] = E_x[\int_0^t dH_s \int_{X \cup \{\delta\}} N(X_s, dy) f(X_s, y)]$$

for any $x \in X$ and any positive Borel function f on $(X \cup \{\delta\}) \times (X \cup \{\delta\})$ (δ is an extra point).

Now we give the main theorem of this section.

Theorem 3.8. *Let ν be the smooth measure corresponding to $H \in \boldsymbol{A}_c^+$. Then we have*
(i) *$\sigma(dx, dy) = N(y, dx)\, \nu(dy)$ on $X \times X - \Delta$,*
(ii) *$\chi(dy) = N(y, \delta)\, \nu(dy)$ on X.*

Proof. (i) Let $u, v \in C_0(X) \cap \boldsymbol{H}$ and $D \subset X$ a relatively compact open set satisfying $\mathrm{supp}[v]\subset D\subset\bar{D}$ and $\mathrm{supp}[u]\subset\mathring{M}$ $(M=D^c)$. We already remarked that the measure $\sigma_u(dy)=I_D(y)\int_X u(x)\,\sigma(dx,dy)$ is of finite energy integral with respect to $\mathbf{a}^D$. Then

$$\begin{aligned}
&\mathbf{a}_\alpha^D(\alpha R_\alpha^D u+H_\alpha^M u-u, v)\\
&= \alpha(u, v)_{L^2}+\mathbf{a}_\alpha^D(H_\alpha^M u-u, v)\\
&= \alpha(u, v)_{L^2}+\mathbf{a}_\alpha(H_\alpha^M u-u, v) \quad (\because\ \ H_\alpha^M u-u\in \boldsymbol{H}^D)\\
&= \alpha(u, v)_{L^2}-\mathbf{a}_\alpha(u, v) \quad (\because\ \ a_\alpha(H_\alpha^M u, v)=0 \ \ (\text{see [11]}))\\
&= -\mathbf{a}(u, v)=\int u(x)\,v(y)\,\sigma(dx, dy)=\int v(y)\,\sigma_u(dy)\\
&= \langle\sigma_u, v\rangle=\mathbf{a}_\alpha^D(U_\alpha^D(\sigma_u), v)\,,
\end{aligned}$$

and hence, for $x\in D$,

$$\begin{aligned}
U_\alpha^D(\sigma_u)(x) &= \alpha R_\alpha^D u(x)+H_\alpha^M u(x)-u(x)\\
&= H_\alpha^M u(x) \quad (\because\ \ \mathrm{supp}[u]\subset\mathring{M})\\
&= E_x[\exp(-\alpha\sigma_M)\,u(X_{\sigma_M})]\\
&= E_x[\sum_{0<s\leq\sigma_M}\exp(-\alpha s)\,u(X_s))]\\
&= E_x[\int_0^{\sigma_M}\exp(-\alpha s)\,Nu(X_s)\,dH_s]\,,
\end{aligned}$$

where

$$Nu(y)=\int_X u(x)\,N(y, dx)\,.$$

Hence σ_u is the smooth measure corresponding to $((Nu)\,H)^D$. On the other hand, by Lemma 3.7, $((Nu)\,\nu)|_D$ is also the measure corresponding to $((Nu)\,H)^D$. Therefore $\sigma_u=((Nu)\,\nu)|_D$. Thus

$$\begin{aligned}
\int_{X\times-\Delta}u(x)\,v(y)\,\sigma(dx, dy) &= \int_X v(y)\,\sigma_u(dy)\\
&= \int_{X\times X-\Delta}u(x)\,v(y)\,N(y, dx)\,\nu(dy)
\end{aligned}$$

and this implies that $\sigma(dx, dy)=N(y, dx)\,\nu(dy)$.

Now we prove (ii). For any $f\in C_0^+(X)$, we have, by Lemma 3.3 and Theorem 3.6,

$$\begin{aligned}
\int_0^t\langle fk, \hat{p}_s 1\rangle\,ds &= E_m[f(X_{\zeta-})\,I_{\{\zeta\leq t\}}]\\
&= E_m[\int_0^t f(X_s)\,N(X_s, \delta)\,dH_s]\\
&= \int_0^t\langle fN(\cdot, \delta)\,\nu, \hat{p}_s 1\rangle\,ds
\end{aligned}$$

which implies that $N(\cdot, \delta)\,\nu = k = \chi$. The proof is complete.

In this section, we have proved that

$$\sigma(dx, dy) = N(y, dx)\,\nu(dy) \text{ on } X \times X - \Delta$$

and

$$\chi(dy) = k(dy) = N(y, \delta)\,\nu(dy) \text{ on } X$$

which are called the *jumping* and *killing measure* of $\boldsymbol{M}$ respectively.

REMARK 3.9. Let $(\hat{H}, \hat{N}(y, dx))$ be a Lévy system of the Hunt process $\hat{\boldsymbol{M}}$ and $\hat{\nu}$ the smooth measure corresponding to $\hat{H} \in \hat{\boldsymbol{A}}_c^+$.

Then it holds that

$$\hat{\sigma}(dx, dy) = \hat{N}(y, dx)\,\hat{\nu}(dy) \text{ on } X \times X - \Delta$$

and

$$\hat{\chi}(dy) = \hat{k}(dy) = \hat{N}(y, \delta)\,\hat{\nu}(dy) \text{ on } X.$$

4. The energy of AF's and a decomposition of AF's of finite energy

In this section, using the notion of the energy of AF's, we give a decomposition of AF's generated by functions in $\boldsymbol{H}$.

DEFINITION 4.1. For any $A \in \boldsymbol{A}$ (=the set of all AF's) we define

$$e(A) = \lim_{\alpha \to \infty} \frac{\alpha^2}{2} E_m\left[\int_0^\infty e^{-\alpha t} A_t^2\, dt\right], \tag{4.1}$$

whenever the finite limit exists. $e(A)$ is called the energy of A. And we define the mutual energy of AF's A and B by $e(A, B) = \frac{1}{2}[e(A+B) - e(A) - e(B)]$.

REMARK 4.2. M. Fukushima [8] defined the energy of AF A by

$$\bar{e}(A) = \lim_{t \downarrow 0} \frac{1}{2t} E_m[A_t^2]. \tag{4.2}$$

If $\bar{e}(A)$ exists, then $e(A)$ is well defined and $e(A) = \bar{e}(A)$.

For any $A \in \boldsymbol{A}_c^+$, the smooth measure corresponding to A is denoted by μ_A. We consider the families

$$\mathcal{M} = \{M \in \boldsymbol{A}: \text{ for q.e. } x \in X,\ E_x[M_t^2] < \infty \text{ and } E_x[M_t] = 0\ (t > 0)\}$$

and

$$\overset{\circ}{\mathcal{M}} = \{M \in \mathcal{M} \colon e(M) < \infty\} .$$

For any $M \in \overset{\circ}{\mathcal{M}}$, it holds that

$$e(M) = \bar{e}(M) = \sup_{t>0} \frac{1}{2t} E_m[M_t^2] = \frac{1}{2} \mu_{\langle M \rangle}(X) \tag{4.3}$$

(see [8]). Here the symbol $\langle M \rangle$ is the quadratic variation of $M \in \overset{\circ}{\mathcal{M}}$ (see [7]).

Lemma 4.3. *Let $A \in \boldsymbol{A}_c^+$. Then*
(i) $\langle \nu, U_A^\alpha f \rangle = \langle f\mu_A, \hat{U}_\alpha \nu \rangle$ *for* $\alpha > 0$, $\nu \subset \boldsymbol{S}_0$ *and* $f \in \boldsymbol{B}^+(X)$.
(ii) $E_\nu[A_t] \leqq e^t \|\hat{U}_1 \nu\|_\infty \mu_A(X)$ *for* $t > 0$ *and* $\nu \in \hat{\boldsymbol{S}}_{00}$.

Proof. (i) For any $f \in \boldsymbol{B}^+(X)$ such that $f\mu_A \in \boldsymbol{S}_0$, it holds that

$$\begin{aligned}\langle \nu, U_A^\alpha f \rangle &= \langle \nu, U_\alpha(f\mu_A) \rangle \quad \text{by Theorem 3.6} \\ &= \mathbf{a}_\alpha(U_\alpha(f\mu_A), \hat{U}_\alpha \nu) \\ &= \langle f\mu_A, U_\alpha \nu \rangle .\end{aligned}$$

Let $f \in \boldsymbol{B}^+(X)$ and $f_n = I_{F_n}(f \wedge n)$, $n = 1, 2, \cdots$, where $\{F_n\}$ is a nest of μ_A. Then

$$\begin{aligned}\langle \nu, U_A^\alpha f \rangle &= \lim_{n\to\infty} \langle \nu, U_A^\alpha f_n \rangle \\ &= \lim_{n\to\infty} \langle f_n \mu_A, \hat{U}_\alpha \nu \rangle \\ &= \langle f\mu_A, \hat{U}_\alpha \nu \rangle .\end{aligned}$$

(ii)

$$\begin{aligned}E_\nu[A_t] &\leqq e^t \langle \nu, E_x[\int_0^\infty e^{-s} dA_s] \rangle \\ &= e^t \langle \nu, U_A^1 1 \rangle \\ &= e^t \langle \mu_A, \hat{U}_1 \nu \rangle \text{ by (i)} \\ &= e^t \|\hat{U}_1 \nu\|_\infty \mu_A(X) .\end{aligned}$$

Using (4.3) and Lemma 4.3 (ii) instead of (5.2.17) in [8], the following theorem can be proved in the same way as Theorem 5.2.1 in [8].

