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CUT-AND-PASTES OF INCOMPRESSIBLE SURFACES IN 3-MANIFOLDS

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1. Introduction

Let M be a compact orientable 3-manifold and F_1 and F_2 properly embedded surfaces in M. If F_1 and F_2 intersect transversely, then by cutting F_1 and F_2 along the intersection and regluing them in a different way, we obtain another embedded surface in M.

DEFINITION. Let F_1 and F_2 be orientable surfaces properly embedded in M intersecting transversely. A *cut-and-paste* (*CP*) *operation* on a component C of $F_1 \cap F_2$ is the following operation in a regular neighborhood of C, $N(C)$: Cut F_1 and F_2 on C and reglue them in a different way. See Figure 1.1.

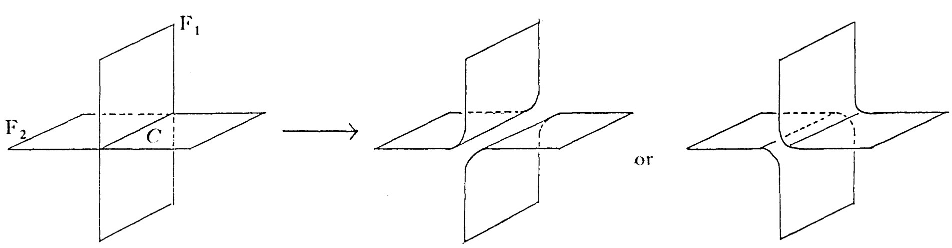

Fig 1.1.

Note that there are two choices in regluing. When we apply a CP operation on each component of $F_1 \cap F_2$, we obtain an embedded surface F in M. We say that F is obtained from F_1 and F_2 by a (way of) CP operation.

Suppose that both F_1 and F_2 are incompressible. In general, a surface which is obtained from F_1 and F_2 by a CP operation is possibly compressible. But we can prove that in certain cases there is a CP operation which yields an

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incompressible surfaces.

Theorem 1. *Let F_1 and F_2 be incompressible surfaces of genus greater than zero properly embedded in M which intersect transversely. If F_1 or F_2 is a torus, then we can obtain an incompressible surface F from F_1 and F_2 by a CP operation.*

Then we show that the assumption of Theorem 1 cannot be omitted in general. In fact, we prove;

Theorem 2. *For any inetgers n_1 and n_2 which are greater than one, there exist a closed orientable 3-manifold M and connected incompressible surfaces F_1 and F_2 properly embedded in M such that they intersect transversely, $g(F_i)=n_i$ ($g(F)$ is the genus of F) and for any surface F obtained from F_1 and F_2 by CP operations, each component of F bounds a handlebody.*

By applying Theorem 1 a number of times, we have the following corollarly.

Corollary 3. *Let $T_1, T_2, \cdots, T_n$ $(n\geq 2)$ be properly embedded incompressible tori in M such that any two of them intersect transversely. Then there exists an incompressible surface F such that $F\subset \cup_{i=1}^{n} T_i \cup N(\cup_{1\leq i\leq j\leq n} T_i \cap T_j)$.*

Let $\mathcal{S}$ be the set of isotopy classes of orientable, incompressible, ∂-incompressible suıfaces in M. And let $\mathcal{S}'$ be the set of isotopy classes of (not necessarily orientable) surfaces S properly embedded in M such that each component of the closure of $\partial N(S)-\partial M$ is incompressible and ∂-incompressible. We call such a surface injective and ∂-injective respectively. Then Oertel [5] defined a function q: $\mathcal{S}\times\mathcal{S}\to$ {finite subset of $\mathcal{S}'$} as follows: Given a pair of isotopy classes of incompressible surfaces, we choose representatives F_1 and F_2 with suitably simplified intersection. Then $q([F_1], [F_2])$ is defined to be the set of isotopy classes of injective surfaces obtained from F_1 and F_2 by CP operations. Oertel showed that the function q is well-defined. In general, for a given pair $[F_1]$, $[F_2]$, $q([F_1], [F_2])$ is possibly an emptyset. But when F_1 or F_2 is a torus, Theorem 1 immediately implies the following:

Corollary 4. *Let $[F_1]$, $[F_2]$ be a pair of isotopy classes of incompressible surfaces in M. If F_1 or F_2 is a torus, then $q([F_1],[F_2])$ is not an emptyset.*

REMARK. When F_1 and F_2 are oriented surfaces, we often use a cut-and-paste operation such that the way of regluing is compatible with orientations on F_1 and F_2. We call this operation an *oriented cut-and-paste (OCP) operation.* We can consider the same problem as Theorem 1 for OCP operations. But there is an example such that we cannot obtain incompressible surfaces from incompressible tori by OCP operations. For example, let M be a Seifert fibered space over S^2 with four singular fibers. Let p be a projection of M to S^2. We consider two incompressible tori T_1 and T_2 such that T_i is a union of regular fibers and $p(T_i)$ $(i=1, 2)$ are as indicated in Figure 1.2. Then we can check that for any orientations of T_1 and T_2, we cannot obtain an incompressible surface from T_1 and T_2 by an OCP operation.

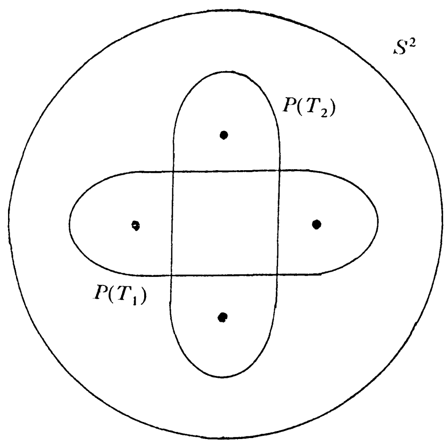

Fig 1.2.

Throughout this paper, we work in the piecewise linear category. For the definition of standard terms of 3-dimensional topology, see [2]. For a subcomplex K of a given H, $N_H(K)$ denotes a reglar neighborhood of K in H. When H is well understood, we often abbreviate $N_H(K)$ to $N(K)$.

2. Proof of Theorem 1

Lemma 2.1. *Let F_1 and F_2 be incompressible surfaces in a 3-manifold M with transverse intersection. Then we can obtain incompressible surfaces $\widetilde{F}_1$ and $\widetilde{F}_2$ by some CP operations on closed curves of $F_1 \cap F_2$ which are inessential on F_1, such that $\widetilde{F}_i$ is homeomorphic to $F_i (i=1, 2)$ and each component of $\widetilde{F}_1 \cap \widetilde{F}_2$ is essential in $\widetilde{F}_1$.*

Proof. If each component of $F_1 \cap F_2$ is an essential curve of F_1, we take $\widetilde{F}_i = F_i$ $(i=1, 2)$. In general, we apply an argument of the proof of [2, Lemma 4, 6].

Let n be the number of components of $F_1 \cap F_2$ which is inessential on F_1. Assume $n \geqq 1$. Let $S = F'_1 \cup F'_2$ be a 2-component 2-manifold such that $F'_i \cong F_i$ $(i=1, 2)$ and f_0: $S \to M$ an immersion such that $f_0|_{F'_i}$: $F'_i \to F_i$ is a homeomorphism. Let $\Sigma_0 = \{x \in S \mid \exists x' \in S$ such that $f_0(x) = f_0(x')\}$. Then $f_0(\Sigma_0) = F_1 \cap F_2$ and Σ_0 consists of closed curves on S. Let Σ'_0 be a subset of Σ_0 which consists of inessential curves on S. Since F_1 and F_2 are incompressible, $C_1 \subset \Sigma'_0$ if and only if $C_2 \subset \Sigma_0'$ for $C_2 \subset \Sigma_0$ with $f_0^{-1}(f_0(C_1)) = C_1 \cup C_2$. Hence Σ_0' consists of $2n$ closed curves.

