

Title	On ∇ -polynomials of links
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Citation	Osaka Mathematical Journal. 1958, 10(2), p. 273-282
Version Type	VoR
URL	https://doi.org/10.18910/11864
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On ∇ -Polynomials of Links

By Fujitsugu HOSOKAWA

In 1934 H. Seifert [3] proved that the Alexander polynomial $\Delta(t)$ of a knot is a symmetric polynomial of even degree and $|\Delta(1)|=1$. And moreover he proved that for a given symmetric polynomial $f(t)$ of even degree, if $|f(1)|=1$, then there exists a knot which has $f(t)$ as its Alexander polynomial.

In 1953 G. Torres [1, 2] proved that for the Alexander polynomial of a link of multiplicity μ , whose components are X_1, X_2, \dots, X_μ , it holds

$$\Delta(t_1, t_2, \dots, t_\mu) = (-1)^\mu t_1^{\nu_1} t_2^{\nu_2} \dots t_\mu^{\nu_\mu} \Delta(t_1^{-1}, t_2^{-1}, \dots, t_\mu^{-1})$$

for suitably chosen integers $\nu_1, \nu_2, \dots, \nu_\mu$. Further he proved that it holds

$$\Delta(t_1, t_2, \dots, t_{\mu-1}, 1) = (t_1^{l_1}, t_2^{l_2}, \dots, t_{\mu-1}^{l_{\mu-1}} - 1) \Delta(t_1, t_2, \dots, t_{\mu-1}) \quad \text{for } \mu > 2$$

and

$$\Delta(t_1, 1) = (t_1^{l_1} - 1) \Delta(t_1) / (t_1 - 1) \quad \text{for } \mu = 2$$

where l_i is the linking number of X_i and X_μ for $i=1, 2, \dots, \mu-1$, and that it holds

$$\begin{aligned} \Delta(1, 1, \dots, 1) &= 0 & \text{for } \mu > 2 & \quad \text{and} \\ \Delta(1, 1) &= l_1 & \text{for } \mu = 2. \end{aligned}$$

In this paper we shall introduce a polynomial, the ∇ -polynomial of a link, deduced simply from the Alexander polynomial. For this ∇ -polynomial $\nabla(t)$ we shall prove the following properties:

- 1) $\nabla(t)$ is a symmetric polynomial of even degree.
- 2) $|\nabla(t)|$ is determined uniquely by the linking numbers between all pairs of μ components of the link of multiplicity μ .
- 3) If a polynomial $f(t)$ is a symmetric polynomial of even degree, then there exists a link of any given multiplicity μ such that its ∇ -polynomial coincides with the polynomial $f(t)$.

1. We shall call a polynomial $f(t)$ *symmetric (skew symmetric)* if $f(t) = t^\nu f(t^{-1})$ ($f(t) = -t^\nu f(t^{-1})$) for a suitably chosen integer ν . Then integer $n-m$ will be called the *reduced degree* of a polynomial

$$f(t) = a_1 t^l + a_{l-1} t^{l-1} + \dots + a_n t^n + \dots + a_m t^m + \dots + a_1 t + a_0$$

if $a_l = a_{l-1} = \dots = a_{n+1} = 0$, $a_n \neq 0$, $a_m \neq 0$ ($n \geq m$) and $a_{m-1} = \dots = a_1 = a_0 = 0$. Then we have the following

Lemma.¹⁾ *Let $f(t)$ and $F(t)$ be symmetric polynomials of even reduced degree and let $g(t)$ and $G(t)$ be skew symmetric polynomials such that*

$$\begin{aligned} F(t) &= tf(t) + (t-1)g(t), \\ G(t) &= (1-t)f(t) + g(t). \end{aligned}$$

Then the difference of reduced degrees of $f(t)$ and $g(t)$ is an odd integer.

2. Let κ be a link of multiplicity μ in a 3-dimensional euclidean space E^3 and let X_1, X_2, \dots, X_μ be the components of κ . The Alexander polynomial $\Delta(t_1, t_2, \dots, t_\mu)$ of κ may be defined as usual [1].