Theorem 4.4. *$\overset{\circ}{\mathcal{M}}$ is a real Hilbert space with inner product e. Moreover, for any e-Cauchy sequence $M^n \in \overset{\circ}{\mathcal{M}}$, there exist a unique $M \in \overset{\circ}{\mathcal{M}}$ and a subsequence n_k such that $\lim_{n\to\infty} e(M^n - M) = 0$ and for q.e. $x \in X$, $P_x(\lim_{n_k\to\infty} M_t^{n_k} = M_t$ uniformly on any finite interval of $t) = 1$.*

Let

$$\mathcal{N}_c = \{N \in \boldsymbol{A} \colon N \text{ is continuous and for q.e. } x \in X, \; E_x[|N_t|] < \infty \; (t > 0) \text{ and } e(N) = 0\} .$$

We define

$$(4.4) \qquad A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0) \quad (t>0)$$

where $\tilde{u}$ is a q.c. version of $u \in \boldsymbol{H}$.

Lemma 4.5. *$A^{[u]}$ is of finite energy and for $u, v \in \boldsymbol{H}$*

$$(4.5) \qquad e(A^{[u]}, A^{[v]}) = \frac{1}{2}\hat{\mathbf{a}}(u, v) + \frac{1}{2}\mathbf{a}(u, v) - \frac{1}{2}\langle \tilde{u}\tilde{v}, \hat{\chi} \rangle .$$

Proof. By the definition of e,

$$\begin{aligned} & e(A^{[u]}) \\ & = \lim_{\alpha\to\infty} \frac{\alpha^2}{2} E_m[\int_0^\infty e^{-\alpha t}(\tilde{u}(X_t) - \tilde{u}(X_0))^2 dt \\ & = \lim_{\alpha\to\infty} \frac{\alpha^2}{2} \int_X (R_\alpha u(x)^2 - 2u(x) R_\alpha u(x) + \frac{1}{\alpha} u(x)^2)\, m(dx) \\ & = \lim_{\alpha\to\infty} \frac{\alpha^2}{2} [\int_X (R_\alpha u(x)^2 - \frac{1}{\alpha} u(x)^2)\, m(dx) + 2\int_X (\frac{1}{\alpha} u(x)^2 - u(x) R_\alpha u(x))\, m(dx)] \\ & = -\frac{1}{2} \lim_{\alpha\to\infty} \alpha \int_X u(x)^2 (1 - \alpha \hat{R}_\alpha 1(x))\, m(dx) + \lim_{\alpha\to\infty} \alpha(u - \alpha R_\alpha u, u)_{L^2} \\ & = -\frac{1}{2} \int_X \tilde{u}(x)^2 \hat{k}(dx) + \mathbf{a}(u, u) \quad \text{by Lemma 2.12 and Lemma 3.3} \\ & = -\frac{1}{2} \int_X \tilde{u}(x)^2 \hat{\chi}(dx) + \mathbf{a}(u, u) \quad \text{by Theorem 3.5}. \end{aligned}$$

The equality (4.5) follows from the definition of the mutual energy. The proof is complete.

REMARK 4.6. Similarly, by the definition of $\bar{e}$, we have

$$(4.6) \qquad \bar{e}(A^{[u]}) = \lim_{t\downarrow 0} \frac{1}{t}(u - p_t u, u)_{L^2} - \frac{1}{2} \lim_{t\downarrow 0} \frac{1}{t} \int_X \tilde{u}(x)^2 (1 - \hat{p}_t 1(x))\, m(dx) .$$

In the symmetric case, it holds that

$$(4.7) \qquad \lim_{t\downarrow 0} \frac{1}{t}(u - p_t u, u)_{L^2} = \mathbf{a}(u, u)$$

and hence

$$(4.8) \qquad \bar{e}(A^{[u]}) = \mathbf{a}(u, u) - \frac{1}{2} \int_X \tilde{u}(x)^2 \hat{\chi}(dx) .$$

But, in the non-symmetric case, we do not know the validity of (4.7), and hence we can not prove (4.8).

Lemma 4.7 (cf. [8]; Lemma 5.1.1). *For any $u\in \boldsymbol{H}$, $\nu\in \boldsymbol{S}_0$, $0<T<\infty$ and $\varepsilon>0$, it holds that*

$$P_\nu(\sup_{0<t\leqq T}|\tilde{u}(X_t)|>\varepsilon)\leqq\frac{Ce^T}{\varepsilon}\mathbf{a}_1(U_1\nu, U_1\nu)^{1/2}\mathbf{a}_1(u, u)^{1/2} \tag{4.9}$$

for some constant $C>0$.

Proof. Let $E=\{x\in X: |\tilde{u}(x)|>\varepsilon\}$. Then the left hand side of (4.9) is dominated by

$$\begin{aligned} &e^T\int_X p(x)\,\nu(dx) \quad (p(x)=E_x[e^{-\sigma_E}]) \\ &=e^T\,\mathbf{a}_1(p, U_1\nu) \\ &\leqq C_1 e^T\,\mathbf{a}_1(p, p)^{1/2}\,\mathbf{a}_1(U_1\nu, U_1\nu)^{1/2} \quad \text{by (a.1) and (a.2)} \\ &=C_1 e^T\,\mathbf{a}_1(u_E^1, u_E^1)^{1/2}\,\mathbf{a}_1(U_1\nu, U_1\nu)^{1/2}, \end{aligned} \tag{4.10}$$

since p is a q.c. version of the capacitary potential u_E^1 of E. By the definition of E and u_E^1, we have

$$\mathbf{a}_1(u_E^1, u_E^1)^{1/2}\leqq\frac{C_2}{\varepsilon}\mathbf{a}_1(|u|, |u|)^{1/2}\leqq\frac{C_2}{\varepsilon}\mathbf{a}_1(u, u)^{1/2},$$

and this completes the proof of Lemma 4.7.

For the AF $A^{[u]}$ generated by $u\in\boldsymbol{H}$, we have the following theorem (see [8]; Theorem 5.2.2 for the symmetric case).

Theorem 4.8. *For any $u\in\boldsymbol{H}$, $A^{[u]}$ admits a unique decomposition;*

$$A^{[u]}=M^{[u]}+N^{[u]},\ M^{[u]}\in\mathring{\mathcal{M}},\ N^{[u]}\in\mathcal{N}_c, \tag{4.11}$$

Moreover, it holds that for u, $v\in\boldsymbol{H}$,

$$e(M^{[u]}, M^{[v]})=\frac{1}{2}\mathbf{a}(u, v)+\frac{1}{2}\hat{\mathbf{a}}(u, v)-\frac{1}{2}\langle\tilde{u}\tilde{v}, \hat{\chi}\rangle \tag{4.12}$$

Proof. The uniqueness is trivial. The equality (4.12) follows from (4.11) and Lemma 4.5. We show the existence of such $M^{[u]}$ and $N^{[u]}$. First we consider the case of $u=R_1f$, $f\in C_0^+(X)$. Then $Lu=u-f$. Let

$$N_t^{[u]}=\int_0^t Lu(X_s)\,ds$$

and

$$M_t^{[u]}=A_t^{[u]}-\int_0^t Lu(X_s)\,ds\,.$$

Then q.e. $x\in X$,

$$E_x[M_t^{[u]}] = p_t u(x) - u(x) - \int_0^t p_s(u-f)\, ds = 0\,.$$

Since

$$(M_t^{[u]})^2 \leqq 3\tilde{u}(X_t)^2 + 3\tilde{u}(X_0)^2 + 3(N_t^{[u]})^2\,,$$

we have, for all $t>0$,

$$E_\nu[(M_t^{[u]})^2 < \infty \quad \text{for any} \quad \nu\in\hat{\boldsymbol{S}}_{00}\,.$$

By Lemma 2.8, for all $t>0$,

$$E_x[(M_t^{[u]})^2] < \infty \ \text{ q.e. } x\in X\,.$$

Since $M_t^{[u]}$ is a martingale, it holds that

$$\text{q.e.} \quad x\in X,\ E_x[(M_t^{[u]})^2] < \infty \quad \text{for all} \quad t>0\,.$$

Thus $M_t^{[u]}\in\mathring{\mathcal{M}}$. Let $Lu=g$. Then

$$\begin{aligned} E_m[(N_t^{[u]})^2] &= 2E_m[\int_0^t g(X_s) \int_s^t g(X_v)\, dvds] \\ &= 2\int_0^t\int_0^{t-s} (\hat{p}_s 1, g p_v g)_{L^2}\, dvds \\ &= 2\int_0^t (\hat{p}_{t-s} 1, g\int_0^s p_v g\, dv)_{L^2}\, ds \\ &\leqq \|g\|_{L^2}^2 \cdot t^2\,. \end{aligned}$$

This implies that $e(N^{[u]})=0$ and hence $N^{[u]}\in\mathcal{N}_c$. Next, for any $u\in\boldsymbol{H}$, there exists $u_n=R_1 f_n$ ($f_n\in C_0(X)$), which converges to u in $\|\ \|_{\boldsymbol{H}}$ as $n\to\infty$. By the uniqueness of the decomposition (4.11) for u_n's, we have

$$\begin{aligned} e(M^{[u_n]}-M^{[u_m]}) &= e(M^{[u_n-u_m]}) \\ &= \mathbf{a}(u_n-u_m, u_n-u_m) - \frac{1}{2}\langle (u_n-u_m)^2, \hat{\mathcal{X}}\rangle \\ &\leqq \mathbf{a}(u_n-u_m, u_n-u_m) \\ &\to 0 \ \text{ as } n, m\to\infty\,. \end{aligned}$$

Hence $\{M^{[u_n]}\}$ is a Cauchy sequence in $(\mathring{\mathcal{M}}, e)$. Then, by Theorem 4.4, there exist a unique $M^{[u]}\in\mathring{\mathcal{M}}$ and a subsequence n_k such that

$$M^{[u_n]} \to M^{[u]} \ \text{ in } (\mathring{\mathcal{M}}, e) \ \text{ as } n\to\infty$$

and q.e. $x\in X$,

(4.13) $P_x(\lim_{n_k\to\infty} M_t^{[u_{n_k}]}=M_t^{[u]}$ uniformly on any finite interval of $t)=1$.

