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# ON THE THEORY OF VARIATION OF STRUCTURES DEFINED BY TRANSITIVE, CONTINUOUS PSEUDOGROUPS

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The theory of deformations of structure was begun some years ago by Kodaira and Spencer [6], who laid the foundations for the theory of variation of complex structure. Later, together with Nirenberg, they established the fundamental existence theorem. On the other hand, it was soon realized that the deformations of complex structure was (at least conceptually) a special case of the theory of variation of the structure defined by a transitive, continuous pseudo-group $\Gamma$ acting on a manifold $X$ (briefly, we say a $\Gamma$-structure on $X$). In the complex analytic case, $\Gamma=\Gamma_{GL(n,C)}$ is the pseudo-group of all local bi-holomorphic transformations in $C^n$. The theory of deformations of other special $\Gamma$-structures has been discussed in [4] and [6]. Following this, Spencer succeeded in [7] in establishing the basic mechanism for deformations of what might be called flat $\Gamma$-structures; i.e. $\Gamma$-structures for which a certain jet bundle has coordinate cross-sections of a suitable type. Furthermore, he proved the existence theorems in case $\Gamma\subset\Gamma_{GL(n,C)}$. On the other hand, in [1] (see also [9]), the deformation theory of certain non-flat $\Gamma$-structures has found important applications, and, in [3], the general theory of a (perhaps non-transitive) $\Gamma$-structure has been developed. The existence theorems for certain special cases have also been given in [3].

The purpose of the present paper is to discuss the deformations of a general $\Gamma$-structure. That this is possible is not surprising, and it may even be possible to extend [7] to the general case. However, we have chosen a somewhat different approach based on the following heurestic consideration: Let $X$ have a $\Gamma$-structure and suppose that this structure is given by a certain "Maurer-Cartain form" $\Phi$ defined on a suitable principal bundle $Q\to X$. Thus, $\Phi$ is a Lie algebra valued one form on $Q$ which satisfies the Maurer-Cartan equation.

$$(1)\quad d\Phi-\frac{1}{2}[\Phi,\Phi]=0\,.$$

In order to vary the $\Gamma$-structure on $X$, we add to $\Phi$ a small tensor $\psi$ of the same type and consider $\Psi=\Phi+\psi$ as an "almost-$\Gamma$-structure". The integrability equation for $\psi$ is the Maurer-Cartan equation (1), which is then written as

(2) $D\Psi-\frac{1}{2}[\Psi, \Psi]=0$ where

(3) $D\Psi=d\Psi-[\Phi, \Psi]$.

With (2) and (3) in mind, an examination of some simple $\Gamma$-structures (e.g., local Lie groups—see § 6 below) easily reveals the following: There exists over $X$ a bundle of Lie algebras $\boldsymbol{L}$, associated to $Q \to X$, and which has the following properties: (i) If $\sum^q$ is the sheaf of germs of $C^\infty$ $\boldsymbol{L}$-valued $q$-forms on $X$, then the operator

(4) $D\sigma=d\sigma-[\Phi, \sigma]$

is defined on germs in the sheaf of graded Lie algebras

$$\textstyle\sum^* = \bigoplus_{q\geqq 0} \sum^q; \text{ (ii) } D(\sum^q)\subset\sum^{q+1},\ D^2=0 \qquad \text{(by (1))},$$

and $D$ satisfies the obvious derivation rule; (iii) if $\boldsymbol{A}^q=H^0(X, \sum^q)$, the tensors $\Phi+\psi(\psi\in\boldsymbol{A}^1)$ give the nearly almost-$\Gamma$-structures, and the integrability equation is just (2); and (iv) if $\Theta$ is the sheaf of germs of infinitesimal $\Gamma$-transformations, there is a natural injection $\Theta\to\sum^0$ such that $0\to\Theta\to\sum^0\xrightarrow{D}\sum^1$ is exact.

In comparing our approach with [5] or [7], it is perhaps helpful to keep in mind the following picture: In the traditional theory of deformations, one *fixes* a manifold $X$ and deforms a structure by peturbing a structure tensor on $X$—this has the effect of fixing the manifold and deforming the principal bundle which gives the almost $\Gamma$-structure. Our method is to fix the bundle and deform the fibering—this happens simply because the Maurer-Cartan form determines the fibering.

Now the above mechanism shows how to take a Maurer-Cartan form and peturb it—this gives a method of deforming an integrable $\Gamma$-structure, where by "integrable", we mean that the Maurer-Cartan equation is satisfied. One actually wants that the resulting structures be locally equivalent to the original by diffeomorphisms. The equations for doing this are very simply derived using the following remark: Suppose, on $U\subseteq\boldsymbol{R}^n$, we have a family of 1-forms $\eta(t)$, $\eta(0)=\eta$, and we wish to find a family of vector fields $\xi(t)$, $\xi(0)=0$, such that $f(t)^*\eta=\eta(t)$ where $f(t)=\exp\xi(t)$. Then the system of partial differential equations is just

(5) $\mathcal{L}_{\xi(t)}\eta(t) = \dfrac{\partial \eta(t)}{\partial t}$.*

In our case, we have locally a family $\omega(t)$ of Lie algebra valued Maurer-Cartan forms which satisfy

(6) $d\omega(t) - \dfrac{1}{2}[\omega(t), \omega(t)] = 0$.

We want then to find vector fields $\theta(t)$ ($=$sections of $\sum^0$) such that $\mathcal{L}_{\theta(t)}\omega(t) = \varphi(t)$ where $\varphi(t) = \dfrac{\partial \omega}{\partial t}(t)$. But $\mathcal{L}_{\theta(t)}\omega(t) = D(t)\theta(t) = d\theta(t) - [\omega(t), \theta(t)]$ (by (4)); also $D(t)\varphi(t) = 0$ by differentiating (6). Thus, in our case, (5) becomes the system

(7) $D(t)\theta(t) = \varphi(t)$

with the integrability condition

(8) $D(t)\varphi(t) = 0$.

The point we should like to make then is that the theory of deformations of transitive structure can be based on the four equations (4), (2), (7), and (8).

The trouble with all of this is that, at the present time, the existence theorems are missing. Namely, in §11 we give a brief discussion of elliptic pseudo-groups (those $\Gamma$-structures for which the system $D: \sum^0 \to \sum^1$ is elliptic) and show that, in this case, one might expect that the $D$-Poincaré lemma (and even the $D(t)$-Poincaré lemma) is true. In fact, for elliptic pseudo-groups, the symbol sequences of $\sum^{q-1} \xrightarrow{D} \sum^q \xrightarrow{D} \sum^{q+1}$ are shown to be exact.

In §12 we discuss some of the ins and outs of the existence theory and give a few relavent examples.

We should like to close this introduction by commenting that most of the material developed below is not new but is meant to provide a modification and amplification of the theory developed by Spencer [7]. Also, the approach to deformation theory and much of the discussion below is taken from a seminar given by the author during the spring 1963.

**1. Transitive continuous pseudo-groups.** Let $M$ be a connected $n$-dimensional manifold. A *pseudo-group* $\Gamma$ of transformations of $M$ is, to begin with, a family of local diffeomorphisms of $M$ which is closed under composition of mappings (where defined) and taking inverses. We say that $\Gamma$ is transitive if, given two sufficiently close points $m$ and $m'$

* The notation "$\mathcal{L}_\gamma(\zeta)$" stands for "the Lie derivative of $\zeta$ along $\gamma$".

in $M$, there exists $f \in \Gamma$ such that $f(m)=m'$. In order to make $\Gamma$ a *transitive continuous pseudo-group*, we may assume that the transformations in $\Gamma$ are locally difined by a regular* system of partial differential equations.

Suppose now that $\Gamma$ is a transitive continuous pseudo-group, and let $e_0$ be a frame at a fixed point $m_0 \in M$. Then the orbit of $e_0$ under the transformations $e_0 \to f_*(e_0)$ $(f \in \Gamma)$ is a principal sub-bundle $P_1 \subset P(M)$** with structure group $G_1 \subset GL(n, \boldsymbol{R})$. Obviously $\Gamma$ induces a transitive continuous pseudo-group $\Gamma_1$ of transformations on $P_1$, and we may then take the $\Gamma_1$-orbit of a fixed frame $e_1$ at $e_0 \in P_1$ to find a principal sub-bundle $P_2 \subset P(P_1)$ with group $G_2$. In this way, we find associated to $\Gamma$ a (in general, infinite) sequence of principal fiberings $G_\mu \to P_\mu \to P_{\mu-1}$, $P_\mu$ being a principal sub-bundle of $P(P_{\mu-1})$. On the other hand, each $P_\mu$ is a principal bundle over $M$ with group $G^\mu$, and the fibering $G^\mu \to P_\mu \to M$ may be thought of as describing the jets up to order $\mu+1$ of the transformations in $\Gamma$. Clearly $G_\mu$ is a closed invariant subgroup of $G^\mu$ and $G^\mu/G_\mu \cong G^{\mu-1}$.

Let $\Theta$ be the sheaf of germs of infinitessimal transformations of $\Gamma$; $\Theta$ consists of the germs of $C^\infty$ vector fields on $M$ whose local one-parameter groups lie in $\Gamma$. The stalk $\Theta_{m_0}$ is a Lie algebra, which is independent of the point $m_0$, and which we thus denote by $\tilde{\mathfrak{g}}_*$. Let $\Theta^\mu_{m_0} \subset \theta_{m_0}$ consist of the germs which vanish to order $\mu$ at $m_0$. Then $\Theta^0_{m_0} = \Theta_{m_0}$ and

$$[\Theta^\mu_{m_0}, \Theta^\nu_{m_0}] \subseteqq \Theta^{\mu+\nu-1}_{m_0}. \tag{1.1}$$

Furthermore, since $\Gamma$ is transitive, $\Theta_{m_0}/\Theta^1_{m_0} \cong \boldsymbol{T}_{m_0}(M)$. We set $\mathfrak{g}^\mu_* = \Theta_{m_0}/\Theta^{\mu+1}_{m_0}$ and $g_* = \lim\limits_{\mu\to\infty} \mathfrak{g}^\mu_m$ ($\mathfrak{g}_*$ is the *formal Lie algebra* of $\Gamma$). Also we define $\mathfrak{g}^\mu = \Theta^1_{m_0}/\Theta^{\mu+1}_{m_0}$ and $\mathfrak{g} = \lim\limits_{\mu\to\infty} \mathfrak{g}^\mu$. Clearly $\mathfrak{g}^1 = \mathfrak{g}_1$ is the linear Lie algebra of $G^1 = G_1$ and $\mathfrak{g}^\mu$ is the Lie algebra of $G^\mu$. There are projections $\mathfrak{g}^\mu_* \to \mathfrak{g}^{\mu-1}_*$ and $\mathfrak{g}^\mu \to \mathfrak{g}^{\mu-1}$, each with kernel $\mathfrak{g}_\mu$ = Lie algebra of $G_\mu$.

The vector space $\mathfrak{g}^\mu_*$ is not (with the bracket inherited from $\mathfrak{g}_*$) a Lie algebra, but, given $\xi$ and $\eta$ in $\mathfrak{g}^\mu_*$, the projection of $[\xi, \eta]$ in $\mathfrak{g}^{\mu-1}_*$ is (by (1.1)) well-defined. Thus we have a pairing

$$[\ \ ,\ \ ]^{\mu-1} : \mathfrak{g}^\mu_* \otimes \mathfrak{g}^\mu_* \to \mathfrak{g}^{\mu-1}_* \tag{1.2}$$

(In general, given $\xi \in \mathfrak{g}^\mu_*$, we denote by $\xi^\nu$ the projection of $\xi$ in $\mathfrak{g}^\nu_*$ for $\nu \leqq \mu$.)

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* This is a precise concept; see [8].