We deffne an immersion $f_1 | S \to M$ as follows; fix a closed curve $C_1^1 \subset \Sigma_0'$ and let $f_0^{-1}(f_0(C_1^1)) = C_1^1 \cup C_2^1$. Let D_i be a disk on S such that $\partial D_i = C_i^1$ and V a solid torus which is a regular neighborhood of $f_0(C_1^1)$. Then $f_0^{-1}(V)$ is a union of two disjoint annuli A_1 and A_2 with $C_i^1 \subset A_i (i=1, 2)$. Put $D_i' = D_i - \text{Int}\, A_i$, $D_i'' = D_i \cup A_i$. There exists disjoint annuli B_1 and B_2 on ∂V with $\partial B_1 = f_0(\partial D_1'' \cup \partial D_2')$ and $\partial B_2 = f_0(\partial D_2'' \cup \partial D_1')$. We define f_1 by putting $f_1|_{S-(D_1'' \cup D_2'')} = f_0|_{S-(D_1'' \cup D_2'')}, f_1(A_i) = B_i, f_1(D_1') \subset f_0(D_2')$ and $f_1(D_2') \subset f_0(D_1')$ so that $\Sigma_1 = \Sigma_0 - \{C_1^1 \cup C_2^1\}$. Then Σ_1' consists of $2(n-1)$ closed curves. Note that $f|_{F_i} (i=1, 2)$ may have self intersections.

For $2 \leq k \leq n$, we define an immersion f_k: $S \to M$ inductively. Assume f_{k-1} was defined, $\Sigma_{k-1} = \{x \in S \,|\, {}^\exists x' \in S$ such that $f_{k-1}(x) = f_{k-1}(x')\}$ consists of closed curves, and for each component $C_1 \subset \Sigma_{k-1}' = \{C \subset \Sigma_{k-1} \,|\, C$ is an inessential curve on $S\}$, $f_{k-1}^{-1}(f_{k-1}(C_1)) = C_1 \cup C_2$ and $C_2 \subset \Sigma_{k-1}'$. Fix a component C_1^k of Σ_{k-1}' and let $f_{k-1}^{-1}(f_{k-1}(C_1^k)) = C_1^k \cup C_2^k$. For $i=1, 2$, let D_i a disk on S such that $\partial D_i = C_i^k$, V a regular neighborhood of $f_{k-1}(C_1^k)$, A_1 and A_2 disjoint annuli of $f^{-1}(V)$ with $C_i^k \subset A_i$, $D_i' = D_i - \text{Int}\, A_i$, $D_i'' = D_i \cup A_i$, $B_1, B_2 \subset \partial V$ annuli with $\partial B_1 = f_{k-1}(\partial D_1'' \cup \partial D_2')$ and $\partial B_2 = f_{k-1}(\partial D_2'' \cup \partial D_1')$.

We divide into two cases a) $D_1 \cap D_2 = \emptyset$ and b) $D_2 \subset \text{Int}\, D_1$.

In case a), we define f_k by putting $f_k|_{S-(D_1'' \cup D_2'')} = f_{k=1}|_{S-(D_2'' \cup D_2'')}, f_k(A_i) = B_i$, $f_k(D_1') \subset f_{k-1}(D_2')$ and $f_k(D_2') \subset f_{k-1}(D_1')$ so that $\Sigma_k = \Sigma_{k-1} - \{C_1^k \cup C_2^k\}$. In case b), put $E = D_1' - \text{Int}\, D_2''$. We define f_k by putting $f_k|_{S-D_1''} = f_{k-1}|_{S-D_1''}, f_k(D_2') \subset f_{k-1}(D_2'), f_k(A_i) = B_i$, and $f_k(E) \subset f_{k-1}(E)$ so that $\Sigma_k = \Sigma_{k-1} - \{C_1^k \cup C_2^k\}$.

In this way, we obtain a sequence of maps $f_0, f_1, \cdots, f_n$ from S to M such that $\Sigma_k = \Sigma_{k-1} - \{C_1^k \cup C_2^k\}$, where $C_1^k, C_2^k \subset \Sigma_{k-1}'$ with $f_{k-1}(C_1^k) = f_{k-1}(C_2^k)$ for $1 \leq k \leq n$.

Since Σ_0' consists of $2n$ components, $\Sigma_n = \Sigma_0 - \Sigma_0'$ and $\Sigma_n' = \emptyset$. Put $f_n(F_i') = \tilde{F}_i$ $(i=1, 2)$. Since the definition of $f_k|_{A_1 \cup A_2}$ corresponds to a CP operation on $f_{k-1}(C_1^k)$ $(1 \leq k \leq n)$, $\tilde{F}_1$ and $\tilde{F}_2$ is obtained from F_1 and F_2 by CP operations on $f_0(\Sigma_0')$, which is equal to the set of inessential curves in $F_1 \cap F_2$. And $\tilde{F}_1 \cap \tilde{F}_2$ consists of essential curves. On the other hand, since $f_k|_{S-(D_1'' \cup D_2'')} = f_{k-1}|_{S-(D_1'' \cup D_2'')}$, for $i=1, 2$, $\tilde{F}_i - \tilde{E}_i = F_i - E_i$ for a union of certain disks E_i ($\tilde{E}_i$, resp.) on F_i ($\tilde{F}_i$, resp.). Hence $\tilde{F}_i$ is incompressible.

This completes the proof of Lemma 2.1.

DEFINITION. Let F_1 and F_2 be properly embedded surfaces in M which intersect transversely. Let F_i' be a closure of a component of $F_i - (F_1 \cap F_2)$ $(i=1, 2)$. We say that F_1 and F_2 have a *semi-product region* between F_1' and F_2' if there exists a map f of a manifold X to M satisfying the following (1)-(4):

(1) $X = W \times [0, 1] - \cup_{i=1}^n \text{Int}\, B_i$, where W is homeomorphic to F_1' and

$B_1, B_2, \cdots, B_n$ are mutual,y disjoint 3-balls in Int $(W\times[0, 1])$.

(2) $f(\partial W\times[0, 1])=\partial F_1'=\partial F_2'$.

(3) $f|_{W\times\{0\}}$ is a homeormophism of $W\times\{0\}$ to F_1' and $f|_{W\times\{1\}}$ is a homeomorphism of $W\times\{1\}$ to F_2'.

(4) $f|_{X-(\partial W\times[0,1])}$ is an embedding.

Lemma 2.2. *Let F_1 and F_2 be properly embedded incompressible surfaces in M which intersect transversely. Suppose that F_1 and F_2 have a semi-product region between F_1' and F_2' ($F_i'\subset F_i$, $i=1, 2$). Then $\hat{F}_i=(F_i-F_i')\cup F_{3-i}'$ is also incompressible ($i=1, 2$).*

Proof. It is enough to prove that $\hat{F}_1=(F_1-F_1')\cup F_2'$ is incompressible. Assume that there exists a compressing disk D of $\hat{F}_1$. Since F_1 and F_2 are incompressible, we may asume that $D\cap F_2'$ consists of some arcs $a_1, a_2, \cdots, a_m$. Using $X=W\times[0, 1]-\cup_{i=1}^{n}\mathrm{Int}\, B_i$ and the map f, we can find a disk D_i in M such that $\partial D_i=a_i\cup b_i$ and $b_i\subset F_1'$ ($i=1, 2, \cdots m$). Let $D'=D\cup_{i=1}^{m}D_i$. Then D' is an immersed disk in M with $\partial D'\subset F_1$. Clearly $\partial D'$ is essential on F_1, contradicting the incompressibility of F_1. Hence $\hat{F}_1$ is incompressible.

This completes the proof of Lemma 2.2.

Proof of Theorem 1. If $F_1\cap F_2$ contains a component C which is inessential on F_1, then we consider incompressible surfaces $\tilde{F}_1$ and $\tilde{F}_2$ in Lemma 2.1. Moreover if $\tilde{F}_1$ and $\tilde{F}_2$ have a smi-product region, we consider incompressible surfaces $\hat{F}_1$ and $\hat{F}_2$ in Lemma 2.2. If Theorem 1 holds for $\hat{F}_1$ and $\hat{F}_2$, we may regard that the obtained surface F is also obtained from F_1 and F_2 by a CP operation by Lemmas 2.1 and 2.2. Hence, without loss of generality, we may assume the following (1)-(3):

(1) F_1 is a torus and F_2 is a surface of genus greater than zero.

(2) Each component of $F_1\cap F_2$ is an essential curve on F_1.