Let $N^{[u]}=A^{[u]}-M^{[u]}$. By virtue of Lemma 4.7 and the definition of $A^{[u]}$, the statement (4.13) for $A^{[u]}$ holds. Hence the statement (4.13) for $N^{[u]}$ holds. This implies that $N^{[u]}$ is a continuous AF. It is clear that q.e. $x\in X$, $E_x[|N_t^{[u]}|]<\infty$ for all $t>0$. Now we prove that $e(N^{[u]})=0$. Since

$$N^{[u]} = A^{[u-u_n]}-(M^{[u]}-M^{[u_n]})+N^{[u_n]},$$

$$\overline{\lim_{\alpha\to\infty}} \frac{\alpha^2}{2} E_m\Big[\int_0^\infty e^{-\alpha t}(N_t^{[u]})^2\,dt\Big]$$
$$\leqq 3e(A^{[u-u_n]})+3e(M^{[u]}-M^{[u_n]})+3e(N^{[u_n]})$$

$\to 0$ as $n\to\infty$, by the same calculus as above. Thus $e(N^{[u]})=0$. The proof is complete.

Next we give a decomposition of $M^{[u]}$ $(u\in \boldsymbol{H})$. Let

$$\mathcal{M}_c = \{M\in\mathcal{M}\colon P_x(M_t(\omega) \text{ is continuous in } t) = 1 \text{ q.e. } x\in X\}$$

and

$$\mathcal{M}_d = \{M\in\mathcal{M}\colon \langle M, L\rangle = 0 \quad\text{for any}\quad L\in\mathcal{M}_c\}\,.$$

P.A. Meyer [7] showed that the martingale AF $M^{[u]}$ has the following decomposition;

$$M^{[u]} = \overset{c}{M}{}^{[u]}+\overset{d}{M}{}^{[u]},\ \overset{c}{M}{}^{[u]}\in\mathcal{M}_c,\ \overset{d}{M}{}^{[u]}\in\mathcal{M}_d \tag{4.14}$$

where

$$\overset{d}{M}{}_t^{[n]} = \widehat{\sum_{0<s\leqq t} \Delta M_s^{[u]}\, I\{\Delta M_s^{[n]}\neq 0\}}\Big(=\widehat{\sum_{0<s\leqq t} (\tilde u(X_s)-\tilde u(X_{s-}))\, I\{X_s\neq X_{s-}\}}\Big) \tag{4.15}$$

and $\overset{c}{M}{}^{[u]}=M^{[u]}-\overset{d}{M}{}^{[u]}$. Here $\Delta M_s^{[u]}=M_s^{[u]}-M_{s-}^{[u]}$ and the right hand side of (4.15) is defined by the following way. Consider the sequence T_n^k, $n=1, 2, \cdots$, of the stopping times defined by

$$T_1^k = \inf\{s\colon \frac{1}{k}<|\tilde u(X_s)-\tilde u(X_{s-})|\leqq\frac{1}{k-1}\}\,,$$

$$T_{n+1}^k = \inf\{s>T_n^k\colon \frac{1}{k}<|\tilde u(X_s)-\tilde u(X_{s-})|\leqq\frac{1}{k-1}\},\ n = 1, 2, \cdots\,.$$

Let

$$A^k = \sum_{n=1}^\infty (\tilde u(X_{T_n^k})-\tilde u(X_{T_n^k-}))\, I_{\{T_n^k\leqq t\}}\,.$$

Since

$$E_x[\sum_{n=1}^{\infty} |\tilde{u}(X_{T_n^k})-\tilde{u}(X_{T_n^k-})| I_{\{T_n^k\leq t\}}]<\infty \text{ q.e. } x\in X\,,$$

there exists a dual predictable projection $(A^k)^p$ of A^k. Let $\widehat{A^k}=A^k-(A^k)^p$. Then $\sum_{k=1}^{n}\widehat{A^k}$ is a Cauchy sequence with respect to the family $\{\eta_{x,t}\colon \eta_{x,t}(M)=E_x[M_t^2]$ for q.e. $x\in X$ and all $t>0\}$ of the semi-norms in $\mathcal{M}$. Then we define the right hand side of (4.15) as $\lim_{n\to\infty}\sum_{k=1}^{n}\widehat{A^k}$.

Moreover $\overset{d}{M}{}^{[u]}$ can be decomposed in the following way;

$$\overset{d}{M}{}^{[u]}=\overset{j}{M}{}^{[u]}+\overset{k}{M}{}^{[u]}\,, \tag{4.16}$$

where

$$\overset{j}{M}{}^{[u]}=\widehat{\sum_{0<s\leq t}(\tilde{u}(X_s)-\tilde{u}(X_{s-}))\, I_{\{\zeta>t\}}} \tag{4.17}$$

and

$$\overset{k}{M}{}^{[u]}=-\widehat{\tilde{u}(X_{\zeta-})\, I_{\{\zeta\leq t\}}}\,.$$

Here the right hand side of (4.17) is defined by the same method as the definition of $\overset{c}{M}{}^{[u]}$.

Recall that μ_A denotes the smooth measure corresponding to $A\in \boldsymbol{A}_c^+$. If $A_t(\omega)=A_t^{(1)}(\omega)-A_t^{(2)}(\omega)$, $A_t^{(1)}$, $A_t^{(2)}\in\boldsymbol{A}_c^+$ and if $\mu_{A^{(i)}}$, $i=1, 2$, are bounded measures, then the measure corresponding to A is defined by $\mu_A=\mu_{A^{(1)}}-\mu_{A^{(2)}}$. In particular, for $M^{[u]}$, $M^{[v]}\in\overset{\circ}{\mathcal{M}}$ $(u, v\in\boldsymbol{H})$, we use the abbreviations $\mu_{\langle u\rangle}$, $\overset{\alpha}{\mu}_{\langle u\rangle}$, $\mu_{\langle u,v\rangle}$ and $\overset{\alpha}{\mu}_{\langle u,v\rangle}$, $\alpha=c, d, j, k$, for $\mu_{\langle M^{[u]}\rangle}$, $\mu_{\langle \overset{\alpha}{M}{}^{[u]}\rangle}$, $\mu_{\langle M^{[u]},M^{[v]}\rangle}$ and $\mu_{\langle \overset{\alpha}{M}{}^{[u]},\overset{\alpha}{M}{}^{[v]}\rangle}$, $\alpha=c, d, j, k$.

Theorem 4.9. *For u, $v\in\boldsymbol{H}$,*

$$\overset{j}{\mu}_{\langle u,v\rangle}(dy)=\int_X(\tilde{u}(x)-\tilde{u}(y))\,(\tilde{v}(x)-\tilde{v}(y))\,\sigma(dx, dy) \tag{4.19}$$

and

$$\overset{k}{\mu}_{\langle u,v\rangle}(dy)=\tilde{u}(y)\,\tilde{v}(y)\,\chi(dy)\,. \tag{4.20}$$

Proof. By Theorem 3.6, for any $f\in\boldsymbol{B}^+(X)$

$$\begin{aligned}\int_X f(y)\,\overset{j}{\mu}_{\langle u,v\rangle}(dy)&=\lim_{t\downarrow 0}\frac{1}{t}\,E_m[(f\langle \overset{j}{M}{}^{[u]}, \overset{j}{M}{}^{[v]}\rangle)_t]\\ &=\lim_{t\downarrow 0}\frac{1}{t}\,E_m[(f([\overset{j}{M}{}^{[u]}, \overset{j}{M}{}^{[v]}])^p)_t]\end{aligned} \tag{4.21}$$

where $[\overset{j}{M}^{[u]}, \overset{j}{M}^{[v]}]_t = \sum_{0<s\leqq t<\zeta} \Delta \overset{j}{M}_s^{[u]} \Delta \overset{j}{M}_s^{[v]}$. Hence the right hand side of (4.21) is equal to

$$\begin{aligned}
&\lim_{t\downarrow 0} \frac{1}{t} E_m[\sum_{0<s\leqq t<\zeta} f(X_{s-}) (\tilde{u}(X_s)-\tilde{u}(X_{s-})) (\tilde{v}(X_s)-\tilde{v}(X_{s-})) I_{\{X_s \neq X_{s-}\}}] \\
&= \lim_{t\downarrow 0} \frac{1}{t} E_m[\int_0^t dH_s \int_X f(X_s) (\tilde{u}(x)-\tilde{u}(X_s)) (\tilde{v}(x)-\tilde{v}(X_s)) N(X_s, dx)] \\
&= \langle \int_X f(\cdot) (\tilde{u}(x)-\tilde{u}(\cdot)) (\tilde{v}(x)-\tilde{v}(\cdot)) N(\cdot, dx) \nu, 1 \rangle
\end{aligned}$$

by Theorem 3.6 (v). This and Theorem 3.8 (i) imply (4.91). (4.20) can be proved by the same calculus as above. The proof is complete.