** For a manifold $X$, $P(X)$ is the principal frame bundle.

We close this paragraph with a simple but suggestive example. Let $A$ be a connected Lie group which acts transitively on $M$—in the obvious way, $A$ induces a transitive continuous pseudo-group of transformations on $M$. Let $B$ be the stability group of $m_0 \in M$; we may identify $\boldsymbol{T}_{m_0}(M)$ with $\mathfrak{a}/\mathfrak{b}$ and $G_1$ is the linear group of all transformations $\mathrm{Ad}\, b: \mathfrak{a}/\mathfrak{b} \to \mathfrak{a}/\mathfrak{b} (b \in \mathfrak{b})$. Let $B_1 \subset B$ be the connected kernel of this representation; in the obvious way, $P_1 = A/B_1$ and $\Gamma_1$ consists of the transformations induced by $A$ acting on $A/B_1$. We may continue this process, getting a sequence of groups $B \supset B_1 \supset \cdots \supset B_\mu \supset \cdots$ and where $P_\mu = A/B_\mu$, $G^\mu = B/B_\mu$, etc. Obviously, this process terminates at some $\mu_0$ where $\mathfrak{b}_{\mu_0} = 0$, and then $P_{\mu_0} = A$ and $G^{\mu_0} = B$. In our notation, $\mathfrak{g}_* = \mathfrak{a}$, $\mathfrak{g} = \mathfrak{b}$, and $\mathfrak{g}_\mu = \mathfrak{b}_\mu/\mathfrak{b}_{\mu-1}$.

Now a pseudo-group $\Gamma$ such that $P_{\mu_0} = P_{\mu_0+1} = \cdots$ for some $\mu_0$ is said to be of *finite type*. It is almost obvious that the most general transitive continuous pseudo-group of finite type is locally equivalent to the example just given (for suitable $A$ and $B$).

**2. The Structure equations.** Let $\Gamma$ be a transitive continuous pseudo-group acting on $M$. The *Maurer-Cartan form* associated to this structure is given by a sequence $\{\Omega^\mu\}$ of forms where $\Omega^\mu$ is a $\mathfrak{g}_*^\mu$-valued form globally defined on $P_{\mu+1}$, and which satisfies the following *Maurer-Cartan equation* on $P_{\mu+2}$:

$$d\Omega^\mu - \frac{1}{2}[\Omega^{\mu+1}, \Omega^{\mu+1}]^\mu = 0\,. \tag{2.1}$$

(The bracket here is that given by (1.2)).

We shall discuss briefly the forms $\Omega^\mu$ and the equation (2.1). On $P(M)$, there is the canonically defined $\boldsymbol{R}^n$-valued form $\omega = (\omega^1, \cdots, \omega^n)$, and $\omega | P_1$ is the first piece $\Omega^0$ of the Maurer-Cartan form. If we have local coordinates $z$ on $M$ and $(z, g)$ $(g \in G_1)$ in $P_1$, then $\omega(z, g) = \theta(z)\cdot g$ where $\theta(z) = (\theta^1(z), \cdots, \theta^n(z))$ is a local co-frame on $M$. We have

$$d\omega = -\omega \wedge g^{-1}dg + d\theta\cdot g\,. \tag{2.2}$$

Let $e_1, \cdots, e_r$ be a basis for the linear Lie algebra $\mathfrak{g}_1 \subset gl(n, \boldsymbol{R})$, and write $e_\rho$ as the matrix $(a^i_{\rho j})$. Then, if $\{\pi_\rho\}$ are the forms on $G_1$ dual to $\{e_\rho\}$, (2.2) becomes (c.f. [2]):

$$d\omega^i = \frac{1}{2}\sum c^i_{jk}\omega^j \wedge \omega^k + \sum a^i_{\rho j}\pi^\rho \wedge \omega^j\,. \tag{2.3}$$

The $c^i_{jk}$ may be globally defined on $P_1$, and, as they are invariant under $\Gamma_1$, the $c^i_{jk}$ are constant and are in fact the first part of the structure constants of the Lie algebra $g_*$.

The forms $\pi^\rho$ are not invariantly defined on $P_1$, but they are global on $P_2$ and there $\Omega^1=(\omega^1, \cdots, \omega^n; \pi^1, \cdots, \pi^r)$ is the $\mathfrak{g}^1_*$-valued Maurer-Cartan form. In addition to (2.3) (which is now global on $P_2$), we have again a local equation

$$d\pi^\rho = \frac{1}{2}\sum a^\rho_{\sigma\tau}\pi^\sigma\wedge\pi^\tau+\frac{1}{2}\sum c^\rho_{ij}\omega^i\wedge\omega^j+\sum a^\rho_{\alpha i}\tilde{\omega}^\alpha\wedge\omega^i \tag{2.4}$$

where the $\{a^\rho_{\sigma\tau}\}$ are the structure constant of $\mathfrak{g}_1$, the $c^\rho_{ij}$ are (because of the transitivity of $\Gamma_2$ on $P_2$) again constants, and the $\tilde{\omega}^\alpha$ are locally the Maurer-Cartan forms on $G_2$. The forms $\{\tilde{\omega}^\alpha\}$ are globally defined on $P_3$, $\Omega^2=(\omega^1, \cdots, \omega^n; \pi^1, \cdots, \pi^r; \tilde{\omega}^1, \cdots, \tilde{\omega}^s)$, and (2.3)-(2.4) may be combined and written on $P_3$ as $d\Omega^1-\frac{1}{2}[\Omega^2, \Omega^2]^1=0$. Clearly we may continue this process and inductively arrive at the $\mathfrak{g}^\mu_*$-valued form $\Omega^\mu$ on $P_{\mu+1}$ and the equation (2.1) on $P_{\mu+2}$.

If $\Gamma$ is of finite type, then, for large $\mu$, $P_\mu=P_{\mu+1}=\cdots=P$ and $\Omega^\mu=\Omega^{\mu+1}=\cdots=\Omega$ is globally defined on $P$. In fact, $P$ is locally a Lie group and $\Omega$ is the usual left-invariant Maurer-Cartan form.

**3. Almost $\Gamma$-structures.** Let $\Gamma$ be a transitive continuous pseudogroup acting on $M$. Then $\Gamma$ defines a sequence of principal fiberings $G_\mu\to P_\mu\to P_{\mu-1}$ where $P_\mu\subset P(P_{\mu-1})$ and $P_0=M$. If now $X$ is a general $n$-manifold, an *almost* $\Gamma$-*structure* on $X$ will be given just such a sequence of principal fiberings $G_\mu\to Q_\mu\to Q_{\mu-1}$ where $Q_\mu\subset P(Q_{\mu-1})$ is a principal sub-bundle and $Q_0=X$.

We now observe that the construction of the Maurer-Cartan forms $\Omega_\mu$ on $P_{\mu+1}$ did not depend on $\Gamma$ but only on the fact that we had the sequence of fiberings $\{G_\mu\to P_\mu\to P_{\mu-1}\}$. Thus, given an almost $\Gamma$-structure on $X$, we may find a sequence of $\mathfrak{g}^\mu_*$-valued forms $\Phi^\mu$ on $Q_{\mu+1}$. We also get equations like (2.3), (2.4), etc., but now the $c^i_{jk}, c^\rho_{ij}, c^\alpha_{ij}, \cdots$ are generally no longer constants, but are functions on the appropriate bundles, called the *structure-functions* of the almost $\Gamma$-structure. We say that structure is *integrable to order* $\mu$ if the first $\mu$ sets of structure functions are constants. This is equivalent to the equation

$$d\Phi^{\mu-1}-\frac{1}{2}[\Phi^\mu, \Phi^\mu]^{\mu-1} = 0 \tag{3.1$^{\mu-1}$}$$

on $Q_{\mu+1}$.

We introduce the following definitions for an almost $\Gamma$-structure:

(A) The structure is *integrable* if $(3.1)^{\mu-1}$ holds for all $\mu$ (by abuse of language, we call this an *integrable* $\Gamma$-*structure*); (B) The structure is *transitive* if the transformations preserving the almost $\Gamma$-structure on $X$ induce a transitive pseudo-group on $Q_\mu$ for each $\mu$; and (C) Finally, we say that we have on $X$ a *$\Gamma$-structure* if, for each $x \in X$, there exists a neighborhood $U$ of $x$, a neighborhood $V_0$ of $m_0 \in M$, and a diffeomorphism $f: U \to V_0$ which preserves the induced almost $\Gamma$-structures*. Clearly (C)$\Rightarrow$(B)$\Rightarrow$(A).

Suppose now that we have on $X$ an almost $\Gamma$-structure which is integrable to order $\mu$, and let $x \in X$ be an arbitrary point. Then there exists neighborhoods $U$ of $x$, $V$ of $m_0 \in M$, and a diffeomorphism $f: U \to V$ such that $f(x) = m_0$ and such that $f$ is, to order $\mu+1$ at $x$, a structure preserving transformation. More precisely, there exist local cross-sections $\psi: V \to P_{\mu+1}$, $\varphi: U \to Q_{\mu+1}$ such that $f^*\psi^*(\Omega^\mu)(x) = \varphi^*(\Phi^\mu)(x)$ and $f^*\psi^*(d\Omega^{\mu-1})(x) = \varphi^*(d\Phi^{\mu-1})(x)$.

**4. Some remarks on the finitude conditions for pseudo-groups.** Suppose that we have on $X$ an almost $\Gamma$-structure and assume that the Maurer-Cartan form $\Phi^\mu$ on $Q_{\mu+1}$ satisfies

$$d\Phi^{\mu-1} - \frac{1}{2}[\Phi^\mu, \Phi^\mu]^{\mu-1} = 0. \tag{9.1}$$

Let $\sigma: X \to Q_{\mu+1}$ be a local cross-section, and set $\sigma^*(\Phi^\mu) = \omega^\mu$. Then we may *locally* define the $\mathfrak{g}_*^\mu$-valued 2-form

$$\Lambda = d\omega^\mu - \frac{1}{2}[\omega^\mu, \omega^\mu]^\mu\,;\ ^{**} \tag{9.2}$$

we assert that $[\omega, \Lambda]^{\mu-1} = 0$ $(\omega = \omega^0)$. Indeed,
$[\omega, \Lambda]^{\omega-1} = [\omega^\mu, d\omega^\mu - \frac{1}{2}[\omega^\mu, \omega^\mu]^\mu]^{\mu-1}$ (since $\Lambda^{\mu-1} = 0$)
$= d[\omega^\mu, \omega^\mu]^{\mu-1} = 0$ (by (9.1)).

On the other hand, suppose that we can find a $\mathfrak{g}_{\mu+1}$-valued one-form $\eta$ such that $-[\omega, \eta]^\mu = \Lambda$. Then $\omega^{\mu+1} = \omega^\mu + \eta$ satisfies $d\omega^\mu - \frac{1}{2}[\omega^{\mu+1}, \omega^{\mu+1}]^\mu = \Lambda - [\omega, \eta]^\mu = 0$, and this implies immediately that $d\Phi^\mu - \frac{1}{2}[\Phi^{\mu+1}, \Phi^{\mu+1}]^\mu = 0$ on $Q_{\mu+2}$.

The above construction suggests that we set $\boldsymbol{\gamma}^{\mu,q} = \mathfrak{g}_\mu \otimes (\Lambda^q \boldsymbol{R}^n)^*$ and define $\delta: \boldsymbol{\gamma}^{\mu,q} \to \boldsymbol{\gamma}^{\mu-1,q+1}$ by $\delta(\eta) = [\omega, \eta]^{\mu-1}$. Then $\delta^2\eta = [\omega, [\omega, \eta]^{\mu-1}]^{\mu-2} = \pm\frac{1}{2}[[\omega, \omega], \eta]^{\mu-2} = 0$, and the resulting cohomology groups are purely algebraic and depend only on $\mathfrak{g}_*$; we set $\{\operatorname{Ker} \delta: \boldsymbol{\gamma}^{\mu,q} \to \boldsymbol{\gamma}^{\mu-1,q+1}\}/\delta(\boldsymbol{\gamma}^{\mu+1,q-1})$

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* This corresponds to "introducing local $\Gamma$-coordinates" in [7].