(3) F_1 and F_2 do not have a semi-product region.

Let N_1 and N_2 be components of $N(F_1)-F_1$. Let F be a surface obtained from F_1 and F_2 by the following CP operation; for each component A of $F_1-\mathrm{Int}\, N(F_1\cap F_2)$, a component of ∂A is regluded to $N_1\cap\partial(F_2-\mathrm{Int}\, N(F_1\cap F_2))$ and the other component of ∂A is reglued to $N_2\cap\partial(F_2-\mathrm{Int}\, N(F_1\cap F_2))$. See Figure 2.1.

We will prove that F is incompressible.

We may assume that F_1 and F intersect transversely and for each component A of $F_1-\mathrm{Int}\, N(F_1\cap F_2)$, $A\cap F$ consists of an essential simple closed curve in A.

Suppose that there exists a compressing disk D of F. Since F_1 is incompressible, we may assume $D\cap F_1$ does not contain a circle component.

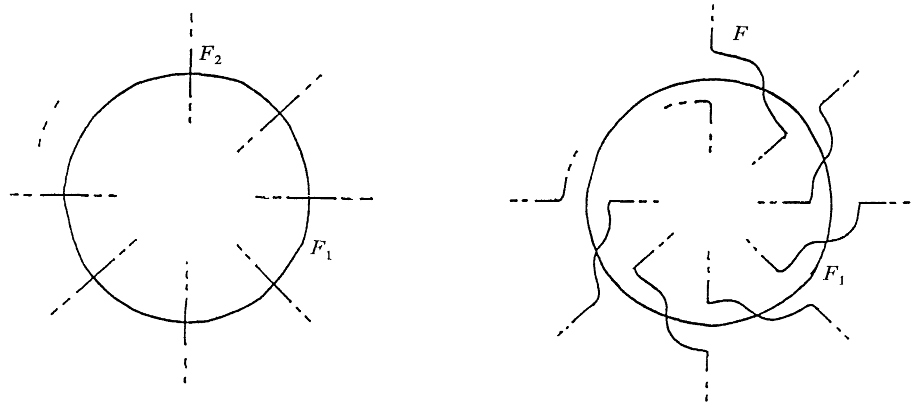

Fig 2.1.

Claim 2.3. $\partial D\cap(F_1\cap F)\neq\emptyset$.

Proof. Suppose that $\partial D\cap(F_1\cap F)=\emptyset$. Then we may assume $\partial D\subset F\cap F_2$. Since F_2 is incompressible, there exists a disk D' on F_2 such that $\partial D=\partial D'$. Since ∂D is an essential curve on F, D' contain a component C of $F_1\cap F_2$. C bounds a disk $D''(\subset D')$ and by the condition (2), C is an essential curve on F_1. It contradicts the incompressibility of F_1. Therefore $\partial D\cap(F_1\cap F)\neq\emptyset$, completing the proof of Claim 2.3.

By Claim 2.3, $D\cap F_1$ consists of some arcs. Let a be an outermost arc of $D\cap F_1$ on D, and $D'\subset D$ an outermost disk such that $\partial D'=a\cup b$ with $b\subset\partial D$. Then using D', we can find a embedded disk E in M such that $\partial E=a'\cup b'$, $a'\subset F_1$, $b'\subset F_2$ with $a\cap a'\neq\emptyset$, $b\cap b'\neq\emptyset$ and $\text{Int}\, E\cap(F_1\cup F_2)=\emptyset$. Let A be a closure of a component of $F_1-(F_1\cap F_2)$ which contains a', and B a closure of a component of $F_2-(F_1\cap F_2)$ which contains b'. By the condition (2), A is an annulus. Consider $E\times[0, 1]$ with $E\times[0, 1]\cap(F_1\cup F_2)=\partial E\times[0, 1]$. Then $E'=(E\times[0, 1]\cup A)-(E\times(0, 1))$ is an embedded disk in M such that $\partial E'\subset F_2$. Since F_2 is incompressible, $\partial E'$ is an inessential curve on F_2. Let E'' be a disk on F_2 with $\partial E''=\partial E'$. If $E''\cap(E\times(0, 1))\neq\emptyset$, then each component of $\partial A(\subset F_1\cap F_2)$ also bounds a disk on F_2. But it contradicts the condition (2). Hence $E''\subset B$ and B is an annulus. Using $A\cup B\cup E\times[0, 1]$, we can see that F_1 and F_2 have a semi-product region between A and B. It contradicts the condition (3). Therefore F is incompressible.

This completes the proof of Theorem 1.

3. Boundary irreducibility of certain 3-manifolds

For the proof of Theorem 2, we construct certain 3-manifolds with incompressible surfaces. A closed orientable surface F properly embedded in a 3-

manifold M is incompressible if and only if $\partial N(F)$ is incompressible in each component of $M-\text{Int}\,N(F)$. In this section, we examine the incompressibility of boundaries of certain 3-manifolds. We say that an orientable 3-manifold M is *∂-irreducible* if M is irreducible and ∂M is incompressible in M.

Suppose that M does not contain a fake 3-ball. Then M is ∂-irreducible iff $\pi_1(M)$ is not a free product or a cyclic group (cf. [2]). Lemma 3.1 shows that for certain one-relator groups, we can examine that the group is a free product or not.

DEFINITION. Let $\langle x_1, x_2, \cdots, x_g\rangle$ be a free group of rank $g(g\geq 2)$ with generators $x_1, x_1, \cdots, x_g$ and H_g a handlebody of genus g. We say that a simple closed curve C on ∂H_g is a *representation curve* of an element $r\in\langle x_1, x_2, \cdots, x_g\rangle$ if $\pi_1(H_g)\cong\langle x_1, x_2, \cdots, x_g\rangle\ni\text{Incl}_*(C)=r$. ($\text{Incl}_*$ is a homomorphism which is induced by the inclusion map.)

Lemma 3.1. *Suppose that r has (at least one) representation curve. Then the following* (1)-(3) *are mutually equivalent*:

(1) *$\langle x_1, x_2, \cdots, x_g: r\rangle$ is not a free product group or a cyclic group.*

(2) *There exists a representation curve C of r on ∂H_g such that ∂H_g-C is incompressible in H_g.*

(3) *For any representation curve C of r, ∂H_g-C is incompressible in H_g.*

Proof. (3)$\Rightarrow$(2) is clear.

(2)$\Rightarrow$(1): Let $M=H_g\cup_C(D^2\times I)$ be a 3-manifold obtained from H_g by attaching a 2-handle $D^2\times I$ along C. By [1], [3] or [6], M is ∂-irreducible. On the other hand, $\pi_1(M)\cong\langle x_1, x_2, \cdots, x_g: r\rangle$. Hence (1) holds.

(1)$\Rightarrow$(3): Suppose that there exists a representation curve C of r such that ∂H_g-C is compressible in H_g. Let B be a compressing disk of ∂H_g-C in H_g. If B is a non-separating disk of H_g, then B is also a non-separating disk of $M=H_g\cup_C(D^2\times I)$. If $H_g-B=V_1\cup V_2$ and V_1 and V_2 are handlebodies, then M is a disk sum of V_1 and $V_2\cup_C(D^2\times I)$. In both cases, $\pi_1(M)\cong\langle x_1, x_2, \cdots, x_g: r\rangle\cong Z*G$ for some group G.

This completes the proof of Lemma 3.1.

Next, we examine the ∂-irreducibility of manifolds which are obtained from handlebodies by Dehn surgeries on links in them. Let V be a handlebody and k a simple closed curve on ∂V. We define *a surgery on pushed k with surgery coefficient p/q* (g.c.d$(p, q)=1$) as follows: Consider an annulus A in V such that $\partial A=k\cup k'$ and $A\cap\partial V=k$ (We say k' is a *pushed k*). There is a neighborhood of k', $N(k')$ such that $N(k')\cap A$ is an annulus. Put $l=\partial N(k')\cap A$ and let m be a meridian of k' on $\partial N(k')$. Remove $\text{Int}N(k')$ and attach a solid torus V' to it so that a meridian m' on $\partial V'$ is attached to a curve C on $\partial N(k')$ with $[C]=p[m]+$

$q[l]\in H_1(\partial N(k'); Z)$.