REMARK 4.10. Since $\overset{d}{\mu}_{\langle u,v\rangle} = \overset{j}{\mu}_{\langle u, v\rangle} + \overset{k}{\mu}_{\langle u,v\rangle}$ $(u, v \in \boldsymbol{H})$, it follows that

$$\overset{d}{\mu}_{\langle u,v\rangle}(dy) = \int_{X \cup \{\delta\}} (\tilde{u}(x)-\tilde{u}(y)) (\tilde{v}(x)-\tilde{v}(y)) N(y, dx) \nu(dy)\,. \tag{4.22}$$

Corollary 4.11. $\frac{1}{2} \overset{c}{\mu}_{\langle u,v\rangle}(X) = N(u, v)$ $\quad (u, v \in \boldsymbol{H})$.

Proof. By (4.12) and Theorem 4.9, we have for any $u, v \in H$,

$$\begin{aligned}
&\frac{1}{2} \mathbf{a}(u, v) + \frac{1}{2} \hat{\mathbf{a}}(u, v) = e(M^{[u]}, M^{[v]}) + \frac{1}{2} \int_X \tilde{u}(x)\, \tilde{v}(x)\, \hat{\chi}(dx) \\
&= \frac{1}{2} \mu_{\langle u,v\rangle}(X) + \frac{1}{2} \int_X \tilde{u}(x)\, \tilde{v}(x)\, \hat{\chi}(dx) \quad \text{by (4.3)} \\
&= \frac{1}{2} \overset{c}{\mu}_{\langle u,v\rangle}(X) + \frac{1}{2} \overset{j}{\mu}_{\langle u,v\rangle}(X) + \frac{1}{2} \overset{k}{\mu}_{\langle u,v\rangle}(X) \\
&\qquad + \frac{1}{2} \int_X \tilde{u}(x)\, \tilde{v}(x)\, \hat{\chi}(dx) \\
&= \frac{1}{2} \overset{c}{\mu}_{\langle u, v\rangle}(X) + \frac{1}{2} \int_{X\times X-\Delta} (\tilde{u}(x)-\tilde{u}(y)) (\tilde{v}(x)-\tilde{v}(y))\, \sigma(dx, dy) \\
&\qquad + \frac{1}{2} \int_X \tilde{u}(x)\, \tilde{v}(x)\, \chi(dx) + \frac{1}{2} \int_X \tilde{u}(x)\, \tilde{v}(x)\, \hat{\chi}(dx)\,.
\end{aligned}$$

Comparing this with (2.14), we get the desired equality. The proof is complete.

5. Derivation property of $\overset{c}{\mu}_{\langle u\rangle}$ and its applications

Lemma 5.1. *For $f, u \in \boldsymbol{H}_b$ (=the set of bounded functions of $\boldsymbol{H}$),*

$$\int_X \tilde{f}(x)\, d\mu_{\langle u\rangle} = 2\mathbf{a}(u, uf) - \mathbf{a}(u^2, f)\,. \tag{5.1}$$

Proof. It is sufficient to show that (5.1) holds for all $f\in \boldsymbol{H}_b^+$. By Theorem 3.6 (ii), we have

$$(f,\ U^{\alpha}_{\langle M^{[u]}\rangle}1)_{L^2}=\langle \mu_{\langle u\rangle},\ \hat{R}_{\alpha}f\rangle\,.$$

Since there exists a subsequence, also written by α, such that $\alpha R_{\alpha}f$ converges to $\tilde{f}$ q.e. as $\alpha\to\infty$,

$$\begin{aligned}(5.2)\qquad \langle \mu_{\langle u\rangle},\ \tilde{f}\rangle &= \lim_{\alpha\to\infty}\alpha(f,\ U^{\alpha}_{\langle M^{[u]}\rangle}1)_{L^2}\\ &= \lim_{\alpha\to\infty}\alpha E_{f\cdot m}\Big[\int_0^{\infty}e^{-\alpha t}\,d\langle M^{[u]}\rangle_t\Big]\\ &= \lim_{\alpha\to\infty}\alpha E_{f\cdot m}\Big[e^{-\alpha t}\langle M^{[u]}\rangle_t\Big|_{t=0}^{\infty}\Big]\\ &\quad +\lim_{\alpha\to\infty}\alpha^2 E_{f\cdot m}\Big[\int_0^{\infty}e^{-\alpha t}\langle M^{[u]}\rangle_t\,dt\Big]\\ &= \lim_{\alpha\to\infty}\alpha^2 E_{f\cdot m}\Big[\int_0^{\infty}e^{-\alpha t}(M_t^{[u]})^2\,dt\Big]\,,\end{aligned}$$

since $E_m[\langle M^{[u]}\rangle_t]=E_m[(M_t^{[u]})^2]\leqq ct$ for some constant $c\geqq 0$. Since

$$\lim_{\alpha\to\infty}\alpha^2 E_{f\cdot m}\Big[\int_0^{\infty}e^{-\alpha t}(N_t^{[u]})^2dt\Big]=0\,,$$

the right hand side of (5.2) is equal to

$$\begin{aligned}&\lim_{\alpha\to\infty}\alpha^2 E_{f\cdot m}\Big[\int_0^{\infty}e^{-\alpha t}(\tilde{u}(X_t)-\tilde{u}(X_0))^2\,dt\Big]\\ &\quad=2\mathbf{a}(u,\ uf)-\mathbf{a}(u^2,f)\end{aligned}$$

by the same calculus as the proof of Lemma 4.5. The proof is complete.

Using the fact $\mu_{\langle u\ v\rangle}=1/2(\mu_{\langle u+v\rangle}-\mu_{\langle u\rangle}-\mu_{\langle v\rangle})$, Lemma 5.1 implies the following corollary.

Corollary 5.2. *For* $f,\ u,\ v\in \boldsymbol{H}_b$,

$$\int_X \tilde{f}\,d\mu_{\langle u,v\rangle}=\mathbf{a}(u,fv)+\mathbf{a}(v,fu)-\mathbf{a}(uv,f)\,.$$

Lemma 5.3. *Let* $u_n\in \boldsymbol{H}$ *converge to* $u\in \boldsymbol{H}$ *with* $\|\ \|_{L^2}$ *as* $n\to\infty$ *and let* $\mathbf{a}(u_n,\ u_n)$ *be uniformly bounded. Then*

(1) $\mathbf{a}(u_n-u,\ v)\to 0$ *as* $n\to\infty$ *for all* $v\in \boldsymbol{H}$.

(2) $\mathbf{a}(v,\ u_n-u)\to 0$ *as* $n\to\infty$ *for all* $v\in \boldsymbol{H}$.

Proof. We only prove (1); The proof of (2) is similar. In the case $v=\hat{R}_1 f$ ($f\in L^2(X,\ m)$), we have

$$\mathbf{a}(u_n-u, \hat{R}_1 f) = (u_n-u, f)_{L^2}-(u_n-u, \hat{R}_1 f)_{L^2} \to 0 \text{ as } n\to\infty .$$

For general $v\in \boldsymbol{H}$ and any $\varepsilon>0$, there exists $w\in\text{Range}(R_1)$ such that $||v-w||_{\boldsymbol{H}} <\varepsilon$. Then

$$\limsup_{n\to\infty} |\mathbf{a}(u_n-u, v)| \leqq \limsup_{n\to\infty} |\mathbf{a}(u_n-u, v-w)| + \limsup_{n\to\infty} |\mathbf{a}(u_n-u, w)| \leqq C\cdot\varepsilon .$$

The proof is complete.

Theorem 5.4 (derivation property of $\overset{c}{\mu}_{\langle u\rangle}$). *It holds that*

$$d\overset{c}{\mu}_{\langle uv,w\rangle} = \tilde{u}\, d\overset{c}{\mu}_{\langle v,w\rangle}+\tilde{v}\, d\overset{c}{\mu}_{\langle u,w\rangle} \quad \textit{for} \quad u, v, w\in \boldsymbol{H}_b . \tag{5.3}$$

Proof. This can be proved by the same method as Y. Le Jan [17] (also cf. [8], [9]), but we give an alternative proof based on the martingale theory. It is sufficient to show that for $u, v\in \boldsymbol{H}_b$,

$$d\overset{c}{\mu}_{\langle u^2,v\rangle} = 2\tilde{u}\, d\overset{c}{\mu}_{\langle u,v\rangle} . \tag{5.4}$$

Let α_n be a sequence such that $u_n=\alpha_n R_{\alpha_n} u$ converges to $\tilde{u}$ q.e. as $\alpha_n\to\infty$. By Theorem 4.8 and its proof, we have the expressions;

$$\begin{aligned} u_n(X_t)-u_n(X_0) &= M_t^{[u_n]}+N_t^{[u_n]} , \\ u_n^2(X_t)-u_n^2(X_0) &= M_t^{[u_n^2]}+N_t^{[u_n^2]} , \end{aligned}$$

where $M^{[u_n]}, M^{[u_n^2]}\in \overset{\circ}{\mathcal{M}}$ and $N^{[u_n]}, N^{[u_n^2]}\in \mathcal{N}_c$. Moreover $N^{[u_n]}$ is of bounded variation. By Ito's formula,

$$u_n^2(X_t)-u_n^2(X_0) = 2\int_0^t u_n(X_{s-})\, dM_s^{[u_n]}+A_t ,$$

where A is an AF of bounded variation. It then follows that

$$\langle \overset{c}{M}{}^{[u_n^2]}, \overset{c}{M}{}^{[v]}\rangle_t+\langle N^{[u_n^2]}, \overset{c}{M}{}^{[v]}\rangle_t = 2\int_0^t u_n(X_s)\, d\langle \overset{c}{M}{}^{[u_n]}, \overset{c}{M}{}^{[v]}\rangle_s . \tag{5.5}$$