** The bracket $[\omega^\mu, \omega^\mu]^\mu$ is defined by an arbitrary splitting of the sequence $\mathfrak{g}^* \to \mathfrak{g}_*^\mu \to 0$.

$=H^{\mu,q}(\mathfrak{g}_*)$. These groups have been introduced by Spencer [7], who proved that $H^{\mu,*}(\mathfrak{g}_*)=0$ for $\mu\geqq\mu_0$.*

We have thus shown above that, if an almost $\Gamma$-structure is integrable to order $\mu$, then the *obstruction* to integrating the structure to order $\mu+1$ is a tensor of type $H^{\mu,2}(\mathfrak{g}_*)$. We consequently see that an almost $\Gamma$-structure which is integrable to order $\mu\geqq\mu_0$ is integrable. (This is, of course, well known and seems to have been done in several places simultaneously and independently.)

As another example of Spencer's finitude theorem, we shall show that there exists an integer $\mu_1$ such that the bundles $\{G^\mu\to Q_\mu\to Q_{\mu-1}\}$ for $\mu\leqq\mu_1$ determine an almost $\Gamma$-structure (in the sense that the bundle $Q_{\mu_1+\nu}$ is the "natural" prolongation of $Q_{\mu_1+\nu-1}$ for $\nu\geqq1$). Denote by $\mathfrak{h}_*$ the formal Lie algebra of the pseudo-group of all local diffeomorphisms. Letting $S^\nu(\boldsymbol{R}^n)$ be the $\nu$-fold symmetric product of $\boldsymbol{R}^n$, there is a natural contraction $\langle\ ,\ \rangle:\mathfrak{h}_*^{\mu+\nu}\otimes S^\nu(\boldsymbol{R}^n)\to\mathfrak{h}_*^\mu$ such that $\langle\mathfrak{g}_*^{\mu+\nu}, S^\nu(\boldsymbol{R}^n)\rangle\subseteqq\mathfrak{g}_*^\mu$. Let $(\mathfrak{g}_*^\mu)^\nu=\{\varphi\in\mathfrak{h}_*^{\mu+\nu}\,|\,\langle\varphi, S^\nu(\boldsymbol{R}^n)\rangle\subseteqq\mathfrak{g}_*^\mu\}$; we assert that there exists a $\mu_1$ such that $(\mathfrak{g}_*^{\mu_1})^\nu=\mathfrak{g}_*^{\mu_1+\nu}$ for all $\nu\geqq0$. For this, it will suffice to prove that $(\mathfrak{g}_*^{\mu_1})^1=\mathfrak{g}_*^{\mu_1+1}$ for some $\mu_1$. But it is immediate that $H^{\mu,1}(\mathfrak{g}_*)\cong(\mathfrak{g}_*^\mu)^1/\mathfrak{g}_*^{\mu+1}$. This algebraic result gives our assertion about the prolongations.

**5. Construction of integrable almost $\Gamma$-structures.** Let $X$ be an $n$-manifold on which we have an integrable $\Gamma$-structure. For $\mu$ sufficiently large, we consider the principal fibering $G^{\mu+1}\to Q_{\mu+1}\to X$. Set $Y=Q_{\mu+1}$. Then there is defined on $Y$ the $\mathfrak{g}_*^\mu$-valued form $\Phi^\mu$ which satisfies the Maurer-Cartan equation. Our purpose now is to prove the converse:

**Theorem.** *Let $\Psi^\mu$ be a $\mathfrak{g}_*^\mu$-valued form on $Y$ which is close to $\Phi^\mu$ and which satisfies the Maurer-Cartan relation*

$$d\Psi^{\mu-1}-\frac{1}{2}[\Psi^\mu, \Psi^\mu]^{\mu-1}=0\,. \tag{5.1}$$

*Then there exist manifolds $R_\mu$ and $Z$, a diagram of fiberings*

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[d] \\ R_\mu \arrow[r] & Z \end{tikzcd}$$

*and a $\mathfrak{g}_*^{\mu-1}$-valued form $\Lambda^{\mu-1}$ on $R_\mu$ with $\pi^*(\Lambda^{\mu-1})=\Psi^{\mu-1}$. Furthermore, there is on $Z$ an integrable $\Gamma$-structure whose $\mu^{\text{th}}$ prolonged bundle is $R_\mu$ and such that $\Lambda^{\mu-1}$ is the Maurer-Cartan form on $R_\mu$.*

Proof. Write $\Psi^\mu=(\Psi_0,\Psi_1,\cdots,\Psi_\mu)$ and set $\Psi_0=(\omega^1,\cdots,\omega^n)$; $\Psi_1=(\pi^1,\cdots,\pi^r)$. Then equations (2.3) and (2.4) hold. Thus the system $\{\omega^j=0\}$ ($j=1,\cdots,n$) is completely integrable and defines a fibering of $Y$. (There is

* This result now has a very simple proof using homological algebra; see [8].

some trouble that the leaves may not be closed—however, this will be automatically satisfied in the applications below.)

Let the quotient space be $Z$; we have $Y \xrightarrow{\{\omega^j=0\}} Z$. Similarly, the distribution on $Y$ given by $\{\omega^j=0;\ \pi^\rho=0\}$ is involutive and we get a fibering $Y \xrightarrow{\left\{\begin{smallmatrix}\omega^j=0\\ \pi^\rho=0\end{smallmatrix}\right\}} R_1$.

In order to show that $R_1$ fibres over $Z$, we must prove that the forms $\omega^1, \cdots, \omega^n$ project from $Y$ down to $R_1$. Thus we must show that $i(\theta)\omega^j=0=\mathcal{L}_\theta\omega^j$ for any vector field $\theta$ tangent to the fibres of $Y\to R_1$ (here $i(\theta)\omega^j=\langle\omega^j, \theta\rangle$). But $\theta$ is tangent to the fibres if, and only if, $\langle\omega^j, \theta\rangle=0=\langle\pi^\rho, \theta\rangle$ $(j=1, \cdots, n;\ \rho=1, \cdots, r)$, and it follows from (2.3) that $\mathcal{L}_\theta\omega^j=0$. Thus we may project $\omega^1, \cdots, \omega^n$ to $R_1$, where the system $\{\omega^j=0\}$ is again completely integrable and defines a fibering $R_1 \xrightarrow{\{\omega^j=0\}} Z$ so that

$$\begin{tikzcd}[column sep=small] & Y \arrow[dl] \arrow[dr] & \\ R_1 \arrow[rr] & & Z \end{tikzcd}$$

commutes.

We assert that $R_1\to Z$ is a principal bundle with group $G_1$. To prove this, we must define on $R_1$ $r$-vector fields $e_1, \cdots, e_r$ such that $\langle\omega^j, e_\rho\rangle=0$ and $[e_\rho, e_\sigma]=\sum_\tau a^\tau_{\rho\sigma}e_\tau$. Let $\varphi: R_1\to Y$ be a local cross-section. Then the $\{\omega^j\}$ and $\{\varphi^*(\pi^\rho)\}$ give a local parallelism, and we define $e_\rho$ by $\langle\omega^j, e_\rho\rangle=0$, $\langle\varphi^*(\pi^\rho), e_\sigma\rangle=\delta^\rho_\sigma$. To prove that the $e_\rho$ are globally defined, we may show that, if $\hat\varphi: R_1\to Y$ is another local cross-section, then $\hat\varphi^*(\pi^\rho)-\varphi^*(\pi^\rho)=\sum_j b^\rho_j\omega^j$. Set $\xi^\rho=\varphi^*(\pi^\rho)$; $\hat\xi^\rho=\hat\varphi^*(\pi^\rho)$. From (2.4), we get $\sum_{k,\rho} a^i_{\rho k}(\xi^\rho-\hat\xi^\rho)\wedge\omega^k=0$ $(i=1, \cdots, n)$. Set $d^i_k=\sum_k a^i_{\rho k}(\xi^\rho-\hat\xi^\rho)$. Then $\sum_k d^i_k\wedge\omega^k=0$ and, by Cartan's lemma, $d^i_k=\sum d^i_{kl}\omega^l$ with $d^i_{kl}=d^i_{lk}$. Thus $\sum_\rho a^i_{\rho k}(\xi^\rho-\hat\xi^\rho)$ $=\sum_l d^i_{kl}\omega^l$ and, since the $r$-matrices $a_\rho=(a^i_{\rho j})$ are linearly independent, we get $\xi^\rho-\hat\xi^\rho=\sum_j b^\rho_j\omega^j$ as desired. Consequently the $\{e_\rho\}$ are defined and we compute $[e_\rho, e_\sigma]$. We have $d\varphi^*(\pi^\rho)=\varphi^*(d\pi^\rho)\equiv\frac{1}{2}\sum a^\rho_{\sigma\tau}\varphi^*(\pi^\sigma)\wedge\varphi^*(\pi^\tau)$ (modulo $\omega^1, \cdots, \omega^n$) and then $\langle d\varphi^*(\pi^\rho), (e_\sigma, e_\tau)\rangle=a^\rho_{\sigma\tau}$ or $[e_\sigma, e_\tau]=\sum a^\rho_{\sigma\tau}e_\rho$.

This proves that $R_1\to Z$ is a principal $G_1$-bundle; in order to complete the Theorem, we need an obvious induction plus the fact that $R_1$ is a sub-bundle of $P(Z)$. Now $\mathcal{L}_{e_\rho}\omega^j=i(e_\rho)d\omega^j+di(e_\rho)\omega^j=\sum a^j_{\rho k}\omega^k$ (by (2.3)), and thus $\omega=(\omega^1, \cdots, \omega^n)$ is an $\boldsymbol{R}^n$-valued form on $R_1$ which satisfies the conditions: (a) the $\omega^i$ are independent at each point and the equations $\{\omega^j=0\}$ define the fibering $R_1\to Z$; (b) $\omega(pg)=\omega(p)g$ $(p\in R_1,\ g\in G_1)$. We shall prove that, under these assumptions, there exists a bundle injection $f: R_1\to P(Z)$ compatible with the linear representation of $G_1$ given on $\mathfrak{g}_1$ by $e_\rho\to(a^i_{\rho j})$.

Indeed, choose a connexion in $G_1 \to R_1 \xrightarrow{\gamma} Z$ given by a field of $G_1$-invariant horizontal spaces $\boldsymbol{H}_p \subset \boldsymbol{T}_p(R_1)$ ($p \in R_1$). For each $p$, the forms $\omega^1(p), \cdots, \omega^n(p)$ give a basis of the dual space $\boldsymbol{H}_p^*$, and we let $f_1(p), \cdots, f_n(p)$ be the corresponding basis of $\boldsymbol{H}_p$. Then $\gamma_*(f_1(p)), \cdots, \gamma_*(f_n(p))$ is a basis of $\boldsymbol{T}_{\gamma(p)}(Z)$, and we let $f(p) \in P(Z)$ be this frame. It is easy to see that $f$ is an injective bundle mapping. Q.E.D.

**6. Deformations of structure; the case of a parallelism.** Because of the theorem in § 5, in order to deform an integrable $\Gamma$-structure on $X$, we may go to a suitable prolonged bundle $Q_{\mu+1}$ and peturb the Maurer-Cartan form $\Phi^{\mu}$ so as to have (5.1) satisfied. In order to carry this out precisely, we begin with the simple case when $M$ is a Lie group $A$ and $\Gamma$ consists of the left translations in $A$.

