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TRANSMUTATION, GENERALIZED TRANSLATION, AND TRANSFORM THEORY. PART II

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1. Introduction. Part I of this series of two papers contains the relevant background and a number of references to which we refer here as $[1-x]$. Additional references added in this paper will be denoted by $[y]$ (references frequently used here from Part I will be listed again). In Part I (ref. [7]) we developed a number of themes in the transmutation framework introduced in [3; 5]. In the present paper we will generalize some constructions of Marčenko [16] (cf. also Koornwinder [14]) in this framework and then, in a sort of canonical manner, develop a procedure for generating Parseval formulas of Gasymov-Marčenko type (cf. [11; 12; 16]). The Parseval formulas will be examined from various points of view and a derivation of the appropriate Gelfand-Levitan equation will also be given (in this connection see also Carroll [6]). Let us mention here also [8; 9] for extensive use of our transmutation framework in studying the interaction of certain scattering theory ideas with the construction of connection formulas of Riemann-Liouville and Weyl type for special functions.

2. Basic constructions. We recall briefly the background ideas from Part I. $P(D)$ and $Q(D)$ will be (second order) linear differential operators acting in spaces E and F with $B: E\to F$ ($B: P\to Q$) a transmutation operator such that $BP=QB$ acting on suitable objects, and $\mathcal{B}=B^{-1}: Q\to P$. As in [3; 4; 7] we consider general eigenfunctions of the form

$$(2.1)\qquad P(D_x)H(x, \mu)=\mu H(x, \mu);\ H(0, \mu)=1;\ H'(0, \mu)=0$$

$$Q(D_y)\theta(y, \nu)=\nu\theta(y, \nu);\ \theta(0, \nu)=1;\ \theta'(0, \nu)=0$$

$$(2.2)\qquad P^*(D_x)\Omega(x, \mu)=\mu\Omega(x, \mu);\ Q^*(D_y)W(y, \nu)=\nu W(y, \nu)$$

where P^* and Q^* denote formal adjoints. We assume either that the spectra $\sigma(P)$ and $\sigma(Q)$ coincide or that, as occurs in typical examples from [6; 7; 8; 9], $\mu=\lambda^2-\rho_P^2$ and $\nu=-\lambda^2-\rho_Q^2$ in which case we shift notation and speak of transmuting $\hat{P}=P+\rho_P^2$ into $\hat{Q}=Q+\rho_Q^2$ (so $\sigma(\hat{P})=\sigma(\hat{Q})$).

EXAMPLE 2.1. The basic example here can be written as $P(D)u=(Au')'/A$ with $\rho_P=\rho_A=1/2 \lim (A'/A)$ as $x\to\infty$ ($A=A_P=\Delta_P$ is a common notation). Set

$\hat{P}=P+\rho_P^2$ and consider eigenfunctions $H=\phi_\lambda^P$ of $\hat{P}(D)u=-\lambda^2 u$. The hypotheses on A are such that $P(D)$ is modeled on the radial part of the Laplace-Beltrami operator on a noncompact Riemannian symmetric space of rank 1. Then eigenfunctions ϕ_λ^P satisfying $\phi_\lambda^P(0)=1$ and $D_x\phi_\lambda^P(0)=0$ correspond to spherical functions. One has $P^*(D)u=(A(u/A)')'$ and $\Omega=\Omega_\lambda^P=A\phi_\lambda^P=\Delta_P\phi_\lambda^P$ satisfies $P^*(D)\Omega_\lambda^P=(-\lambda^2-\rho_P^2)\Omega_\lambda^P$ or $\hat{P}^*(D)\Omega_\lambda^P=-\lambda^2\Omega_\lambda^P$. Many such examples are discussed in [3; 4; 6; 7; 8; 9; 14] and in Chebli [1–18; 1–19] and Flensted-Jensen [1–25]; we will not dwell on this for the moment. Our constructions will be based on the physically important case $A=x^{2m+1}$ where $P(D)=P_m(D)=D^2+((2m+1)/x)D$ and $\rho_P=0$. In this case $P^*(D)u=u''-(2m+1)(u/x)'$ and for basic eigenfunctions we take ($\mu\sim-\lambda^2$)

$$H(x, \mu) = 2^m\Gamma(m+1)(\lambda x)^{-m}J_m(\lambda x) = R^m(x, \lambda); \tag{2.3}$$
$$\Omega(x, \mu) = 2^{-2m}\Gamma(m+1)^{-2}(\lambda x)^{2m+1}H(x, \mu)\,.$$

This choice of Ω for P_m^* was made earlier in [3; 4] for purposes of symmetry and we will retain it now for uniformity of notation; note however $\Omega\neq AH$ ($\Omega=AH$ is a more natural choice of Ω in general—see Remark 4.1 in [7]). Let us record here that in fact $\Omega=R_0(\lambda)A(x)H$ where $A=x^{2m+1}$ and $R_0=c_m^2\lambda^{2m+1}$ ($c_m=1/2^m\Gamma(m+1)$) is the density of the associated spectral measure $d\nu_P=R_0 d\lambda$. With the above choice of Ω we change the associated spectral measure to $d\nu_P(\lambda)=d\lambda$.

REMARK 2.2. We recall some notation for transforms based on a transmutation $B: P\to Q$ with eigenfunctions $H=\phi_\lambda^P$, $\Omega=\Omega_\lambda^P$, $\theta=\phi_\lambda^Q$, and $W=\Omega_\lambda^Q$. Thus

$$\mathsf{P}f(\lambda)=\hat{f}(\lambda)=\langle\Omega(x, \mu), f(x)\rangle=\int_0^\infty \Omega(x, \mu)f(x)dx \tag{2.4}$$

$$\mathbf{P}F(x)=\langle F(\lambda), H(x, \mu)\rangle_\nu=\int_0^\infty F(\lambda)H(x, \mu)d\nu_P(\lambda) \tag{2.5}$$

where $\mu\sim-\lambda^2$ in general and $d\nu_P$ can be given an explicit form in terms of $|c(\lambda)|^{-2}$($c(\lambda)$ is the Harish-Chandra or Jost function in our examples $P(D)u=(Au')'/A$. However when (complex) potentials $q(x)$ are added to $P(D)$ the spectral pairing may not be given in terms of a measure and we will have a generalized spectral function R_P such that for suitable $F(\lambda)$

$$\mathbf{P}F(x)=\langle F, H\rangle_\nu=\langle R_P, F(\lambda)H(x, \mu)\rangle_\lambda \tag{2.6}$$

where the last bracket is a distribution pairing in λ (cf. [11; 16; 17]). We remark that it is necessary to study this situation in physics (cf. Chadan-Sabatier [1–17], Coudray-Coz [1–21], Newton [1–39]) and we refer to [10; 17] for nonselfadjoint operators, spectral singularities, etc. In any event we will have the following collection of maps and properties, where $\langle\Omega, 1\rangle_\nu=\delta(x)$ and $\langle W, 1\rangle_\omega=\delta(y)$:

$$(2.7)\qquad \mathcal{P}f(\lambda) = \hat{\hat{f}}(\lambda) = \langle f(x), H(x, \mu)\rangle;\ \boldsymbol{P}F(x) = \langle F(\lambda), \Omega(x, \mu)\rangle_\nu;$$
$$\boldsymbol{\mathcal{P}}F(x) = \langle F(\lambda), H(x, \mu)\rangle_\omega$$

$$(2.8)\qquad \mathbf{Q}g(\lambda) = \tilde{g}(\lambda) = \langle g(y), W(y, \mu)\rangle;\ \mathcal{Q}g(\lambda) = \bar{g}(\lambda) = \langle g(y), \Theta(y, \mu)\rangle;$$
$$\mathbf{Q}G(y) = \langle G(\lambda), \Theta(y, \mu)\rangle_\omega;\ \boldsymbol{Q}G(y) = \langle G(\lambda), W(y, \mu)_\omega\rangle;$$
$$\boldsymbol{\mathcal{Q}}G(y) = \langle G(\lambda), \Theta(y, \mu)\rangle_\nu$$

$$(2.9)\qquad \mathbf{P} = \mathbf{P}^{-1};\ \mathbf{Q} = \boldsymbol{Q}^{-1};\ \boldsymbol{P} = \mathcal{P}^{-1};\ \boldsymbol{Q} = \mathcal{Q}^{-1};\ \mathbf{P}^* = \boldsymbol{P};$$
$$\mathbf{Q}^* = \boldsymbol{Q};\ \boldsymbol{\mathcal{P}}^* = \mathcal{P};\ \boldsymbol{\mathcal{Q}}^* = \mathcal{Q};\ B^* = (\boldsymbol{\mathcal{Q}}\mathcal{P})^* = \boldsymbol{P}\mathcal{Q};$$
$$\mathcal{B}^* = (\boldsymbol{\mathcal{P}}\mathbf{Q})^* = Q\mathcal{P};\ \boldsymbol{\mathcal{Q}}^{-1} = \mathcal{P}\boldsymbol{\mathcal{P}}\mathbf{Q};\ \boldsymbol{\mathcal{P}}^{-1} = \mathbf{Q}\boldsymbol{\mathcal{Q}}\mathcal{P}$$

(see here [3; 4; 5; 7; 8; 9; 14] and Chebli [1–18; 1–19], Flensted-Jensen [1–25] for details).

Let us recall here also the expressions for the kernels of $B = \boldsymbol{\mathcal{Q}}\mathbf{P}$ and $\mathcal{B} = \boldsymbol{\mathcal{P}}\mathbf{Q}$ which we write as $Bf(y) = \langle \beta(y, x), f(x)\rangle$ and $\mathcal{B}g(x) = \langle \gamma(x, y), g(y)\rangle$. Thus

$$(2.10)\qquad \beta(y, x) = \langle \Omega(x, \mu), \Theta(y, \mu)\rangle_\nu;\ \gamma(x, y) = \langle H(x, \mu), W(y, \mu)\rangle_\omega$$

In certain cases it is possible and convenient to work with the kernels in the form $\beta(y, x) = \delta(x-y) + L(y, x)$ and $\gamma(x, y) = \delta(x-y) + K(x, y)$. In general β and γ are distributions and a decomposition of this sort with L and K functions is only possible in certain circumstances.

We give now a key theorem (cf. [5]), generalizing a result of Marcenko [16] (cf. also Koornwinder [14]). The proof is very simple but the theorem is extremely important in working with Paley-Wiener and Parseval type theorems.

Theorem 2.3. *Let* $\breve{f}(y) = (\mathcal{B}^*f)(y) = \langle \gamma(x, y), f(x)\rangle$ *and* $\check{g}(x) = (B^*g)(x) = \phi$ $\langle \beta(y, x), g(y)\rangle$. *Then*

$$(2.11)\qquad \mathcal{Q}\breve{f}(\lambda) = \mathcal{P}f(\lambda);\ \mathcal{P}\check{g}(\lambda) = \mathcal{Q}g(\lambda).$$

Proof. From (2.9) $\mathcal{B}^* = \boldsymbol{Q}\mathcal{P}$ and $B^* = \boldsymbol{P}\mathcal{Q}$ so $\mathcal{Q}\breve{f} = \mathcal{Q}\mathcal{B}^*f = \mathcal{Q}\boldsymbol{Q}\mathcal{P}f = \mathcal{P}f$ and $\mathcal{P}\check{g} = \mathcal{P}\boldsymbol{P}\mathcal{Q}g = \mathcal{Q}g$ since $\mathcal{Q} = \boldsymbol{Q}^{-1}$ and $\mathcal{P} = \boldsymbol{P}^{-1}$.

This proof uses the transforms indicated and thus depends on spectral data. Let us give an alternative proof of Theorem 2.3 independent of any spectral information or transform theory.

Second proof: The operator $B: P \to Q$ can often be constructed by solving $P(D_x)\phi(x, y) = Q(D_y)\phi(x, y)$ with $\phi(x, 0) = f(x)$ and $\phi_y(x, 0) = 0$; then $Bf(y)\phi = \phi(0, y)$ and similar constructions yield $\mathcal{B} = B^{-1}$ (cf. [3; 4; 5; 7] and Carroll-Showalter [1–14], Lions [1–35]). In particular B and $\mathcal{B}$ can often be constructed using Riemann functions in a manner which yields relevant properties of β or L (resp. γ or K) quite readily (cf. [18; 19; 21] and Braaksma [1–1], Braaksma-deSnoo [1–2], Levitan [1–33], Lions [1–35]). We know $\Theta = BH$ and $H = \mathcal{B}\Theta$ from [3; 4; 7] so define then $\breve{f}(y) = (\mathcal{B}^*f)(y)$ and $\check{g}(x) = (B^*g)(x)$ as in Theorem 2.3 and write for example (formally)

$$(2.12)\qquad \rho f(\lambda)=\langle H(x,\mu),f(x)\rangle=\langle\mathcal{B}(\cdot,\mu)\theta(x),f(x)\rangle=\langle\theta(y,\mu),\mathcal{B}^*f(y)\rangle=\langle\theta(y,\mu),\breve{f}(y)\rangle=\mathcal{Q}\breve{f}(\lambda)\,.$$

Similarly $\rho\breve{g}(\lambda)=\mathcal{Q}g(\lambda)$ and we have an alternative proof of Theorem 2.3 independent of any spectral data or a priori transform theory. *QED*

REMARK 2.4. Recall now the P_m spaces $E=\{f;\,x^{m+1/2}f(x)\in L^2\}$, $\boldsymbol{E}=E'=\{f;\,x^{-m-1/2}f(x)\in L^2\}$, $\hat{E}=\mathbf{P}E=\{f;\,\lambda^{-m-1/2}f(\lambda)\in L^2\}$, and $\hat{E}'=\hat{\hat{\boldsymbol{E}}}=\rho\boldsymbol{E}=\{f;\,\lambda^{m+1/2}f(\lambda)\in L^2\}$ (from [3; 5; 7]). For general P one can also envision a framework where $\hat{E}=\mathbf{P}E$, $\boldsymbol{E}=E'$, $\rho\boldsymbol{E}=\hat{E}'=\hat{\hat{\boldsymbol{E}}}$, etc. and similarly the Q-operators involve $\tilde{F}=\mathbf{Q}F$, $F'=\boldsymbol{F}$, $\mathcal{Q}\boldsymbol{F}=\bar{\boldsymbol{F}}=\tilde{F}'$, etc. For a transmutation B adapted to such a $(P-Q)$-framework (by which we mean a situation as in Theorem 4.3 of [7] or Theorem 4 of [5] whose properties are summarized in (2.9)) one has from [3; 5; 7] $B=\mathcal{Q}\mathbf{P}\colon E\to F$, $B^*=\mathbf{P}\mathcal{Q}^*$, and $R(\mathcal{Q}^*)\subset\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$. Thus $\rho\breve{g}=\rho B^*g=\rho\mathbf{P}\mathcal{Q}^*g=\mathcal{Q}^*g\subset\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$ and similarly for $\mathcal{B}=\mathbf{P}\mathbf{Q}$ with $\mathcal{B}^*=\mathbf{Q}\rho^*$ and $R(\mathbf{P}^*)\subset\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$ we have $\mathcal{Q}\breve{f}=\mathcal{Q}\mathcal{B}^*f=\mathcal{Q}\mathbf{Q}\rho^*f=\rho^*f\subset\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$. Hence for f, g such that $\breve{f}$ and $\breve{g}$ make sense we have ρf and $\mathcal{Q}g$ in $\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$ and as an adjunct to theorem 2.3 we state

Proposition 2.5. *Given a transmutation B adapted to a $(P$-$Q)$-framework as in Part I, theorem 4.3, $f\in\boldsymbol{E}$ and $g\in\boldsymbol{F}$ as in Theorem 2.3 we have ρf and $\mathcal{Q}g$ in $\hat{\hat{\boldsymbol{E}}}\cap\bar{\boldsymbol{F}}$.*

REMARK 2.6. We recall that the operators $\mathbf{P}$, ρ, etc. will have realizations in various spaces so we are not always concerned with "pinning down" the P and Q operators in any one framework; similarly B can act in various spaces. When a framework is to be specified we refer to $E=E_A=\{f;\,A^{1/2}f\in L^2\}$, $E'=E'_A=\{f;\,A^{-1/2}f\in L^2\}$, and set $\hat{E}=\mathbf{P}E$.