Lemma 3.2. *Let V be a handlebody of genus greater than one and $C_1, C_2, \cdots, C_n$ $(n\geq 1)$ mutually disjoint simple closed curves on ∂V. If $\partial V-\cup_{i=1}^n C_i$ is incompressible in V and $|p_i|\geq 2$ $(i=1, 2, \cdots, n)$, then the manifold M which obtained from V by surgeries on pushed $C_1, C_2, \cdots, C_n$ with surgery coefficient $p_1/q_1, p_2/q_2, \cdots, p_n/q_n$ is ∂-irreducible.*

Proof. Let $V_1, V_2, \cdots, V_n$ be solid tori and m_i and l_i meridian and longitude on ∂V_i. Consider a simple closed curve C_i'' on ∂V_i such that $[C_i'']=r_i[m_i]+p_i[l_i]\in H_1(\partial V_i; Z)$, for integers r_i and s_i with $p_i s_i-q_i r_i=1$. Then we can regard M as the 3-manifold obtained form V and $V_1, V_2, \cdots, V_n$ by identifying $N_{\partial V_i}(C_i'')$ to $N_{\partial V}(C_i)$.

Since $|p_i|>0$ and $\partial V-\cup_{i=1}^n C_i$ is incompressible in V, M is irreducible. We will prove that ∂M is incompressible in M. Note that since $|p_i|\geq 2$, for any compressing disk D of V_i, $\#(\partial D\cap N_{\partial V_i}(C_i''))\geq 2$. Suppose that there exists a compressing disk D of ∂M in M. Since $\partial V-\cup_{i=1}^n C_i$ is incompressible in V, D must intersect with $\cup_{i=1}^n N_{\partial V}(C_i)$ in at least one arc. We may assume D has a minimal number of components in all such disks. By standard innermost circle and outermost arc arguments, we may assume $D\cap(\cup_{i=1}^n N_{\partial V}(C_i))$ consists of some essential arcs in $N_{\partial V}(C_i)$. Let a be an outermost arc of $D\cap(\cup_{i=1}^n N_{\partial V}(C_i))$ on D, D' an outermost disk on D with $\partial D'=a\cup b, b\subset\partial D$ and $a\subset N_{\partial V}(C_j)$ $(1\leq j\leq n)$. By the minimality of the number of intersections, $\partial D'$ is an essential curve on ∂V or ∂V_j. Since $\partial D'$ intersects with $N_{\partial V}(C_j)$ in an arc, D' is contained in V. But it contradicts the following Claim 3..3.

Claim 3.3. *If $\partial V-\cup_{i=1}^n C_i$ is incompressible in V, then for any compressing disks D of V, $\#(\partial D\cap(\cup_{i=1}^n C_i))\geq 2$.*

Proof of Claim 3.3. Suppose that there exists a compressing disk D of ∂V such that ∂D intersects with $\cup_{i=1}^n C_i$ in a point $p\in C_j (1\leq j\leq n)$. Consider a regular neighborhood of D, $D\times[0, 1]\subset V$ such that $D\times[0, 1]\cap\partial V=\partial D\times[0, 1]$ and $(\partial D\times[0, 1])\cap(\cup_{i=1}^n C_i)=p\times[0, 1]$. Then $D'=\partial(N(C_j)\cup(D\times[0, 1]))-\text{Int}(\partial N(C_j)\cap\partial V)\cup(\partial D\times(0, 1))$ is a compressing disk of $\partial V-\cup_{i=1}^n C_i$, a contradiction.

Hence Claim 3.3 holds.

This completes the proof of Lemma 3.2

To know the incompressibility of $\partial V-\cup_{i=1}^n C_i$ in V, we use the following Lemma 3.4.

Let H_g be a handlebody of genus $g (g\geq 2)$ and $\{D_1, D_2, \cdots, D_{3g-3}\}$ a set of mutually disjoint non-parallel compressing disks in H_g. Then each component of $H_g-\cup_{i=1}^{3g-3}(D_i\times(0, 1))$ is a 3-ball B such that $\partial B-\text{Int}(\partial H_g\cap\partial B)$ consists of three disks D_1', D_2', D_3' and D_i' is parallel to D_j for some $1 \leq j \leq 3g-3$ in H_g ($i=1, 2, 3$). Let $C_1, C_2, \cdots, C_n$ be mutually disjoint simple closed curves on ∂H_g. We may assume each component of $(D_i \times [0, 1]) \cap C_j$ is an essential arc on $\partial D_i \times [0, 1]$. We say that $C = \cup_{i=1}^n C_i$ is *full with respect to* $D_1, D_2, \cdots, D_{3g-3}$ if for any component B of $H_g - \cup_{i=1}^{3g-3}(D_i \times (0, 1))$, C satisfies the following conditions (1), (2);

(1) each component of $C \cap \partial B$ is an arc connecting D_i' and D_j' for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

(2) for any pair of D_i' and D_j' ($i \neq j$, and $i, j \in \{1, 2, 3\}$), there is a sub arc a of C on ∂B connecting D_i' and D_j'.

Lemma 3.4. ([3, Lemma 6.1]). *Let $\{C_1, C_2, \cdots, C_n\}$ be a set of mutually disjoint simple closed curves on ∂H_g. If there exists a set of mutually disjoint non-parallel compressing disks $\{D_1, D_2, \cdots, D_{3g-3}\}$ of H_g such that $C = \cup_{i=1}^n C_i$ is full with respect to $D_1, D_2, \cdots, D_{3g-3}$, then $\partial H_g - C$ is incompressible in H_g.*

Let N be a ∂-irreducible 3-manifold with boundary and $\{C_1, C_2, \cdots, C_n\}$ a set of mutually disjoint non-parallel simple closed curves such that $\partial N - \cup_{i=1}^n C_i$ is incompressible in N. We consider a manifold M which is obtained from N by attaching 2-handles along $C_1, C_2, \cdots, C_n$. In the case that $n=1$, M is ∂-irreducible by [1], [3], or [6]. But in general cases, M may not be ∂-irreducible. The following Lemma 3.5 gives a sufficient condition for M to be ∂-irreducible.

Let C be a simple closed curve on a surface F and a an arc on F with $a \cap C = \partial a$. We say that a is *an inessential arc relative to C* if there exists a disk D on F such that $\partial D = a \cup b$ with $b \subset C$. If a is not an inessential arc relative to C, then we say that a is an essential arc relative to C.

Lemma 3.5. *Let $\{C_1, C_2, \cdots, C_n\}$ ($n \geq 1$) be a set of mutually disjoint simple closed curves on ∂N. Suppose that there exists a set of mutually disjoint properly embedded disks $\{D_1, D_2, \cdots, D_n\}$ which satisfies the following conditions* (1)-(3);

(1) *each component of $N - \cup_{i=1}^n(D_i \times (0, 1))$ is ∂-irreducible,*

(2) *if $i \neq j$, then $D_i \cap C_j = \emptyset$,*

(3) *if $i=j$, then $\sharp(D_i \cap C_j)=2$, the algebraic intersection number of ∂D_i and C_j on ∂N is 0, and each component of $C_i - (C_i \cap \partial D_i)$ is an essential arc relative to ∂D_i.*

Then the manifold M which is obtained from N by attaching 2-handles along $C_1, C_2, \cdots, C_n$ is ∂-irreducible.

Proof. Put $\bar{D} = \cup_{i=1}^n D_i$ and $\bar{C} = \cup_{i=1}^n C_i$. Let $\bar{D} \times [0, 1]$ be a regular neighborhood of $\bar{D}$. We may assume that each component of $(\partial \bar{D} \times [0, 1]) \cap \bar{C}$ is an essential arc on a component of $\partial \bar{D} \times [0, 1]$. Let N' be a component of $N - (\bar{D} \times (0, 1))$. We abbreviate $D_i \times \{0\}$ and $D_i \times \{1\}$ on $\partial N'$ to D_i for simplicity. Then $\partial N'$ is a union of some D_i's and $N' \cap \partial N$.

Claim 3.6. *Let a be a component of* $C_i-(C_i\cap(D_i\times(0,1))$ *and* N' *the component of* $N-(\bar{D}\times(0,1))$ *which contains a. Then a is an essential arc relative to* ∂D_i *on* $\partial N'$.