Now we shall define the ∇ -polynomial $\nabla(t)$ of a link of multiplicity μ as follows:

$$\begin{aligned} \nabla(t) &= \Delta(t, \dots, t) / (1-t)^{\mu-2} && \text{for } \mu \geq 2, \\ \nabla(t) &= \Delta(t) && \text{for } \mu = 1, \end{aligned}$$

where $\Delta(t, \dots, t)$ is the reduced polynomial²⁾ obtained by putting $t_1 = t_2 = \dots = t_\mu = t$ in the Alexander polynomial $\Delta(t_1, t_2, \dots, t_\mu)$ of κ .

G. Torres showed that

$$\Delta(t_1, t_2, \dots, t_\mu) = (-1)^\mu t_1^{\nu_1} t_2^{\nu_2} \dots t_\mu^{\nu_\mu} \Delta(t_1^{-1}, t_2^{-1}, \dots, t_\mu^{-1})$$

for suitably chosen integers $\nu_1, \nu_2, \dots, \nu_\mu$ [1].

From this it is easy to see that the ∇ -polynomial is symmetric. Then we have the following

Theorem 1. *The ∇ -polynomial of a link κ is a symmetric polynomial of even degree.*

Proof. We have only to prove that the degree of the ∇ -polynomial is even. To prove this we shall use a mathematical induction on the multiplicity of κ . In the case of multiplicity 1, i.e. in the case of a knot, our theorem is evidently true [3].

Now suppose that our theorem is true for the case of multiplicity $\mu-1$.

Let $\bar{\kappa}$ be a regular projection³⁾ of a link κ of multiplicity μ on a plane E^2 in E^3 and let $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_\mu$ be the projections of X_1, X_2, \dots, X_μ respectively on E^2 . We may confine κ in a cube Q in E^3 . We choose

1) See [4].

2) A reduced polynomial is by definition a polynomial with non vanishing constant term.

3) See p. 60 of [1].

arbitrarily two components \bar{X}_i and \bar{X}_j of $\bar{\kappa}$ and take out parts of arcs of \bar{X}_i and \bar{X}_j from the cube Q , running parallel to each other in the same direction as shown in Fig. 1.

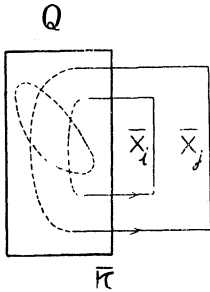


Fig. 1.

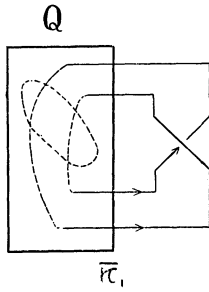


Fig. 2.

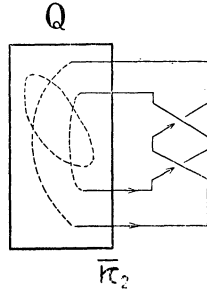


Fig. 3.

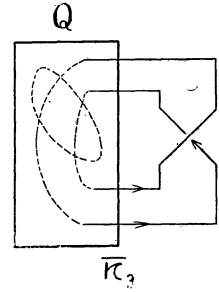


Fig. 4.

From this $\bar{\kappa}$ we introduce new links⁴⁾ $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$ as defined in Fig. 2, Fig. 3 and Fig. 4 respectively [4]. Let κ_1 , κ_2 and κ_3 be links in E^3 such that they have $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$ as their regular projections. Then it is clear that κ_1 and κ_3 are links of multiplicity $\mu-1$ and κ_2 is a link of multiplicity μ . Let $\nabla(t)$, $\nabla_1(t)$, $\nabla_2(t)$ and $\nabla_3(t)$ be the ∇ -polynomials of κ , κ_1 , κ_2 and κ_3 respectively.

Here we shall define $f(t)$, $g(t)$, $F(t)$ and $G(t)$ as follows:

$$\begin{aligned} f(t) &= \pm t^{p_1} \nabla_1(t), \\ g(t) &= \pm t^p (1-t) \nabla(t), \\ F(t) &= \pm t^{p_3} \nabla_3(t), \\ G(t) &= \pm t^{p_2} (1-t) \nabla_2(t) \end{aligned} \quad \text{and}$$

for suitably chosen integers p , p_1 , p_2 and p_3 .