We now prove that

$$\langle N^{[u_n^2]}, \overset{c}{M}{}^{[v]}\rangle = 0 \quad P_m\text{-a.e.} , \tag{5.6}$$

which implies that $\langle N^{[u_n^2]}, \overset{c}{M}{}^{[v]}\rangle=0$ P_x-a.e., q.e. x. To see this it suffices to show that

$$\langle N^{[u_n^2]}\rangle = 0 \quad P_m\text{-a.e.} \tag{5.7}$$

Write N for $N^{[u_n^2]}$. Since

$$E_x[\langle N\rangle_t] = \lim_{k\to\infty} \sum_{0\leqq i<2^k} E_x(N_{(i+1)t/2^k}-N_{it/2^k})^2 \text{ q.e. } x$$

and since

$$\sum_{0\leqq i<2^k} E_m(N_{(i+1)t/2^k}-N_{it/2^k})^2=\sum_{0\leqq i<2^k} E_m[E_{X_{it/2^k}}[N^2_{t/2^k}]]$$
$$\leqq 2^k\, E_m[N^2_{t/2^k}]\,,$$

we have

$$E_m[\langle N\rangle_t]\leqq \liminf_{k\to\infty} 2^k E_m[N^2_{t/2^k}]\,.$$

Hence

$$\int_0^\infty e^{-t}E_m[\langle N\rangle_t]\,dt\leqq \liminf_{k\to\infty} 2^k E_m[\int_0^\infty e^{-t}N^2_{t/2^k}\,dt]$$
$$=\liminf_{k\to\infty}(2^k)^2E_m[\int_0^\infty e^{-2^kt}\,N_t^2\,dt]=0\,,$$

which proves (5.7). By (5.5), (5.6) and Theorem 3.6, we have

$$d\overset{c}{\mu}_{\langle u_n^2,v\rangle}=2u_n\,d\overset{c}{\mu}_{\langle u_n,v\rangle}\,. \tag{5.8}$$

Now we prove that for $f\in C_0(X)\cap \boldsymbol{H}$,

$$\lim_{n\to\infty}\int_X f\,d\overset{c}{\mu}_{\langle u_n^2,v\rangle}=\int_X f\,d\overset{c}{\mu}_{\langle u^2,v\rangle}\,. \tag{5.9}$$

In fact, for $f\in C_0(X)\cap \boldsymbol{H}$,

$$|\int_X f\,d\mu_{\langle u_n^2,v\rangle}-\int_X f\,d\mu_{\langle u^2,v\rangle}|$$
$$\leqq|\mathbf{a}(u_n^2-u^2,fv)|+|\mathbf{a}(v,fu_n^2-fu^2)|+|\mathbf{a}(u_n^2\,v-u^2v,f)|$$

by Corollary 5.2, and the right hand side converges to 0 as $n\to\infty$ by virtue of Lemma 5.3. Thus we have

$$\lim_{n\to\infty}\int_X f\,d\mu_{\langle u_n^2,v\rangle}=\int_X f\,d\mu_{\langle u^2,v\rangle}\,. \tag{5.10}$$

On the other hand, by Remark 4.10,

$$\int_X f\,d\overset{d}{\mu}_{\langle u_n^2,v\rangle}=\int_{X\times X-\Delta}f(y)\,(u_n(x)^2-u_n(y)^2)\,(\tilde v(x)-\tilde v(y))\,\sigma(dx,dy)$$
$$+\int_X f(y)\,u_n(y)^2\,\tilde v(y)\,\chi(dy)\,. \tag{5.11}$$

Here

$$\int_{X\times X-\Delta}f(y)\,(u_n(x)^2-n_n(y)^2)\,(\tilde v(x)-\tilde v(y))\,\sigma(dx,dy)$$

$$-\int_{X\times X-\Delta} f(y)\,(\tilde{u}(x)^2-\tilde{u}(y)^2)\,(\tilde{v}(x)-\tilde{v}(y))\,\sigma(dx, dy)$$

$=\mathrm{I}+\mathrm{II}$,
where

$$\mathrm{I}=\int_{X\times X-\Delta} f(y)\,(u_n(x)+u_n(y))\,[(u_n(x)-\tilde{u}(x))\,(u_n(y)-\tilde{u}(y))] \times(\tilde{v}(x)-\tilde{v}(y))\,\sigma(dx, dy)$$

and

$$\mathrm{II}=\int_{X\times X-\Delta} f(y)\,(u_n(x)+u_n(y)-\tilde{u}(x)-\tilde{u}(y))\,(\tilde{u}(x)-\tilde{u}(y)) \times(\tilde{v}(x)-\tilde{v}(y))\,\sigma(dx, dy)\,.$$

Firstly we have

$$\begin{aligned}|\mathrm{I}| &\leqq 2\|f\|_\infty\|u\|_\infty\Big[\int_{X\times X-\Delta}((\tilde{u}-u_n)(x)-(\tilde{u}-u_n)(y))^2\,\sigma(dx, dy)\Big]^{1/2}\\ &\times\Big[\int_{X\times X-\Delta}(\tilde{v}(x)-\tilde{v}(y))^2\,\sigma(dx, dy)\Big]^{1/2}\\ &\leqq 2\|f\|_\infty\|u\|_\infty\,\mathbf{a}(u-u_n, u-u_n)^{1/2}\,\mathbf{a}(v, v)^{1/2}\\ &\to 0 \text{ as } n\to\infty\,.\end{aligned}$$

Secondly, since

$$\int_{X\times X-\Delta}|\tilde{u}(x)-\tilde{u}(y)|\,|\tilde{v}(x)-\tilde{v}(y)|\,\sigma(dx, dy)<\infty\,,$$

the measure $\mu(dx, dy)=|\tilde{u}(x)-\tilde{u}(y)|\,|\tilde{v}(x)-\tilde{v}(y)|\,\sigma(dx, dy)$ on $X\times X-\Delta$ is bounded. Let N be a properly exceptional set such that $u_n(x)\to\tilde{u}(x)$ as $n\to\infty$ for all $x\in X-N$. Then, since $\sigma(dz, dy)=N(y, dx)\,\nu(dy)=\hat{N}(x, dy)\,\hat{\nu}(dx)=\hat{\sigma}(dy, dx)$, we have $\mu(X\times N)=\mu(N\times X)=0$. Hence

$$(u_n(x)-u_n(y)-\tilde{u}(x)-\tilde{u}(y))\to 0 \text{ as } n\to\infty\ \mu\text{-a.e.}$$

This implies that $|\mathrm{II}|\to 0$ as $n\to\infty$. Using the same method, the second term of the right hand side of (5.11) converges to

$$\int_X f(y)\,\tilde{u}(y)^2\,\tilde{v}(y)\,\chi(dy)$$

as $n\to\infty$. Thus

$$\lim_{n\to\infty}\int_X f\,d\mu^{d}_{\langle u_n^2, v\rangle}=\int_{X\times X-\Delta} f(y)\,(\tilde{u}(x)^2-\tilde{u}(y)^2)\,(\tilde{v}(x)-\tilde{v}(y))\sigma(dx, dy) +\int_X f(y)\,\tilde{u}(y)^2\,\tilde{v}(y)\,\chi(dy) \tag{5.12}$$

$$= \int_X f \, d\overset{d}{\mu}_{\langle u^2, v\rangle} .$$

Then (5.9) holds by (5.10) and (5.12). From the fact that

$$(5.13) \qquad |\overset{c}{\mu}_{\langle u_n, v\rangle} - \overset{c}{\mu}_{\langle u, v\rangle}|(X) \leqq \overset{c}{\mu}_{\langle u_n - u\rangle}(X)^{1/2} \, \overset{c}{\mu}_{\langle v\rangle}(X)^{1/2} ,$$

it is easy to show that

$$\lim_{n\to\infty} \int_X f u_n \, d\overset{c}{\mu}_{\langle u_n, v\rangle} = \int_X f \tilde{u} \, d\overset{c}{\mu}_{\langle u, v\rangle} .$$

Combining this with (5.8) and (5.9), we have (5.4). The proof is complete.

Corollary 5.5. *Let v, $w \in \boldsymbol{H}$ and $v = r =$ constant m-a.e. on an open set $D \subset X$. Then*

$$\overset{c}{\mu}_{\langle v, w\rangle} = 0 \text{ on } D .$$

Proof. Let $v \in \boldsymbol{H}_b$. Then for any $u \in C_0(X) \cap \boldsymbol{H}$ such that

$$\begin{aligned} r \, d\overset{c}{\mu}_{\langle u, w\rangle} &= d\overset{c}{\mu}_{\langle ru, w\rangle} \quad \text{by Theorem 5.4,} \\ &= d\overset{c}{\mu}_{\langle vu, w\rangle} \\ &= \tilde{v} \, d\overset{c}{\mu}_{\langle u, w\rangle} + u \, d\overset{c}{\mu}_{\langle v, w\rangle} \\ &= r \, d\overset{c}{\mu}_{\langle u, w\rangle} + u \, d\overset{c}{\mu}_{\langle v, w\rangle} \quad \text{on } D . \end{aligned}$$

Hence

$$u \, d\overset{c}{\mu}_{\langle v, w\rangle} = 0 \text{ on } D .$$

Since u is arbitrary, we have

$$\overset{c}{\mu}_{\langle v, w\rangle} = 0 \text{ on } D .$$

Let $v \in \boldsymbol{H}$ and $v_n = ((-n) \vee v) \wedge n$. Then since $v_n \in \boldsymbol{H}_b$ and it converges to v with $\| \ \|_H$, it follows by (5.13) that

$$\overset{c}{\mu}_{\langle v_n, w\rangle} \to \overset{c}{\mu}_{\langle v, w\rangle} \quad \text{vaguely as} \quad n \to \infty ,$$

which completes the proof.