3. Parseval formulas. We will sketch first the kind of procedure followed by Marčenko [16] to obtain Parseval formulas for operators $D^2-q(x)$. Then we will show how to generalize this formally to deal with operators having singularities of the type arising in $P(D)u=(Au')'/A-q(x)$. Precise results can then be obtained for $A=x^{2m+1}$, where further information is available, and this gives an independent derivation of Gasymov's Parseval formula for this case (see [11; 12]). The rigorous extension of this technique to general A as in Chebli [1–18; 1–19] is in progress. The type of Parseval formula in question can be written

$$(3.1)\qquad \langle R,\rho f\rho g\rangle_\lambda=\langle A^{-1/2}f,\,A^{-1/2}g\rangle$$

which reduces to the Marčenko case for $A=1$ and is equivalent to to Gasymov's formula for $A=x^{2m+1}$ (where Gasymov works with $l=m-1/2$ integral).

REMARK 3.1. Consider the case $P(D)=D^2-q(x)$ of Marčenko [16] and suppose first that spectral information is known (i.e. the ν pairing). Let δ_n be a sequence of functions, $\delta_n \in E$ if possible, $\delta_n(x)=0$ for $x \geqslant 1/n$, $\int_0^\infty \delta_n(x)dx=1$, $\delta_n(x) \geqslant 0$ ($x \in [0, 1/n]$), and $\delta_n(x) \to \delta$ in say $\mathcal{E}'$. Set $R_n = \mathbf{P}\delta_n$ and $U_n(x, y) = T_x^y \delta_n(x)$. Then (cf. [3; 7]).

$$U_n(x, y) = \langle H(y, \mu), R_n(\lambda)H(x, \mu)\rangle_\nu \tag{3.2}$$

Multiply by suitable $f, g \in \boldsymbol{E}$ and integrate to obtain

$$\langle g(y), \langle U_n(x, y), f(x)\rangle\rangle = \langle R_n(\lambda), \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\nu. \tag{3.3}$$

Given that $T_x^y\delta(x)$ makes sense we have formally $\langle U_n(x, y), f(x)\rangle \to \langle T_x^y\delta(x), f(x)\rangle = (\delta * f)(y) = f(y)$ (cf. [3; 7]) so the left side of (3.3) tends to $\langle f(y), g(y)\rangle = \int_0^\infty f(y)g(y)dy$. On the other hand from (2.4) $R_n(\lambda) = \mathbf{P}\delta_n(\lambda) \to R(\lambda) = \Omega(0, \mu)$ which we call $\mathbf{P}\delta(\lambda)$ if this makes sense and is nonzero. Hence we can state.

Theorem 3.2. *If the ν spectral pairing is known for $\hat{\boldsymbol{E}} - \hat{\hat{\boldsymbol{E}}}$ and $T_x^y\delta(x)$ makes sense acting as indicated then the spectral function $R^\nu(\lambda) = \Omega(0, \mu)$ yields a Parseval formula*

$$\langle f, g\rangle = \langle R^\nu(\lambda), \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\nu. \tag{3.4}$$

Note that when a singularity is present as in our operators P based on A and $\Omega = AH$ then $\Omega(0, \mu) = 0$. This also occurs for Ω as in (2.3) and Example 3.5 of Part I shows that $T_x^y\delta(x) = 0$ in such a case also. With operators such as $D^2 - q$ however $\Omega(0, \mu)$ is a sensible function and $T_x^y\delta(x)$ will make sense.

REMARK 3.3. In general the idea is to discover the ν pairing and if one has a transmutation $B: P \to Q$ where the Q theory is known then the ν pairing can be obtained by a variation on the above argument (cf. Marčenko [16]). With the operator $P(D) = D^2 - q(x)$ (for suitable q) one transmutes P into $Q = D^2$ of course and we sketch here a version of Marčenko's argument in our framework. It is convenient to use the representation $\beta(y, x) = \delta(x-y) + L(y, x)$ and $\gamma(x, y) = \delta(x-y) + K(x, y)$ here where K and L will be functions. In particular $L(y, x) = 0$ for $x > y$ and $K(x, y) = 0$ for $y > x$ (such triangularity properties are proved in a general way in Carroll-Gilbert [8; 9]). Let L and K be obtained via Riemann functions as in [16] so that no spectral theory is assumed (B^{-1} refers to spaces like $\mathcal{E} = C^\infty$—not L^2). We can write $\overleftarrow{f}(y) = \mathcal{B}^* f(y) = f(y) + \int_y^\infty K(\xi, y) \times f(\xi)d\xi$ with $\check{g}(x) = g(x) + \int_x^\infty L(\xi, x)g(\xi)d\xi$. Let $K^2(\sigma)$ denote L^2 functions f vanishing for $x > \sigma$ and $CK^2(\sigma)$ their cosine transform $\mathcal{Q}f(\lambda)$; from the definitions $f, g \in K^2(\sigma)$ implies $\overleftarrow{f}, \check{g} \in K^2(\sigma)$ (since K and L are triangular). From (2.12) we see then that $\mathcal{P}f(\lambda) \in CK^2(\sigma)$ and $CK^2(\sigma)$ can be characterized as the set of even entire functions $G(\lambda)\in L^2$ for $\lambda\in \boldsymbol{R}$ satisfying $|G(\lambda)|\leq c\exp\sigma|\mathrm{Im}\,\lambda|$ for $\lambda\in\boldsymbol{C}$. Let $Z(\sigma)$ denote the space of even entire functions $G(\lambda)\in L^1$ for $\lambda\in\boldsymbol{R}$ satisfying this same type of estimate for $\lambda\in\boldsymbol{C}$. Let $Z=\bigcup Z(\sigma)$ and $CK^2=\bigcup CK^2(\sigma)$ (countably normed—cf. [1–28]) so we have $Z\subset CK^2$. Note that $F, G\in CK^2$ implies $FG\in Z$ which is the kind of situation one wants in (3.1) (i.e. we will have the product $\mathscr{P}f(\lambda)\mathscr{P}g(\lambda)\in Z$). Following a procedure indicated in part already and extended below it can be shown that the spectral function R of (3.1) lies in Z'. First we go back to $U_n(x,y)=T_x^y\delta_n(x)$ which we write in the somewhat different form $U_n(x,y)=\int\tilde{R}_n(\lambda)H(x,\mu)H(y,\mu)d\lambda$. Then $U_n(x,0)=\delta_n(x)=\int\tilde{R}_n(\lambda)H(x,\mu)d\lambda$ $(\sim\mathbf{P}R_n(x))$. But $\Theta=BH$ so we want $(B\delta_n)(y)=\delta_n(y)+\int_0^y L(y,x)\delta_n(x)dx=\int\tilde{R}_n(\lambda)\Theta(y,\mu)d\lambda$ $(\sim\mathbf{Q}R_n(y))$. Thus we pass the determination of $\tilde{R}_n$ from the P theory to the (known) Q theory but without introducing ν the pairing used in specifying $\mathbf{P}$ and $\mathscr{Q}$ before; thus the use of B here bypasses the spectral theory for P. Now $\delta_n\in E$, $B\delta_n\in F$, and we suppose an inversion for the Θ transform is known relative to the λ pairing. For example assume the $\tilde{F}-\tilde{F}'=\bar{\boldsymbol{F}}$ or ω pairing can be passed to λ as $d\lambda=\omega(\lambda)d\lambda$. Then it follows that $\int\tilde{R}_n(\lambda)\Theta(y,\mu)d\lambda=\int\tilde{R}_n(\lambda)\Theta(y,\mu)\omega(\lambda)d\omega=\mathbf{Q}(\tilde{R}_n\omega)(y)\in F$ so $\tilde{R}_n$can be determined as $\tilde{R}_n(\lambda)\omega(\lambda)\in\tilde{F}$ by

$$\tilde{R}_n(\lambda)\omega(\lambda)=\mathbf{Q}B\delta_n=\mathbf{Q}[\delta_n(y)+\int_0^y L(y,x)\delta_n(x)dx]\,. \tag{3.5}$$

When $Q(D)=D^2$, $\Theta(y,\mu)=\mathrm{Cos}\,\lambda y$, $F=\boldsymbol{F}$, $W(y,\mu)=\frac{2}{\pi}\mathrm{Cos}\,\lambda y$, we have $\omega(\lambda)=1$ and (3.5) works. Once $\tilde{R}_n$ is thus determined we multiply $U_n(x,y)$ by $f(x)g(y)$ and integrate to obtain as in (3.3)

$$\langle g(y),\langle U_n(x,y),f(x)\rangle\rangle=\langle\tilde{R}_n(\lambda),\mathscr{P}f(\lambda)\mathscr{P}g(\lambda)\rangle_\lambda\,. \tag{3.6}$$

Using Riemann functions again it can be shown that (cf. [13; 16; 1–33])

$$U_n(x,y)=\frac{1}{2}[\delta_n(x+y)+\delta_n(x-y)]+\int_{x-y}^{x+y}\beta(x,y,t)\delta_n(t)dt \tag{3.7}$$

where $\theta_n(x,y)=\int_{x-y}^{x+y}\beta(x,y,t)\delta_n(t)dt$ can be estimated and $\iint f(x)g(y)\theta_n(x,y)\times dxdy\to 0$ as $n\to\infty$. Since f, g are even the left side of (3.6) tends to $\langle f(x),g(x)\rangle$ and we write (formally)

$$\begin{aligned} R=\lim\tilde{R}_n&=\mathbf{Q}[\delta(y)+L(y,0)]\\ &=\frac{2}{\pi}\{1+C[L(y,0)]\}\in Z' \end{aligned} \tag{3.8}$$

since $\mathbf{Q}$ is based on $W=\frac{2}{\pi}$ Cos λy (C denotes the cosine transform). Consequently (3.1) will follow.

REMARK 3.4. As indicated before when singularities are present the above argument breaks down at several points (e.g. (3.7) is inaccurate). The formal change needed is basically to replace $\delta(x)$ by $\delta_A(x)=\delta(x)/A(x)$, acting on suitable objects, and rephrase the argument. This applies whether we take $\Omega=AH$ or $\Omega=R_0AH$ as in(2.3). For simplicity take $\Omega=AH$ with $P(D)u=(Au')'/A$ and observe that from (2.4) formally $\mathbf{P}\delta_A(\lambda)=\hat{\delta}_A(\lambda)=1$ so that from (2.5) $\delta_A(x)=\langle H(x, \mu), 1\rangle_\nu$ (or $\delta(x)=\langle\Omega(x, \mu), 1\rangle_\nu$). Further from [7] $[T_x^y\delta_A(x)]^\wedge=H(y, \mu)$ where T_x^y denotes the generalized translation associated equivalently with P or $\hat{P}=P+\rho_P^2$. Consider as in (3.2) $\delta_n(x)\rightarrow\delta(x)$ and set $\delta_n^A(x)=\delta_n(x)/A(x)$ with $U_n^A(x, y)=T_x^y\delta_n^A(x)$; however in order to work in E_A for example let $\delta_n\in C_0^\infty$ (see Remark 3.5 for technical comments). Then

$$U_n^A(x, y) = \langle H(y, \mu), R_n^A(\lambda)H(x, \mu)\rangle_\nu \tag{3.9}$$

where $R_n^A(\lambda)=\mathbf{P}\delta_n^A\rightarrow 1$. Hence the analogue of (3.3) is

$$\begin{aligned}\langle g(y), \langle U_n^A(x, y), f(x)\rangle\rangle &= \langle R_n^A(\lambda), \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\nu \rightarrow \\ \langle 1, \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\nu &= \int_0^\infty \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)d\nu_P(\lambda)\end{aligned} \tag{3.10}$$

since the ν pairing is given by a measure in this situation. Now, writing $f_A=f/A\in E_A$,

$$\begin{aligned}\langle U_n^A(x, y), f(x)\rangle &= \langle T_x^y\delta_n^A(x), f(x)\rangle = \int_0^\infty T_x^y\delta_n^A(x)f(x)dx= \\ &\int_0^\infty T_x^y\delta_n^A(x)f_A(x)A(x)dx = (\delta_n^A * f_A)(y)\end{aligned} \tag{3.11}$$

where we recall from [14] and Flensted-Jensen [1–25] that a generalized convolution is given by $(f*g)(x)=\int T_x^yf(x)g(y)A(y)dy=\int T_x^yf(y)g(y)A(y)dy$ (cf. also Part I). Further one can prove for $f, g\in E_A$ for example that $f*g=g*f$ (cf. Theorem 3.6). Then the left side of (3.10) is formally

$$\begin{aligned}\langle g(y), \langle U_n^A(x, y), f(x)\rangle\rangle &= \langle g(y), (f_A*\delta_n^A\rangle)(y)\rangle = \\ \langle g(y), \int_0^\infty \delta_n^A(x)T_x^yf_A(x)A(x)dx &\rightarrow \langle g(y), T_x^yf_A(x)\,|_{x=0}\rangle = \\ \langle g(y), f_A(y)\rangle &= \langle A^{-1/2}g, A^{-1/2}f\rangle\end{aligned} \tag{3.12}$$

and hence (3.10) yields a Parseval formula of the form (3.4).