Let $\mathcal{S}^q$ be the sheaf on $A$ of vector-valued $q$-forms. If $e_1, \cdots, e_n$ are a basis for the left-invariant vector fields on $A$, then any germ $\varphi$ in $\mathcal{S}^q$ may be written $\varphi = \sum_{\rho=1}^{n} e_\rho \otimes \varphi^\rho$, $\varphi^\rho$ being a germ of an ordinary $q$-form. We define $[\ ,\ ]: \mathcal{S}^p \otimes \mathcal{S}^q \to \mathcal{S}^{p+q}$ by $[e_\rho \otimes \varphi^\rho, e_\sigma \otimes \psi^\sigma] = [e_\rho, e_\sigma] \otimes \varphi^\rho \wedge \psi^\sigma$. There are then the usual identities:

$$[\varphi, \psi] = (-1)^{pq+1}[\psi, \varphi] \qquad (\varphi \in \mathcal{S}^p,\ \psi \in \mathcal{S}^q); \tag{6.1}$$

$$(-1)^{pr}[\varphi, [\psi, \eta]] + (-1)^{qr}[\eta, [\varphi, \psi]] + (-1)^{pq}[\psi, [\eta, \varphi]] = 0 \qquad (\varphi \in \mathcal{S}^p,\ \psi \in \mathcal{S}^q,\ \eta \in \mathcal{S}^r); \tag{6.2}$$

$$d[\varphi, \psi] = [d\varphi, \psi] + (-1)^p[\varphi, d\psi] \qquad (\varphi \in \mathcal{S}^p), \tag{6.3}$$

where $d(\sum e_\rho \otimes \varphi^\rho) = \sum e_\rho \otimes d\varphi^\rho$.

For later use, we observe from (6.2) that, if $\varphi \in \mathcal{S}^1$, we have

$$[\varphi, [\psi, \eta]] = [[\varphi, \psi], \eta] + (-1)^p[\psi, [\varphi, \eta]] \qquad (\varphi \in \mathcal{S}^p); \tag{6.4}$$

$$[[\varphi, \varphi], \psi] = 2[\varphi, [\varphi, \psi]]. \tag{6.5}$$

Let now $\omega^1, \cdots, \omega^n$ be the forms dual to $e_1, \cdots, e_n$, and set $\Omega = \sum e_\rho \otimes \omega^\rho$. Then

$$d\Omega - \frac{1}{2}[\Omega, \Omega] = 0. \tag{6.6}$$

Denote by $\boldsymbol{T}^q$ the global sections of $\mathcal{S}^q$. The nearby almost $\Gamma$-structures ($=$parallelisms close to $e_1, \cdots, e_n$) are given by forms $\Phi = \Omega + \varphi$ where $\varphi \in \boldsymbol{T}^1$ is small; the integrability condition is just

$$d(\Omega + \varphi) - \frac{1}{2}[\Omega + \varphi, \Omega + \varphi] = 0. \tag{6.7}$$

Using (6.6) and (6.1), (6.7) reduces to

$$D\varphi - \frac{1}{2}[\varphi, \varphi] = 0 \tag{6.8}$$

where

$$D\varphi = d\varphi - [\Omega, \varphi]. \tag{6.9}$$

**Theorem.** *If we define* $D\colon \mathcal{S}^q \to \mathcal{S}^{q+1}$ *by* $D(\varphi) = d\varphi - [\Omega, \varphi]$, *then* $D$ *has the following properties*:

(i) $D^2 = 0$ *and* $D[\varphi, \psi] = [D\varphi, \psi] + (-1)^p[\varphi, D\psi]$ $\quad(\varphi \in \mathcal{S}^p)$;

(ii) *The kernel of* $D$ *on* $\mathcal{S}^0$ *is just* $\Theta$;

(iii) *The sheaf sequence*

$$0 \to \Theta\ \mathcal{S}^0 \xrightarrow{D} \mathcal{S}^1 \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{S}^q \xrightarrow{D} \cdots \tag{6.10}$$

*is exact*; *and*

(iv) *The equation* (6.8) *for small* $\varphi \in \boldsymbol{T}^1$ *gives the deformations of the integrable* $\Gamma$*-structure.*

Proof. Using (6.6), (6.5), and (6.3), we get: $D^2\varphi = D(d\varphi - [\Omega, \varphi]) = -d[\Omega, \varphi] - [\Omega, d\varphi - [\Omega, \varphi]] = -\frac{1}{2}[[\Omega, \Omega], \varphi] + [\Omega, d\varphi] - [\Omega, d\varphi] + [\Omega, [\Omega, \varphi]] = 0$. The derivation rule follows from (6.3) and (6.4).

To establish (ii), it will suffice to prove that, for a germ $\theta \in \mathcal{S}^0$,

$$D\theta = \sum e_\rho \otimes \mathcal{L}_\theta \omega^\rho. \tag{6.11}$$

Indeed, if $\{c^\rho_{\sigma\tau}\}$ are the structure constants of $\mathfrak{a}$,
$\sum e_\rho \otimes \mathcal{L}_\theta \omega^\rho = \sum e_\rho \otimes (i(\theta)d\omega^\rho + di(\theta)\omega^\rho) = \sum e_\rho \otimes (c^\rho_{\sigma\tau}\theta^\sigma\omega^\tau + d\theta^\sigma)$
$= \sum e_\rho \otimes d\theta^\rho - \sum c^\rho_{\sigma\tau} e_\rho \otimes \omega^\sigma \theta^\tau = d\theta - [\Omega, \theta] = D\theta.$

The $D$-Poincaré lemma follows from the usual one together with the following remark: If (locally) $f_1, \cdots, f_r$ are a basis for the right-invariant vector fields, and if we write $\varphi = \sum f_\rho \otimes \varphi^\rho$, then

$$D\varphi = \sum f_\rho \otimes d\varphi^\rho. \tag{6.12}$$

The theorem now follows.

**7. Deformation of structures of finite type.** Let $A$ and $B$ be connected Lie groups with $B \subset A$ a closed subgroup. Let $M = A/B$ and let $\Gamma$ be the transitive continuous pseudo-group on $M$ induced by the action of $A$ on $A/B$. From the principal fibering $B \to A \to M$, we may construct the vector bundle $\boldsymbol{T}(A)/B \to M$ (i.e. we identify tangent vectors to $A$ which are right translates by an element of $B$), and we denote by

$\sum^q$ the sheaf on $M$ of germs of $q$-forms with values in this bundle.

Now it is easy to see that

$$\boldsymbol{T}(A)/B = A\times_B\mathfrak{a}\,,^* \tag{7.1}$$

where $B$ acts on $\mathfrak{a}$ by the adjoint action. Thus we have an injection $\sum^q\to\mathcal{S}^q$

**Lemma.** $[\sum^p, \sum^q]\subseteqq\sum^{p+q}$ *and* $D\sum^q\subseteqq\sum^{q+1}$.

Proof. We choose a basis $e_1, \cdots, e_n$ for $\mathfrak{a}$ such that $e_1, \cdots, e_r$ gives a basis for $\mathfrak{b}\subset\mathfrak{a}$. Also, we let $1\leqq i,\ j\leqq n$ and $1\leqq\rho,\ \sigma,\ \tau\leqq r$ be our ranges of indices. Then $\sum^q$ consists of the germs $\varphi\in\mathcal{S}^q$ which satisfy

$$i(e_\rho)\varphi = 0 \tag{7.2}$$

$$\mathcal{L}_{e_\rho}\cdot\varphi = [e_\rho, \varphi] \tag{7.3}$$

where $i(e_\rho)\varphi=\sum\limits_{i=1}^{n} e_i\otimes\langle\varphi^i, e_\rho\rangle$ and $[e_\rho, \varphi]=\sum c^i_{\rho j}e_i\otimes\varphi^j$.

Now let $\varphi\in\sum^p$, $\psi\in\sum^q$. Then $i(e_\rho)[\varphi, \psi]=[i(e_\rho)\varphi, \psi]+[\varphi, i(e_\rho)\psi]=0$. Also, $\mathcal{L}_{e_\rho}[\varphi, \psi]=[\mathcal{L}_{e_\rho}\varphi, \psi]+[\varphi, \mathcal{L}_{e_\rho}\psi]=[[e_\rho, \varphi], \psi]+[\varphi, [e_\rho, \psi]]=[e_\rho, [\varphi, \psi]]$ [by (6.2)), and consequently $[\sum^p, \sum^q]=\sum^{p+q}$.

On the other hand, for $\varphi\in\sum^p$, we have:
$i(e_\rho)D\varphi=i(e_\rho)d\varphi-i(e_\rho)[\Omega, \varphi]=\mathcal{L}_{e_\rho}\varphi-di(e_\rho)\varphi-[i(e_\rho(\Omega, \varphi]=[e_\rho, \varphi]-[e_\rho, \varphi]=0$. Also, $\mathcal{L}_{e_\rho}D\varphi=\mathcal{L}_{f_\rho}d\varphi-\mathcal{L}_{e_\rho}[\Omega, \varphi]=d[\mathcal{L}_{e_\rho}, \varphi]-[\mathcal{L}_{e_\rho}\Omega, \varphi]-[\Omega, \mathcal{L}_{e_\rho}\varphi]=d[e_\rho, \varphi]-[e_\rho, \Omega], \varphi]-\Omega, [e_\rho, \varphi]]=[e_\rho, d\varphi]-[e_\rho, [\Omega, \varphi]]=[e_\rho, D\varphi]$.

Q.E.D.

We now phrase the above in a manner suggestive of the general case. Let $X$ be a manifold on which we have an integrable $\Gamma$-structure, where $\Gamma$ is a transitive continuous pseudo-group of finite type acting on $M$. Then, for some $\mu'$, $Q_{\mu'}=Q_{\mu'+1}=\cdots=Q$ and $G^{\mu'}=G^{\mu'+1}=\cdots=G$, and we denote by $\sum^q$ the sheaf over $X$ of germs of $q$-forms with values in the vector bundle $\boldsymbol{T}(Q)/G$ $(=Q\times_G\mathfrak{g}_*)$. Then there is an algebraic bracket $[\ \ ,\ \ ]:\sum^p\otimes\sum^q\to\sum^{p+q}$ which makes $\sum^*=\bigoplus\limits_{q\geqq 0}\sum^q$ a sheaf of graded Lie algebras. The operator $D$ defined on $\sum^q$ by $D\varphi=d\varphi-[\Phi, \varphi]$ ($\Phi=$ Maurer-Cartan form on $Q$) maps $\sum^q$ into $\sum^{q+1}$, and satisfies $D^2=0$ together with the derivation law in $\sum^*$.

The sheaf $\Theta$ of infinitessimal transformations preserving the integrable $\Gamma$-structure on $X$ may be naturally lifted or prolonged to a subsheaf of $\sum^0$, and we have the

**Theorem.** *The sequence*

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* $\mathfrak{a}$=left-invariant vector-fields on $A$.

(7.4) $$0 \to \Theta \to \sum^0 \xrightarrow{D} \sum^1 \xrightarrow{D} \cdots \to \sum^q \xrightarrow{D} \cdots$$

*is exact. Furthermore, if* $C^q = H^0(X, \sum^q)$, *the integrable* $\Gamma$*-structures close to* $\Phi$ *are given by the small* $\varphi \in C^1$ *which satisfy the differential equation*

(7.5) $$D\varphi - \frac{1}{2}[\varphi, \varphi] = 0\,.$$

**8. The general case, a first resolution.** Let $\Gamma$ be a transitive continuous pseudo-group of transformations on $M$, and let $X$ be a manifold with an integrable $\Gamma$-structure. We denote the $\mu^{\text{th}}$ prolonged bundle over $X$ by $Q_\mu$ so that we have a principal fibering $G^\mu \to Q_\mu \to X$. Consider the bundle $\boldsymbol{T}(Q_\mu)/G^\mu$ and let $\sum^{\mu,q}$ be the sheaf on $X$ of germs of $q$-forms with values in this bundle.