REMARK 5.6. Let u_1, $u_2 \in \boldsymbol{H}$ satisfy that $u_1 - u_2 = r =$ constant m-a.e. on an open set $D \subset X$. Then, by Corollary 5.5, $I_D \, \overset{c}{\mu}_{\langle u_1 - u_2\rangle} = 0$. Since $I_D \, \overset{c}{\mu}_{\langle u_1 - u_2\rangle}$ is the smooth measure corresponding to $\langle \overset{c}{M}{}^{[u_1 - u_2]} \rangle_{\sigma_M \wedge t} \in \boldsymbol{A}_c^+(D)$ $(M = D^c)$, we have

$$(5.14) \qquad \overset{c}{M}{}_t^{[u_1]} - \overset{c}{M}{}_t^{[u_2]} = \overset{c}{M}{}_t^{[u_1 - u_2]} = 0 \quad \text{for all} \quad t < \sigma_{X-D} .$$

DEFINITION 5.7. For AF A, we define the support of A by

$$\text{supp}[A] = \{x \in X - N: P_x(R=0) = 1\},$$

where N is the exceptional set of A and $R = \inf\{t: A_t \neq 0\}$.

Theorem 5.8. *It holds that for $u \in \boldsymbol{H}$,*

$$\text{supp}[\overset{c}{M}{}^{[u]}] \subset \overline{\text{supp}}[u] \quad q.e.,$$

where $\overline{\text{supp}}[u] = \{x \in X$: *for some neighbourhood U of x, u is constant m-a.e. on* $U\}^c$.

Proof. Let $\{O_p\}_{p=1}^{\infty}$ be a sequence of open sets of X satisfying the following conditions;

(i) $\bigcup_{p=0}^{\infty} O_p = X - \overline{\text{supp}}[u]$,

(ii) $u = r_p =$ constant m-a.e. on O_p.

By Remark 5.6, we have

$$\overset{c}{M}{}_t^{[u]} = \overset{c}{M}{}_t^{[0]} \quad (t < \sigma_{X-O_p}),$$

where 0 is the zero valued function. It is trivial that $\overset{c}{M}{}_t^{[0]} = 0$ for any $t < \sigma_{X-O_p}$. Hence

$$O_p \subset (\text{supp}[\overset{c}{M}{}^{[u]}])^c \text{ q.e.} \quad \text{for any } p.$$

Since p is arbitrary, this completes the proof.

Theorem 5.9. *It holds that for $u, v \in \boldsymbol{H}$*

$$\text{supp}[\langle \overset{c}{M}{}^{[u]}, \overset{c}{M}{}^{[v]} \rangle] \subset \overline{\text{supp}}[u] \cap \overline{\text{supp}}[v] \quad q.e..$$

Proof. Since

$$|\langle \overset{c}{M}{}^{[u]}, \overset{c}{M}{}^{[v]} \rangle_t| \leqq (\langle \overset{c}{M}{}^{[u]} \rangle_t)^{1/2} (\langle \overset{c}{M}{}^{[v]} \rangle_t)^{1/2},$$

we have

$$\begin{aligned} \text{supp}[\langle \overset{e}{M}{}^{[u]}, \overset{c}{M}{}^{[v]} \rangle] &\subset \text{supp}[\langle \overset{c}{M}{}^{[u]} \rangle] \cap \text{supp}[\langle \overset{c}{M}{}^{[v]} \rangle] \\ &= \text{supp}[\overset{c}{M}{}^{[u]}] \cap \text{supp}[\overset{c}{M}{}^{[v]}] \\ &\subset \overline{\text{supp}}[u] \cap \overline{\text{supp}}[v], \end{aligned}$$

by Lemma 5.8. The proof is complete.

Corollary 5.10 (stronger local property of the form N). *The symmetric form N satisfies the following condition*;

$$N(u, v) = 0 \quad \textit{for all} \quad u, v \in \boldsymbol{H} \textit{ such that } \overline{\text{supp}}[u] \cap \overline{\text{supp}}[v] = \phi.$$

Proof. Since $\overline{\text{supp}}[u] \cap \overline{\text{supp}}[v] = \phi$, by Theorem 5.9, it holds that supp

$[\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle]=\phi$. Hence there exists a properly exceptional set N such that $\operatorname{supp}[\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle]\subset N$. Let $R(\omega)=\inf\{t: \langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle_t(\omega)\neq 0\}$. Then $P_x(R(\omega)>0)=1$ and $P_x[\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle_R=0]=1$. This implies that $E_x[\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle_t]=0$ for all $t>0$ and all $x\in X-N$ (see P.A. Meyer [13]), i.e., $\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle_t=0$ for all $t>0$. Hence $N(u, v)=\frac{1}{2}\overset{c}{\mu}_{\langle u,v\rangle}(X)=e(\langle \overset{c}{M}^{[u]}, \overset{c}{M}^{[v]}\rangle)=0$. The proof is complete.

6. Examples

As was stated in Section 1, we extend slightly the notion of Dirichlet form according to H. Kunita [10]. A form $\mathbf{a}$ is said to be a *general Dirichlet form* if it satisfies ($\mathbf{a}$.2), ($\mathbf{a}$.3), ($\mathbf{a}$.4) and the following, weaker than ($\mathbf{a}$.1), condition; ($\mathbf{a}$.1)$'$ There exists a constant $\alpha_0\geqq 0$ such that $\mathbf{a}_\alpha$ is coercive for any $\alpha>\alpha_0$. All the results in the first half of Section 2 concerning the Dirichlet form $\mathbf{a}$ hold for the general Dirichlet form $\mathbf{a}$ if $\alpha>\alpha_0$, $\mathbf{a}_1$, u_A^1 and $U_1\mu$ there are replaced by $\alpha>\alpha_0$, $\mathbf{a}_{\alpha_0+1}$, $u_A^{\alpha_0+1}$ and $U_{\alpha_0+1}\mu$([4], [12]). It is easy to see that the Beurling-Deny formula (2.8) also holds for the general Dirichlet form. Since any regular general Dirichlet space also admits an associated Hunt process (see [7]) and our arguments in preceding sections only involve $\mathbf{a}_\alpha$ for α large enough, all of the previous assertions for the Dirichlet form persist to hold for the general Dirichlet form. In this section we will give several examples of regular general Dirichlet spaces.

[I]. Let D be a relatively compact open subset of R^d and $H^1(D)$ the Sobolev space of order 1, i.e.,

$$H^1(D) = \{u\in L^2(D): \frac{\partial u}{\partial x_i}\in L^2(D),\ 1\leqq i\leqq d\},$$

where the derivatives $\frac{\partial u}{\partial x_i}$ are taken in the sense of Schwartz distributions. We also consider the space $H_0^1(D)$ the closure of $C_0^\infty(D)$ in $H^1(D)$. Let dx be the Lebesgue measure on D and $L^2=L^2(D, dx)$. We define the norm on $H^1(D)$ by

$$||u||_{H^1} = ||u||_{L^2}+||u_x||_{L^2} \qquad \text{for} \quad u\in H^1(D),$$

where

$$||u_x||_{L^2} = \Big(\sum_{i=1}^d \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^2}^2\Big)^{1/2}.$$

Consider the following formal generator

$$L^0u = \sum_{i,j=1}^d \frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^d b_i\frac{\partial u}{\partial x_i}-cu, \tag{6.1}$$

where a_{ij}, b_i, $i, j=1, 2, \cdots, d$, and c are bounded measurable functions on D. The bilinear form $\mathbf{a}^0$ corresponding to L^0 is given by

$$\mathbf{a}^0(u, v) = \sum_{i,j=1}^{d} \int_D a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^{d} \int_D b_i v \frac{\partial u}{\partial x_i} dx + \int_D cuvdx \tag{6.2}$$

for $u, v \in H^1(D)$. It is clear that $\mathbf{a}^0$ satisfies the condition ($\mathbf{a}$.2). Moreover we assume that L^0 is uniformly elliptic, i.e., there exists a constant $\nu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) y_i y_j \geqq \nu |y|^2 \qquad \text{for all} \quad x \in D \tag{6.3}$$

((a_{ij}) is not necessarily symmetric). Then $\mathbf{a}^0$ satisfies the condition ($\mathbf{a}$.1)′. In fact, for any $\alpha > 0$,

$$\begin{aligned} &\mathbf{a}^0(u, u) + \alpha(u, u)_{L^2} \\ &\geqq \nu ||u_x||_{L^2}^2 - \sqrt{d}\, ||b||_\infty ||u_x||_{L^2} ||u||_{L^2} + (\alpha - ||c||_\infty) ||u||_{L^2}^2 \end{aligned} \tag{6.4}$$