REMARK 3.5. That T_x^y can be extended to $\delta_A(x)$ is clear (recall that

$[T_x^y\delta_A(x)]^\wedge = H(y, \mu))$. The manner in which one represents this in the argument of Remark 3.4 is basically a matter of choosing a point of departure. Thus if we work in $E_A = E$ (which is a convenient place to prove Theorem 3.6 below) then $\delta_n^A(x) = \delta_n(x)/A(x)$ must be chosen accordingly. In the precise form which is possible for $A = x^{2m+1}$ the continuity of $T_x^y\colon \mathcal{E}^0(\boldsymbol{R}_+^1) \to \mathcal{E}^0(\boldsymbol{R}_+^2)$ is available (cf. [15; 18]—$\mathcal{E}^0$ denotes C^0 functions with the topology of uniform convergence on compact sets). Hence for various arguments we will be able to work in the dual $\mathcal{M}$ of $\mathcal{E}^0$ ($\mathcal{M}$ being Radon measures of compact support) and in this context it will be convenient to approximate δ by $\delta_n \in C_0^\infty$ (see Section 4 for more details). In particular let $L_0^1 \subset \mathcal{M}$ be L^1 functions with compact support and let δ_n be a δ approximation as in Remark 3.1. Then choose $\delta_n^k \in C_0^\infty$ which converge to δ_n in L_0^1 as a $C_0^\infty\delta$ approximation in $\mathcal{E}'$ (i.e. in $\mathcal{M}$). We denote such δ_n^k by δ_n now so that $\delta_n^A(x) \in E_A$.

Now for $\breve{f} \in E_A$ set $f = A\breve{f}$ so that $f \in E'_A$ (cf. Remark 2.6—$A^{1/2}\breve{f} = A^{-1/2}f$). Then we have

Theorem 3.6. *Given a v pairing, for $\breve{f}, \breve{g} \in E_A$, $f = A\breve{f}$, $g = A\breve{g} \in E'_A$ one has*

$$(3.13)\qquad \langle T_x^y\breve{f}, g\rangle = \int_0^\infty T_x^y\breve{f}(x)\breve{g}(x)A(x)dx = (T_x^y\breve{f}, \breve{g}) = (f, T_x^y\breve{g}) = \int_0^\infty T_x^y\breve{g}(x)\breve{f}(x)A(x)dx = \langle T_x^y\breve{g}, f\rangle\,.$$

Proof. We generalize and recast an angument of Levitan [15] in our framework. Thus for $f \in E = E_A$ let

$$(3.14)\qquad \tilde{\varphi}(y, \mu) = \langle T_x^y f(x), \Omega(x, \mu)\rangle = \boldsymbol{P}T_x^y f(x)(\lambda)$$

Then $P(D_y)\tilde{\varphi} = -\lambda^2\tilde{\varphi}$ with $\tilde{\varphi}(0, \mu) = \boldsymbol{P}f(\lambda)$ and $\tilde{\varphi}_y(0, \mu) = 0$ (since $D_yT_x^yf(x) = 0$ at $y = 0$); here $\tilde{\varphi}(y, \cdot) \in \hat{E}$ and $\tilde{\varphi}(\cdot, \mu) \in \mathcal{E}$. On the other hand observe that $\psi(x, y, \mu) = H(x, \mu)H(y, \mu)$ satisfies $P(D_x)\psi = P(D_y)\psi$ with $\psi(x, 0) = H(x, \mu)$ and $\psi_y(x, 0) = 0$ so

$$(3.15)\qquad H(x, \mu)H(y, \mu) = T_x^yH(x, \mu)$$

(cf. Part I—$H(\cdot, \mu) \in \mathcal{E}$ for example). Consider then (with $H(y, \cdot)$ a multiplier in $\hat{E}$)

$$(3.16)\qquad \omega(y, \mu) = H(y, \mu)\langle f(x), \Omega(x, \mu)\rangle = H(y, \mu)\boldsymbol{P}f(\lambda)$$

Clearly $P(D_y)\omega = -\lambda^2\omega$ with $\omega(0, \mu) = \boldsymbol{P}f(\lambda)$ and $\omega_y(0, \mu) = 0$ ($\omega(\cdot, \mu) \in \mathcal{E}$). By uniqueness $\tilde{\varphi}(y, \mu) = \omega(y, \mu)$ while $\omega(y, \mu)$ can be written as (from (3.15))

$$(3.17)\qquad \begin{aligned} \omega(y, \mu) &= \langle f(x), H(y, \mu)H(x, \mu)A(x)\rangle \\ &= \langle f(x), T_x^yH(x, \mu)A(x)\rangle \end{aligned}$$

From (3.14) and (3.17) we obtain then

$$(3.18) \qquad \langle T_x^y f(x), H(x, \mu)A(x)\rangle = \langle f(x), A(x)T_x^y H(x, \mu)\rangle$$

Now let $g(x)=\langle H(x, \mu), G(\lambda)\rangle_\lambda = \mathbf{P}G(x)\in E$ and take ν brackets (assumed to exist in general and in fact known explicitly here already) in (3.18) with $G(\lambda)$ to obtain (3.13) for $f, g\in E$, i.e.

$$(3.19) \qquad \langle T_x^y f(x), g(x)A(x)\rangle = \langle f(x), A(x)T_x^y g(x)\rangle$$

Note here that $\langle G(\lambda), T_x^y H(x, \mu)\rangle_\nu = T_x^y\langle G(\lambda), H(x, \mu)\rangle_\nu = T_x^y g(x)$ since if

$$(3.20) \qquad \varphi(x, y) = \langle G(\lambda), T_x^y H(x, \mu)\rangle_\nu$$

then $P(D_x)\varphi = P(D_y)\varphi$ with $\varphi(x, 0)=\langle G(\lambda), H(x, \mu)\rangle = g(x)$ (and $\varphi_y(x, 0)=0$) so $\varphi(x, y)=T_x^y g(x)$. Also from $T_x^y f(x)\in E$ and $g(x)\in E$ one has $A^{1/2}(x)T_x^y f(x)\in L^2$ and $A^{1/2}(x)g(x)\in L^2$ so $\int T_x^y g(x)g(x)A(x)\, dx$ for example makes sense. Now in order to get $\boldsymbol{E}=E'$ into the picture let $f, g\in \boldsymbol{E}=E'$. Then note $f\in E' \sim A^{-1}(x)f\in E$ so writing $f(x)=A(x)\breve{f}(x)$ we have $\breve{f}(x)\in E$. Hence write (3.19) now as

$$(3.21) \qquad (T_x^y \breve{f}(x), \breve{g}(x)) = (\breve{f}(x), T_x^y \breve{g}(x))$$

for a (real) scalar product $(\breve{f}, \breve{g})=\int f(x)\breve{g}(x)A(x)dx$ and we can write for $f\in E'$, $\breve{g}\in E$,

$$(3.22) \qquad \langle \breve{g}, f\rangle = \int_0^\infty \breve{g}(x)f(x)dx = \int_0^\infty \breve{g}(x)\breve{f}(x)A(x)dx = (\breve{g}, \breve{f}).$$

Actually (,) could be a complex scalar product here since $T_x^y f(x)$ is real for $f(x)$ real; this may not be true in later sections. *QED*

REMARK 3.7. Problems modeled on the functions A introduced in Example 2.1, and discussed briefly with some specific examples in Part I (for which the preceding analysis based on Remark 3.4 is in fact correct), are treated more extensively in our transmutation framework in Carroll-Gilbert [8; 9]. Properties of the corresponding $H(x, \mu)=\phi_\lambda^P(x)$ etc. are obtained in [14; 1–25] for A of the form $(e^x-e^{-x})^{2\alpha+1}(e^x+e^{-x})^{2\beta+1}$ or $(e^x-e^{-x})^p(e^{2x}-e^{-2x})^q$. More general A as well as perturbations of $P(D)u=(Au')'/A$ by a potential are treated in [1–18; 1–19] and the transmutation method for such A is developed in a forthcoming book [24]. Thus at this point we restrict our investigation of general A in asserting only that the preceding argument expounded via Remark 3.4 is valid for A of the type in [14; 1–25; 8; 9] and leads to the following theorem. Note that the theorem is not at all new or surprising ($d\nu_P$ is explicitly known) and has only been proved formally; it is the methodology which is being summarized in its statement (cf. Remark 3.4 and remarks after Lemma 4.2).

Theorem 3.8. *For $P(D)=(Au')'/A$ with A as in* [8; 9; 14; 1–25] *the above procedure yields the Parseval formula for suitable $f, g \in \boldsymbol{E}$*

$$\langle A^{-1/2}f, A^{-1/2}g\rangle = \int_0^\infty \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)d\nu_P(\lambda)\,. \tag{3.23}$$

REMARK 3.9. Note that, without specifying spaces, the formula (3.18) leads one to write

$$(T_x^y)^*\Omega(x, \mu) = A(x)T_x^y H(x, \mu)\,. \tag{3.24}$$

4. Parseval formulas for $A=x^{2m+1}$. There remains of course the extension and modification of the argument of Remark 3.3 to discover the pairing for the more general A of [1–18; 1–19] via a transmutation $B: P\to Q$ where the Q theory is known. Then the ν pairing for $P(D)+q$ can be obtained by a transmutation $P(D)+q\to P(D)$ for example using the same method. One wants to isolate the essential features of such arguments in order to arrive at a minimal collection of properties to study by hard analysis. As a step in this direction we examine the case $A=x^{2m+1}$ in detail. Most of the technique will clearly generalize. First let us mention that the arguments of Levitan [15] on which the proof of Theorem 3.6 is based can be used to prove (cf. [15]).

Theorem 4.1. *For continuous f such that $\int x^{2m+1}f(x)dx<\infty$ and $g\in C^0\cap L^\infty$ one has for the T_x^y associated with $P_m(D)$*

$$\int_0^\infty T_x^y f(x)g(x)x^{2m+1}dx = \int_0^\infty T_x^y g(x)f(x)x^{2m+1}dx\,. \tag{4.1}$$

We take now $P=P_m$ and Ω as in (2.3)(i.e. $\Omega=R_0AH$ for $A=x^{2m+1}$ and $R_0=c_m^2\lambda^{2m+1}$). Then $\delta_A(x)=\delta(x)/x^{2m+1}$ and $\mathcal{P}\delta_A(x)=c_m^2\lambda^{2m+1}=R_0(\lambda)$. With this normalization for Ω recall that $d\nu_P=d\lambda$ and $R^A(\lambda)=R_0(\lambda)$ so that the Parseval formula of type (3.23) which arises is ($\langle\,,\rangle_\nu=\langle\,,\rangle_\lambda$)

$$\langle x^{-m-1/2}f, x^{-m-1/2}g\rangle = \langle R_0, \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\lambda = \int_0^\infty R_0(\lambda)\mathcal{P}f(\lambda)\mathcal{P}g(\lambda)d\lambda \tag{4.2}$$

(if $\Omega=AH$ recall $d\nu_P=R_0d\lambda$ and $R^A=1$ as in (3.10)).

Now no transmutation is needed to produce (4.2). Formally we can derive it via Remark 3.4 and a study of T_x^y as in Remark 3.5 and Theorem 3.6 (this is made rigorous below). It is also interesting however to see how (4.2) can be derived via a transmutation of $P=P_m$ into $Q=D^2$. This will serve as a model for producing Parseval formulas for $P=P_m-q$ via a transmutation with D^2 by displaying in skeletal form how the different order of singularity affects the transmutation kernels etc. Another method on which we prefer to rely for such

P is then developed where the Parseval formula for $P=P_m-q$ is obtained via transmutation into $Q=P_m$.

First let us deal with the limiting passage in (3.10) and (3.12) for $A=x^{2m+1}$. As mentioned in Remark 3.5 one knows T_x^y: $\mathcal{E}^0(\boldsymbol{R}_+^1)\to\mathcal{E}^0(\boldsymbol{R}_+^2)$ is continuous and we assume Theorem 4.1 is known (as well as Theorem 3.6 for $E=E_A=\{f;\ x^{m+1/2}f(x)\in L^2\}$). Recall also the notation for $\mathcal{M}$ and L_0^1 from Remark 3.5 and set $\tilde{E}=\{\tilde{\phi};\ x^{2m+1}\tilde{\phi}\in L_0^1\}$ (one is thinking of $\delta_n/x^{2m+1}=\tilde{\phi}\in\tilde{E}$ where $\delta_n\in L_0^1$ is a δ approximation in $\mathcal{M}$). For $\tilde{\phi}\in\tilde{E}$ set $x^{2m+1}\tilde{\phi}=\phi\in L_0^1$ and approximate ϕ by C_0^∞ functions ϕ_k (recall that C_0^∞ is dense in L^1 and supp $\phi\subset[0, x_\phi]$). Then $\tilde{\phi}_k=\phi_k/x^{2m+1}\in\tilde{E}$ is continuous and (4.1) can be invoked for $g\in C^0\cap L^\infty$; thus $\int T_x^y\tilde{\phi}_k(x)g(x)x^{2m+1}dx=\int T_x^yg(x)\tilde{\phi}_k(x)x^{2m+1}dx=\int T_x^yg(x)\phi_k(x)dx\to\int T_x^yg(x)\phi(x)dx$ and one extends (4.1) to $\tilde{\phi}\in\tilde{E}$ by this limiting procedure. In order to provide a representation for the limiting values note that for $g\in\mathcal{E}^0$ and $\tilde{\phi}\in\tilde{E}$ the map

$$(4.3)\qquad g\to\int_0^\infty\tilde{\phi}x^{2m+1}T_x^yg(x)dx=M_y(g)\colon\ \mathcal{E}^0\to\boldsymbol{C}$$

is continuous so we can write

$$(4.4)\qquad M_y(g)=\langle\Phi_y, g\rangle$$

for $\Phi_y\in\mathcal{M}$ and we set $\Phi_y(x)=x^{2m+1}T_x^y\tilde{\phi}(x)$ to determine $T_x^y(x)\tilde{\phi}$. Thus the version of (4.1) obtained by limiting procedures from (4.1) can be written ($\tilde{\phi}\in\tilde{E}, g\in\mathcal{E}^0$)

$$(4.5)\qquad\int_0^\infty T_x^yg(x)\tilde{\phi}(x)x^{2m+1}dx=\langle T_x^yg(x), \tilde{\phi}(x)x^{2m+1}\rangle=\langle x^{2m+1}T_x^y\tilde{\phi}(x), g(x)\rangle$$

($\langle\,,\rangle$ denotes $\mathcal{E}^0-\mathcal{M}$ duality). Now let $\tilde{\phi}_n(x)x^{2m+1}=\delta_n(x)\in L_0^1$ where $\delta_n\to\delta$ in $\mathcal{M}$. The left side of (4.5) tends to $T_x^yg(x)|_{x=0}=g(y)$ in $\mathcal{E}^0-\mathcal{M}$ duality and henec in $\mathcal{M}$

$$(4.6)\qquad x^{2m+1}T_x^y(\delta_n(x)/x^{2m+1})\to\delta(x-y)=x^{2m+1}T_x^y(\delta(x)/x^{2m+1})\,.$$

We summarize this in

Lemma 4.2. *For* $\tilde{\phi}\in\tilde{E}$ *and* $g\in\mathcal{E}^0$ (4.5) *holds in* $\mathcal{E}^0-\mathcal{M}$ *duality and extends* (4.1). *By limiting procedures we then arrive at* (4.6).