Proof. Note that since $\mathrm{Int}_{\partial N}[\partial D_i, C_i]=0$, ∂a is contained in one component of $\partial N'-\partial N$. Assume that a is an inessential arc relative to ∂D_i on $\partial N'$. Then $a\cup b$ $(b\subset\partial D_i)$ bounds a disk D on $\partial N'$. We may assume $a\cup b$ is an "innermost" curve on $\partial N'$, i.e. D does not contain any other D_j. Hence D is contained in $\partial N'\cap\partial N$ and a is an inessential arc relative to ∂D_i on ∂N. It contradicts to the condition (3). Therefore a is an essential arc relative to ∂D_i on $\partial N'$.

This completes the proof of Claim 3.6.

We say that a closed curve J on ∂N is *$\bar{C}$-inessential* if J bounds a disk on ∂N or J and some components of $\bar{C}$ bounds a planar surface on ∂N. If J is not $\bar{C}$-inessential, we say that J is C-essential.

Suppose that M is not ∂-irreducible, i.e. there exists an essential sphere or a disk F in M. By standard innermost circle and outermost arc arguments, we may assume that F intersects the 2-handles in horizontal disks. Hence $S=F\cap N$ is a planar surface such that at most one component of ∂S is a $\bar{C}$-essential curve and other components are parallel to a component of $\bar{C}$. We will prove that there does not exist such a planar surface S.

The next claim gives a proof of this assertion in a very special case (the case of S a disk).

Claim 3.7. *There does not exist a disk S such that* ∂S *is* $\bar{C}$*-essential.*

Proof. Assume that there exists such a disk S. We suppose that $\#(S\cap\bar{D})$ is minimal over all such disks. Suppose that $\#(S\cap\bar{D})\geq 1$. Then there is an outermost arc a on S and an outermost disk D on S such that $\partial D=a\cup b$, $b\subset\partial S$. Let D_i $(1\leq i\leq n)$ be the disk which contains a and N' the component of $N-(\bar{D}\times(0,1))$ which contains D. By the ∂-irreducibility of N', there exists a 3-ball B in N' such that $\partial B=D\cup D'\cup D_i'$, where $D'\subset\partial N'\cap\partial N$ and $D_i'\subset D_i$. By Claim 3.6, D' does not intersect $\bar{C}$. Hence by using B, we can obtain a disk S' such that $\partial S'$ is $\bar{C}$-essential and $\#(S'\cap\bar{D})<\#(S\cap\bar{D})$, a contradiction.

Hence $\#(S\cap\bar{D})=0$. Then S is contained in a component N' of $N-(\bar{D}\times(0,1))$. Since N' is ∂-irreducible, there is a disk E on $\partial N'$ such that $\partial E=\partial S$ and E contains some D_i's. Then a component d of $C_i-(\partial D_i\times(0,1))$ intersects E. By Claim 3.6, d is an essential arc relative to D_i on N'. Hence d intersects $\partial E=\partial S$. It contradicts the choice of S. Hence there does not exist a disk in N whose boundary is $\bar{C}$-essential.

This completes the proof of Claim 3.7.

By Claim 3,7, if there exists such a planar surface S, then $\#(\partial S)\geq 2$ and

$\partial S \cap \bar{D} \neq \emptyset$. Let S be a planar surface in N such that at most one component J of ∂S is $\bar{C}$-essential, and that each component J' of $\partial S - J$ is parallel to a component C_i of $\bar{C}$. We assume that $\#(S \cap \bar{D})$ is minimal over all such planar surfaces. Let J be a component of ∂S (if exists) which is $\bar{C}$-essential and D_i a component of $\bar{D}$ intersecting $\partial S - J$. Let $K_1, K_2, \cdots, K_n$ be the components of $\partial S - J$ which are parallel to C_i and we suppose that $K_1, K_2, \cdots, K_n$ are contained in $N_{\partial N}(C_i)$ in this order. Since each component of $N - (\bar{D} \times (0, 1))$ is ∂-irreducible, by using standard innermost circle and outermost arc arguments, we may assume $S \cap D_i$ consists of arcs. Let a be an outermost arc of $S \cap D_i$ on D_i and D an outermost disk on D_i with $D \cap S = a$. Put $\partial a = p_1 \cup p_2$. Then we have the following four possible cases.

(a) Both p_1 and p_2 are on J.

(b) $p_1 \in J$ and $p_2 \in \partial S - J$.

(c) $p_1 \in K_j$ and $p_2 \in K_{j+1}$ $(1 \leq j \leq n-1)$.

(d) p_1 and p_2 are on the same component K $(=K_1$ or $K_n)$ of $\partial S - J$.

Let $S' = (S \cup D \times [0, 1]) - D \times (0, 1)$. Then S' is a planar surface. In Case (a), S' has two components, at least one component S'' of S' has a $\bar{C}$-essential curve in $\partial S''$ and $\#(S'' \cap \bar{D}) < \#(S \cap \bar{D})$. It contradicts the choice of S. In Case (b), clearly a component of S' is $\bar{C}$-essential and $\#(S' \cap \bar{D}) < \#(S \cap \bar{D})$, a contradiction. In case (c), $\partial S'$ has a component $L = (K_j \cup K_{j+1} \cup b \times [0, 1]) - b \times (0, 1)$, where $b = \partial D - a$. L bounds a disk B on ∂N. By capping off S' by B and pushing B into N, we obtain a planar surface S'' such that $\#(S'' \cap \bar{D}) < \#(S \cap \bar{D})$, a contradiction. In Case (d), S' consists of two components. Let S'' be a component of S' which does not contain J. Let J' be a component of $\partial S''$ which consists of a subarc of K and a copy of $\partial D - a$.

Claim 3.8. *J' is $\bar{C}$-essential.*

Proof. Assume that J' is $\bar{C}$-inessential. If J' bounds a disk D on ∂N, then a subarc of K is an inessential arc relative to ∂D_i. It contradicts the condition (3). Hence J' bounds a planar surface P on ∂N with some C_j's, say $C_1, C_2, \cdots, C_l$. Note that $J' \cap (\cup_{i=1}^{n} \partial D_i) = \emptyset$. By conditions (2) and (3), for $j = 1, 2, \cdots, l$, a subarc of ∂D_j, d_j is contained in P and d_j is an essential arc relative to C_j. Hence $P - \cup_{j=1}^{l} d_j$ consists of l components $P_1, P_2, \cdots, P_l$ and for each $j = 1, 2, \cdots, l$, $\chi(P_j) \leq 0$. But $1 - l = \chi(P) = \Sigma_{j=1}^{l} \chi(P_j) - l \leq -l$, a contradiction.

This completes the proof of Claim 3.8.

By Claim 3.8 and the fact $\#(S'' \cap \bar{D}) < \#(S \cap \bar{D})$, we have a contradiction.

Hence in any cases it contradicts the choice of S. Therefore $M = N \cup_{\bar{C}} (D^2 \times I)$ is ∂-irreducible.

This completes the proof of Lemma 3.5.

4. Proof of Theorem 2

Proof of Theorem 2. We consider the following two cases and construct a 3-manifold M and incompressible surfaces F_1 and F_2 in M which satisfy the conditions in Theorem 2:

(I) $n_1=n_2\geq 2$.

(II) $n_1>n_2\geq 2$.

Case (I) $n_1=n_2\geq 2$.

We put $n=n_1=n_2$. Let H_1 and H_2 be handlebodies with $g(H_i)=n$ $(i=1, 2)$ and $C_{i,1}, C_{i,2}, \cdots, C_{i,n+1}$ simple closed curves on ∂H_i as indicated in Figure 4.1 (a) (in Figure 4.1, $n=4$). For each $C_{i,j}$, we consider a simple closed curve $C'_{i,j}$ in H_i such that there exists an embedded annulus A and $\partial A=C_{i,j}\cup C'_{i,j}$. $C'_{i,j}$ is a pushed $C_{i,j}$ in the sense of Section 3. Let $F_{i,2}$ be a properly embedded surface in H_i with $F_{i,2}\cap(\cup_{j=1}^{n} C_{i,j})=\emptyset$ $(i=1, 2)$ as indicated in Fugure 4.1 (b). $F_{1,2}(F_{2,2}$, resp.) consists of $[(n+1)/2]$ $([n/2]$, resp.) components, where $[x]$ is the greatest integer which is less than or equal to x.