Since

$$(\nu + \sqrt{d}\, ||b||_\infty) ||u_x||_{L^2} ||u||_{L^2} \leqq \frac{\nu}{2} ||u_x||_{L^2}^2 + \frac{1}{2\nu} (\nu + \sqrt{d}\, ||b||_\infty)^2 ||u||_{L^2}^2 ,$$

the right hand side of (6.4) is not smaller than

$$\frac{\nu}{2} ||u_x||_{L^2}^2 + (\alpha - ||c||_\infty - \frac{1}{2\nu} (\nu + \sqrt{d}\, ||b||_\infty)^2) ||u||_{L^2}^2 + \nu ||u_x||_{L^2} ||u||_{L^2} . \tag{6.5}$$

Choose $\alpha_0 = \frac{\nu}{2} + \frac{1}{2\nu} (\nu + \sqrt{d}\, ||b||_\infty)^2 + ||c||_\infty$. Then

$$\mathbf{a}^0(u, u) + \alpha(u, u) \geqq \frac{\nu}{2} (||u_x||_{L^2}^2 + ||u||_{L^2}^2) = \frac{\nu}{2} ||u||_{H^1}^2$$

for any $\alpha > \alpha_0$, and hence $\mathbf{a}_\alpha^0$ is coercive for any $\alpha > \alpha_0$. Now we assume that

$$c \geqq 0 . \tag{6.6}$$

Since

$$\mathbf{a}^0(T_1 u, u - T_1 u) = \int_{u \geqq 1} c(u-1)\, dx \geqq 0 , \tag{6.7}$$

$\mathbf{a}^0$ satisfies the condition ($\mathbf{a}$.3). Now we assume that

$$c + \sum_{i=1}^{d} (b_i)_{x_i} \geqq 0 \quad \text{(in the sense of distributions).} \tag{6.8}$$

Then

$$\mathbf{a}^0(u-T_1u,\ T_1u) = -\sum_{i=1}^{d}\int_{u\geqq 1} b_i\,\frac{\partial(u-1)}{\partial x_i}\,dx + \int_{u\geqq 1} c(u-1)\,dx\,,$$

i.e., $\mathbf{a}^0$ satisfies the condition (**a**.4). If $D=R^d$, then $H^1(D)$ is regular.

Summarizing above we have the following theorem.

Theorem 6.1. *Let* $\mathbf{a}^0$ *be the bilinear form given by* (6.2). *If the conditions* (6.3), (6.6) *and* (6.8) *are satisfied, then* $(\mathbf{a}^0, H^1(D))$ *and* $(\mathbf{a}^0, H_0^1(D))$ *are general Dirichlet spaces on* $L^2(D, dx)$. *Moreover the latter one is regular.*

REMARK ([19]). Let

$$L^\infty + L^n = \{b\colon b = b^1+b^2,\ b^1\in L^\infty,\ b^2\in L^n\}\,.$$

Theorem 6.1 still holds if the boundedness assumption for b_i and c is replaced by the weaker condition that $b_i\in L^\infty+L^d$, $i=1, 2, \cdots, d$ and $c\in L^\infty+L^{d/2}$.

Let $D=R^d$ and consider the following differential operator given formally by

$$L = L^0 + \sum_{i\neq j}^{d} \delta(x_i-x_j)\,\frac{\partial}{\partial x_i}\,. \tag{6.9}$$

Then (6.9) can be rewritten as

$$L = L^0 + \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\,H(x_j-x_i)\,\frac{\partial}{\partial x_i}\,, \tag{6.10}$$

where $H(y)=1/2(I_{\{y>0\}}-I_{\{y<0\}})$. Now we have for any $u,\ v\in C_0^\infty(R^d)$,

$$\Big(\frac{\partial}{\partial x_j}\,H(x_j-x_i)\,\frac{\partial u}{\partial x_i},\ v\Big)_{L^2} = -(H(x_j-x_i)\,\frac{\partial^2 u}{\partial x_j\,\partial x_i},\ v)_{L^2}$$
$$-\Big(H(x_j-x_i)\,\frac{\partial u}{\partial x_i},\ \frac{\partial v}{\partial x_j}\Big)_{L^2}\,,$$

and the first term of the right hand side is antisymmetric in i, j because $H(y) = -H(-y)$. Hence the bilinear form $\mathbf{a}$ corresponding to L is given by, for $u,\ v\in H^1(R^d)$,

$$\mathbf{a}(u, v) = \mathbf{a}^0(u, v) + \sum_{i,j=1}^{d}\int_{R^d} H(x_j-x_i)\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\,dx\,, \tag{6.11}$$

and accordingly

$$\mathbf{a}(u, v) = \sum_{i,j=1}^{d}\int_{R^d} \tilde{a}_i\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\,dx - \sum_{i=1}^{d}\int_{R^d} b_i\,v\,\frac{\partial u}{\partial x_i}\,dx + \int_{R^d} cuvdx\,, \tag{6.12}$$

where $\tilde{a}_{ij}=a_{ij}+H(x_j-x_i)$, $i, j=1, 2, \cdots, d$. These $\tilde{a}_{ij}$, $i, j=1, 2, \cdots, d$, are bounded measurable functions on R^d satisfying the condition (6.3). Therefore, by Theorem 6.1, the bilinear form $\mathbf{a}$ given by (6.11) is a general Dirichlet form which is regular and consequently generates a diffusion process on R^d by virtue of S. Carrillo Menendez [5]. The present argument can be applied to the form corresponding to formal generator

$$L = L^0 + \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} d_{ij} \frac{\partial}{\partial x_i},$$

if the condition

$$\sum_{i,j=1}^{d} d_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (u \in C_0^\infty(D))$$

holds. Under a more general condition that

$$\sum_{i,j=1}^{d} \int d_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

H. Osada [16] has given a specific construction of the diffusion by means of the associated transition density function.

[II]. Next we consider the case with boundary conditions. Let $D=\{x=(\xi, x_d) \in R^d: \xi \in R^{d-1}, x_d \in R^1 \text{ and } x_d>0\}$, $\partial D = \{x \in R^d: x_d=0\}$ and $\bar{D}=D \cup \partial D$, and we consider the formal generator L^0 with the following boundary condition

$$\sum_{i=1}^{d} a_{id} \frac{\partial u}{\partial x_i} + \sum_{i=1}^{d-1} \beta_i \frac{\partial u}{\partial x_i} = 0, \tag{6.13}$$

where β_i, $i=1, 2, \cdots, d-1$ are bounded measurable functions on ∂D with bounded derivatives of first order. Then the bilinear form corresponding to L^0 with boundary condition (6.13) is given by, for $u, v \in H^1(D)$,

$$\mathbf{a}(u, v) = \mathbf{a}^0(u, v) - \sum_{i=1}^{d-1} \int_{\partial D} \beta_i(\xi)\, v(\xi, 0) \frac{\partial u(\xi, 0)}{\partial \xi_i} d\xi. \tag{6.14}$$

The term $\int_{\partial D} \beta_i(\xi)\, v(\xi, 0) \frac{\partial u(\xi, 0)}{\partial \xi_i} d\xi$ in (6.14) makes sense by the following argument. Let γ be a trace operator of ∂D (see S. Mizohara [14]). Then for $u \in H^1(D)$, $u(\xi, 0)=\gamma u(\xi) \in H^{1/2}(\partial D, d\xi)$, where $H^{1/2}(\partial D, d\xi)=\{f \in L^2(\partial D): (1+|z|)^{1/2} \hat{f}(z) \in L^2(\partial D)$, $\hat{f}(z)$ is the Fourier transform of $f(\xi)\}$, and $\frac{\partial u(\xi, 0)}{\partial \xi_i} = \frac{\partial \gamma u(\xi)}{\partial \xi_i} \in H^{-1/2}(\partial D, d\xi)$, where $H^{-1/2}(\partial D, d\xi)$ is the dual space of $H^{1/2}(\partial D, d\xi)$. Then $\int_{\partial D} \beta_i(\xi)\, v(\xi, 0) \frac{\partial u(\xi, 0)}{\partial \xi_i} d\xi$ is understood as the pairing $\langle \frac{\partial u(\xi, 0)}{\partial \xi_i}, \beta_i(\xi)$

$v(\xi, 0)\rangle$ of $\dfrac{\partial u(\xi, 0)}{\partial \xi_i}$ and $\beta_i(\xi)\, v(\xi, 0) \in H^{1/2}(\partial D, d\xi)$.

Theorem 6.2. *Let* $\mathbf{a}^0$ *be the form given in Theorem* 6.1 *and* β_i *satisfy the condition stated after* (6.13), *Moreover we assume that*

$$\sum_{i=1}^{d-1} (\beta_i)_{\xi_i} \geqq 0 \quad (\textit{in the sense of distributions}). \tag{6.15}$$

Then the bilinear form $\mathbf{a}$ *given by* (6.14) *is a regular general Dirichlet form on* $\bar{D}$.

