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On $\nabla$-Polynomials of Links

By Fujitsugu Hosokawa

In 1934 H. Seifert [3] proved that the Alexander polynomial $\Delta(t)$ of a knot is a symmetric polynomial of even degree and $|\Delta(1)|=1$. And moreover he proved that for a given symmetric polynomial $f(t)$ of even degree, if $|f(1)|=1$, then there exists a knot which has $f(t)$ as its Alexander polynomial.

In 1953 G. Torres [1, 2] proved that for the Alexander polynomial of a link of multiplicity $\mu$, whose components are $X_1, X_2, \ldots, X_\mu$, it holds

$$\Delta(t_1, t_2, \ldots, t_\mu) = (-1)^\mu t_1^n t_2^n \cdots t_\mu^n \Delta(t^{-1}_1, t^{-1}_2, \ldots, t^{-1}_\mu)$$

for suitably chosen integers $\nu_1, \nu_2, \ldots, \nu_\mu$. Further he proved that it holds

$$\Delta(t_1, t_2, \ldots, t_{\mu-1}, 1) = (t_1^{\nu_1}, t_2^{\nu_2}, \ldots, t_{\mu-1}^{\nu_{\mu-1}} - 1) \Delta(t_1, t_2, \ldots, t_{\mu-1}) \text{ for } \mu > 2$$

and

$$\Delta(t_1, 1) = (t_1^{\nu_1} - 1) \Delta(t_1)/(t_1 - 1) \text{ for } \mu = 2$$

where $l_i$ is the linking number of $X_i$ and $X_\mu$ for $i = 1, 2, \ldots, \mu - 1$, and that it holds

$$\Delta(1, 1, \ldots, 1, 1) = 0 \text{ for } \mu > 2 \text{ and } \Delta(1, 1) = l_1 \text{ for } \mu = 2.$$

In this paper we shall introduce a polynomial, the $\nabla$-polynomial of a link, deduced simply from the Alexander polynomial. For this $\nabla$-polynomial $\nabla(t)$ we shall prove the following properties:

1) $\nabla(t)$ is a symmetric polynomial of even degree.
2) $|\nabla(t)|$ is determined uniquely by the linking numbers between all pairs of $\mu$ components of the link of multiplicity $\mu$.
3) If a polynomial $f(t)$ is a symmetric polynomial of even degree, then there exists a link of any given multiplicity $\mu$ such that its $\nabla$-polynomial coincides with the polynomial $f(t)$.

1. We shall call a polynomial $f(t)$ symmetric (skew symmetric) if $f(t) = t^n f(t^{-1})$ (or $f(t) = -t^n f(t^{-1})$) for a suitably chosen integer $\nu$. Then integer $n-m$ will be called the reduced degree of a polynomial

$$f(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_m t^m + \cdots + a_{m-1} t + a_0$$
Lemma. Let $f(t)$ and $F(t)$ be symmetric polynomials of even reduced degree and let $g(t)$ and $G(t)$ be skew symmetric polynomials such that

$$F(t) = tf(t) + (t-1)g(t),$$
$$G(t) = (1-t)f(t) + g(t).$$

Then the difference of reduced degrees of $f(t)$ and $g(t)$ is an odd integer.

2. Let $\kappa$ be a link of multiplicity $\mu$ in a 3-dimensional euclidean space $E^3$ and let $X_1, X_2, \ldots, X_\mu$ be the components of $\kappa$. The Alexander polynomial $\Delta(t_1, t_2, \ldots, t_\mu)$ of $\kappa$ may be defined as usual [1].

Now we shall define the $\nabla$-polynomial $\nabla(t)$ of a link of multiplicity $\mu$ as follows:

$$\nabla(t) = \Delta(t_1, \ldots, t_\mu)/(1-t)^{\mu-2}$$

for $\mu \geq 2$,

$$\nabla(t) = \Delta(t)$$

for $\mu = 1$,

where $\Delta(t_1, \ldots, t_\mu)$ is the reduced polynomial obtained by putting $t_1 = t_2 = \ldots = t_\mu = t$ in the Alexander polynomial $\Delta(t_1, t_2, \ldots, t_\mu)$ of $\kappa$.