Now to derive (4.2) we consider (3.11)–(3.12) for $A=x^{2m+1}$, $\delta_n^A(x)=\delta_n(x)/A(x)\in E_A=E$, and $f\in E'=\boldsymbol{E}$ (i.e. use C_0^∞ approximations to δ). The $f_A*\delta_n^A=\delta_n^A*f_A$ interchange is then justified by Theorem 3.6 actually and the limit in (3.12) is correct if $T_x^yf_A(x)\in C^0$. Hence approximate $\tilde{f}=A^{1/2}f_A=A^{-1/2}f\in L^2$ by C_0^∞ functions $\tilde{f}_k$ in L^2 so $f_A^k=A^{-1/2}\tilde{f}_k$ is continuous. Therefore for $g\in\boldsymbol{E}$ (3.12) yields ($f_k=Af_A^k=A^{1/2}\tilde{f}_k$)

(4.7) $\langle g(y), \langle T_x^y \delta_n^A(x), f_k(x)\rangle\rangle \to \langle A^{-1/2}g, A^{-1/2}f_k\rangle = \langle A^{-1/2}g, \tilde{f}_k\rangle \to \langle A^{-1/2}g, A^{-1/2}f\rangle$.

Note that Lemma 4.2 says

(4.8) $A(x)T_x^y\delta_A(x) = \delta(x-y)$

and this can be applied directly in (3.11) (to continuous $f_A(x)$ at least) without using Theorem 3.6. Thus it appears that we can use either Theorem 3.6 or Lemma 4.2 in order to obtain (4.7). Note however that the existence of a ν pairing is used in proving Theorem 3.6. Also observe that a corresponding Lemma 4.2 for general A however involves knowing that $T_x^y: \mathcal{E}^0 \to \mathcal{E}^0$ is continuous for the associated T_x^y. It remains to show that (cf. (3.10) and recall that $\langle\,,\,\rangle_\nu = \langle\,,\,\rangle_\lambda$)

(4.9) $\langle R_n^A(\lambda), \mathcal{P}f_k(\lambda)\mathcal{P}g(\lambda)\rangle_\nu \to \int_0^\infty R_0(\lambda)\mathcal{P}f(\lambda)\mathcal{P}g(\lambda)d\lambda$

where (cf. (2.3)) $R_n^A(\lambda) = \mathbf{P}\delta_n^A(\lambda) = \langle\delta_n^A(x), \Omega(x,\mu)\rangle = c_m^2\lambda^{2m+1}\langle\delta_n(x), H(x,\mu)\rangle$. Now $\mathcal{P}: \boldsymbol{E} \to \hat{\hat{\boldsymbol{E}}} = \hat{E}' = \{\hat{\hat{f}}; \lambda^{m+1/2}\hat{\hat{f}}(\lambda) \in L^2\}$ is continuous and we denote by $\hat{\hat{\boldsymbol{E}}}_\pi$ the space $\hat{\hat{\boldsymbol{E}}}_\pi = \{h; \lambda^{2m+1}h(\lambda) \in L^1\}$ so that for $f, g \in \boldsymbol{E}$ we have $\mathcal{P}f\mathcal{P}g \in \hat{\hat{\boldsymbol{E}}}_\pi$. Further from $\tilde{f}_k \to \tilde{f} = A^{-1/2}f$ in L^2we have $f_k = A^{1/2}\tilde{f}_k \to f$ in $\boldsymbol{E}$ by definitions. Since $\mathcal{P}$ is continuous we have $\mathcal{P}f_k \to \mathcal{P}f$ in $\hat{\hat{\boldsymbol{E}}}$ and hence $\mathcal{P}f_k\mathcal{P}g \to \mathcal{P}f\mathcal{P}g$ in $\hat{\hat{\boldsymbol{E}}}_\pi$. If we show that $R_n^A(\lambda) \to R_0$ weakly (weak *) in $\hat{\hat{\boldsymbol{E}}}'_\pi = \{\psi; \lambda^{-2m-1}\psi \in L^\infty\}$ then (4.9) holds. To do this note first that

(4.10) $|R_n^A(\lambda)\lambda^{-2m-1}| = c_m^2|\langle\delta_n(x), H(x,\mu)\rangle| \leqslant c_m^2 h\int|\delta_n(x)|dx = c_m^2 h$

where $|H(x,\mu)| \leqslant h$ and we can take $\delta_n \geqslant 0$ with $\int\delta_n dx = 1$. Hence $R_n^A \in \hat{\hat{\boldsymbol{E}}}'_\pi$. On the other hand

(4.11) $|R_0 - R_n^A|\lambda^{-2m-1} = c_m^2(1 - \langle\delta_n, H\rangle)$.

We know $H(\cdot,\mu) \in \mathcal{E}$ so we need only show $\langle\delta_n, H\rangle \to 1$ weakly in L_λ^∞ as $\delta_n \to \delta$ in $\mathcal{E}'$ (or $\mathcal{M}$). For λ fixed $\langle\delta_n, H\rangle \to 1$ since $H(0,\mu) = 1$ and for fixed $f \in L^1$ we have $\langle f, (1 - \langle\delta_n, H\rangle)\rangle \to 0$ by dominated convergence in L^1. Hence we have proved

Theorem 4.3. *For $P_m = D^2 + ((2m+1)/x)D$ the Parseval formula* (4.2) *holds for $f, g \in \boldsymbol{E}$.*

REMARK 4.4. We emphasize again that this procedure is given in detail to indicate clearly some of the ingredients which go into general Parseval formulas in our framework. In this direction let us also consider a derivation based on a transmutation $B: P = P_m \to Q = D^2$. Thus (cf. Remark 3.3 now) write $U_n(x,y)$

$=T_x^y\delta_n^A(x)$ in the form

(4.12) $\quad U_n(x, y) = \langle R_n^\omega(\lambda), H(x, \mu)H(y, \mu)\rangle_\omega$

where $\langle\, ,\rangle_\omega = \langle\, ,\rangle_\lambda$ in fact ($\delta_n^A(x) = \delta_n(x)/x^{2m+1}$ as before). Then

(4.13) $\quad \delta_n^A(x) = \langle R_n^\omega(\lambda), H(x, \mu)\rangle_\omega = \boldsymbol{\rho} R_n^\omega$

($\boldsymbol{\rho}$ suitably extended) and we operate on (4.13) with $B = \mathcal{2}\mathbf{P}$ to get

(4.14) $\quad (B\delta_n^A)(y) = \mathbf{Q}R_n^\omega$

since $\mathcal{2}^{-1} = \mathbf{P}\boldsymbol{\rho}\mathbf{Q}(\mathcal{2}\mathbf{P}\boldsymbol{\rho} = \mathbf{Q}^{-1} = \mathbf{Q})$. Thus the determination of the spectral function R is passed to the Q theory and

(4.15) $$R_n^\omega(\lambda) = \mathbf{Q}B\delta_A^n = \frac{2}{\pi}\int_0^\infty (B\delta_n^A)(y)\,\mathrm{Cos}\,\lambda y dy\,.$$

We know from multiplying (4.12) by $f(x)g(y)$ and integrating as before that (4.2) should follow and since $d\omega = d\lambda$ this means that $R_n^\omega(\lambda) \to R_0(\lambda)$. Thus in order to apply the transmutation method of determining the spectral function via the Q theory we are led in general to deal with distribution arguments since (4.15) involves generalized cosine transforms of functions like λ^{2m+1}. This is connected with the kernel $\beta(y, x)$ of B being a distribution of order >0 and basically arises out of the different type of singularity in P_m and D^2. Since this method will be important in determining Parseval formulas for $Pu = (Au')'/A$ with general A as in [1–18; 1–19] we will give some further discussion here of (4.15) and related formulas. The groundwork for this was developed in Part I, Section 4 (Theorem 4.6) where a formula for $B\delta_A$ was derived. We recall this here (cf. equation (4.23) in [7]).

(4.16) $$(B\delta_A)(y) = \beta_m y^{-2m-2} = c_m^2\int_0^\infty \lambda^{2m+1}\,\mathrm{Cos}\,\lambda y d\lambda$$

where $\beta_m = 2\Gamma(1/2)/\Gamma(m+1)\Gamma(-1/2-m)$ and y^{-2m-2} is to be interperted as $y_+^{-2m-2} + y_-^{-2m-2}$ where $y_\pm^{-\alpha}$ denotes the standard pseudofunction of Schwartz (cf. [1–43]). Various formulas for the kernel $\beta(y, x)$ of B were given in [7] (cf. also [3; 4]) but there is no need to repeat these here. Now we need only show that $R_n^\omega(\lambda) \to R_0(\lambda)$ weakly in $\hat{\hat{E}}'$ in order to pass from (4.12) to (4.2) (since the passage $\langle g(y), \langle U_n(x, y). f(x)\rangle\rangle \to \langle x^{-m-1/2}g, x^{-m-1/2}f\rangle$ has already been established). For this let us show first.

Lemma 4.5. *The image of* $\tilde{E} = \{\phi;\ \phi x^{2m+1} \in L_0^1$ *under* B *consists of distributions* $B\phi$ *of the form* ($\psi = \phi x^{2m+1}$, $\eta = y^2$, $\xi = x^2$)

(4.17) $$\frac{1}{\sqrt{\eta}}(B\phi)(\sqrt{\eta}) = \frac{\Gamma(1/2)}{\Gamma(m+1)}\left(Y_{-m-1/2} * \frac{\psi\sqrt{\xi}}{\sqrt{\xi}}\right).$$

Proof. The formula (4.12) of Part I for $B\phi$ becomes here

$$(4.18)\qquad (B\phi)(y)=\gamma_m^n y\Big(\frac{1}{y}D_y\Big)^n\int_0^y \psi(x)(y^2-x^2)^{n-m-3/2}dx$$

where $\gamma_m^n=\Gamma(1/2)/2^{n-1}\Gamma(m+1)\Gamma(n-m-1/2)$ and $-1/2<m<n-1/2$. One can take $n>m+3/2$ to insure that everything makes sense for $\psi\in L^1$. Now set $x=\sqrt{\xi}$, $y=\sqrt{\eta}$, and $(1/y)D_y=2D_\eta$ and recall the definitions of the pseudo-functions Y_α from [7] (based on Schwartz [1–43] and Gelfand-Šilov [1–28]). Write (4.18) now as

$$(4.19)\qquad \frac{1}{\sqrt{\eta}}(B\phi)(\sqrt{\eta})=\frac{\Gamma(1/2)D_\eta^n}{\Gamma(m+1)}\int_0^\eta \frac{\psi(\sqrt{\xi})(\eta-\xi)^{n-m-3/2}d\xi}{\sqrt{\xi}\,\Gamma(n-m-1/2)}$$
$$=\frac{\Gamma(1/2)}{\Gamma(m+1)}D_\eta^n\Big(Y_{n-m-1/2}*\frac{\psi\sqrt{\xi}}{\sqrt{\xi}}\Big)$$

and recall that $D(S*T)=DS*T$ with $D_\eta^n Y_{n-m-1/2}=Y_{-n}*Y_{n-m-1/2}=Y_{-m-1/2}$ to obtain (4.17). QED

Next in order to describe the R_n^ω of (4.15) we want to characterize the cosine transforms of $B\hat{E}$ where $B\varphi$ is given by Lemma 4.5. First observe that we already know the answer since $R_n^\nu=R_n^A=\mathbf{P}\delta_n^A$ must coincide with R_n^ω (recall $\langle\,,\rangle_\omega=\langle\,,\rangle_\nu=\langle\,,\rangle_\lambda$ here and compare (4.12) with (3.9)). Hence for $\phi x^{2m+1}=\psi$ one has ($c_m=1/2^m\Gamma(m+1)$)

$$(4.20)\qquad R_n^\nu=\langle\delta_n^A(x),\,\Omega(x,\mu)\rangle=c_m^2\lambda^{2m+1}\int\delta_n(x)H(x,\mu)dx$$
$$=c_m\lambda^{2m+1}\int\delta_n(x)(\lambda x)^{-m}J_m(\lambda x)dx$$

and we want to arrive at this formula without knowledge of the ν pairing or the identification $R_n^\nu=R_n^\omega$. Thus write from (4.15) and (4.17) ($\psi=\phi x^{2m+1}$)

$$(4.21)\qquad R_\phi^\omega(\lambda)=\frac{2}{\pi}\int_0^\infty(B\phi)(y)\,\mathrm{Cos}\,\lambda y\,dy=\frac{1}{\pi}\int_0^\infty(B\phi)(\sqrt{\eta})\,\mathrm{Cos}\,\lambda\sqrt{\eta}\,\frac{d\eta}{\sqrt{\eta}}$$
$$=\frac{\Gamma(1/2)}{\pi\Gamma(m+1)}\int_0^\infty\Big(Y_{-m-1/2}*\frac{\psi(\sqrt{\xi})}{\sqrt{\xi}}\Big)(\eta)\,\mathrm{Cos}\,\lambda\sqrt{\eta}\,d\eta\,.$$