Put $M=H'_1\cup_f H'_2$, where H'_i is obtained from H_i by performing 2-surgery on $C'_{i,j}$ (; pushed $C_{i,j}$), $(i=1, 2, j=1, 2, \cdots, n+1)$, and f is a homeomorphism of $\partial H'_2$ to $\partial H'_1$ such that $f(\partial F_{2,2})=\partial F_{1,2}$ and $f^{-1}(C_{1,2k+1})$ and $f(C_{2,2k})$ $(k=1, 2, \cdots, [n/2])$ are as indicated in Figure 4.1 (c).

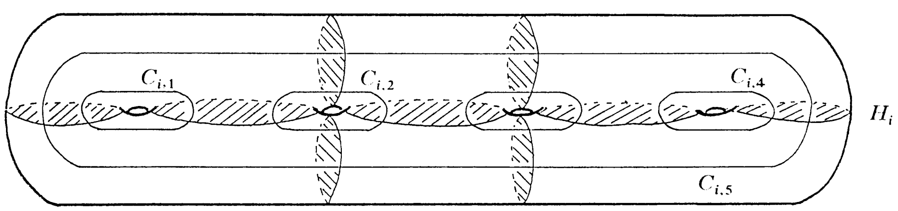

Fig 4.1. (a)

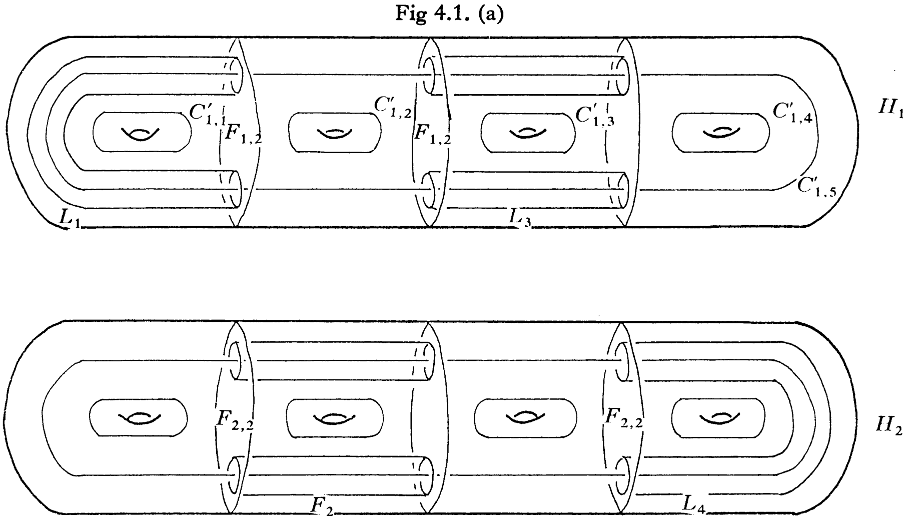

Fig 4.1. (b)

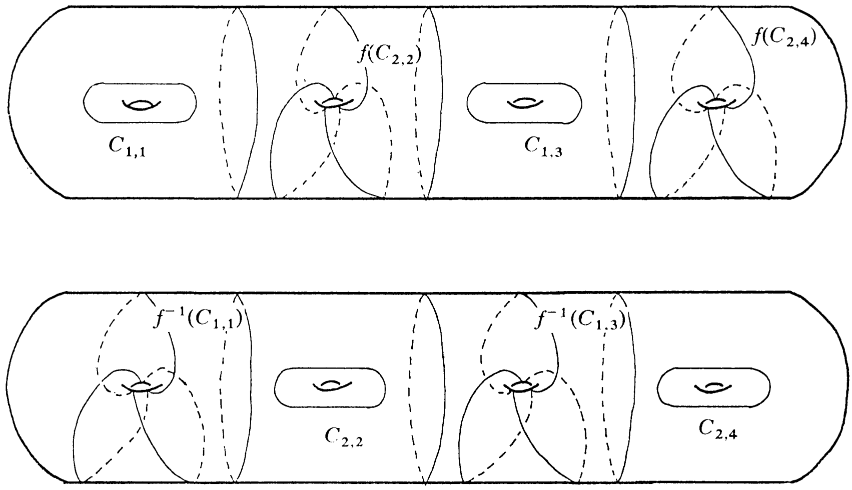

Fig 4.1. (c)

Then M is an orientable closed 3-manifold, $F_1=\partial H'_1$ and $F_2=F_{1,2}\cup F_{2,2}$ are embedded surfaces of genus n, and F_1 and F_2 intersect transversely.

For any orientation of F_1 and F_2, an OCP operation produces two genus n surfaces or two genus two surfaces and $n-2$ genus three surfaces. In both cases, these surfaces bound handlebodies. Let $L_i (i=1, 2, \cdots, n)$ be a closure of a component of $H_j-F_{j,2}$ $(j\equiv i \bmod 2)$ which contains $C'_{j,i}$. And let L'_i be a manifold which is obtained form L_i by 2-surgery on $C'_{j,i}$. Then L'_1 (L'_n, resp.) has a compressing disk D_1 (D_n, resp.) and L'_i $(i=2, 3, \cdots, n-1)$ has compressing disks $D_{i,1}$ and $D_{i,2}$ which are indicated in Figure 4.2.

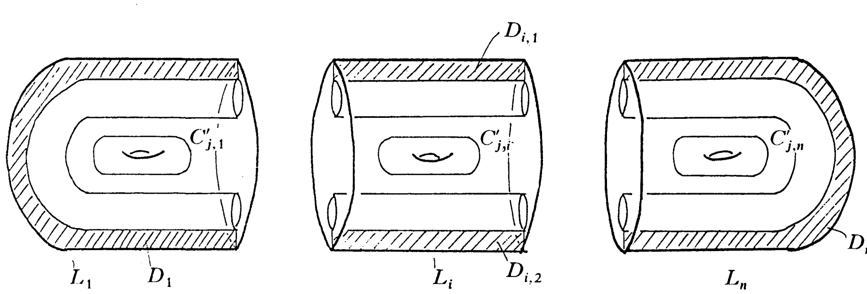

Fig 4.2.

Note that L'_i $(i=1, 2, \cdots, n)$ is a handlebody and $L'_i-D_i\times(0, 1)$ $(i=1, n)$ and $L'_i-\cup_{l=1}^{2}(D_{i,l}\times(0, 1))$ $(i=2, 3, \cdots, n-1)$ are solid tori. Suppose that F is a surface which is obtained from F_1 and F_2 by a CP operation which cannot be realized by an OCP operation. Then each component of F bounds a handlebody L'_i or a manifold which is homeomrophic to $\tilde{L}=\cup_{i=h}^{k} L'_i\cup(\cup_{j=h}^{k-1} N(L'_j\cap L'_{j+1}))$ $(1\leq h<k\leq n$, if $h=1$ ($k=n$, resp.) then $k<n$ ($1<h$, resp.)). If $1<h<k<$

n, then for $l=1, 2$, $\tilde{D}_l=(\cup_{i=h}^{k} D_{i,l})\cup(\cup_{j=h}^{k-1} E_{i,l})$, where $E_{i,l}$ is a meridional disk of $N(L_i'\cap L_{i+1}')$ such that $N(L_i'\cap L_{i+1}')\cap(D_{i,l}\cup D_{i+1,l})\subset E_{i,l}$, is a compressing disk of $\tilde{L}$. And we can see that $\tilde{L}-(\cup_{l=1}^{2}\tilde{D}_l\times(0, 1))$ is obtained from solid tori $L_j'-\cup_{l=1}^{2}(D_{j,l}\times(0, 1))$ $(j=h, h+1, \cdots, k)$ by identifying disks on boundaries of these solid tori. Hence $\tilde{L}$ is a handlebody. If $h=1$ ($k=n$, resp.), $\tilde{D}=D_1\cup\cup_{l=1}^{2}((\cup_{j=2}^{k} D_{j,l})\cup(\cup_{i=1}^{k-1} E_{i,l}))(=\cup_{l=1}^{2}((\cup_{j=h}^{n-1} D_{j,l})\cup E_{j,l})\cup D_n$, resp.) $(D_{i,1}=D_{i,2}=D_i$ for $i=1, 2)$ is a compressing disk of $\tilde{L}$. And $\tilde{L}-\tilde{D}\times(0, 1)$ is obtained from solid tori $L_1'-D_1\times(0, 1)$ $(L_n'-D_n\times(0, 1)$, resp.) and $L_j'-\cup_{l=1}^{2} D_{i,l}\times(0, 1)$ $(j=2, 3, \cdots, k, j=h, h+1, \cdots, n-1$, resp.) by identifying disks on boundaries of these solid tori. Hence $\tilde{L}$ is a handlebody.