From the assumption of induction $f(t)$ and $F(t)$ are symmetric polynomials of even reduced degree and $g(t)$ and $G(t)$ are skew symmetric polynomials and from the constructions of κ_1 , κ_2 and κ_3 we have [4]

$$\begin{aligned} F(t) &= tf(t) + (t-1)g(t), \\ G(t) &= (1-t)f(t) + g(t). \end{aligned}$$

Hence, $f(t)$, $g(t)$, $F(t)$ and $G(t)$ satisfy the conditions of Lemma. Therefore, the difference of the reduced degrees of $f(t)$ and $g(t)$ is an odd integer. Since the reduced degree of $f(t)$ is even, the reduced degree of $g(t)$ is odd. By the definition of $g(t)$ the degree of $\nabla(t)$ is even. Thus, the proof of our theorem is complete.

4) A knot is considered as a link of multiplicity 1.

3. Given a link κ of multiplicity μ , κ can be spanned by an orientable surface F of genus h , represented by a disk to which is attached $2h + \mu - 1$ bands $B_1, B_2, \dots, B_{2h+\mu-1}$, whose corresponding projection is named by Torres the Seifert projection.

Let B_1, B_2, \dots, B_{2h} be canonical bands and let $B_{2h+1}, \dots, B_{2h+\mu-1}$ be extra bands⁵⁾. Let $a_1, a_2, \dots, a_{2h+\mu-1}$ be simple closed curves which are drawn along each $B_1, B_2, \dots, B_{2h+\mu-1}$. The curves a_1, a_2, \dots, a_{2h} are oriented such that a_{2i-1} crosses a_{2i} from left to right ($i=1, 2, \dots, h$). The curves $a_{2h+1}, \dots, a_{2h+\mu-1}$ have the same orientation as the orientation of X_1 as shown in Fig. 5.

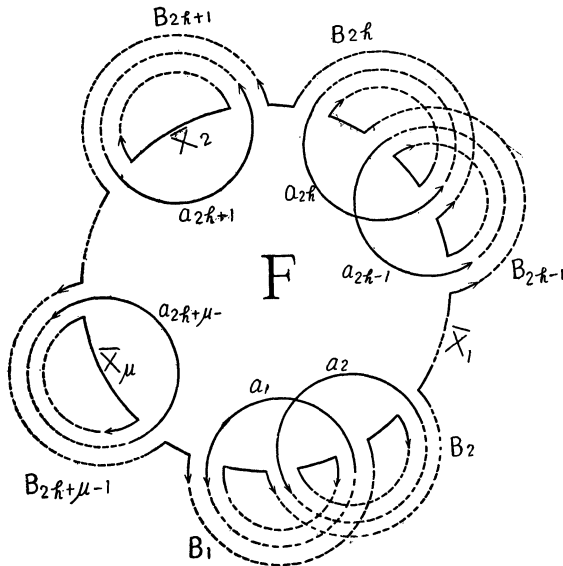


Fig. 5.

Then it is clear that a_i intersects a_j if and only if $i=2k-1$ and $j=2k$ ($1 \leq k \leq h$). If a_{2k-1} intersects a_{2k} , we can lift a_{2k-1} in a neighborhood of the intersection and let the new curve be denoted by a'_{2k-1} . Let $v_{i,j}$ ($i, j=1, 2, \dots, 2h + \mu - 1$) be equal to the number of times that a_i crosses over a_j from left to right minus the number of times that a_i crosses over a_j from right to left. Then we have easily the following relations:

$$v_{i,j} = v_{j,i} = \text{link}(a_i, a_j) \quad \text{if } a_i \cap a_j = \phi$$

$$v_{2k-1,2k} = v_{2k,2k-1} + 1 = \text{link}(a'_{2k-1}, a_{2k}) \quad \text{if } 1 \leq k \leq h,$$

where $\text{link}(a_i, a_j)$ denotes the linking number of a_i and a_j .