Then writing formally $Y_\alpha*(\psi(\sqrt{\xi})/\sqrt{\xi})=2\int_0^y Y_\alpha(y^2-x^2)\psi(x)dx=2\langle(y^2-x^2)_+^{\alpha-1},\psi(x)\rangle/\Gamma(\alpha)$ ($\alpha=-m-1/2$) and setting $k_m=2\Gamma(1/2)/\pi\Gamma(\alpha)\Gamma(m+1)$ we obtain

$$(4.22)\qquad R_\phi^\omega(\lambda)=k_m\int_0^\infty\langle(y^2-x^2)_+^{\alpha-1},\,\psi(x)\rangle\,\mathrm{Cos}\,\lambda\sqrt{\eta}\,d\eta$$
$$=\frac{k_m\lambda}{\alpha}\int_0^\infty\psi(x)\Big(\int_x^\infty(y^2-x^2)_+^\alpha\,\mathrm{Sin}\,\lambda y\,dy\Big)dx$$

where some basically routine calculation has been omitted in (4.22). Now the last integral in (4.22) can be evaluated by a formula in Bryčkov-Prudnikov [1–3]. Indeed setting $1/2\int_{-\infty}^{\infty} \operatorname{sgn}(y)(y^2-x^2)_+^{\alpha} \operatorname{Sin} \lambda y \, dy = \Xi(x, \lambda)$ we have

$$\Xi(x, \lambda) = \frac{\sqrt{\pi}\,\Gamma(\alpha+1) \operatorname{sgn} \lambda}{2} \left(\frac{|\lambda|}{2x}\right)^{-\alpha-1/2} J_{-\alpha-1/2}(x|\lambda|) . \tag{4.23}$$

Since $\alpha = -m-1/2$ we obtain for $\lambda \geq 0$ the formula

$$R_{\phi}^{\omega}(\lambda) = \tilde{k}_m \int_0^{\infty} \lambda^{m+1} \psi(x) x^{-m} J_m(\lambda x) dx \tag{4.24}$$

where $\tilde{k}_m = k_m \sqrt{\pi}\,\Gamma(-m+1/2)/2^{m+1}(-m-1/2) = c_m$ which has the desired form (4.20). Thus

Lemma 4.6. *The image of $\hat{E}$ under* **Q***B consists of elements of the form* (4.24) ($\psi = \phi x^{2m+1} \in L_0^1$) *and lies in $\hat{\hat{E}}'_{\pi}$. If $\phi_n \to \delta / x^{2m+1}$ via $\psi_n = \delta_n \to \delta$ in $\mathcal{E}'$ then $R_{\phi_n}^{\omega} = R_n^{\omega} \to R_0$ in $\hat{\hat{E}}'_{\pi}$ weakly.*

Proof. The formula (4.24) has been established and the remaining statements can be proved exactly as in the proof of Theorem 4.3. QED

Using the background discussion for Theorem 4.3 with Lemmas 4.5 and 4.6 we can summarize by stating

Theorem 4.7. *The Parseval formula* (4.2) *can be proved via a transmutation $B: P_m \to D^2$ as indicated.*

5. Parseval formulas with singularities and potential. Having "discovered" the ν pairing for $P_m(D)$ via a transmutation with $Q(D) = D^2$ for example let us turn to $P(D) = P_m(D) - q(x)$ and set $Q(D) = P_m(D)$. There will be some interplay here with $\tilde{P}_m(D) = D^2 - (m^2 - 1/4)/x^2$ and we observe that $x^{m+1/2} P_m(D)\psi = \tilde{P}_m(D)\{x^{m+1/2}\psi\}$ ($\tilde{P}_m$ is the form usually studied in quantum mechanics). It will be convenient to use here some results of Braaksma [1–1], Braaksma-deSnoo [1–2], Gasymov [11; 12], Siersma [18], Staševskaya [19; 20], Volk [21], et al., where transmutation kernels connecting $\tilde{P} = \tilde{P}_m - q$ and $\tilde{P}_m$ are constructed using Riemann functions (cf. also [1; 2]). In the present situation where there is a singularity of the same order of magnitude ($1/x^2$) in $\tilde{P}$ and $\tilde{P}_m = \tilde{Q}$ (with the same coefficient) it is possible (for suitable q) to transmute $\tilde{P}$ into $\tilde{Q}$ via formulas $\tilde{B}\tilde{P} = \tilde{Q}\tilde{B}$ with inverse $\tilde{B}^{-1} = \tilde{\mathfrak{B}}$ where

$$\tilde{B}\tilde{f}(y) = \tilde{f}(y) + \int_0^y \tilde{L}(y, x) \tilde{f}(x) dx ; \tag{5.1}$$

$$\tilde{\mathfrak{B}}\tilde{g}(y) = \tilde{g}(x) + \int_0^x \tilde{K}(x, y) \tilde{g}(y) dy .$$

Let us set $L(y,x)=y^{-m-1/2}\tilde{L}(y,x)x^{m+1/2}$ and $K(x,y)=x^{-m-1/2}\tilde{K}(x,y)y^{m+1/2}$ with $\tilde{f}(y)=y^{m+1/2}f(y)$ and $\tilde{g}(x)=x^{m+1/2}g(x)$. Further let $B=y^{-m-1/2}\tilde{B}x^{m+1/2}(x\to y)$ and $\mathcal{B}=x^{-m-1/2}\tilde{\mathcal{B}}y^{m+1/2}(y\to x)$. We are thinking here of P and Q in $E=F=\{f;\, x^{m+1/2}f=\tilde{f}\in L^2\}$. Then (5.1) is equivalent to

$$(5.2)\qquad Bf(y)=f(y)+\int_0^y L(y,x)f(x)dx\,;$$

$$\mathcal{B}g(x)=g(x)+\int_0^x K(x,y)g(y)dy$$

and $BP=QB$ with $B^{-1}=\mathcal{B}$. We emphasize that the transmutation operators exist even though the spectra of P and Q are not the same.

REMARK 5.1. There are various hypotheses on $q(x)$ which are used in literature mentioned above (cf. also Chadan-Sabatier [1–17] for hypotheses in physics). Regarding behavior near $x=0$ we mention for example Siersma [18] where it is assumed that: $n-1/2<\mathrm{Re}\,m<n+1/2$ (for $m\neq 0$) or $m=n+1/2$; $M=\max(2,n)$; $\alpha>0$ (where in addition $\alpha>3/2-\mathrm{Re}\,m$ for $n=1$ and $\alpha>1/2-\mathrm{Re}\,m$ for $n=0$); and $q\in C^M(0,a]$ with $D^kq(x)=0(x^{\alpha-k-1})$ as $x\to 0$ for $0\le k\le M$. Then (working on $[0,a]$, $a<\infty$ arbitrary) there exists a continuous $\tilde{L}(y,x)$ such that $\tilde{B}$ given by (5.1) is a transmutation operator $\tilde{P}\to\tilde{Q}$ in L^2. The domain of $\tilde{P}$ and $\tilde{Q}$ involves here $\tilde{f}\sim x^{m+1/2}(1+0(1))$ and $D_x(x^{-m-1/2}\tilde{f}(x))\sim 0(x^{-\gamma})$ as $x\to 0$ where $\gamma=1+\mathrm{Re}\,m-|\mathrm{Re}\,m|$ ($\gamma=1$ for $\mathrm{Re}\,m\ge 0$). We prefer to leave $D(\tilde{P})$ and $D(\tilde{Q})$ unspecified in noting that various realizations are possible ($x^{-m-1/2}\tilde{f}(x)\sim 0(1)$ is retained however). Further $|\tilde{L}(y,x)|\le Ky^{\alpha}(x/y)^{\mathrm{Re}\,m+1/2}$ $(0\le x\le y)$ so that (taking m real for simplicity) $|\tilde{L}(y,x)x^{-m-1/2}|\le Ky^{\alpha-m-1/2}$ and it will make sense to talk about $\tilde{l}(y)=\lim\tilde{L}(y,x)x^{-m-1/2}$ as $x\to 0$; $\tilde{l}(y)$ will come up later in our development as it does in Gasymov [11; 12]. For smoothness, Gasymov [11; 12] takes $l=m-1/2$ integral and assumes $q(x)$ has l locally summable derivatives with $q^{(l)}\in L^2_{\mathrm{loc}}$ in which case, in particular, it follows that $L_1(y,x)=y^l\tilde{L}(y,x)x^{-l-1}=y^{m-1/2}\tilde{L}(y,x)x^{-m-1/2}$ $(0\le x\le y)$ has $l+1$ locally summable derivatives and $\tilde{L}$ has l continuous derivatives. Here $\tilde{L}(y,x)x^{-l-1}=y^{-l}L_1(y,x)$ so $\tilde{l}(y)=y^{-l}L_1(y,0)$ and $L_1(y,x)$ is continuous in (y,x) down to $x=0$. Staševskaya [19] assumes $\int_0^a|xq(x)|^{2+\varepsilon}dx<\infty$ but the results are generally weaker regarding properties of $\tilde{L}$. Volk [21] assumes $q\in C^0[0,a]$ which is stronger than necessary. The behavior of $q(x)$ at ∞ does not play a role in constructing $\tilde{L}$ or $\tilde{K}$ since basically we are dealing with hyperbolic problems having compact domains of dependence. It will come up later however in [6] when we consider Jost solutions and the Marčenko equation; it also plays a role in determining the number of bound states (i.e. eigenvalues).

We go now to the construction of Parseval formulas based on the technique indicated in Remark 3.3, so this can be considered as an extension of Marčenko's procedure. Along the way we will indicate some of Gasymov's development for comparison (where $l-m-1/2$ is an integer). Recall that we are basically interested in the "structure" of such theorems and will confine our attention to a singular term $(m^2-1/4)/x^2$ in $\tilde{P}(D)$. The machinery extends then in an obvious manner to other operators with comparable singularity (in particular to $P(D)u=(Au')'/A$) and indicates a "canonical" direction for more general operators. Moreover in view of the various types of detail available (see e.g. Remark 5.1) for different hypotheses on q we do not make an explicit choice of such hypotheses and only note when necessary that the properties we want are available for suitable q. In this section then $\tilde{P}(D)=\tilde{P}_m(D)-q(x)$ (q suitable) and $\tilde{Q}(D)=\tilde{P}_m(D)$ so we write out

$$\Theta(y,\mu)=c_m^{-1}(\lambda y)^{-m}J_m(\lambda y)\ ; \tag{5.3}$$
$$W(y,\mu)=c_m^2(\lambda y)^{2m+1}\Theta(y,\mu)$$

($c_m=1/2^m\Gamma(m+1)$). We will assume m real, $m\geq-1/2$, but $q(x)$ may be complex; m complex, $\operatorname{Re} m>-1/2$, could be included but we omit this for convenience. The function $H(x,\mu)$ is now a solution of $P(D)H=\mu H$, $H(0,\mu)=1$, $H'(0,\mu)=0$ whose explicit form is not known. Thus we are obliged to work with a ω pairing (ν is unknown) and think of $U_n(x,y)$ in the form (4.12) ($d\omega=d\lambda$) with $R_n^\omega(\lambda)=\mathbf{Q}B\varphi_n$ ($\phi_n=\delta_n^A$); we only need $BH=\Theta$—no spectral comparison is required. From the previous development we know here that $\phi_n(x)=\delta_n^A(x)=\delta_n(x)/x^{2m+1}$ with $\delta_n\to\delta$ is the right kind of object to introduce in dealing with the Parseval formula for $P(D)$. Now take B in the form (5.2) so that

$$\begin{aligned} B\varphi_n(y) &= \varphi_n(y)+\int_0^y L(y,x)\varphi_n(x)dx \\ &= \varphi_n(y)+\int_0^y y^{-m-1/2}\tilde{L}(y,x)x^{-m-1/2}\delta_n(x)dx. \end{aligned} \tag{5.4}$$

Then formally as $\delta_n\to\delta$ and $\varphi_n\to\varphi=\delta/x^{2n+1}$ we have $B\varphi_n\to B\varphi$ where

$$B\varphi(y)=\varphi(y)+\tilde{l}(y)y^{-m-1/2} \tag{5.5}$$

(cf. Remark 5.1 for $\tilde{l}(y)$). Hence formally $R_n^\omega=\mathbf{Q}B\varphi_n\to R=\mathbf{Q}B\varphi$ with (cf. (5.3) and (3.8))

$$\begin{aligned} R(\lambda) &= \langle W(y,\mu), B\varphi(y)\rangle \\ &= c_m^2\lambda^{2m+1}+c_m^2\lambda^{2m+1}\int_0^\infty y^{m+1/2}\tilde{l}(y)\Theta(y,\mu)dy \end{aligned} \tag{5.6}$$

Thus $R=R_0+R_q$ where R_q measures the effect of q.

REMARK 5.2. Note that $R(\lambda)$ could have genuine distribution components arising from R_q. For example a conceivable $\tilde{l}$ is $\tilde{l}(y)=D_a^p[(ay)^{1/2}J_m(ay)]$ in which case (cf. [23])

$$(5.7)\qquad R_q = c_m\lambda^{m+1/2}\int_0^\infty \tilde{l}(y)(\lambda y)^{1/2}J_m(\lambda y)dy = c_m\lambda^{m+1/2}(-1)^p\delta^{(p)}(\lambda-a)\,.$$

Now to model a Parseval formula on the procedure of Remark 3.3 we take (4.12), multiply it by suitable $f, g\in \boldsymbol{E}$, and integrate to obtain

$$(5.8)\qquad \langle g(y), \langle U_n(x, y), f(x)\rangle\rangle = \langle R_n^\omega(\lambda), \rho f(\lambda)\rho g(\lambda)\rangle_\omega\,.$$

Here f, g will have to be selected so that $\rho f\rho g\in W\subset \hat{\hat{\boldsymbol{E}}}_\pi$ with $R_n^\omega\to R$ in W' ($\hat{\hat{\boldsymbol{E}}}_\pi$ itself will not do in general since $R\notin \hat{\hat{\boldsymbol{E}}}'_\pi$—cf. Remark 5.2). The correct spaces W were found by Gasymov [11; 12] and are defined below (W is analogous to the Z of Remark 3.3). Then a version of Lemma 4.2 (i.e. $U_n(x, y)=T_x^y\phi_n(x)\to T_x^y\phi(x)=\delta(x-y)/x^{2m+1}$ for $T_x^y\sim P$) must be obtained in order to get (4.7). Alternatively a version of Theorem 3.6 can be envisioned and we remark that the arguments used in proving Theorem 3.6 remain valid, given a ν pairing with suitable Ω. Thus (in passing).