Therefore any surface obtained from F_1 and F_2 by a CP operation bounds handlebodies.

We will prove the incompressibility of F_1 and F_2. For the incompressibility of F_1, note that $\cup_{j=1}^{n+1} C_{i,j}$ is full with respect to a set of compressing disks of H_i which are indicated in Figure 4.1 (a). Hence by Lemmas 3.2 and 3.4, $\partial H_i'$ is incompressible in H_i' $(i=1, 2)$, and F_1 is incompressible in M.

Note that we can regard L_i as $F'\times[0, 1]/\{(x, t)\sim(x, t')\,|\,x\in\partial F', t, t'\in[0, 1]\}$, where $F'=F_1\cap L_i$, and $F'\times 1=F_{j,2}\cap L_i (j\equiv i \bmod 2)$. Let M_1 be the closure of the component of $M-F_2$ which contains $C_{1,2}$. Then by the above fact, M_1 is obtained from a handlebody $V=(H_1-\cup_{k=1}^{[n+1/2]} L_{2k-1})\cup(\cup_{k=1}^{[2/n]} L_{2k})$ by 2-surgeries on $C_{1,2k}'$, pushed $C_{2,2k}''$(i.e. $C_{2,2k}'$) $(1\leq k\leq[n/2])$ and $C_{1,n+1}'$ as indicated in Fugure 4.3.

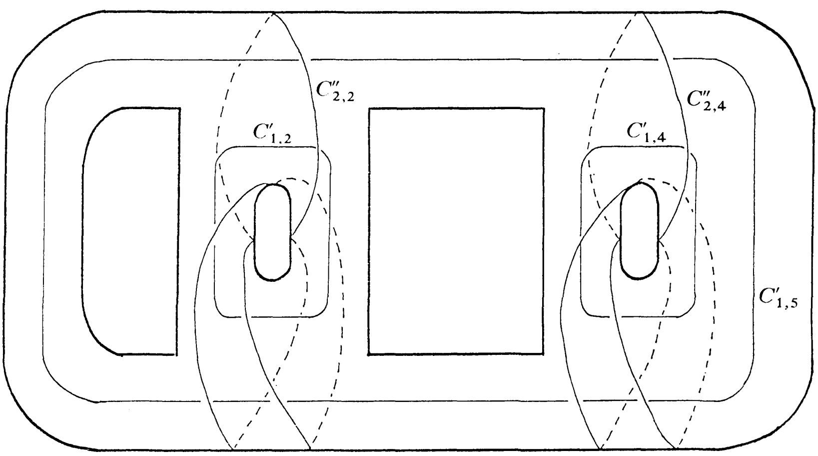

Fig 4.3.

By the same way as the above, the closure M_2 of the other component of $M-F_2$ is also obtained from a handlebody of genus n by 2-surgeries on such closed curves. We consider the following two cases:

(a) $n=2$.

(b) $n\geq 3$.

(a) $n=2$. Since M_2 is homeomorphic to M_1, it is enough to prove the incompressibility of F_2 in M_1.

We use Lemma 3.1. We have

$$\pi_1(M_1)=\langle a, b, c, d, e, f \mid c^2ab=1, d^2b=1, e^2bcdb(cd)^2=1, f=d^{-2}cd\rangle,$$
$$=\langle d, e, f \mid e^2f^2d^2f=1\rangle,$$

where a, b, c, d, e are represented by curves which are indicated in Figure 4.4.

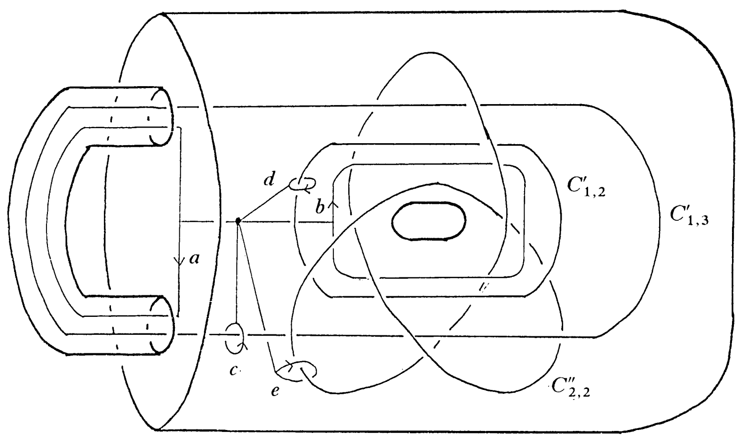

Fig 4.4.

Let $r=e^2f^2d^2f$. We have a representation curve C of r on a handlebody V as indicated in Figure 4.5.

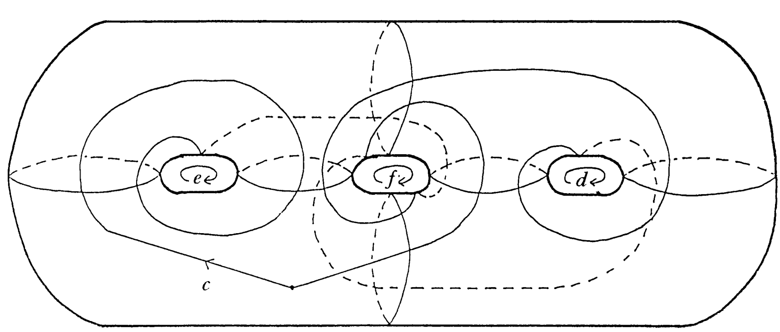

Fig 4.5.

C is full with respect to a set of compressing disks whose boundaries are indicated in Figure 4.5. Hence by Lemma 3.4, $\partial H-C$ is incompressible in H and by Lemma 3.1, $\pi_1(M_1)$ is not a free product group or a cyclic group. Therefore F_2 is incompressible in M_1.

(b) $n\geq 3$. We prove the incompressibility of F_2 in M_1.

We use Lemma 3.2. Recall that M_1 is obtained from a handlebody V by 2-surgeries on $C'_{1,2k}$, pushed $C''_{2,2k}(1\leq k\leq[n/2])$ and $C'_{1,n+1}$. Note that a manifold V' which is obtained from V by 2-surgeries on $C'_{1,2k}(1\leq k\leq[n/2])$ and $C'_{1,n+1}$ is a handlebody. Hence we may regard that M_1 is obtained from the handlebody V' by 2-surgeries on pushed $C''_{2,2k}(1\leq k\leq[n/2])$. We consider a set of compressing disks $D_1, D_2, \cdots, D_{3n-3}$ of V' such that $D_1, D_2, \cdots, D_{n-1}$ separates H' into $[n/2+1]$ solid tori and each of which contains $C'_{1,j}$. See Figure 4.6.