G. Torres showed that

$$\Delta(t_1, t_2, \ldots, t_\mu) = (-1)^\mu t_1^{\nu_1} t_2^{\nu_2} \cdots t_\mu^{\nu_\mu} \Delta(t_1^{-1}, t_2^{-1}, \ldots, t_\mu^{-1})$$

for suitably chosen integers $\nu_1, \nu_2, \ldots, \nu_\mu$ [1].

From this it is easy to see that the $\nabla$-polynomial is symmetric. Then we have the following

Theorem 1. The $\nabla$-polynomial of a link $\kappa$ is a symmetric polynomial of even degree.

Proof. We have only to prove that the degree of the $\nabla$-polynomial is even. To prove this we shall use a mathematical induction on the multiplicity of $\kappa$. In the case of multiplicity 1, i.e. in the case of a knot, our theorem is evidently true [3].

Now suppose that our theorem is true for the case of multiplicity $\mu-1$.

Let $\kappa$ be a regular projection of a link $\kappa$ of multiplicity $\mu$ on a plane $E^2$ in $E^3$ and let $X_1, X_2, \ldots, X_\mu$ be the projections of $X_1, X_2, \ldots, X_\mu$ respectively on $E^2$. We may confine $\kappa$ in a cube $Q$ in $E^2$. We choose

---

1) See [4].
2) A reduced polynomial is by definition a polynomial with non vanishing constant term.
3) See p. 60 of [1].
arbitrarily two components $X_i$ and $X_j$ of $\bar{X}$ and take out parts of arcs of $X_i$ and $X_j$ from the cube $Q$, running parallel to each other in the same direction as shown in Fig. 1.

From this $\bar{X}$ we introduce new links $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ as defined in Fig. 2, Fig. 3 and Fig. 4 respectively [4]. Let $\kappa_1$, $\kappa_2$ and $\kappa_3$ be links in $E^3$ such that they have $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ as their regular projections. Then it is clear that $\kappa_1$ and $\kappa_3$ are links of multiplicity $\mu - 1$ and $\kappa_2$ is a link of multiplicity $\mu$. Let $\nabla(t)$, $\nabla_1(t)$, $\nabla_2(t)$ and $\nabla_3(t)$ be the $\nabla$-polynomials of $\kappa$, $\kappa_1$, $\kappa_2$ and $\kappa_3$ respectively.

Here we shall define $f(t)$, $g(t)$, $F(t)$ and $G(t)$ as follows:

$$f(t) = \pm t^p \nabla(t),$$
$$g(t) = \pm t^p (1-t) \nabla(t),$$
$$F(t) = \pm t^{p_1} \nabla_1(t),$$
$$G(t) = \pm t^{p_2} (1-t) \nabla_2(t)$$

for suitably chosen integers $p$, $p_1$, $p_2$ and $p_3$.

From the assumption of induction $f(t)$ and $F(t)$ are symmetric polynomials of even reduced degree and $g(t)$ and $G(t)$ are skew symmetric polynomials and from the constructions of $\kappa_1$, $\kappa_2$ and $\kappa_3$ we have [4]

$$F(t) = tf(t) + (t-1)g(t),$$
$$G(t) = (1-t)f(t) + g(t).$$

Hence, $f(t)$, $g(t)$, $F(t)$ and $G(t)$ satisfy the conditions of Lemma. Therefore, the difference of the reduced degrees of $f(t)$ and $g(t)$ is an odd integer. Since the reduced degree of $f(t)$ is even, the reduced degree of $g(t)$ is odd. By the definition of $g(t)$ the degree of $\nabla(t)$ is even. Thus, the proof of our theorem is complete.

---

4) A knot is considered as a link of multiplicity 1.
3. Given a link \( \kappa \) of multiplicity \( \mu \), \( \kappa \) can be spanned by an orientable surface \( F \) of genus \( h \), represented by a disk to which is attached \( 2h+\mu-1 \) bands \( B_1, B_2, \ldots, B_{2h+\mu-1} \), whose corresponding projection is named by Torres the Seifert projection.

Let \( B_1, B_2, \ldots, B_{2h} \) be canonical bands and let \( B_{2h+1}, \ldots, B_{2h+\mu-1} \) be extra bands\(^5\). Let \( a_1, a_2, \ldots, a_{2h+\mu-1} \) be simple closed curves which are drawn along each \( B_1, B_2, \ldots, B_{2h+\mu-1} \). The curves \( a_1, a_2, \ldots, a_{2h} \) are oriented such that \( a_{2i-1} \) crosses \( a_{2i} \) from left to right \((i=1, 2, \ldots, h)\). The curves \( a_{2h+1}, \ldots, a_{2h+\mu-1} \) have the same orientation as the orientation of \( X_i \) as shown in Fig. 5.