Then the ∇ -polynomial $\nabla(t)$ of κ can be written as follows [1]:

5) See p. 63 of [1].

$$\begin{aligned}
 & \pm t^n \nabla(t) = M_{h,\mu}(t) \\
 & = \begin{array}{c}
 v_{1,1}(1-t) \quad v_{1,2}(1-t) + t \quad \cdots \quad v_{1,2h-1}(1-t) \quad v_{1,2h}(1-t) \\
 v_{1,2}(1-t) - 1 \quad v_{2,2}(1-t) \quad \cdots \quad v_{2,2h-1}(1-t) \quad v_{2,2h}(1-t) \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 v_{2h-1,1}(1-t) \quad v_{2h-1,2}(1-t) \quad \cdots \quad v_{2h-1,2h-1}(1-t) \quad v_{2h-1,2h}(1-t) + t \\
 v_{2h,1}(1-t) \quad v_{2h,2}(1-t) \quad \cdots \quad v_{2h-1,2h}(1-t) - 1 \quad v_{2h,2h}(1-t) \\
 \hline
 v_{2h+1,1} \quad v_{2h+1,2} \quad \cdots \quad v_{2h+1,2h-1} \quad v_{2h+1,2h} \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 v_{2h+\mu-1,1} \quad v_{2h+\mu-1,2} \quad \cdots \quad v_{2h+\mu-1,2h-1} \quad v_{2h+\mu-1,2h}
 \end{array} \\
 & \begin{array}{c}
 v_{1,2h+1}(1-t) \quad \cdots \quad v_{1,2h+\mu-1}(1-t) \\
 v_{2,2h+1}(1-t) \quad \cdots \quad v_{2,2h+\mu-1}(1-t) \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 v_{2h-1,2h+1}(1-t) \quad \cdots \quad v_{2h-1,2h+\mu-1}(1-t) \\
 v_{2h,2h+1}(1-t) \quad \cdots \quad v_{2h,2h+\mu-1}(1-t) \\
 \hline
 v_{2h+1,2h+1} \quad \cdots \quad v_{2h+1,2h+\mu-1} \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 v_{2h+\mu-1,2h+1} \quad \cdots \quad v_{2h+\mu-1,2h+\mu-1}
 \end{array}
 \end{aligned}$$

(*)

$$\begin{aligned}
 &= \begin{vmatrix} v_{2h+1, 2h+1} & \cdots & v_{2h+1, 2h+\mu-1} \\ \vdots & & \vdots \\ v_{2h+\mu-1, 2h+1} & \cdots & v_{2h+\mu-1, 2h+\mu-1} \end{vmatrix} \\
 &= \begin{vmatrix} -\sum_{k=1, k \neq 2}^{\mu} l_{2, i} & l_{2, 3} & \cdots & l_{2, \mu} \\ l_{3, 2} & -\sum_{k=1, k \neq 3}^{\mu} l_{3, k} & \cdots & l_{3, \mu} \\ \vdots & \vdots & \ddots & \vdots \\ l_{\mu, 2} & l_{\mu, 3} & \cdots & -\sum_{k=1}^{\mu-1} l_{\mu, k} \end{vmatrix}.
 \end{aligned}$$

Since it is clear that the rank of matrix A is $\mu-1$, the $(\mu-1)$ -minor determinants of A coincide with $\pm \nabla(1)$. Thus the proof is complete.

Finally, we shall prove the following theorem.

Theorem 3. *Let $f(x)$ be a symmetric polynomial of even degree whose constant term is different from zero. Then there exists a link of any given multiplicity μ whose ∇ -polynomial is $f(x)$.*

Proof. In order to construct a link κ of multiplicity μ whose ∇ -polynomial is $f(x)$, we shall determine $v_{i, j}$ ($i, j = 1, 2, \dots, 2h + \mu - 1$) in (*).

To begin with, let

$$\begin{aligned}
 &v_{1, 4}, v_{3, 6}, \dots, v_{2h-3, 2h} \quad \text{and} \\
 &v_{2h+2, 2h+2}, \dots, v_{2h+\mu-1, 2h+\mu-1} \quad \text{and} \quad v_{2, 2h+1}
 \end{aligned}$$

be equal to 1, let

$$v_{1, 3}, v_{3, 5}, \dots, v_{2h-3, 2h-1} \quad \text{and} \quad v_{1, 1}$$

be undetermined, let

$$v_{2h+1, 2h+1}$$

be equal to $-f(1)$, and let the others be equal to zero.