Proof. For any $u \in C_0^\infty(\bar{D})$, by integration by parts, we have

$$2\int_{\partial D} \beta_i(\xi)\, u(\xi, 0)\, \frac{\partial u(\xi, 0)}{\partial \xi_i}\, d\xi = -\int_{\partial D} \frac{\partial \beta_i(\xi)}{\partial \xi_i}\, u(\xi, 0)^2\, d\xi\,.$$

Thus

$$\Big| \sum_{i=1}^{d-1} \int_{\partial D} \beta_i(\xi)\, u(\xi, 0)\, \frac{\partial u(\xi, 0)}{\partial \xi_i}\, d\xi \Big| \leqq C_1 \|u(\xi, 0)\|^2_{L^2(\partial D)}\,, \tag{6.16}$$

for some constant $C_1 > 0$. By Theorem 3.16 in [14], it holds that for any $\varepsilon > 0$,

$$\|u(\xi, 0)\|^2_{L^2(\partial D)} \leqq \varepsilon \|u_x\|^2_{L^2(D)} + C(\varepsilon) \|u\|^2_{L^2(D)}\,,$$

where $C(\varepsilon)$ is a constant depending on $\varepsilon > 0$. Hence for any $\alpha \geqq 0$,

$$\begin{aligned} \mathbf{a}(u, u) + \alpha(u, u)_{L^2} \geqq (\nu - C_1 \varepsilon)\, \|u_x\|^2_{L^2(D)} - \sqrt{d}\, \|b\|_\infty \|u_x\|_{L^2} \|u\|_{L^2} \\ + (\alpha - \|c\|_\infty - C_1\, C(\varepsilon))\, \|u\|^2_{L^2} \end{aligned}$$

which implies that $\mathbf{a}_\alpha$ is coercive on $C_0^\infty(\bar{D})$ for α large enough. Similarly $\mathbf{a}$ satisfies ($\mathbf{a}$.2) for any $u, v \in C_0^\infty(\bar{D})$. Since $C_0^\infty(\bar{D})$ is dense in $H^1(D)$ with respect to $\|\ \|_{H^1}$, $\mathbf{a}$ satisfies the conditions ($\mathbf{a}$.1)$'$ and ($\mathbf{a}$.2) for $u, v \in H^1(D)$. Since $\mathbf{a}(T_1 u, u - T_1 u) = \mathbf{a}^0(T_1 u, u - T_1 u)$, $\mathbf{a}$ satisfies ($\mathbf{a}$.3). It holds that

$$\mathbf{a}(u - T_1 u, T_1 u) = \mathbf{a}^0(u - T_1 u, T_1 u) - \sum_{i=1}^{d-1} \int_{u \geqq 1} \beta_i(\xi)\, \frac{\partial (u(\xi, 0) - 1)}{\partial \xi_i}\, d\xi \geqq 0\,,$$

by Theorem 6.1 and the condition (6.15). This completes the proof.

Now we give an alternative proof of ($\mathbf{a}$.1)$'$ and ($\mathbf{a}$.2) for the form $\mathbf{a}$ of Theorem 6.2, following the idea of S. Agmon [1]. By integration by parts, we have

$$\begin{aligned} &\int_{\partial D} \beta_i(\xi)\, \frac{\partial u(\xi, 0)}{\partial x_i}\, v(\xi, 0)\, d\xi \\ &= \int_D \frac{\partial}{\partial x_d} (\beta_i(\xi)\, v(x))\, \frac{\partial u(x)}{\partial x_i}\, dx + \int_D \beta_i(\xi)\, v(x)\, \frac{\partial^2 u(x)}{\partial x_d\, \partial x_i}\, dx \end{aligned}$$

$$= \int_D \beta_i(\xi) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_d} dx - \int_D \frac{\partial}{\partial x_i} (\beta_i(\xi) v(x)) \frac{\partial u(x)}{\partial x_d} dx$$
$$= \int_D \beta_i(\xi) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_d} dx + \int_D (-\beta_i(\xi)) \frac{\partial u(x)}{\partial x_d} \frac{\partial v(x)}{\partial x_i} dx$$
$$+ \int_D \frac{\partial}{\partial x_i} (-\beta_i(\xi)) \frac{\partial u(x)}{\partial x_d} v(x) dx .$$

Hence the bilinear form $\mathbf{a}$ given by (6.14) is represented by

$$\mathbf{a}(u, v) = \sum_{i,j=1}^{d} \int_D \tilde{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^{d} \int_D \tilde{b}_i \frac{\partial u}{\partial x_i} v dx + \int_D cuv dx ,$$

where

$$\tilde{a}_{id}(x) = a_{id}(x) - \beta_i(\xi),\ i = 1, 2, \cdots, d-1 ,$$
$$\tilde{a}_{di}(x) = a_{di}(x) + \beta_i(\xi),\ i = 1, 2, \cdots, d-1 ,$$
$$\tilde{a}_{ij}(x) = a_{ij}(x), \text{ otherwise}$$

and

$$\tilde{b}_i(x) = b_i(x) \quad i = 1, 2, \cdots, d-1 ,$$
$$\tilde{b}_d(x) = b_d(x) + \sum_{i=1}^{d-1} \frac{\partial}{\partial x_i} \beta_i(\xi) .$$

The coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$, $i, j=1, 2, \cdots, d$, are bounded measurable and $\tilde{a}_{ij}$ satisfy the condition (6.3). Thus, by the proof of Theorem 6.1, the bilinear form $\mathbf{a}$ given by (6.14) satisfies ($\mathbf{a}$.1)′ and ($\mathbf{a}$.2).

REMARK. Let $L^0 = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. Then the process $X=(X^1, X^2, \cdots, X^d)$ associated with the general Dirichlet form $\mathbf{a}$ given by (6.14) is expressed by

$$X_t^i = x^i + B_t^i + \int_{\partial D} \beta_i(\tilde{X}_s)\, d\phi_s,\ i = 1, 2, \cdots, d-1 ,$$
$$X_t^d = x^d + B_t^d + \phi_t$$

where $x^i = X_0^i$, $i=1, 2, \cdots, d$, $B=(B^1, B^2, \cdots, B^d)$ is d-dimensional Brownian motion, $X_t = (\tilde{X}_t, X_t^d)$ and ϕ_t is the local time of B_t^d (see M. Tsuchiya [21]).

[III]. Finally we consider an example with non-local form which is a slight generalization of Example IV. 3.2 in [5]. Let $\boldsymbol{H} = H^1(R^d)$ and a_{ij}, $i, j = 1, 2, \cdots, d$, be as in [I]. Let ϕ be a positive measurable function on $R^d \times R^d$ such that

$$\int_{R^d \times R^d} \phi(x, y)^2\, dy dx < \infty$$

and we define the functions c_1 and c_2 on R^d by

$$c_1(x) = \int_{R^d} \phi(x, y)\, dy \quad \text{and} \quad c_2(x) = \int_{R^d} \phi(y, x)\, dy\,.$$

Assume that c_1 and c_2 are bounded. Let c be some bounded measurable function satisfying $c \geqq c_1 \vee c_2$. Consider the operator

$$Lu(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u(x)}{\partial x_i}\right) + \int_{R^d} \phi(x, y)\, u(y)\, dy - c(x)\, u(x)\,. \tag{6.17}$$

Then the associated bilinear form $\mathbf{a}$ is defined by

$$\begin{aligned} \mathbf{a}(u, v) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}\, dx \\ - \int_{R^d \times R^d} \phi(x, y)\, u(y)\, v(x)\, dy dx + \int_{R^d} c(x)\, u(x)\, v(x)\, dx\,. \end{aligned} \tag{6.18}$$

Then it is easy to see that the form $\mathbf{a}$ satisfies $(\mathbf{a}.1)'$ and $(\mathbf{a}.2)$. Furthermore, $\mathbf{a}$ has another expressions;

$$\begin{aligned} \mathbf{a}(u, v) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}\, dx \\ + \int_{R^d \times R^d} u(y)\, (v(y) - v(x))\, \phi(x, y)\, dx dy \\ + \int_{R^d} u(x)\, v(x)\, (c(x) - c_2(x))\, dx \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} \mathbf{a}(v, u) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial v(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j}\, dx \\ + \int_{R^d \times R^d} u(x)\, (v(x) - v(y))\, \phi(x, y)\, dy dx \\ + \int_{R^d} u(x)\, v(x)\, (c(x) - c_1(x))\, dx\,. \end{aligned} \tag{6.20}$$

By (6.20), we have

$$\begin{aligned} \mathbf{a}(T_1 u, u - T_1 u) = \int_{\{x \in R^d\,:\, u(x) < 0\} \times R^d} -u(x)\, T_1 u(y)\, \phi(x, y)\, dy dx \\ + \int_{\{x \in R^d\,:\, u(x) > 1\} \times R^d} (u(x) - 1)\, (1 - T_1 u(y))\, \phi(x, y)\, dy dx \\ + \int_{\{x \in R^d\,:\, u(x) > 1\}} (u(x) - 1)\, (c(x) - c_1(x))\, dx\,. \end{aligned} \tag{6.21}$$

Since $0 \leqq T_1 u(y) \leqq 1$ for all $y \in R^d$ and $c \geqq c_1$ and ϕ is positive on $R^d \times R^d$, (6.21) implies that $\mathbf{a}(T_1 u,\ u-T_1 u)\geqq 0$. By the same method, (6.19) implies that $\mathbf{a}(u-T_1 u,\ T_1 u)\geqq 0$. Hence $\mathbf{a}$ given by (6.18) is a general Dirichlet form on $\boldsymbol{H}\times\boldsymbol{H}$. Moreover, the equalities (6.19) and (6.20) show that the measure $\sigma(dx, dy)=\phi(x, y)\, dxdy$ is the jumping measure of $\mathbf{a}$ and $\chi(dx)=(c(x)-c_1(x))\, dx$ (resp. $\hat{\chi}(dx)=(c(x)-c_2(x))\, dx$) is the killing measure of $\mathbf{a}$ (resp. $\hat{\mathbf{a}}$). And

$$N(u, v)=\sum_{i,j=1}^{d}\int_{R^d}\frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx$$

is a symmetric form satisfying the stronger local property.

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