Then it is clear that \( a_i \) intersects \( a_j \) if and only if \( i=2k-1 \) and \( j=2k \)(\( 1 \leq k \leq h \)). If \( a_{2k-1} \) intersects \( a_{2k} \), we can life \( a_{2k-1} \) in a neighborhood of the intersection and let the new curve be denoted by \( a'_{2k-1} \). Let \( v_{i, j} (i, j=1, 2, \ldots, 2h+\mu-1) \) be equal to the number of times that \( a_i \) crosses over \( a_j \) from left to right minus the number of times that \( a_i \) crosses over \( a_j \) from right to left. Then we have easily the following relations:

\[
\begin{align*}
v_{i, j} & = v_{j, i} = \text{link} (a_i, a_j) \quad \text{if} \quad a_i \cap a_j = \emptyset \\
v_{2k-1, 2k} & = v_{2h, 2h-1} + 1 = \text{link} (a'_{2k-1}, a_{2k}) \quad \text{if} \quad 1 \leq k \leq h,
\end{align*}
\]

where \( \text{link} (a_i, a_j) \) denotes the linking number of \( a_i \) and \( a_j \).

Then the \( \nabla \)-polynomial \( \nabla (t) \) of \( \kappa \) can be written as follows \([1]\):
\[ \pm t^n \nabla(t) = M_{h,u}(t) \]

\[
\begin{align*}
&= \begin{pmatrix}
\begin{array}{cccc}
v_{1,1}(1-t) & v_{1,2}(1-t) + t & \cdots & v_{1,2h-1}(1-t) \\
v_{1,2}(1-t) - 1 & v_{2,2}(1-t) & \cdots & v_{2,2h-1}(1-t) \\
\vdots & \vdots & \ddots & \vdots \\
v_{2h-1,1}(1-t) & v_{2h-1,2}(1-t) & \cdots & v_{2h-1,2h-1}(1-t) \\
v_{2h,1}(1-t) & v_{2h,2}(1-t) & \cdots & v_{2h,2h-1}(1-t) - 1 \\
v_{2h+1,1} & v_{2h+1,2} & \cdots & v_{2h+1,2h-1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{2h+u-1,1} & v_{2h+u-1,2} & \cdots & v_{2h+u-1,2h-1} \\
\end{array}
\end{pmatrix} \\
&= \begin{pmatrix}
\begin{array}{cccc}
v_{1,2h+1}(1-t) & \cdots & v_{1,2h+u-1}(1-t) \\
v_{2,2h+1}(1-t) & \cdots & v_{2,2h+u-1}(1-t) \\
\vdots & \ddots & \vdots \\
v_{2h-1,2h+1}(1-t) & \cdots & v_{2h-1,2h+u-1}(1-t) \\
v_{2h,2h+1}(1-t) & \cdots & v_{2h,2h+u-1}(1-t) \\
v_{2h+1,2h+1} & \cdots & v_{2h+1,2h+u-1} \\
\vdots & \ddots & \vdots \\
v_{2h+u-1,2h+1} & \cdots & v_{2h+u-1,2h+u-1}
\end{array}
\end{pmatrix}
\end{align*}
\]
Conversely if \( v_{i,j} \) \( (i, j = 1, 2, \ldots, 2h + \mu - 1) \) are given, then the \( \nabla \)-polynomial, \( \nabla(t) \), and a link of multiplicity \( \mu \), which has \( \nabla(t) \) as the \( \nabla \)-polynomial, can evidently be determined.