Theorem 5.3. *Assume there is a ν pairing with $\Omega(x, \mu)=k_m(\lambda x)^{2m+1}H(x, \mu)$ and $\delta(x)=\langle\Omega(x, \mu), 1\rangle_\nu$. Let $f, g\in \boldsymbol{E}$, $f=x^{2m+1}\check{f}$, and $g=x^{2m+1}\check{g}$ ($\check{f}, \check{g}\in E$). Then* (3.13) *holds (i.e. $\langle T_x^y\check{f}, g\rangle=\langle f, T_x^y\check{g}\rangle$).*

Proof. Existence and uniqueness theorems for $P(D_x)U=P(D_y)U$, $U(x,0)=f(x)$, $U_y(x, 0)=0$ follow from [18; 1–1; 1–2] and determine $U(x, y)=T_x^yf(x)$ (cf. also [11; 12; 19; 20]). One takes H as indicated above ($P(D)H=\mu H$, $H(0, \mu)=1$, $H'(0, \mu)=0$) and the ν pairing with $\langle\Omega(x, \mu), 1\rangle_\nu=\delta(x)$ is assumed so by Part I $\mathsf{P}=\mathsf{P}^{-1}$, etc. Here $P^*(D)\Omega=\mu\Omega$ with $P^*(D)$ the real formal adjoint $P^*(D)\Omega=\Omega''-(2m+1)(\Omega/x)'-q(x)\Omega$. The explicit $\Omega=k_m(\lambda x)^{2m+1}H$ is used to obtain (3.18)–(3.19). QED

Now the main ingredient used in proving Lemma 4.2 was the fact that, for the $T_x^y\sim P_m(D)$, $T_x^y\colon \mathcal{E}^0(\boldsymbol{R}_+^1)\to\mathcal{E}^0(\boldsymbol{R}_+^2)$ was continuous and this will hold also for the T_x^y associated with $P(D)$. For example if $m>-1/2$ ($m\neq 0$) and say $q\in C^0[0, a]$ one obtains from Siersma [18]

$$(5.9)\qquad T_x^yf(x) = \int_{x-y}^{x+y}\beta(x, y, \xi)f(\xi)d\xi$$

where β is continuous and for $x-y<\xi<x+y$ $(0<y\le x)$ there is a bound $|\beta(x, y, \xi)|\le M(\xi/x)^{m+1/2}y^{-2m}[y^2-(x-\xi)^2]^{m-1/2}$. However observe that

$$(5.10)\qquad \int_{x-y}^{x+y}\left(\frac{\xi}{x}\right)^{m+1/2} y^{-2m}[y^2-(x-\xi)^2]^{m-1/2}d\xi$$
$$= \frac{1}{y}\int_{-y}^{y}\left(1-\frac{z}{x}\right)^{m+1/2}\left(1-\frac{z^2}{y^2}\right)^{m-1/2}dz$$
$$\leq \frac{2^{m+3/2}}{y}\int_0^y\left(1-\frac{z^2}{y^2}\right)^{m-1/2}dz = K_m$$

since $1-\frac{z}{x}\leq 2$ and $\int_0^y\left(1-\frac{z^2}{x^2}\right)^{m-1/2}dz = y\int_0^1(1-\eta^2)^{m-1/2}d\eta = yB(1/2, m+1/2)/2 = y\Gamma(1/2)\Gamma(m+1/2)/2\Gamma(m+1)$ (cf. [1–36]); thus $K_m = 2^{m+1/2}\Gamma(1/2)\Gamma(m+1/2)/\Gamma(m+1)$. Consequently T_x^y given by (5.9) maps $L^\infty(\boldsymbol{R}_+^1)\to L^\infty(\boldsymbol{R}_+^2)$ and $\mathcal{E}^0(\boldsymbol{R}_+^1)\to \mathcal{E}^0(\boldsymbol{R}_+^2)$ continuously. The argument of Lemma 4.2 can then be repeated to obtain

Lemma 5.4. *The formula* (4.5) *holds for* $T_x^y \sim P(D)$, $f\in E$, *and* $g\in\mathcal{E}^0$ *and* (4.6) *determines* $x^{2m+1}T_x^y(\delta(x)/x^{2m+1})$.

Hence, as in the proof of Theorem 4.3, the calculation based on (3.11)–(3.12) is valid for $f, g\in \boldsymbol{E}$ and leads to (4.7). It remains to examine the convergence $R_n^\omega\to R$ (cf. (5.6)). At this point we will introduce some spaces utilized by Gasymov [11]. Recall first from Remark 3.3 that $K^2(\sigma)$ denotes L^2 functions vanishing for $x>\sigma$ ($L^2=L^2(0,\infty)$) and set $K^2=\cup K^2(\sigma)$. For f such that $x^{-m-1/2}f(x)\in K^2(\sigma)$ consider

$$(5.11)\qquad F(\lambda) = \mathcal{Q}f(\lambda) = \langle f(x), \Theta(x,\mu)\rangle = \tilde{f}(\lambda)$$
$$= c_m^{-1}\int_0^\infty f(x)(\lambda x)^{-m}J_m(\lambda x)dx\,.$$

Gasymov calls $c_mF(\lambda)$ the Fourier-Bessel transform and notes that $F\in W_m^2$ where (cf. Remark 5.9)

DEFINITION 5.5. Let W_m^2 be the space of even entire functions satisfying a) $|F(\lambda)|\leq c|\lambda|^{-m-1/2}\exp\sigma|\operatorname{Im}\lambda|$ for $|\lambda|$ large (some σ—here σ is related to f) and also b) $\int_0^\infty|\lambda|^{2m+1}|F(\lambda)|^2d\lambda<\infty$. One says that a sequence $F_n(\lambda)\to 0$ in W_m^2 if a) holds for a fixed σ in the form $|F_n(\lambda)|\leq c\exp\sigma|\operatorname{Im}\lambda|$ and $\int_0^\infty|F_n(\lambda)|^2\lambda^{2m+1}d\lambda\to 0$. Let W_m^1 denote even entire functions satisfying a) $|F(\lambda)|\leq|\lambda|^{-2m}\exp\sigma|\operatorname{Im}\lambda|$ for $|\lambda|$ large (some σ) and b) $\int_0^\infty|\lambda|^{2m+1}|F(\lambda)|d\lambda<\infty$. A sequence $F_n(\lambda)\to 0$ in W_m^1 if $|F_n(\lambda)|\leq c\exp\sigma|\operatorname{Im}\lambda|$ for a fixed σ and $\int_0^\infty|F_n(\lambda)|\lambda^{2m+1}d\lambda\to 0$.

We note that if $F\in W_m^1$ then F is bounded for λ real so $|F(\lambda)|^2\lambda^{2m+1}\leq c|F(\lambda)|\lambda^{2m+1}$ and $F\in W_m^2$ will follow. W_m^1 will serve as the space W alluded to earlier. From [11] then we have

Lemma 5.6. *$W_m^1 \subset W_m^2$ is dense and $F, G \in W_m^2$ implies $FG \in W_m^1$.*

Now recall the format of Remark 3.3 and observe the difference in notation $\breve{f} = \mathcal{B}^* f$ and $f = x^{2m+1}\breve{f}$. Let us proceed (up to a point) as in Remark 3.3. From (5.2) we have $\mathcal{B}^* f(y) = f(y) + \int_y^\infty K(x,y) f(x)dx$. If $f \in \boldsymbol{F}(=\boldsymbol{E})$ and $f(x)x^{-m-1/2} \in K^2$ then from (5.11) $\mathcal{Q}f = F = \breve{f} \in W_m^2$ and $f = \boldsymbol{Q}F$. If $F \in W_m^1$ and $f = \boldsymbol{Q}F$ we say $x^{-m-1/2} f \in K_m^1$; in this event supp f is compact (cf. [11]). Now let $f \in \boldsymbol{E} = E'$ so $\breve{f} = \mathcal{B}^* f \in \boldsymbol{F} = F'$ and supp $f \subset [0, \sigma]$ implies supp $\breve{f} \subset [0, \sigma$; consequently $x^{-m-1/2} f \in K^2(\sigma)$ implies that $x^{-m-1/2}\breve{f} \in K^?(\sigma)$. By Theorem 2.3 $\mathcal{Q}\breve{f} = \rho f \in W_m^2$ (similarly $\mathcal{Q}\breve{g} = \rho g \in W_m^2$); hence $\rho f \rho g \in W_m^1$ by Lemma 5.6. Recall now (cf. (4.12)) $U_n(x, y) = T_x^y \varphi_n(x) = \langle R_n^\omega, H(x, \mu)H(y, \mu)\rangle_\omega$ so that $\varphi_n(x) = \langle R_n^\omega, H(x, \mu)\rangle_\omega$ and $B\varphi_n(y) = \langle R_n^\omega, \Theta(y, \mu)\rangle_\omega = \mathbf{Q}R_n^\omega(y)$ $(R_n^\omega = \mathbf{Q}B\varphi_n)$. Again we will have an equation (5.8) of the form $\langle f(y), \langle U_n(x, y), g(x)\rangle\rangle = \langle R_n^\omega, \rho f \rho g\rangle_\omega$ and the left side tends to $\langle y^{-m-1/2} f, y^{-m-1/2} g\rangle$ (i.e. to (4.7)—using Lemma 5.4 for $\breve{g} \in \mathcal{E}^0$ and then passing to $\breve{g} \in L^2$ as in the proof of Theorem 4.3). Consider now a function $H \in W_m^1$, $H = \mathcal{Q}h$, so that $x^{-m-1/2} h \in K_m^1$ and in the formula $\langle R_n^\omega, \Theta\rangle_\omega = \mathbf{Q}R_n^\omega = B\varphi_n = \varphi_n(y) + \int_0^y L(y, x)\varphi_n(x)dx$ (cf. (5.4)) multiply by $h(y)$ to obtain

$$\text{(5.12)} \qquad \begin{aligned} \langle R_n^\omega(\lambda), H(\lambda)\rangle_\omega &= \langle \varphi_n(y), h(y)\rangle + \langle h(y), \int_0^y L(y, x)\varphi_n(x)dx\rangle \\ &= \langle \delta_n(y), y^{-2m-1}h(y)\rangle + \langle y^{-m-1/2}h(y), \int_0^y \tilde{L}[(y, x)x^{-m-1/2}]\delta_n(x)dx\rangle . \end{aligned}$$

We can suppose supp $\delta_n(x) \subset [0, 1/n]$ for example and all the terms in (5.12) make sense (recall from Remark 5.1 that $\tilde{L}(y, x)x^{-m-1/2} = \tilde{l}(y, x)$ is continuous in x with $\tilde{l}(y, x) \to \tilde{l}(y)$ as $x \to 0$—also supp h is compact). In this respect we note that since$h = \boldsymbol{Q}H$ one has

$$\text{(5.13)} \qquad \begin{aligned} h(y)y^{-2m-1} &= \langle W(y, \mu), H(\lambda)\rangle / y^{2m+1} \\ &= c_m^2 \int_0^\infty \lambda^{2m+1} H(\lambda)\Theta(y, \mu)d\lambda . \end{aligned}$$

But for $H \in W_m^1$, $\lambda^{2m+1}H(\lambda) \in L^1$ and since $z^{-m}J_m(z) = \alpha_m \int_0^{\pi/2} \operatorname{Cos}(z \operatorname{Cos}\theta) \operatorname{Sin}^{2m}\theta d\theta$ for $\alpha_m = 2^{1-m}/\sqrt{\pi}\,\Gamma(m+1/2)$ (cf. [22]) it follows that $|\Theta(y, \mu)| \leq c(m)$ and consequently $\Theta(y, \cdot) \in L_\lambda^\infty$. Therefore $h(y)y^{-2m-1}$ is well defined (and continuous) with

$$\text{(5.14)} \qquad \lim_{y \to 0} h(y)/y^{2m+1} = \lim_{n \to \infty} \langle \varphi_n(y), h(y)\rangle = c_m^2 \int_0^\infty \lambda^{2m+1} H(\lambda)d\lambda .$$

In particular we can define $R_0 \in (W_m^1)'$ by (cf. [11])

$$\langle R_0, \bar{h}\rangle = \langle c_m^2, \lambda^{2m+1}, \bar{h}(\lambda)\rangle = \lim_{y\to 0} h(y)/y^{2m+1} \tag{5.15}$$

(clearly $R_0 = c_m^2\lambda^{2m+1} \in (W_m^1)'$). Now from (5.13) if a sequence $H_p \to 0$ in W_m^1 then $h_p(y)/y^{2m+1} \to 0$ in L^∞ say. Hence for $\psi \in \hat{E} = \{\psi;\, x^{2m+1}\psi \in L_0^1\}$ we have $\langle \psi, h_p\rangle = \langle x^{2m+1}\psi, h_p/x^{2m+1}\rangle \to 0$. Thus $h \in \hat{E}'$ and the map $H \to h\colon W_m^1 \to \hat{E}'$ is continuous (sequential limits as indicated in Definition 5.5 are quite sufficient here). In particular the first term in (5.12) is well defined for $\varphi_n \in \hat{E}$ and one can determine then $R_0^n\colon \bar{h} \to \langle \varphi_n, h\rangle \in (W_m^1)'$ which we write as

$$\langle R_0^n, \bar{h}\rangle = \langle \varphi_n, h\rangle \tag{5.16}$$

(cf. [11] for an essentially equivalent version). In view of (5.15) we have then $R_0^n \to R_0$ in $(W_m^1)'$ weakly.