We wish to define on $\sum^{\mu,*} = \bigoplus_{q\geqq 0} \sum^{\mu,q}$ a bracket product. In order to do this, we first observe that the Lie algebra structure on $\mathfrak{g}_*$ induces a well-defined bracket

(8.1) $$[\ \ ,\ \ ]\colon \mathfrak{g}^{\mu+\nu} \otimes \mathfrak{g}_*^\mu \to \mathfrak{g}_*^{\mu+\nu-1}\,,$$

and consequently, for $\nu \geqq 1$ (which will henceforth be assumed), a bracket

(8.2) $$[\ \ ,\ \ ]\colon \mathfrak{g}^{\mu+\nu} \otimes \mathfrak{g}_*^\mu \to \mathfrak{g}_*^\mu\,.$$

Thus the adjoint representation of $G^{\mu+\nu}$ on $\mathfrak{g}_*^\mu$ is defined, and it is easy to see that (c.f. (7.1))

(8.3) $$\boldsymbol{T}(Q_\mu)/G^\mu \cong Q_{\mu+\nu} \times_{G^{\mu+\nu}} \mathfrak{g}_*^\mu\,.$$

From (1.2) and (8.3), there is then naturally defined

(8.4) $$[\ \ ,\ \ ]\colon \sum^{\mu,p} \otimes \sum^{\mu,q} \to \sum^{\mu-1,p+q}$$

which satisfies (6.1) and (6.2).

The sections of $\sum^{\mu,q}$ are given by the $\mathfrak{g}_*^\mu$-valued $q$-forms $\varphi$ on $Q_{\mu+\nu}$ which satisfy

(8.5) $$i(e)\varphi = 0\,, \quad \text{and}$$

(8.6) $$\mathcal{L}_e\varphi = [e, \varphi]$$

where $e \in \mathfrak{g}^{\mu+\nu}$ acts on $Q_{\mu+\nu}$ on the right. Now the $\mathfrak{g}_*^\mu$-valued Maurer-Cartan form $\Phi^\mu$ is defined on $Q_{\mu+\nu}$, and so, for a germ $\varphi$ in $\sum^{\mu,q}$, the expression

(8.7) $$D\varphi = d\varphi^{\mu-1} - [\Phi^\mu, \varphi]^{\mu-1}$$

makes sense and is a $\mathfrak{g}_*^{\mu-1}$-valued $(q+1)$-form on $Q_{\mu+\nu}$. From the Lemma in §7 and (8.5), (8.6), we see that $D\varphi$ is a section of $\sum^{\mu-1,q+1}$. Thus we have defined

$$D: \textstyle\sum^{\mu,q} \to \sum^{\mu-1,q+1}, \tag{8.8}$$

where $D^2=0$ and $D$ is compatible with the bracket (8.4).

We now investigate the kernel of $D$ on $\sum^{\mu,0}$ for $\mu$ sufficiently large. A germ $\theta$ in $\sum^{\mu,0}$ is a vector field on $Q_\mu$, and we assert that

$$D\theta = \mathcal{L}_\theta \Phi^{\mu-1}, \tag{8.9}$$

where $\Phi^{\mu-1}$ is the Maurer-Cartan form on $Q_\mu$. Symbolically, this goes as follows: $\mathcal{L}_\theta\Phi^{\mu-1}=di(\theta)\Phi^{\mu-1}+i(\theta)d\Phi^{\mu-1}=d(\theta^{\mu-1})+i(\theta)\left(\frac{1}{2}[\Phi^\mu, \Phi^\mu]^{\mu-1}\right)= d\theta^{\mu-1}-[\Phi^\mu, \theta]^{\mu-1}=D\theta$. This formal computation is easily justified.

Now the sheaf $\Theta$ on $X$ of germs of infinitesimal transformations preserving the integrable $\Gamma$-structure induces naturally a sheaf $\Theta^\mu\subset\sum^{\mu,0}$ on $Q_\mu$, and from (8.4) we see that $D(\Theta^\mu)=0$. Conversely, let $\mu\geqq\mu_1$ (§4) and suppose that $\theta\in\sum^{\mu,0}$ is a germ for which $D\theta=0$. Then, by (8.9), $\theta$ preserves the fibering $G^\eta\to Q_\mu\to X$ and induces an infinitesimal transformation $\theta_0$ on $X$ whose local one-parameter group preserves all of the bundles $G^\eta\to Q_\eta\to X$ $(\eta\leqq\mu)$. Thus $\theta_0$ preserves the integrable $\Gamma$-structure (since these bundles determine it).*

In conclusion then, for $\mu$ sufficiently large, we have over $X$ a sheaf sequence

$$0\to\Theta\to\textstyle\sum^{\mu+1,0}\xrightarrow{D}\sum^{\mu,1}\xrightarrow{D}\cdots\to\sum^{\mu-q+1,q}\xrightarrow{D}, \tag{8.10}$$

which is exact at the second stage, and which is such that the small $\varphi\in C^{\mu,1}=H^0(X,\sum^{\mu,1})$ which satisfy

$$D\varphi^{\mu-1}-\frac{1}{2}[\varphi, \varphi]^{\mu-1}=0 \tag{8.11}$$

give the integrable $\Gamma$-structures close to $\Phi^\mu$.

**9. Deformations of structure; the general case.** For the purpose of existence theory, the sequence (8.10) has the defect that the $\mu$ index

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* It is perhaps worthwhile to illustrate this process in local coordinates. Let $(\omega^1,\cdots,\omega^n;\ \pi^1,\cdots,\pi^r)$ be as in §2 and let $\theta=(\theta,\cdots,\theta^n)$ be a local vector field on $X$. Then $\mathcal{L}_\theta\omega^i=\sum a_j{}^i\omega^j$ $(=d\theta^i-\sum c^i_{jk}\omega^j\theta^k)$ for some matrix function $a_j{}^i(x)$. If $a_j{}^i\in\mathfrak{g}_1$, then $\theta$ induces a vector field $\theta^{(1)}$ (=first prolongation of $\theta$) on $P^1$. In fact, $a_j{}^i(x)=\sum a_\rho(x)\,a_{\rho j}{}^i$ and $\theta^{(1)}$ has components $(\theta^1\cdots,\theta^n;\ a_1,\cdots,a_r)$; observe that $\theta^{(1)}\in\sum^{1,0}$. Furthermore, $D\theta^{(1)}$ has components $d\theta^i-\sum c^i_{jk}\omega^j\theta^k-\sum a_\rho a_{\rho j}{}^i\omega^j=0$. Conversely, given $\theta^{(1)}$ on $P_1$ with $D\theta^{(1)}=0$, we see that $\theta^{(1)}$ is the prolongation of a vector field $\theta$ on $X$ which preserves $G_1\to P_1\to X$.

is dropped by one each time (the top term in $\varphi$ goes undifferentiated in $D$). One remedy for this is to pass to the projective limit: Set $Q=\lim_{\mu\to\infty} Q_\mu$ and $\sum^q=\lim_{\mu\to\infty}\sum^{\mu,q}$. Then $D:\sum^q\to\sum^{q+1}$ is a well-defined operator and $D^2=0$. Furthermore, just as in (6.12), the $D$-Poincaré lemma is the $d$-Poincaré lemma on $Q$, and is hence *formally* true. If $X$ and $Q$ are real-analytic, and if $\sum^*=\bigoplus_{q\geqq 0}\sum^q$ consists of real analycic germs only, then the sequence

$$0\to\Theta\to\textstyle\sum^0\xrightarrow{D}\sum^1\xrightarrow{D}\cdots\to\sum^q\xrightarrow{D}\sum^{q+1}\xrightarrow{D}\cdots \tag{9.1}$$

is exact, and, since $H^q(X,\sum^*)=0$ $(q>0)$ (by Theorems $A$ and $B$ and the general Künneth relation), the groups $H^q(X,\Theta)$ are represented by the global $D$-cohomology groups.

However, even in this real-analytic case, the available techniques fail to prove the fundamental theorem: If $X$ is compact and has an integrable $\Gamma$-structures and if $H^2(X,\Theta)=0$, then the nearby integrable $\Gamma$-structures are effectively parametrized by a neighborhood of the origin in $H^1(X,\Theta)$ (indeed, it is not clear that $\dim H^1(X,\Theta)<\infty$). Some of these difficulties may be circumvented by the following device taken from [7]: Define $\mathcal{S}^{\mu,q}$ to be the set of all pairs $\psi=(\varphi,\xi)$ for $\varphi$ a germ in $\sum^{\mu,q}$ and $\xi=[\Omega^{\mu+1},\varphi^{\mu+1}]^\mu$ where $\varphi^{\mu+1}$ is a germ in $\sum^{\mu+1,q}$ which, under the projection $\sum^{\mu+1,q}\to\sum^{\mu,q}$, goes onto $\varphi$. Thus we have an exact sequence

$$0\to\Lambda^{\mu+1,q}\to\textstyle\sum^{\mu+1,q}\xrightarrow{\pi}\mathcal{S}^{\mu,q}\to 0 \tag{9.2}$$

defined by $\pi(\varphi)=(\varphi^\mu,[\Omega^{\mu+1},\varphi]^\mu)$, and $\Lambda^{\mu+1,q}=\{\varphi\in\sum^{\mu+1,q}\,|\,\varphi^\mu=0$ and $[\omega,\varphi]=0\}$.

With (8.7) in mind, we define $\boldsymbol{D}$ on $\mathcal{S}^{\mu,q}$ by

$$\boldsymbol{D}(\varphi,\xi)=(d\varphi-\xi,\,-d\xi)\,. \tag{9.3}$$

Then it is easy to verify that $\boldsymbol{D}(\mathcal{S}^{\mu,q})\subseteqq\mathcal{S}^{\mu,q+1}$ and $\boldsymbol{D}^2=0$. Also, with (8.11) as motivation, we define a bracket $[\ ,\ ]:\mathcal{S}^{\mu,p}\otimes\mathcal{S}^{\mu,q}\to\mathcal{S}^{\mu,p+q}$ by

$$[\psi,\tilde\psi]=([\varphi^{\mu+1},\tilde\varphi^{\mu+1}]^\mu,\,2[\xi^{\mu+1},\tilde\varphi^{\mu+1}]^\mu)\qquad(\psi=(\varphi,\xi),\ \tilde\psi=(\tilde\varphi,\tilde\xi))\,. \tag{9.4}$$

Let now $\mu$ be sufficiently large and set $\boldsymbol{A}^q=H^0(X,\mathcal{S}^{\mu,q})$. If $\psi=(\varphi,\xi)$ is a small element in $\boldsymbol{A}^1$, and if $\boldsymbol{D}\psi-\frac{1}{2}[\psi,\psi]=0$, then from (9.3) and (9.4) it follows that

$$D\varphi^\mu-\frac{1}{2}[\Omega,\varphi^{\mu+1}]^\mu=0\,, \tag{9.5}$$

which implies that $\Omega^{\mu}+\varphi^{\mu}$ gives an integrable $\Gamma$-structure close to the structure given by $\Omega^{\mu}$.

**Theorem.** *Let $X$ be a compact manifold on which we have an integrable $\Gamma$-structure, and let $\Theta$ be the sheaf of germs of infinitessimal $\Gamma$-transformations. Then there exists on $X$ sheaves of graded Lie algebras $\mathcal{S}^q (0 \leqq q \leqq n)$, differential operators $\boldsymbol{D}: \mathcal{S}^q \to \mathcal{S}^{q+1}$, and an injection (prolongation) $\Theta \to \mathcal{S}^0$ such that:*

(i) *In the sequence*

$$0 \to \Theta \to \mathcal{S}^0 \xrightarrow{\boldsymbol{D}} \mathcal{S}^1 \xrightarrow{\boldsymbol{D}} \cdots \to \mathcal{S}^{n-1} \xrightarrow{\boldsymbol{D}} \mathcal{S}^n \to 0, \tag{9.6}$$

$$\boldsymbol{D}^2 = 0 \quad \text{and} \quad 0 \to \Theta \to \mathcal{S}^0 \xrightarrow{\boldsymbol{D}} \mathcal{S}^1 \quad \text{is exact};$$

(ii) *$\boldsymbol{D}$ is compatible with the graded Lie algebra structures, and the nearly integrable $\Gamma$-structures are given by the small sections $\psi \in \boldsymbol{A}^1 = H^0(X, \mathcal{S}^1)$ which satisfiy*

$$\boldsymbol{D}\psi - \frac{1}{2}[\psi, \psi] = 0. \tag{9.7}$$

**10. Deformations of transitive $\Gamma$-structures and of $\Gamma$-structures.** In §9 we have established the mechanism for deforming an *integrable* $\Gamma$-structure on a manifold $X$. If the original structure is either transitive or a $\Gamma$-structure (i.e. "local coordinates exist"—c.f. §3),* it may be asked whether or not the same holds for the new structure. This is a local problem and we now derive the relavant systems of partial differential equations.