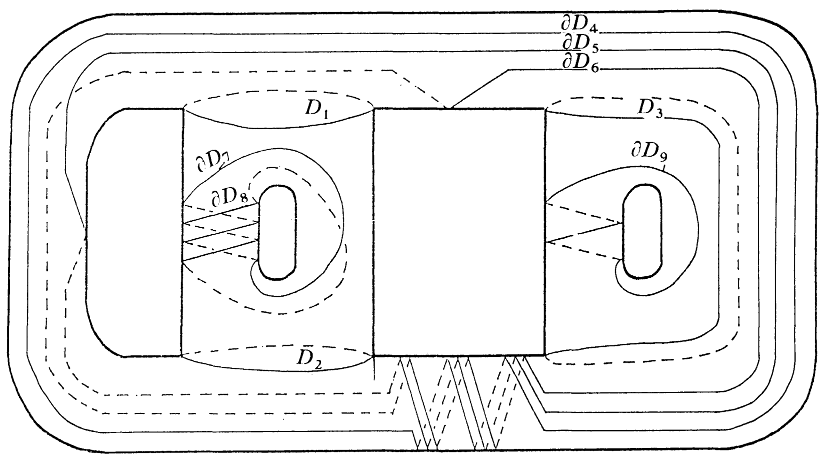

Fig 4.6.

Then we can check that $\cup_{k=1}^{[n/2]} C''_{2,2k}$ is full with respect to $D_1, D_2, \cdots, D_{3n-3}$. Hence by Lemmas 3.2 and 3.4, F_2 is incompressible in M_1.

We can prove the incompressibility of F_2 in M_2 in the same way as the above. This completes the proof of Case (I).

Case (II) $n_1>n_2\geq 2$.

Let H_1 and H_2 be handlebodies of genus n_1. We consider $n+1$ surgery

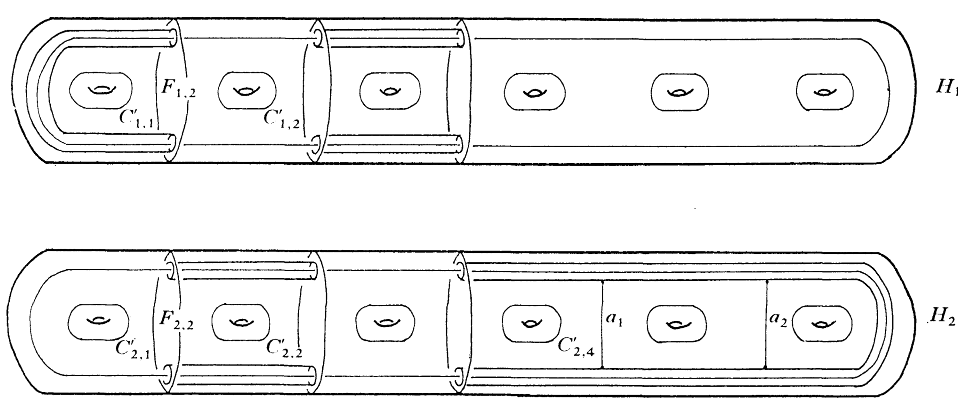

Fig 4.7. (a)

curves as in the proof of Case (I) and properly embedded surfaces $F_{i,2}$ in H_1 and H_2 as indicated in Figure 4.7 (a). Put $M=H_1'\cup_f H_2'$. Here H_i' is obtained from H_i $(i=1, 2)$ by performing 2-surgeries on those curves and f is a homeomorphism of $\partial H_2'$ to $\partial H_1'$ such that

(1) $f(\partial F_{2,2})=\partial F_{1,2}$,

(2) $f^{-1}(C_{1,j})$ $(j\equiv 1 \bmod 2,\ 1\leq j\leq n_2-1$ or $j=n_1)$ and $f(C_{2,k})$ $(k\equiv 2 \bmod 2,$ $2\leq k\leq n_2-1$ or $k=n_1)$ are as indicated in Figure 4.7 (b),

(3) $f(C_{2,i})$ $(i=n_2, n_2+1, \cdots, n_1-1)$ is parallel to $C_{1,i}$.

(In Figure 4.7. $n_1=6$ and $n_2=4$.)

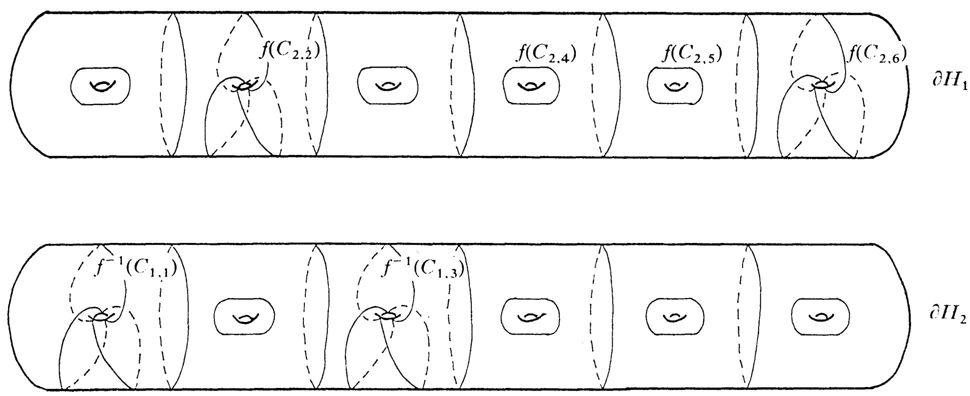

Fig 4.7. (b)

Then M is an orientable closed 3-manifold, and $F_1=\partial H_1'$ and $F_2=F_{1,2}\cup F_{2,2}$ are properly embedded surfaces such that $g(F_1)=n_1$ and $g(F_2)=n_2$.

In the same way as in the proof of Case (I), we can prove that for any surface F obtained from F_1 and F_2 by CP operations, each component of F bounds a handlebody, and F_1 is incompressible in M.

We will prove the incompressibility of F_2 in M. Let M_1 be a clsoure of a component of $M-F_2$ which does not contain C_{1,n_2}. Then M_1 is the same manifold as obtained in the proof of Case (I). Hence F_2 is incompressible in M_1.

Let L be the closure of a component of $H_i-F_{i,2}$ $(i\equiv n_2 \bmod 2)$ which contains C_{i,n_2}'. We consider properly embedded arcs $a_1, a_2, \cdots, a_m (m=n_1-n_2)$ in L as indicated in Figure 4.7 (a). Let $N_L(a_i)=a_i\times D^2$. Then $L-\cup_{i=1}^m a_i\times \mathrm{Int}\, D^2$ has a form $F'\times[0, 1]/\{(x, t)\sim(x, t')\,|\, x\in\partial F', t, t'\in[0, 1]\}$, where $F'=F_1\cap L$. Using this fact, we can see that M_2 is obtained from handlebody V of genus n_2 by performing 2-surgeries on closed curves indicated in Figure 4.8, and by attaching 2-handles $N_L(a_i)=a_i\times D^2$ $(i=1, 2, \cdots, m)$ so that $a_i'=p_i\times\partial D^2$ $(p_i\in a_i)$ is identified with such curves as that indicated in Figure 4.8.

Consider properly embedded disks $D_1, D_2, \cdots, D_m$ as indicated in Figure 4.8. Then the closure of each component of $M_2-\cup_{i=1}^m D_i$ is a manifold which is obtained from solid torus by performing 2-surgeries on two parallel curves which

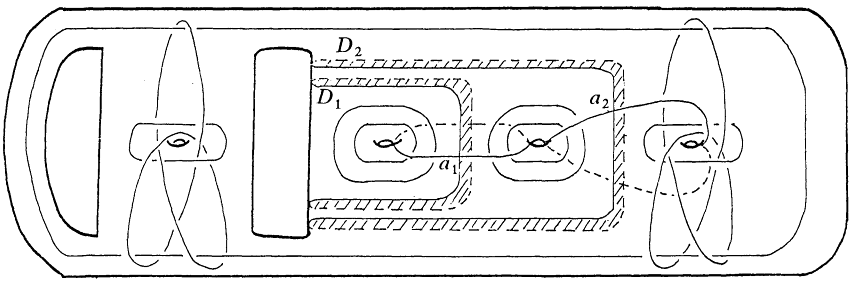

Fig 4.8.

are parallel to a core of solid torus, or the same manifold as that was obtained in the proof of Case (I). In both cases the manifolds are ∂-irreducible. Hence $a_1', a_2', \cdots, a_m'$ and $D_1, D_2, \cdots, D_m$ satisfy the assumption of Lemma 3.5. Therefore M_2 is ∂-irreducible and F_2 is incompressible in M_2.

Hence F_2 is incompressible in M, completing the proof of Case (II).

This completes the proof of Theorem 2.

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