Now we have

**Theorem 2.** Let \( \kappa \) be a link of multiplicity \( \mu \) and let \( X_1, X_2, \ldots, X_u \) be the components of \( \kappa \). Let \( l_{i,j} \) be the linking number of \( X_i \) and \( X_j \), i.e.

\[
l_{i,j} = \text{link}(X_i, X_j) \quad (i \neq j, \ i, j = 1, 2, \ldots, \mu).
\]

Let \( \nabla(t) \) be the \( \nabla \)-polynomial of \( \kappa \). Then \( \pm \nabla(1) \) is equal to the \((\mu - 1)\)-minor determinant of the following matrix \( A \):

\[
A = \begin{bmatrix}
- \sum_{k=2}^{\mu} l_{1,k} & l_{1,2} & \cdots & l_{1,j} & \cdots & l_{1,\mu} \\
l_{2,1} & - \sum_{k=2}^{\mu} l_{2,k} & \cdots & l_{2,j} & \cdots & l_{2,\mu} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
l_{j,1} & l_{j,2} & \cdots & - \sum_{k=2}^{\mu} l_{j,k} & \cdots & l_{j,\mu} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
l_{\mu,1} & l_{\mu,2} & \cdots & l_{\mu,j} & \cdots & - \sum_{k=2}^{\mu} l_{\mu,k}
\end{bmatrix}
\]

Proof. From the definition of \( v_{i,j} \) it is easy to see that

\[
l_{i,j} = v_{2h+i-1,2h+j-1} = v_{2h+j-1,2h+i-1} = l_{j,i} \quad \text{if} \ i, j > 1 \ i \neq j
\]

\[
l_{1,i} = -(v_{2h+i-1,2h+1} + \cdots + v_{2h+i-1,2h+i-1} + \cdots + v_{2h+i-1,2h+\mu-1}) \quad \text{if} \ i > 1.
\]

Hence, we have

\[
v_{2h+i-1,2h+i-1} = -(l_{1,i} + l_{2,i} + \cdots + l_{i-1,i} + l_{i+1,i} + \cdots + l_{\mu,i}).
\]

If we set \( t = 1 \) in (*) , we have

\[
\pm \nabla(1) = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

\[
\begin{array}{c|c}
0 & 1 \\
-1 & 0
\end{array}
\]

\[
\begin{array}{c|c}
0 & 1 \\
-1 & 0
\end{array}
\]

\[
\begin{array}{c|c}
0 & 1 \\
-1 & 0
\end{array}
\]

\[
\begin{array}{c|c}
0 & 1 \\
-1 & 0
\end{array}
\]

**\( v_{2h+1,2h+1} \) \cdots \( v_{2h+1,2h+\mu-1} \)
\]

\[
\begin{array}{c|c}
\vdots
\end{array}
\]

\[
\begin{array}{c|c}
\vdots
\end{array}
\]

\[
\begin{array}{c|c}
\vdots
\end{array}
\]

**\( v_{2h+\mu-1,2h+1} \) \cdots \( v_{2h+\mu-1,2h+\mu-1} \)**
Since it is clear that the rank of matrix $A$ is $\mu - 1$, the $(\mu - 1)$-minor determinants of $A$ coincide with $\pm \nabla(1)$. Thus the proof is complete.

Finally, we shall prove the following theorem.

**Theorem 3.** Let $f(x)$ be a symmetric polynomial of even degree whose constant term is different from zero. Then there exists a link of any given multiplicity $\mu$ whose $\nabla$-polynomial is $f(x)$.

Proof. In order to construct a link $\kappa$ of multiplicity $\mu$ whose $\nabla$-polynomial is $f(x)$, we shall determine $v_{i,j}$ ($i, j = 1, 2, \ldots, 2h + \mu - 1$) in (*).

To begin with, let $v_{1,4}, v_{3,6}, \ldots, v_{2h-3,2h}$ and $v_{2h+2,2h+2}, \ldots, v_{2h+\mu-1,2h+\mu-1}$ and $v_{2,2h+1}$ be equal to 1, let $v_{1,3}, v_{3,5}, \ldots, v_{2h-3,2h-1}$ and $v_{1,1}$ be undetermined, let $v_{2h+1,2h+1}$ be equal to $-f(1)$, and let the others be equal to zero.