Then we have

$$M_{h,\mu}(t) = \left| \begin{array}{cccc|ccc|ccc} v_{1,1}(1-t) & t & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 & 0 & (1-t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & t & v_{2h-3,2h-1}(1-t) & (1-t) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & v_{2h-3,2h-1}(1-t) & 0 & 0 & t & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & (1-t) & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 1 & \dots & 0 & 0 & 0 & 0 & -f(1) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{array} \right|.$$

Interchange the second column with the $(2h+1)$ -th column and add the $(2h+1)$ -th row multiplied by $-t$ to the first row. Then, we have

$$-M_{h,\mu}(t) = \bar{M}_{h,\mu}(t) = \left| \begin{array}{cccc|ccc|ccc} v_{1,1}(1-t) & f(1)t & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1-t & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & t & v_{2h-3,2h-1}(1-t) & (1-t) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & v_{2h-3,2h-1}(1-t) & 0 & 0 & 0 & 0 & t & \dots & 0 \\ 0 & 0 & \dots & (1-t) & 0 & -1 & 0 & 0 & 0 & \dots & 0 \end{array} \right|.$$

Now we shall prove our theorem by a mathematical induction on the degree of $f(t)$.

If the degree of $f(x)$ is 2, i.e. if

$$f(t) = c_0t^2 + c_1t + c_0,$$

then we set $h=1$.

Since

$$\bar{M}_{1,\mu}(t) = \left| \begin{array}{cc|c} v_{1,1}(1-t) & f(1)t & \\ -1 & (1-t) & \end{array} \right| = v_{1,1}t^2 - (2v_{1,1} - f(1))t + v_{1,1},$$

if we set $v_{1,1} = c_0$, then $|\nabla(t)| = |\bar{M}_{1,\mu}(t)| = |f(t)|$.

Suppose that our theorem is proved for the case where the degree of $f(t)$ is $2(n-1)$.

Then let

$$f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \dots + c_{n-1} t^{n+1} + c_n t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0.$$

We set $h = n$.

Let $\bar{M}'_{n,\mu}(t)$ be the $(2n-1)$ -minor determinant which is deduced from $\bar{M}_{n,\mu}(t)$ by striking out $(2n-1)$ -th row and $(2n-1)$ -th column. By a simple calculation we have

$$\bar{M}'_{n,\mu}(t) = -(1-t)^2 \bar{M}'_{n-1,\mu}(t)$$

and since $\bar{M}'_{1,\mu}(t) = (1-t)$, we have further

$$\bar{M}'_{n,\mu}(t) = (-1)^{n-1} (1-t)^{2n-1}.$$

Developing $\bar{M}_{n,\mu}(t)$ at the $(2n-1)$ -th column, we have

$$\begin{aligned} \bar{M}_{n,\mu}(t) &= v_{2n-3,2n-1} t(1-t)^2 \bar{M}'_{n-1,\mu}(t) + \{t \bar{M}_{n-1,\mu}(t) - (1-t)^2 v_{2n-3,2n-1} \bar{M}'_{n-1,\mu}(t)\} \\ &= t \bar{M}_{n-1,\mu}(t) - (1-t)^3 v_{2n-3,2n-1} \bar{M}'_{n-1,\mu}(t) \\ &= t \bar{M}_{n-1,\mu}(t) + (-1)^{n-1} v_{2n-3,2n-1} (1-t)^{2n}. \end{aligned}$$

In order that

$$\bar{M}_{n,\mu}(t) = f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \dots + c_1 t + c_0,$$

it is sufficient that it holds

$$(**) \quad \bar{M}_{n-1,\mu}(t) = \frac{c_0 t^{2n} + c_1 t^{2n-1} + \dots + c_1 t + c_0 + (-1)^n v_{2n-3,2n-1} (1-t)^{2n}}{t}.$$

If we set $v_{2n-3,2n-1} = (-1)^{n+1} c_0$, then the degree of the right hand side of $(**)$ is $2(n-1)$. Hence, by the assumption of induction we can determine $v_{1,1}, v_{1,3}, \dots, v_{2n-5,2n-3}$ satisfying $(**)$. Therefore, $v_{1,1}, v_{1,3}, \dots, v_{2n-3,2n-1}$ can be determined such that $|\nabla(t)| = |\bar{M}_{n,\mu}(t)| = |f(t)|$.

Hence, we have a link of multiplicity μ whose ∇ -polynomial is $f(t)$, and the proof is complete.

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(Received September 29, 1958)

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