Let $U \Subset \boldsymbol{R}^n$ be a domain and consider over $U$ a family of bundles $Q_{\mu+1}(t) \to U$ coming from almost $\Gamma$-structures. Let $\Phi(t)$ be the $\mathfrak{g}_*^{\mu}$-valued Maurer-Cartan form on $Q_{\mu+1}(t)$, and assume the integrability equations

$$d\Phi(t)^{\mu-1} - \frac{1}{2}[\Phi(t), \Phi(t)]^{\mu-1} = 0. \tag{10.1}$$

Let $\sigma(t): U \to Q_{\mu+1}(t)$ be local cross-sections which define trivializations $Q_{\mu+1}(t) \cong U \times G^{\mu+1}$. If we set $\omega(t) = \sigma(t)^* \Phi(t)$, then the Maurer-Cartan form $\Phi(t)$ is given on $U \times G^{\mu+1}$ by $\omega(t) +$ (Maurer-Cartan form on $G^{\mu+1}$). The bundle $\boldsymbol{T}(Q_{\mu}(t))/G^{\mu} \cong \mathfrak{g}_*^{\mu} \times U$ and so the sections of $\sum^{\mu,0}$ are given by the $\mathfrak{g}_*^{\mu}$-valued functions on $U$.

Now let $\omega = \omega(0)$ and assume that this structure is transitive. Thus there exist "sufficiently many" local solutions of

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* It may well be that this is always the case.

(10.2) $$D\theta = d\theta^{\mu-1} - [\omega, \theta]^{\mu-1} \qquad \text{(c.f. §8)},$$

for local $\mathfrak{g}^{\mu}_{*}$-valued functions $\theta$. The condition that the peturbed structures be transitive is simply that there exist local solutions of

(10.3) $$D(t)\theta(t) = d\theta(t)^{\mu-1} - [\omega(t), \theta(t)]^{\mu-1} = 0.$$

We now give a system of partial differential equations for the existence of bundle diffeomorphisms $f(t)\colon Q_{\mu}(t) \to Q_{\mu}$ $(=Q_{\mu}(0))$ such that $f(t)^{*}\Phi = \Phi(t)$. Before doing this we recall that, associated to a vector field $\xi$ on $V \subseteq \boldsymbol{R}^{N}$, there exists a local 1-parameter group $h(s) = \exp(s, \xi)$ acting on $W \subseteq V$. Similiarly, given a section $\theta$ of $\sum^{\mu,0}$ $(=\mathfrak{g}^{\mu}_{*}$-valued function on $U)$, there is a corresponding local 1-parameter group $h(s) = \exp(s, \theta)$ of bundle diffeomorphisms of $U \times Q_{\mu}$.* We now seek a 1-parameter family $\theta(t)$ of sections $\sum^{\mu,0}$ with $\theta(0)=0$ and such that

(10.4) $$g(t)^{*}\Phi = \Phi(t)$$

where $g(t) = \exp(t, \theta(t))$.

**Theorem.** *If we set* $\varphi(t) = \dfrac{\partial\omega(t)}{\partial t}$, *then* (10.4) *holds if, and only if,*

(10.5) $$D(t)\theta(t) = d\theta(t)^{\mu-1} - [\omega(t), \theta(t)]^{\mu-1} = \varphi(t)^{\mu-1}.$$

Proof. By (8.9), $D(t)\theta(t) = \mathcal{L}_{\theta(t)}\omega(t)^{\mu-1}$, and we shall prove the Theorem in the following form: Let $W \subseteq \boldsymbol{R}^{N}$ be a domain, $\eta(t)$ a family of 1-forms on $W$, and $\xi(t)$ a family of vector fields with $\xi(0)=0$. Then $f(t)^{*}\eta = \eta(t)$ if, and only if, $\mathcal{L}_{\xi(t)}\eta(t) = \dfrac{\partial\eta(t)}{\partial t}$ where $f(t) = \exp(t, \xi(t))$.

Assume $f(t)^{*}\eta = \eta(t)$, and write $\eta(t) = \sum_{j=1}^{N} \eta_{j}(t)dx^{j}$, $f(t) = (f^{1}(t), \cdots, f^{N}(t))$, $\xi(t) = (\xi^{1}(t), \cdots, \xi^{N}(t))$. It is easy to see that $f(t)$ is defined by

(10.6) $$\frac{\partial f^{i}(t)}{\partial t} = \sum \frac{\partial f^{i}(t)}{\partial x^{j}}\xi^{j}(t).$$

By assumption, $\sum \eta_{i}(f(t))\dfrac{\partial f^{i}(t)}{\partial x^{j}} = \eta_{j}(t)$ where $\eta_{i}(f(t)) = \eta_{i}(t)\circ f(t)$. Using (10.6), we get $\dfrac{\partial}{\partial t}(\sum \eta_{i}(f(t))\dfrac{\partial f^{i}(t)}{\partial x^{t}} = \sum \eta_{l}(t)\dfrac{\partial \xi^{l}(t)}{\partial x^{j}} + \sum \eta_{i}(f(t))\dfrac{\partial^{2} f^{i}}{\partial x^{j}\partial x^{l}}\xi^{l}(t) + \sum \partial\eta_{i}\dfrac{(f(t))}{\partial x^{k}}\dfrac{\partial f^{k}(t)}{\partial x^{l}}\dfrac{\partial f^{i}(t)}{\partial x^{j}}\xi^{l}(t)$. On the other hand, $\mathcal{L}_{\xi(t)}\eta(t) = \sum(\mathcal{L}_{\xi(t)}\eta_{j}(t)\partial x^{j} + \sum \eta_{j}(t)\mathcal{L}_{\xi(t)}dx^{j} = \sum \xi^{l}(t)\dfrac{\partial\eta_{j}(t)}{\partial x^{l}}dx^{j} + \sum \eta_{j}(t)\dfrac{\partial \xi^{j}(t)}{\partial x^{l}}dx^{l}$. Substituting $\dfrac{\partial \eta_{j}(t)}{\partial x^{l}} =$

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* Indeed, $h(s) = (\exp(s, \theta^{0}), \exp(s\theta'))$ where $\theta'$ is the projection of $\theta$ into $\mathfrak{g}^{\mu}$ and $\exp(s\theta')$ is the exponential mapping in $G^{\mu}$.

$\sum \partial\eta_i \frac{(f(t))}{\partial x^k} \frac{\partial f^i(t)}{\partial x^l} \frac{\partial f^i(t)}{\partial x^j} + \sum \eta_i(f(t)) \frac{\partial^2 f^i(t)}{\partial x^j \partial x^l}$ in the above expression for $\frac{\partial}{\partial t}\left(\sum \eta_i(f(t)) \frac{\partial f^i(t)}{\partial x^j}\right) = \frac{\partial \eta_j(t)}{\partial t}$, we get $\mathcal{L}_{\xi(t)}\eta(t) = \frac{\partial \eta(t)}{dt}$. Since this argument is reversible (using the fact that $f(t)_* \xi(t) = \xi(t)$), the Theorem follows.

Q.E.D.

The equations (10.3) and (10.5) present the same analytical difficulty as the sequence (8.10). Again then, we define over $U$ a family of sheaves $\mathcal{S}_t^q$ where a germ in $\mathcal{S}_t^q$ is represented by a pair $(\varphi, \xi)$, $\varphi$ is a germ in $\sum^{\mu,q}$ ($=\mathfrak{g}_*^{\mu}$-valued $q$-forms) and $\xi = [\omega(t), \varphi^{\mu+1}]^{\mu}$ for some germ $\varphi^{\mu+1}$ in $\sum^{\mu+1,q}$ which projects onto $\varphi$. We define $\boldsymbol{D}(t): \mathcal{S}_t^q \to \mathcal{S}_t^{q+1}$ by

$$\boldsymbol{D}(t)(\varphi, \xi) = (d\varphi - \xi, -d\xi). \tag{10.7}$$

Then, by (10.1), $\boldsymbol{D}(t)^2 = 0$ and we get a sequence

$$0 \to \Theta_t \to \mathcal{S}_t^0 \xrightarrow{\boldsymbol{D}(t)} \mathcal{S}_t^1 \to \cdots \to \mathcal{S}_t^q \xrightarrow{\boldsymbol{D}(t)} \mathcal{S}_t^{q+1} \to \cdots \tag{10.8}$$

which may be viewed as a peturbation of (9.6). For example, from (10.3), we see that $\Theta_t$ is just the sheaf of germs of infinitessinal automorshisms preserving $\omega(t)$.

If now $\eta(t)$ is the germ $\left(\frac{\partial \omega(t)^{\mu-1}}{\partial t}, \left[\omega(t), \frac{\partial \omega(t)}{\partial t}\right]^{\mu-1}\right)$ in $\mathcal{S}_t^1$, we get from $d\omega(t)^{\mu-1} - \frac{1}{2}[\omega(t), \omega(t)]^{\mu-1} = 0$ that $\boldsymbol{D}(t)\eta(t) = 0$. If we can find $\xi(t)$ in $\mathcal{S}_t^0$ such that $\boldsymbol{D}(t)\xi(t) = \eta(t)$, then $\xi(t) = (\theta(t)^{\mu-1}, [\omega(t), \theta(t)]^{\mu-1})$ and we have that $\boldsymbol{D}(t)\theta(t) = \frac{\partial \omega(t)^{\mu-1}}{\partial t}$, i.e. (10.5). Thus, in order to insure that our family of integrable $\Gamma$-structures is in fact a family of $\Gamma$-structures, it is sufficient to have the $\boldsymbol{D}(t)$-Poincaré lemma in (10.8).

REMARK. For some $\Gamma$, it is possible to see directly that a transitive $\Gamma$-structure is a $\Gamma$-structure. For eample, suppose that we have a transitive almost-complex structure, and let $\theta^1, \cdots, \theta^n$ be local vector fields, of type (1, 0), which are a frame and are also infinitessimal automorphisms. Then, if $\theta$ is any other $\Gamma$ vector field, $\theta = \sum f_j \theta^j$ for some complex functions $f_j$. Now, for any function $g$, $\sum f_j \bar{\partial} L_{\theta j} g = \sum f_j L_{\theta j} \bar{\partial} g = L_\theta \bar{\partial} g = \bar{\partial} L_\theta g = \sum \bar{\partial} f_j \mathcal{L}_{\theta j} g + \sum f_j \bar{\partial} \mathcal{L}_{\theta j} g$ and so $\sum \bar{\partial} f_j \mathcal{L}_{\theta j} g = 0$ for all $g$, i.e. $\bar{\partial} f_j = 0$ ($j = 1, \cdots, n$). Thus we have local holomorphic functions, which is in this case the same as a $\Gamma$-structure.

**11. Elliptic pseudo-groups.** Suppose that we have on $X$ an integrable $\Gamma$-structure. The sheaf $\Theta$ of infinitessimal $\Gamma$-transformations is a sub-sheaf of $\mathcal{J}$, the germs of all $C^\infty$ vector fields, which is defined by a system of linear partial differential equations. We say that $\Gamma$ is *elliptic* if this system is elliptic.