Then we have
$M_{h,\mu}(t) =$

$$
\begin{array}{cccccc}
 v_{1,1}(1-t) & t & \cdots & 0 & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & t & v_{2h-3,2h-1}(1-t) (1-t) \\
0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & \cdots & v_{2h-3,2h-1}(1-t) & 0 & 0 \\
0 & 0 & \cdots & (1-t) & 0 & -1 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{array} = (1-t) 0 \cdots 0 \\
(1-t) 0 \cdots 0 \\
\vdots \\
0 0 \cdots 0 \\
0 0 \cdots 0 \\
0 0 \cdots 0 \\
0 0 \cdots 0 \\
\end{array}
$$

Interchange the second column with the $(2h+1)$-th column and add the $(2h+1)$-th row multiplied by $-t$ to the first row. Then, we have

$-M_{h,\mu}(t) = \overline{M}_{h,\mu}(t) =$

$$
\begin{array}{cccccc}
 v_{1,1}(1-t) & f(1)t & \cdots & 0 & 0 & 0 \\
-1 & 1-t & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & t & v_{2h-3,2h-1}(1-t) (1-t) \\
0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & \cdots & v_{2h-3,2h-1}(1-t) & 0 & 0 \\
0 & 0 & \cdots & (1-t) & 0 & -1 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{array} = -f(1) 0 \cdots 0 \\
-1 0 \cdots 0 \\
\vdots \\
0 1 \cdots 0 \\
0 0 \cdots 0 \\
0 \cdots \cdots \\
0 \cdots \cdots \\
\end{array}
$$

Now we shall prove our theorem by a mathematical induction on the degree of $f(t)$.

If the degree of $f(x)$ is 2, i.e. if

$$f(t) = c_0 t^2 + c_1 t + c_0,$$

then we set $h = 1$.

Since

$$\overline{M}_{1,\mu}(t) = \begin{vmatrix} v_{1,1}(1-t) & f(1)t \\ -1 & (1-t) \end{vmatrix} = v_{1,1} t^2 - (2v_{1,1} - f(1)) t + v_{1,1},$$

if we set $v_{1,1} = c_0$, then $|\nabla(t)| = |\overline{M}_{1,\mu}(t)| = |f(t)|$.

Suppose that our theorem is proved for the case where the degree of $f(t)$ is $2(n-1)$. 

Then let
\[ f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_{n-1} t^{n+1} + c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0. \]

We set \( h = n \).

Let \( M'_{n,\mu}(t) \) be the \((2n-1)\)-minor determinant which is deduced from \( M_{n,\mu}(t) \) by striking out \((2n-1)\)-th row and \((2n-1)\)-th column. By a simple calculation we have
\[ \tilde{M}'_{n,\mu}(t) = -(1-t)^n \tilde{M}'_{n-1,\mu}(t) \]
and since \( \tilde{M}'_{1,\mu}(t) = (1-t) \), we have further
\[ \tilde{M}'_{n,\mu}(t) = (-1)^{n-1}(1-t)^{2n-1}. \]

Developing \( \tilde{M}_{n,\mu}(t) \) at the \((2n-1)\)-th column, we have
\[ \tilde{M}_{n,\mu}(t) = v_{n-3,2n-1} t (1-t)^2 \tilde{M}'_{n-1,\mu}(t) + \{ t \tilde{M}_{n-1,\mu}(t) - (1-t)^2 v_{2n-3,2n-1} \tilde{M}'_{n-1,\mu}(t) \} \]
\[ = t \tilde{M}_{n-1,\mu}(t) - (1-t)^2 v_{2n-3,2n-1} \tilde{M}'_{n-1,\mu}(t) \]
\[ = t \tilde{M}_{n-1,\mu}(t) + (-1)^{n-1} v_{2n-3,2n-1} (1-t)^{2n}. \]

In order that
\[ \tilde{M}_{n,\mu}(t) = f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_1 t + c_0, \]
it is sufficient that it holds
\[(***) \quad \tilde{M}_{n-1,\mu}(t) = \frac{c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_1 t + c_0 + (-1)^n v_{2n-3,2n-1} (1-t)^{2n}}{t}. \]

If we set \( v_{2n-3,2n-1} = (-1)^{n+1} c_0 \), then the degree of the right hand side of (***) is \(2(n-1)\). Hence, by the assumption of induction we can determine \( v_{1,1}, v_{1,3}, \ldots, v_{2n-5,2n-3} \) satisfying (**). Therefore, \( v_{1,1}, v_{1,3}, \ldots, v_{2n-5,2n-3} \) can be determined such that \( |\nabla(t)| = |\tilde{M}_{n,\mu}(t)| = |f(t)| \).

Hence, we have a link of multiplicity \( \mu \) whose \( \nabla \)-polynomial is \( f(t) \), and the proof is complete.

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