Theorem 5.7. *One can write $R_n^\omega = R_0^n + R_q^n$ in* (5.12) *and $R_n^\omega \to R_0 + R_q = R$ weakly in $(W_m^1)'$ where R is given by* (5.6) *as $R = R_0 + R_q$, $R_0 = c_m^2\lambda^{2m+1}$, $\langle R_q, \bar{h}\rangle = \int_0^\infty h(y)y^{-m-1/2}\tilde{l}(y)dy$ for $\bar{h} \in W_m^1$ ($\bar{h} = \mathcal{Q}h$) and formally, as a distribution,*

$$R_q = c_m^2\lambda^{2m+1}\int_0^\infty y^{m+1/2}\tilde{l}(y)\Theta(y, \mu)dy\,. \tag{5.17}$$

Proof. Writing $y^{m+1/2}\tilde{L}(y, x)x^{-m-1/2} = y^{m+1/2}\tilde{l}(y, x)$ the remaining term in (5.12) becomes

$$\Xi_n = \langle h(y)y^{-2m-1}, \int_0^y y^{m+1/2}\tilde{l}(y, x)\delta_n(x)dx\rangle\,. \tag{5.18}$$

Now as noted in Remark 5.1 it is appropriate to assume $y^{m+1/2}\tilde{l}(y, x) \le K$ for $0 \le x \le y$ ($y \le \sigma$ say) so we write $y^{m+1/2}\tilde{l}(y, x) \in L_{\text{loc}}^\infty$; here supp $h \subset [0, \sigma]$ will be compact and one can assume $y \le \sigma$ in this discussion. The function $\psi_n(y) = \int_0^y y^{m+1/2}\tilde{l}(y, x)\delta_n(x)dx$ is continuous since $\delta_n \in L^1$ and $L_1(y, x) = y^{m-1/2}\tilde{l}(y, x)$ is continuous in (y, x) (cf. Remark 5.1). Hence if $H_p = \bar{h}_p \to 0$ in W_m^1 then as above $h_p(y)/y^{2m+1} \to 0$ in L^∞ so there exists $R_q^n \in (W_m^1)'$ such that $\Xi_n = \langle h(y)y^{-2m-1}, \psi_n(y)\rangle = \langle R_q^n, \bar{h}\rangle$. Now as $n \to \infty$ $\psi_n(y) \to \psi(y) = y^{m+1/2}\tilde{l}(y)$ pointwise boundedly—hence in L^1 by dominated convergence—and therefore $\langle R_q^n, \bar{h}\rangle = \Xi_n \to \langle h(y)y^{-2m-1}, y^{m+1/2}\tilde{l}(y)\rangle = \Xi$. But as above $\Xi = \langle R_q, \bar{h}\rangle$ ($y^{m+1/2}\tilde{l}(y)$ is continuous) and we have then $R_q^n \to R_q$ in $(W_m^1)'$ weakly. As explicit formula for R_q^n is unnecessary and for R_q formally the last expression in (5.6) is required. We note in this respect that given $\bar{h} \in W_m^1$ with supp $h \subset [0, \sigma]$, where (by (5.13)) $h(y)y^{-2m-1} = c_m^2\int_0^\infty \lambda^{2m+1}\bar{h}(\lambda)\Theta(y, \mu)d\lambda$, we have formally (cf. also (5.6) and (3.8))

$$(5.19)\quad \langle R_q, \bar{h}\rangle = \int_0^\infty h(y)y^{-2m-1}y^{m+1/2}\tilde{l}(y)dy$$
$$= c_m^2\int_0^\infty\left(\int_0^\infty \lambda^{2m+1}\bar{h}(\lambda)\Theta(y,\mu)d\lambda\right)y^{m+1/2}\tilde{l}(y)dy$$
$$= c_m^2\int_0^\infty \lambda^{2m+1}\bar{h}(\lambda)\left(\int_0^\infty y^{m+1/2}\tilde{l}(y)\Theta(y,\mu)dy\right)d\lambda$$

which gives (5.17). We emphasize that in equation (5.19) in general R_q is a distribution. Q.E.D.

Thus for $f, g\in \boldsymbol{E}$ with $x^{-m-1/2}f$ and $x^{-m-1/2}g\in K$ we have proved the Parseval formula

$$(5.20)\quad \langle y^{-m-1/2}f(y), y^{-m-1/2}g(y)\rangle = \langle R, \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\omega$$

for $R\in W'=(W_m^1)'$ (this coincides in form with (3.1) since $d\omega=d\lambda$). We state this formally as

Theorem 5.8. *Let $f, g\in \boldsymbol{E}$ with compact supports. Then there exists a generalized spectral function $R=R_0+R_q\in W'=(W_m^1)'$, where $R_0=c_m^2\lambda^{2m+1}$ and R_q is determined by Theorem* 5.7, *such that the Parseval formula* (5.20) *holds* $(d\omega=d\lambda)$.

REMARK 5.9. In Chebli [1–18; 1–19] operators of the form $P(D)u=(Au')'/A-q(x)u$ are considered for real A and q with A'/A generally of the form a/x near $x=0$ and various hypotheses on q at 0 and ∞ (cf. also [14] and [1–25] for special A with $q=0$). Paley-Wiener type theorems are obtained there using analyticity properties of transforms $\mathcal{P}f(\lambda)$, $\boldsymbol{P}f(\lambda)$, etc. and the analysis there should lead to the construction of suitable spaces W for general Parseval formulas (as in Definition 5.5). In particular (cf. Remark 4.1 in Part I) given a spectral measure $d\nu=\hat{\nu}^2(\lambda)d\lambda$ for the principal part $(Au')'/A$ of $P(D)u$ the function $\hat{\nu}^2(\lambda)$ should play the role of the weight function λ^{2m+1} in W_m^2 or W_m^1. The technique of utilizing $\mathcal{Q}\breve{g}=\mathcal{P}g$ for $\breve{g}=\mathcal{B}^*g$ (cf. Theorem 2.3), which we extracted from Marčenko [16], is also used in Koornwinder [14] for studying Paley-Wiener type theorems and this is analyzed in Carroll-Gilbert [8; 9].

6. The Gelfand-Levitan equation.

We will give a sketch here of Gasymov's proof of the Parseval formula in [11] since it can be recast in our framework in a meaningful way and brings the Gelfand-Levitan equation into the picture (the Marčenko equation will be studied in [6]). The discussion will be formal in general but precision can easily by supplied following Sections 2–5. First one observes that if there is a Parseval formula of the form (5.20) say, then, for $f_i\in \boldsymbol{E}$, $\tilde{f}_i=x^{-m-1/2}f_i\in K^2$, $F_i=\mathcal{P}f_i=\mathcal{Q}\mathcal{B}^*f_i\in W_m^2$, and $F_1F_2\in W_m^1$ with $f=\boldsymbol{Q}(F_1F_2)$, the action of R is specified formally in $(W_m^1)'$ by the rule

$$(6.1)\qquad \langle R, F_1F_2\rangle_\omega = \lim_{y\to 0}\frac{f(y)}{y^{2m-1}} + \int_0^\infty y^{-m-1/2}f(y)\tilde{l}(y)dy\,.$$

The point here is to deduce this without recourse to T_x^y and then to show that this formal stipulation allows us to determine R.

Lemma 6.1. *Given* (5.20) *and* $\tilde{l}(y)=\lim \tilde{L}(y, x)x^{-m-1/2}$ *as before it follows that* (6.1) *holds formally and describes the action of* R *on* F_1F_2.

Proof. Note that $\langle R, \mathcal{P}f(\lambda)\mathcal{P}g(\lambda)\rangle_\omega=\langle R, \int f(x)H(x,\mu\mu)dx \int g(y)H(y,\mu)dy\rangle_\omega$ $=\iint f(x)g(y)\langle R, H(x, \mu)H(y, \mu)\rangle_\omega dxdy$ so formally $\langle R, H(x, \mu)H(y, \mu)\rangle_\omega=$ $\delta(y-x)/y^{2m+1}$ (equivalently $\delta(x-y)/x^{2m+1}$). Now recall $\Theta=BH$ and take B in the form (5.2) so $H(y, \mu)=\Theta(y, \mu)-\int_0^y L(y, t)H(t, \mu)dt$. Put this in the expression for $\langle R, HH\rangle_\omega$ to obtain

$$\begin{aligned}(6.2)\qquad \langle R, H(x, \mu)\Theta(y, \mu)\rangle_\omega &= \delta(y-x)/y^{2m+1}\\ &\quad +\langle R, H(x, \mu)\int_0^y L(y, t)H(t, \mu)dt\rangle\\ &= \frac{\delta(y-x)}{y^{2m+1}}+\int_0^y L(y, t)\frac{\delta(t-x)}{t^{2m+1}}dt\\ &= \frac{\delta(y-x)}{y^{2m+1}}+y^{-m-1/2}\tilde{L}(y, x)x^{-m-1/2}\,.\end{aligned}$$

Let now $x\to 0$ in (6.2) to obtain

$$(6.3)\qquad \langle R, \Theta(y, \mu)\rangle_\omega = \frac{\delta(y)}{y^{2m+1}}+y^{-m-1/2}\tilde{l}(y)\,.$$

Multiply (6.3) now by f as in (6.1) with $F=\mathcal{Q}f$; this gives (6.1) upon integration (with $F=F_1F_2$). Q.E.D.

Equation (6.1) shows what R must do acting on F_1F_2 and we now refer to Section 5 to confirm that there is an element $R=R_0+R_q\in(W_m^1)'$ which fulfills this. Thus by (5.15) $\langle R_0, \bar{h}\rangle=\lim H)y)/y^{2m+1}$ and as in (9.15) $\langle R_q, \bar{h}\rangle=$ $\int_0^\infty h(y)y^{-m-1/2}\tilde{l}(y)dy$. Hence

Lemma 6.2. *The formal requirement* (6.1) (*with* $d\omega=d\lambda$) *is fulfilled by choosing* $R=R_0+R_q\in(W_m^1)'$ *and this determines* R.

Since $F_i=\mathcal{Q}g_i$ for $g_i=\mathcal{B}^*f_i$ the Parseval formula for Q transforms derived in Section 4 allows us to say that

$$(6.4)\qquad \lim_{y\to 0}\frac{f(y)}{y^{2m+1}} = c_m^2\int_0^\infty F_1(\lambda)F_2(\lambda)\lambda^{2m+1}d\lambda = \langle x^{-m-1/2}g_1(x),\ x^{-m-1/2}g_2(x)\rangle\,.$$

Let us write out this last term using the relation $g_i(x)=f_i(x)+\int_x^\infty K(\xi, x)f_i(\xi)d\xi$ to get $\int_0^\infty x^{-2m-1}g_1(x)g_2(x)dx=\int_0^\infty x^{-2m-1}f_1(x)f_2(x)dx+\int_0^\infty x^{-2m-1}f_1(x)\cdot\left(\int_x^\infty K(\xi, x)\times f_2(\xi)d\xi\right)dx+\int_0^\infty x^{-2m-1}f_2(x)\left(\int_x^\infty K(\xi, x)f_1(\xi)d\xi\right)dx+\int_0^\infty x^{-2m-1}\cdot\left(\int_x^\infty K(\xi, x)f_1(\xi)d\xi\right)\times\left(\int_x^\infty K(\eta, x)f_2(\eta)d\eta\right)dx=\int_0^\infty x^{-2m-1}f_1(x)f_2(x)dx+I_1$ (we will compress some calculations here). Next set $I_2=\langle R-R_0, F_1F_2\rangle=\int_0^\infty y^{-m-1/2}f(y)\tilde{l}(y)dy=\int_0^\infty [\tilde{l}(y)/y^{m+1/2}]\boldsymbol{Q}\{\mathcal{Q}[f_1(x)+\int_x^\infty K(\xi, x)f_1(\xi)d\xi]\cdot\mathcal{Q}[f_2(x)+\int_x^\infty K(\eta, x)f_2(\eta)d\eta]\}$ dy where $\mathcal{Q}g_i(\lambda)=\int_0^\infty g_i(x)\Theta(x, \mu)dx$ and $\boldsymbol{Q}F(y)=\int_0^\infty F(\lambda)W(y, \mu)d\lambda$. We consider the term

$$\begin{aligned}(6.5)\qquad &\int_0^\infty \tilde{l}(y)y^{-m-1/2}\int_0^\infty W(y, \mu)\mathcal{Q}f_1(\lambda)\mathcal{Q}f_2(\lambda)d\lambda dy\\ &=\int_0^\infty \tilde{l}(y)y^{-m-1/2}\int_0^\infty W(y, \mu)\int_0^\infty f_1(x)\Theta(x, \mu)dx\int_0^\infty f_2(\xi)\Theta(\xi, \mu)d\xi d\lambda dy\\ &=\int_0^\infty\int_0^\infty f_1(x)f_2(\xi)\int_0^\infty\int_0^\infty \frac{\tilde{l}(y)}{y^{m+1/2}}W(y, \mu)\Theta(x, \mu)\Theta(\xi, \mu)dyd\lambda d\xi dx\end{aligned}$$

and set formally ($W(y, \mu)=c_m^2(\lambda y)^{2m+1}\hat{R}^m(y, \lambda)$ and $\Theta(x, \mu)=c_m^{-1}(\lambda x)^{-m}J_m(\lambda x)=\hat{R}^m(x, \lambda)$)

$$\begin{aligned}(6.6)\qquad F(x, \xi) &=\int_0^\infty\int_0^\infty \frac{\tilde{l}(y)}{y^{m+1/2}}W(y, \mu)\Theta(x, \mu)\Theta(\xi, \mu)dyd\lambda\\ &=c_m^{-1}x^{-m-1/2}\int_0^\infty(\lambda x)^{1/2}\left[\int_0^\infty\tilde{l}(y)(\lambda y)^{1/2}J_m(\lambda y)dy\right]J_m(\lambda x)\frac{J_m(\lambda\xi)}{(\lambda\xi)^m}d\lambda\,.\end{aligned}$$

When $\xi\to 0$, $c_m^{-1}(\lambda\xi)^{-m}J_m(\lambda\xi)\to 1$ and we have

$$(6.7)\qquad F(x, \xi)\to x^{-m-1/2}\boldsymbol{H}_m[\boldsymbol{H}_m[\tilde{l}(y)]]=\frac{\tilde{l}(x)}{x^{m+1/2}}\,.$$

Further (formally) $Q(D_x)F=Q(D_\xi)F$ so that we can write

$$(6.8)\qquad F(x, \xi)=S_x^\xi[\tilde{l}(x)x^{-m-1/2}]$$

where S is the generalized translation associated with Q (cf. Part I). Since $F(t, \tau)=F(\tau, t)$, I_2 can be written (with (6.5) as a model) $I_2=\iint f_1(x)f_2(\xi)F(x, \xi)\times dxd\xi+\iint f_1(x)f_2(\xi)\int_0^\xi K(\xi, s)F(s, x)dsdxd\xi+\iint f_1(x)f_2(\xi)\int_0^x K(x, s)F(s, \xi)dsdxd\xi+\iint f_1(x)f_2(\xi)\int_0^x\int_0^\xi K(x, t)K(\xi, \tau)F(\tau, t)dtd\tau dxd\xi\,.$

In order to deal with I_2 further we need a few facts relating K and L.