To be more precise, for a vector bundle $\boldsymbol{E}\rightarrow X$ denote by $J^{\mu}(\boldsymbol{E})$ the bundle of jets up to order $\mu$ of $C^{\infty}$ sections of $\boldsymbol{E}$, and let $\mathcal{J}^{\mu}(\boldsymbol{E})$ be the sheaf of sections of $J^{\mu}(\boldsymbol{E})$. There is a natural prolongation mapping $i^{\mu}: \mathcal{T}\rightarrow\mathcal{J}^{\mu}(\boldsymbol{T})$, and there is a sub-bundle $\boldsymbol{Q}\subset J^{\mu}(\boldsymbol{T})$ $(\mu\geqq\mu_0)$ such that a germ $\theta$ in $\boldsymbol{T}$ is in $\Theta$ if, and only if, $i^{\mu}(\theta)$ is a section of $\boldsymbol{Q}$.* We set $\boldsymbol{R}=J^{\mu}(\boldsymbol{T})/\boldsymbol{Q}$; there is a natural linear mapping $A: J^{\mu}(\boldsymbol{T})\rightarrow\boldsymbol{R}$ which may be considered as a system of linear partial differential equations defining $\Theta\subset\mathcal{T}$.

For each covector $\eta\in\boldsymbol{T}_x^*$, there is defined the *symbol* $\sigma_A(\eta)=\sigma(\eta)$ of the system; $\sigma(\eta)\in\mathrm{Hom}(\boldsymbol{T}_x, \boldsymbol{R}_x)$, and we say that $A\in\mathrm{Hom}(J^{\mu}(\boldsymbol{T}), \boldsymbol{R})$ defines an *elliptic system* if $\sigma(\eta)$ is injective for each $\eta\neq 0$. We shall explicity write out $\sigma(\eta)$ below.

In §9 we have defined the operators $\boldsymbol{D}: \mathcal{S}^q\rightarrow\mathcal{S}^{q+1}$; by introducing metrics, we may define the adjoint operators $\boldsymbol{D}^*: \mathcal{S}^q\rightarrow\mathcal{S}^{q-1}$ $(q\geqq 1)$ and, as usual, the Laplacians

$$\boldsymbol{D}\boldsymbol{D}^*+\boldsymbol{D}^*\boldsymbol{D}=\boldsymbol{L}: \mathcal{S}^q\rightarrow\mathcal{S}^q \qquad (q\geqq 1)\,. \tag{11.1}$$

(For $q=0$, $\boldsymbol{L}=\boldsymbol{D}^*\boldsymbol{D}$).

**Theorem.** *$\Gamma$ is elliptic if, and only if, the Laplacians* (11.1) *are elliptic.***

Proof. The main trouble is notational, and we shall show that $\Gamma$ elliptic implies that $\boldsymbol{L}$ is elliptic for $q=0,1$. This is sufficient for applications to (10.3), and the reader may easily supply the general argument. We assume that $\mu$ is such that $H^{\nu,*}(\mathfrak{g}_*)=0$ for $\nu\geqq\mu-1$, and we set $\Xi^{\mu,q}=\{\varphi\in\sum^{\mu,q}: \varphi^{\mu-1}=0\}$ so that the sequence $0\rightarrow\Lambda^{\mu,q}\rightarrow\Xi^{\mu,q}\xrightarrow{\delta}\Lambda^{\mu-1,q+1}\rightarrow 0$ is exact. If we define $\varpi: \mathcal{S}^{\mu,q}\rightarrow\sum^{\mu,q}$ by $\varpi(\varphi, \xi)=\varphi$, then we have

$$0\rightarrow\Lambda^{\mu,q+1}\rightarrow\mathcal{S}^{\mu,q}\xrightarrow{\varpi}\textstyle\sum^{\mu,q}\rightarrow 0\,. \tag{11.3}$$

For a covector $\eta$, the symbol mapping of $\boldsymbol{D}$,

$$\sigma_D(\eta): \mathcal{S}^{\mu,q}\rightarrow\mathcal{S}^{\mu,q+1} \tag{11.4}$$

is defined and $\sigma_D(\eta)^2=0$. Also defined are $\sigma_{D^*}(\eta): \mathcal{S}^{\mu,q}\rightarrow\mathcal{S}^{\mu,q-1}$ and $\sigma_L(\eta)$: $\mathcal{S}^{\mu,q}\rightarrow S^{\mu,q}$. Since $\sigma_L(\eta)=\sigma_D(\eta)\sigma_{D^*}(\eta)+\sigma_{D^*}(\eta)\sigma_D(\eta)$, it follows that $\boldsymbol{L}$ is elliptic if, and only if, the symbol sequence

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* In fact, $\boldsymbol{Q}$ corresponds to the subsheaf $\Sigma^{\mu,0}$ of $\mathcal{J}^{\mu}(\boldsymbol{T})$.

** Quillen has a more general result in his Harvard thesis.

$$\mathcal{S}^{q-1} \xrightarrow{\sigma_D(\eta)} \mathcal{S}^q \xrightarrow{\sigma_D(\eta)} \mathcal{S}^{q+1} \tag{11.5}$$

is exact. Furthermore, we easily see that $\sigma_D(\eta)(\Lambda^{\mu,q+1}) \subset \Lambda^{\mu,q+2}$ in (11.3) and that (11.5) is exact if, and only if

$$\Lambda^{\mu,q} \xrightarrow{\sigma_D(\eta)} \Lambda^{\mu,q+1} \xrightarrow{\sigma_D(\eta)} \Lambda^{\mu,q+2} \tag{11.6}$$

is exact where, for $\lambda \in \Lambda^{\mu,*}$, $\sigma_D(\eta)\lambda = \eta \wedge \lambda$. (Remark: If we denote the algebraic cohomology groups arising from (11.6) by $H_\eta^{\mu,q+1}(\Lambda)$, then from (11.2) and the fact that $H_\eta^{\mu,q}(\Xi)=0$ always, we see that $H_\eta^{\mu,q}(\Lambda) \cong H_\eta^{\mu+1,q}(\Lambda) \cong \cdots$. This is an algebraic proof of the statement that a prolongation of an elliptic equation is elliptic.)

We now introduce local coordinates $(x^1, \cdots, x^n)$ so that $\eta = dx^1$. As is customary, we let $\boldsymbol{p}=(p_1, \cdots, p_n)$ represent a vector of non-negative integers; $|\boldsymbol{p}| = p_1 + \cdots + p_n$, $x^{\boldsymbol{p}} = (x^1)^{p_1} \cdots (x^n)^{p_n}$, etc. Also we set $\boldsymbol{p}+1_k = (p_1, \cdots, p_k+1, \cdots, p_n)$. We let $e_1, \cdots, e_N$ be a local basis for $\boldsymbol{R}$, and we agree on the ranges of indices $1 \leqq i, j, k, \cdots, \leqq n$; $1 \leqq r, s, t \leqq N$. A local section $\varphi$ of $\mathcal{J}^\mu(\boldsymbol{T})$ is written as $\varphi = \sum_{|\boldsymbol{p}| \leqq \mu} \varphi_{\boldsymbol{p}}^j \frac{\partial}{\partial x^j} \otimes x^{\boldsymbol{p}}$, and $A \in \mathrm{Hom}(J^\mu(\boldsymbol{T}), \boldsymbol{R})$ is locally written as

$$(A(\varphi))^s = \sum_{\boldsymbol{p},j} a_{\boldsymbol{p}j}^s \varphi_{\boldsymbol{p}}^j . \tag{11.7}$$

For a covector $\xi = (\xi^1, \cdots, \xi^n)$ and $\psi = (\psi^j) \in \boldsymbol{T}$, the symbol mapping is given by

$$(\sigma(\xi)\psi)^s = \sum_{|\boldsymbol{p}|=\mu} a_{\boldsymbol{p}j}^s \xi^{\boldsymbol{p}} \psi^j \tag{11.8}$$

We now prove that $0 \to \Lambda^{\mu,1} \xrightarrow{\sigma_D(\eta)} \Lambda^{\mu,2} \xrightarrow{\sigma_D(\eta)} \Lambda^{\mu,3}$ is exact. If $\lambda \in \Lambda^{\mu,1}$, $\lambda = \delta\gamma$ for some $\gamma \in \Xi^{\mu+1,0}$. Since $\Xi^{r,0} \subset \sum^{r,0} \subset \mathcal{J}^r(\boldsymbol{T})$, we write $\gamma = \{\gamma_{\boldsymbol{q}}^i ; |\boldsymbol{q}| = \mu+1\}$. Then, with a suitable choice of coordinates, $(\delta\gamma)_{\boldsymbol{p}} = \sum_k \gamma_{\boldsymbol{p}+1_k} dx^k$.* Furthermore,

$$(\sigma_D(\eta)(\delta\gamma)_{\boldsymbol{p}} = dx^1 \wedge (\delta\gamma)_{\boldsymbol{p}} = \sum_{k>1} \gamma_{\boldsymbol{p}+1_k} dx^1 \wedge dx^k . \tag{11.9}$$

If $\sigma_D(\eta)\gamma = 0$, then from (11.9) we get $\gamma_{\boldsymbol{q}} = 0$ for $\boldsymbol{q} \neq (\mu+1, 0, \cdots, 0)$. On the other hand, $A(\gamma) = 0$ and we then get from (11.7) that

$$\sum a_j^s \gamma^j = 0 \quad (a_j^s = a_{(\mu+1,0,\cdots,0)j}^s ; \ \gamma^j = \gamma_{(\mu+1,0,\cdots,0)}^j) . \tag{11.10}$$

But (11.10) contradicts the injectivity of the symbol (11.8) for $\xi = (1, 0, \cdots, 0)$.

* This representation is only valid at a point.

Now let $\lambda \in \Lambda^{\mu,2}$ and assume that $dx^1 \wedge \lambda = 0$. We write $\lambda = \delta\varphi$ for $\varphi \in \Xi^{\mu+1,1}$, and then we may express $\varphi$ as $\varphi = \sum_{k=1}^{n} {}_k\varphi dx^k$ (${}_k\varphi \in \Xi^{\mu+1,0}$). Clearly this is possible. We write explicitly ${}_k\varphi = \{{}_k\varphi_q\,;\, |q| = \mu+1\}$ so that

$$(\delta\varphi)_p = \sum_{k>l} ({}_k\varphi_{p+1_l} - {}_l\varphi_{p+1_k}) dx^k \wedge dx^l\,. \tag{11.11}$$

From (11.11) and $dx^1 \wedge \delta\varphi = 0$, it follows that

$$ {}_k\varphi_{p+1_l} - {}_l\varphi_{p+1_k} = 0\,; \qquad 1 < k < l,\ |p| = \mu\,. \tag{11.12}$$

We wish to find $\tau = \{\tau_q\} \in \Xi^{\mu+1,0}$ such that $dx^1 \wedge \delta\tau = \delta\varphi$, which is explicitly written as

$$\tau_{p+1_k} = {}_l\varphi_{p+1_k} - {}_k\varphi_{p+1_l}\,. \tag{11.13}$$

Now it follows immediately from (11.12) and (11.13) that we may *consistently* define, for $q \neq (\mu+1, 0, \cdots, 0)$,

$$\tau_q = {}_l\varphi_{p+1_k} - {}_k\varphi_{p+1_l} \qquad (q = p+1_k)\,. \tag{11.14}$$

To complete the proof, it will suffice to define $\tau_{(\mu+1,0\cdots,0)}$ so that $A(\tau)=0$ ($\tau = \{\tau_q\}$). In the notation of (11.10), we wish to solve

$$\sum a^s_j \tau^j = - \sum_{q \neq (\mu+1,0,\cdots,0)} a^s_{qj} \tau^j_q \tag{11.15}$$

From (11.14) it follows that the right hand side of (11.15) satisfies $N-n$ independent linear relations which are also satisfied by vectors $a(\zeta)^s = \sum a^s_j \zeta^j$. Since the symbol $\sigma(\xi)$ ($\xi = (1, 0, \cdots, 0)$) is injective, we may then (uniquely) solve (11.15). Q.E.D.

EXAMPLE. The following example is due to Singer. Let $V$ be a (real) vector space and $G \subset GL(V)$ a linear Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V) = V \otimes V^*$. We let $X = V$ and $\Gamma = \Gamma_G$ be the pseudo-group determined by the flat $G$-structure on $V$; that is, a local diffeomorphism of $V$ is in $\Gamma_G$ if, and only if, the jacobian matrix $J(f)_x \in G$ for all $x$. (We make the identifications $T = V \times V$; $J^1(T) = V \times (V \otimes V^*)$; etc.)

**Proposition.** *$\Gamma_G$ is elliptic if, and only if, $\mathfrak{g}$ contains no linear transformatiohs of rank* 1.

Proof. A germ $\theta \in \mathcal{G}$ is in $\Theta$ if, and only if, $j^1(\theta)_x \in \mathfrak{g}$ for all $x$; thus $Q \subset J^1(T)$ is $V \times \mathfrak{g}$ and $A$ is essentially given as the projection $\tilde{\omega} : \mathfrak{gl}(V) \to \mathfrak{gl}(V)/\mathfrak{g}$. For $\xi \in V^*$ (=cotangent vector), the symbol of $A$, $\sigma(\xi) \in \mathrm{Hom}\,(V, \mathfrak{gl}(V)/\mathfrak{g})$, is given by $\sigma(\xi)\psi = \tilde{\omega}(\psi \otimes \xi)$. The symbol mapping fails to be injective if, and only if, $\psi \otimes \xi \in \mathfrak{g} \subset V \otimes V^*$ for some $\psi$, $\xi$, which in turn happens if, and only if, $\mathfrak{g}$ contains a linear transformation of rank 1. Q.E.D.

## 12. Remarks and examples

(i) *Remark on Elliptic Systems.* We consider the following problem: Let $X$ be a manifold, $\boldsymbol{E} \to X$ a vector bundle, $\boldsymbol{E}^q = \boldsymbol{E} \otimes \Lambda^q \boldsymbol{T}^*$, and $A^q(\boldsymbol{E})$ the sheaf of germs of $C^\infty$ sections of $\boldsymbol{E}$. Suppose that we are given a linear differential operator $D\colon A^q(\boldsymbol{E}) \to A^{q+1}(\boldsymbol{E})$ whose principal part is $d$. Thus, locally, $D\varphi = d\varphi - A \wedge \varphi$ where $A$ is a local section of $\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{E}^1)$. Suppose that, for each $q$, we are given a locally free subsheaf $S^q \subset A^q(\boldsymbol{E})$ such that $D(S^q) \subset S^{q+1}$ and $D^2\colon S^q \to S^{q+2}$ is zero. Finally, we assume that the symbol sequences

$$S^q \xrightarrow{\sigma_\eta(D)} S^{q+1} \xrightarrow{\sigma_\eta(D)} S^{q+2} \qquad (\eta \in \boldsymbol{T}^* - 0) \quad (q \geqq 0) \tag{12.1}$$

are exact. Question: Is the Poincaré lemma for

$$0 \to \Theta \to S^0 \xrightarrow{D} S^1 \xrightarrow{D} \cdots \to S^q \xrightarrow{D} S^{q+1} \to \cdots \tag{12.2}$$

true? This is a special case of the general problem of solving "regular" over-determined elliptic systems, but is one which has arisen several times in differential geometry.

EXAMPLE 1. We shall check that the $D$-Poincaré lemma for $\boldsymbol{D}\colon \mathcal{S}^q \to \mathcal{S}^{q+1}$ as given in the Theorem of §9 falls in the above class. (More generally, the $\boldsymbol{D}(t)$-Poincaré lemma for the operators in (10.8) will be of the form (12.2).) By (11.3) we have a diagram

$$\begin{tikzcd}
\Sigma^{\mu,q-1} \arrow[r, dashed, "\Delta"] \arrow[d, dashed, bend right] & \Sigma^{\mu,q} \arrow[r, dashed, "\Delta"] \arrow[d, dashed, bend right] & \Sigma^{\mu,q+1} \arrow[d, dashed, bend right] \\
\mathcal{S}^{\mu,q-1} \arrow[r, "\boldsymbol{D}"] \arrow[u] \arrow[d, dashed, bend right] & \mathcal{S}^{\mu,q} \arrow[r, "\boldsymbol{D}"] \arrow[u] \arrow[d, dashed, bend right] & \mathcal{S}^{\mu,q+1} \arrow[u] \arrow[d, dashed, bend right] \\
\Lambda^{\mu,q} \arrow[r, dashed, "d"] \arrow[u] & \Lambda^{\mu,q+1} \arrow[r, dashed, "d"] \arrow[u] & \Lambda^{\mu,q+2} \arrow[u]
\end{tikzcd} \tag{12.3}$$

without the broken arrows and the operators $d$ and $\Delta$. The broken arrows mean that locally, after trivializing all bundles, we split the vertical sequences. The operators $\Delta$ and $d$ (= exterior differentiation) are then induced by commutativity. Now the top row in (12.3) is exact because (locally) $\Sigma^{\mu,q}$ = germs of $\mathfrak{g}^\mu_*$-valued $q$-forms and $\Delta$ has the exterior derivative $d$ as its principal part.

As for the bottom row (whose exactness would obviously give the $\boldsymbol{D}$-Poincaré lemma), we have locally a diagram

$$\begin{array}{ccccc} \Xi^{\mu,q} & \xrightarrow{d} & \Xi^{\mu,q+1} & \xrightarrow{d} & \Xi^{\mu,q+2} \\ \uparrow & & \uparrow & & \uparrow \\ \Lambda^{\mu,q} & \xrightarrow{d} & \Lambda^{\mu,q+1} & \xrightarrow{d} & \Lambda^{\mu,q+2} \end{array} \tag{12.4}$$

where the $\Xi^{\mu,p}$ are just (locally) vector-valued $p$-forms. If $\Gamma$ is elliptic, then the symbol sequence of the bottom row in (12.4) is exact (c.f. §11) and we are precisely in the situation (12.1).

EXAMPLE 2. Suppose we have given in $\boldsymbol{E}$ a connexion $\omega$ with covariant differential $D: A^q(\boldsymbol{E}) \to A^{q+1}(\boldsymbol{E})$. For $\varphi \in A^q(\boldsymbol{E})$, $D^2\varphi = \Omega\varphi$ where $\Omega \in \boldsymbol{A}^2(\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{E}))$ is the curvature. If we let $S^q \subset A^q(\boldsymbol{E})$ be the germs $\varphi$ with $\Omega \circ \varphi = 0$, then $D(S^q) \subset S^{q+1}$ since, for $\varphi \in S^q$, $\Omega D\varphi = D(\Omega\varphi) - (D\Omega)\varphi = 0$ ($D\Omega = 0$ by the Bianchi identity). Then the sequence

$$0 \to \Theta \to S^0 \xrightarrow{D} S^1 \xrightarrow{D} S^2 \to \tag{12.5}$$

has $D^2 = 0$. If the connexion $\omega$ in $\boldsymbol{E}$ has the property that the sheaves $S^q$ are locally free and that the symbol sequences in $S^{q-1} \xrightarrow{D} S^q \xrightarrow{D} S^{q+1}$ are exact ($q \geqq 1$), then we say that $\omega$ is *regular*.

Unfortunately, the condition of regularity may be quite restrictive; for example, if $\boldsymbol{E} = \boldsymbol{T}$ and $\omega$ is a Riemannian connexion of constant curvature, then the symbol sequence of $S^0 \xrightarrow{D} S^1 \xrightarrow{D} S^2$ is *not* exact (although that of $S^{q-1} \xrightarrow{D} S^q \xrightarrow{D} S^{q+1}$ is exct for $q > 1$).

The reason that one would like the $D$-Poincaré lemma for (12.5) is to say what elements $\varphi$ in $\boldsymbol{A}^1(\boldsymbol{E})$ (=global sections of $A^1(\boldsymbol{E})$) are of the form $\varphi = D\xi$ for $\xi \in \boldsymbol{A}^0(\boldsymbol{E})$. To treat this question, we may introduce the sheaves $R^q$ by $R^q = A^q(\boldsymbol{E})$ ($q = 0, 1$) and $R^q = A^q(\boldsymbol{E})/\Omega \circ A^{q-2}(\boldsymbol{E})$ ($q \geqq 2$). Then $D$ induces $D: R^q \to R^{q+1}$ and we have

$$0 \to \Theta \to R^0 \xrightarrow{\boldsymbol{D}} R^1 \xrightarrow{\boldsymbol{D}} R^2 \to \tag{12.6}$$

with $\boldsymbol{D}^2 = 0$. Furthermore, it is easy to check that $D$-Poincaré lemma in (12.5) is equivalent to corresponding statement for $\boldsymbol{D}$ in (12.6). Then $\varphi$ above should be a global section of $R^1$ which satisfies $\boldsymbol{D}\varphi = 0$ and we are looking for a global section $\xi$ of $R^0$ with $\boldsymbol{D}\xi = \varphi$.

EXAMPLE 3. *Deformations of Structures of Finite Type.* Let $A, B$ be connected Lie groups with $B \subset A$ a closed subgroup. If $X = A/B$, then the action of $A$ induces a transitive continuous pseudo-group $\Gamma$ on $X$ which is elliptic. Furthermore, the analytical difficulties disappear and the $D$-Poincaré lemma in (7.4) holds. We shall give an interpretation of the groups $H^q(X, \Theta)$ as calculated from the resolution (7.4).

If $C^\infty(A)=C^\infty$ functions on $A$, then $\mathfrak{b}$ acts on $C^\infty(A)$ by right translation and there is an isomorphism

$$C^q \cong \{C^\infty(A)\otimes\mathfrak{a}\otimes\Lambda^q(\mathfrak{a}/\mathfrak{b})^*\}^{\mathfrak{b}} \tag{12.7}$$

Now the right hand side of (12.7) is the Lie algebra cochain group $C^q(\mathfrak{a}, \mathfrak{b}; C^\infty(A)\otimes\mathfrak{a})$ and the whole point is that, under the isomorphism (12.7), $D: C^q \to C^{q+1}$ goes into the boundary operator $\delta: C^q(\mathfrak{a}, \mathfrak{b}; C^\infty(A)\otimes\mathfrak{a}) \to (C^{q+1}(\mathfrak{a}, \mathfrak{b}; C^\infty(A)\otimes\mathfrak{a})$. Furthermore, $A$ acts on $C^q$, this action commutes with $D$, and under the isomorphism (12.7) this corresponds to $A$ acting on $C^\infty(A)$ by left-translation. With this understood, we get then an isomorphism of $A$-modules

$$H^q(X, \Theta) \cong H^q(\mathfrak{a}, \mathfrak{b}; C^\infty(A)\otimes\mathfrak{a}). \tag{12.8}$$

For instance, if $E$ is a finite dimensional $A$-module which is a direct summand of $C^\infty(A)$, we have the reciprocity formula

$$\mathrm{Hom}_A(E, H^q(X, \Theta)) \cong H^q(\mathfrak{a}, \mathfrak{b}; \mathrm{Hom}(E, \mathfrak{a})). \tag{12.9}$$

For example, taking $E=\mathfrak{a}$, we find that $\mathfrak{a}\otimes H^1(\mathfrak{a}, \mathfrak{b})$ is a direct summand of $H^1(X, \Theta)$ where $A$ acts by $Ad\otimes 1$. The deformations arising from this space have been constructed in [3], §10. Also, taking $E=$trivial $A$-module in (12.8), we find that $H^1(\mathfrak{a}, \mathfrak{b}; \mathfrak{a})$ is a direct summand of $H^1(X, \Theta)$ where $A$ acts trivially. This vector space gives rise to deformations of the $\Gamma$-structure constructed as "outer automorphisms of $\mathfrak{a}$ preserving $\mathfrak{b}$/inner automorphisms of $\mathfrak{a}$ preserving $\mathfrak{b}$".

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