First recal (using (5.2)) $H(x, \mu)=\mathcal{B}\theta=\theta(x, \mu)+\int_0^x K(x, t)\theta(t, \mu)dt$ and $\theta(y, \mu)=BH=H(y, \mu)+\int_0^y L(y, \xi)H(\xi, \mu)d\xi$. The relation $\mathcal{Q}\mathcal{Z}=I$ also says that $\langle W(y, \mu), \theta(x, \mu)\rangle_\omega=\delta(x-y)$. Writing out $\theta(y, \mu)=BH=B\mathcal{B}\theta=(\mathcal{B}\theta)(y, \mu)+\int_0^y L(y, \xi)(\mathcal{B}\theta)(\xi, \mu)d\xi$ one obtains then (since $K(\xi, y)=0$ for $y>\xi$)

$$(6.9) \qquad K(x, y)+L(x, y)+\int_y^x L(x, \xi)K(\xi, y)d\xi = 0 .$$

We will state the next relation as a theorem because of its general importance. Equation (6.11) is the Gelfand-Levitan equation and we give a derivation below in our framework. First set $F(x, \xi)=(x\xi)^{-m-1/2}\tilde{F}(x, \xi)$ with $\tilde{K}$ and $\tilde{L}$ as before.

Theorem 6.3. *The following formulas hold under the hypotheses indicated in Section* 5, *a somewhat neater formulation being given in* (6.21).

$$(6.10) \qquad \tilde{F}(\xi, t)+\int_0^t \tilde{K}(t, s)\tilde{F}(s, \xi)ds = \tilde{L}(\xi, t)-\tilde{K}(t, \xi) ;$$

$$F(\xi, t)+\int_0^t K(t, s)F(s, \xi)ds = t^{-2m-1}L(\xi, t)-\xi^{-2m-1}K(t, \xi)$$

For $\xi<t$, $L(\xi,t)=0$ in (6.10) *and we have an integral equation for $K(t, \xi)$* (*phrased in (x, y) variables for convenience later, $y<x$*)

$$(6.11) \qquad y^{-2m-1}K(x, y)+F(x, y)+\int_0^x K(x, t)F(t, y)dt = 0 ;$$

$$\tilde{K}(x, y)+\tilde{F}(x, y)+\int_0^x \tilde{K}(x, y)\tilde{F}(t, y)dt = 0 .$$

We defer the proof of Theorem 6.5 for a moment (see Remark 6.5) in order to return to I_2. Thus, using the relations above between K and L we have $I_2=\iint \tilde{f}_1(x)\tilde{f}_2(\xi)\tilde{F}(x, \xi)dxd\xi+\iint \tilde{f}_1(x)\tilde{f}_2(\xi)[-\tilde{F}(x, \xi)+\tilde{L}(x, \xi)-\tilde{K}(\xi, x)]dxd\xi$ $+\iint \tilde{f}_1(x)\tilde{f}_2(\xi)[-\tilde{F}(\xi, x)+\tilde{L}(\xi, x)-\tilde{K}(x, \xi)]dxd\xi+\iint \tilde{f}_1(x)\tilde{f}_2(\xi)\int_0^x \tilde{K}(x, t)[-\tilde{F}(t, \xi)$ $+\tilde{L}(t, \xi)-\tilde{K}(\xi, t)dtdxd\xi=\iint \tilde{f}_1(x)\tilde{f}_2(\xi)[\tilde{L}(x, \xi)-\tilde{K}(\xi, x)]dxd\xi+\iint \tilde{f}_1(x)\tilde{f}_2(\xi)\cdot$ $\int_0^x \tilde{K}(x, s)[\tilde{L}(s, \xi)-\tilde{K}(\xi, s)]dsdxd\xi$. Note here for example one can write $\iint f_1(x)f_2(\xi)\int_0^\xi K(\xi, s)F(s, x)dsdxd\xi=\iint \tilde{f}_1(x)\tilde{f}_2(\xi)(x\xi)^{m+1/2}\cdot \int_0^\xi \xi^{-m-1/2}\tilde{K}(\xi, s)s^{m+1/2}$ $\times(sx)^{-m-1/2}\tilde{F}(s, x)dsdxd\xi=\iint \tilde{f}_1(x)\tilde{f}_2(\xi)\int_0^\xi \tilde{K}(\xi, s)\tilde{F}(s, x)dsdxd\xi$ etc. Now $K(x, y)$ and $L(x, y)$ vanish for $y>x$ so from (6.9) we can write $\int_0^x K(x, s)L(s, \xi)ds=0$ for $\xi>x$ and one can show as in (6.9) that for $\xi<x$ $\int_\xi^x K(x, s)L(s, \xi)ds=-L(x, \xi)$

$-K(x, \xi)$. The same formulas then hold for $\tilde{K}$ and $\tilde{L}$. Consequently in the last expression for I_2 we have for example $\iint \tilde{f}_1(x)\tilde{f}_2(\xi)\int_0^x \tilde{K}(x, s)\tilde{L}(s, \xi)ds = \int \tilde{f}_1(x)\int_0^x \tilde{f}_2(\xi)[-\tilde{L}(x, \xi)-\tilde{K}(x, \xi)]d\xi dx$ and hence I_2 becomes $I_2 = -\int_0^\infty \tilde{f}_1(x) \times \int_x^\infty \tilde{f}_2(\xi)\tilde{K}(x, \xi)d\xi dx - \int_0^\infty \tilde{f}_1(x)\int_0^x \tilde{f}_2(\xi)\tilde{K}(x, \xi)d\xi dx - \int_0^\infty d\xi \int_0^\xi \tilde{f}_1(x)\tilde{f}_2(\xi)dx \int_0^x \tilde{K}(x, s) \times \tilde{K}(\xi, s)ds$. Now look at I_1 and write for example $\int x^{-2m-1}f_1(x)\Big(\int_x^\infty K(\xi, x)f_2(\xi) \times d\xi\Big)dx = \int \tilde{f}_1(x)\int_x^\infty x^{-m-1/2}K(\xi, x)\xi^{m+1/2}\tilde{f}_2(\xi)d\xi dx = \int \tilde{f}_1(x)\Big(\int_x^\infty \tilde{K}(\xi, x)\tilde{f}_2(\xi)d\xi\Big)dx$. It follows that $I_1+I_2=0$ and $\langle R, F_1F_2\rangle = \langle \tilde{f}_1, \tilde{f}_2\rangle$ which is the desired Parseval formula. We summarize this in

Theorem 6.4. *The Parseval formula* (5.20) *with* $R=R_0+R_q$ *as before* (*and* $d\omega=d\lambda$) *can be established as above* (*without recourse to* T_x^y).

REMARK 6.5. We will prove Theorem 6.3 now in our framework of spaces and maps. The proof is modeled on a procedure of Marčenko [16] but our representation in terms of generalized translation exhibits the facts more meaningfully. Recall first

$$R-R_0 = c_m^2\lambda^{2m+1}\int_0^\infty x^{m+1/2}\tilde{l}(x)\hat{R}^m(x, \lambda)dx = \langle W(x, \mu),\ \hat{l}(x)\rangle = \mathbf{Q}[\hat{l}(x)] \tag{6.12}$$

where $\hat{l}(x) = \tilde{l}(x)x^{-m-1/2}$. Note that when $m=-1/2$, $c_m=\sqrt{2/\pi}$ and $R_0 = c_m^2\lambda^{2m+1} = \dfrac{2}{\pi}$ with $W(y, \mu) = \dfrac{2}{\pi}\operatorname{Cos} y\lambda$; further $R-\dfrac{2}{\pi} = C\Big[\dfrac{2}{\pi}L(y, 0)\Big]$ where C denotes the cosine transform. Recall also that $L(y, t) = y^{-m-1/2}\tilde{L}(y, t)\cdot t^{m+1/2}$ and $\tilde{l}(y) = \lim \tilde{L}(y, t)t^{-m-1/2}$ as $t\to 0$ so that $\hat{l}(y) = \tilde{l}(y)/y^{m+1/2} = \lim L(y, t)/t^{2m+1}$ ($=L(y, 0)$ when $m=-1/2$). Now write $H(x, \mu) = (\mathcal{B}\theta)(x, \mu) = \theta(x, \mu) + \int_0^x K(x, t)\theta(t, \mu)dt$ and consider the product $(R-R_0)H(x, \mu)$ in $(W_m^1)'$. Let us ask for $\phi(\xi, x)$ such that $(R-R_0)\theta(x, \mu) = \mathbf{Q}[\phi(y, x)]$. Formally this says that

$$\begin{aligned}\phi(y, x) = \mathbf{Q}[(R-R_0)\theta(x, \mu)] &= \langle \theta(y, \mu), \theta(x, \mu)\langle W(\xi, \mu), \hat{l}(\xi)\rangle\rangle \\ &= \langle \gamma(y, x, \xi), \hat{l}(\xi)\rangle = S_x^y\hat{l}(x)\end{aligned} \tag{6.13}$$

where $\gamma(y, x, \xi)$ is the kernel of S_x^y given in Part I as $\gamma(y, x, \xi) = \int \theta(y, \mu)\theta(x, \mu) \times W(\xi, \mu)d\omega$. Hence one can say (from $H=\theta+\int K\theta$)

$$\begin{aligned}(R-R_0)H(x, \mu) &= \mathbf{Q}[S_x^y\hat{l}(x)] + \int_0^x K(x, t)\mathbf{Q}[S_t^y\hat{l}(t)]dt \\ &= \mathbf{Q}[S_x^y\hat{l}(x) + \int_0^x K(x, t)S_t^y\hat{l}(t)dt]\,.\end{aligned} \tag{6.14}$$

Now consider $F(\lambda) = \mathcal{Q}f(\lambda) \in W_m^1$ so that from Theorem 2.3

$$(6.15)\qquad \begin{aligned} F(\lambda) &= \mathcal{Q}f(\lambda) = \langle f(x), \theta(x, \mu)\rangle = \langle f(x), (BH)(x, \mu)\rangle \\ &= \langle H(t,\mu), B^*f(t)\rangle = \langle f(t)+\int_t^\infty f(x)L(x, t)dx, H(t, \mu)\rangle \\ &= \mathcal{P}(B^*f)(\lambda) . \end{aligned}$$

Recall if $g=\mathcal{B}^*f$ then $\mathcal{Q}g=\mathcal{Q}(\boldsymbol{Q}\mathcal{P})f=\mathcal{P}f$ and now for $f=B^*f$ we have $\mathcal{P}h=\mathcal{P}(\boldsymbol{P}\mathcal{Q})f=\mathcal{Q}f$. Suppose we have a Parseval formula (5.20) for $f, g\in \boldsymbol{E}=\boldsymbol{F}$ suitable. In a standard way now this extends to say $g(x)=\delta(x-y)$ with $\mathcal{P}g=H(y, \mu)$ and one has $\langle \mathcal{P}f, H(y, \mu)R\rangle_\omega=y^{-2m-1}f(y)$. Since $F(\lambda)=\mathcal{P}(B^*f)$ we have then

$$(6.16)\qquad \begin{aligned} \langle F(\lambda), H(x, \mu)R\rangle_\omega &= x^{-2m-1}(B^*f)(x) \\ &= x^{-2m-1}f(x)+x^{-2m-1}\int_x^\infty f(y)L(y, x)dy . \end{aligned}$$

On the other hand R_0 is the spectral function for Q so that $\langle \mathcal{Q}f\mathcal{Q}g, R_0\rangle_\omega=\langle x^{-m-1/2}f, x^{-m-1/2}g\rangle$. Hence $\langle \mathcal{Q}f, \theta(x, \mu)R_0\rangle_\omega=x^{-2m-1}f(x)$ and

$$(6.17)\qquad \begin{aligned} \langle F(\lambda), H(x, \mu)R_0\rangle_\omega &= \langle \mathcal{Q}f, H(x, \mu)R_0\rangle_\omega \\ &= \langle \mathcal{Q}f, [\Theta(x, \mu)+\int_0^x K(x, t)\Theta(t, \mu)dt]R_0\rangle_\omega \\ &= x^{-2m-1}f(x)+\int_0^x K(x, t)t^{-2m-1}f(t)dt . \end{aligned}$$

Consequently

$$(6.18)\qquad \begin{aligned} \langle F(\lambda), (R-R_0)H(x, \mu)\rangle_\omega &= \Xi \\ &= x^{-2m-1}\int_x^\infty f(y)L(y, x)dy-\int_0^x f(t)K(x, t)t^{-2m-1}dt . \end{aligned}$$

Since $K(x, t)=0$ for $t>x$ and $L(y, x)=0$ for $x>y$ we can write these as integrals over $(0, \infty)$ and obtain

$$(6.19)\qquad \Xi = \int_0^\infty f(y)[x^{-m-1}L(y, x)-y^{-2m-1}K(x, y)]dy .$$

Now $\langle \mathcal{Q}f, (R-R_0)H(x, \mu)\rangle_\omega=\langle f, \mathcal{Q}^*[(R-R_0)H(x, \mu)]\rangle$ and here $\mathcal{Q}^*=\mathbf{Q}$ so that we can write

$$(6.20)\qquad (R-R_0)H(x, \mu) = \mathbf{Q}[x^{-2m-1}L(y, x)-y^{-2m-1}K(x, y)] .$$

Equating (6.20) and (6.14) we get (6.10) in the form

$$(6.21)\qquad x^{-2m-1}L(y, x)-y^{-2m-1}K(x, y) = S_x^y\hat{l}(x)+\int_0^x K(x, t)S_t^y\hat{l}(t)dt$$

from which the Gelfand-Levitan equation (6.11) follows. Q.E.D.

REMARK 6.6. The importance and use of the Gelfand-Levitan equation in quantum physics is well known and we will not comment on this here (cf. [1–17; 1–39]). For connections of the Gelfand-Levitan equation with transmutation and special functions see [3; 6; 16; 24].

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